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A New Optimal Control Formulation to Ensure the Stability of NMPC Systems ⋆

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Abstract: Nonlinear model predictive control (NMPC) has been considered as a promising control algorithm, since a rigorous model can be explicitly employed and state variable restrictions ensured by formulating inequality constraints. However, due to the nonlinear problem formulation, it is difficult to analyze the stability properties of NMPC systems. In this study we propose a new formulation of the optimal control problem to ensure the stability of NMPC systems. Auxiliary state variables and linear state equations will be introduced and their eigenvalues optimized in the dynamic optimization framework. System state variables will be constrained by these auxiliary variables so that they will conform to the stability properties of the auxiliary variables. Features of this stabilization approach are analyzed. Results of case studies indicate satisfactory effectiveness of the proposed NMPC.

Keywords: Optimal control; Nonlinear model predictive control; Stability.

1. INTRODUCTION

Model predictive control (MPC) or receding horizon control (RHC) is considered as one of the most important control algorithms. Recently, more and more attention have been paid to nonlinear MPC (NMPC), since a rigorous model can be explicitly employed and restrictions of state variables ensured by adding inequality constraints (Camacho and Bordons (1998); Diehl et al. (2002); Hong et al. (2006); Qin and Badgwell (2003)). For NMPC, a discretiztion scheme will be used and thus the optimal control problem is converted to a nonlinear programming (NLP) problem which will then be solved usually by the method of sequential quadratic programming (SQP). For more details the reader may refer to (Camacho and Bordons (1998); Qin and Badgwell (2003)). The stability analysis of NMPC systems has long been a challenge due to the nonlinear problem formulation. The most straightforward way to ensure stability is to use stability constraints (besides the input and state constraint) by introducing e.g. terminal equality constraints (Keerthi and Gilbert (1988); Michalska and Mayne (1993)) or terminal cost functions (Chmielewski and Manousiouthakis (1996); Mayne et al. (2000); Rawlings and Muske (1993)) into the finite horizon open-loop optimization problem. It can be shown that a stability for a NMPC system can not be guaranteed, if a finite prediction horizon is employed (Bequette (1991); Bitmead et al. (1990)). But if a terminal constraint is introduced, an asymptotic stability can be achieved (Genceli and Nikolaou (1993); Meadows et al. (1995)). Although using the terminal equality constraint to guarantee stability is an intuitive approach, it also increases significantly the on-line computation expense necessary to solve the open-loop optimization problem, and often causes feasibility problems (DeNicolao et al. (1998)).

Chen and Allgöwer (Chen and Allgöwer (1998)) considered infinite prediction horizon. Thus the nonlinear system can be approximated by a linear system at the stationary point. Rawlings and Muske (Rawlings and Muske (1993)) introduced a Lyapunov stable MPC. They considered infinite and quasi-infinite NMPC and assumed constant weights on inputs and outputs. Zheng (Zheng (1997)) showed that constrained NMPC can be stabilized by using time-varying weights. Santos and Biegler (Santos et al. (2008)) developed a strategy to build stability bounds on model mismatch for a class of NMPC algorithms. De Olivera and Morari (De Oliveira and Morari (2000)) proposed a contractive algorithm to stabilize NMPC by introducing an additional state constraint. It is called contractive MPC (CNTMPC) and is also a Lyapunov based approach. The contractive constraint renders the system exponentially stable in the state feedback case and uniformly asymptotically stable in the case of output feedback (De Oliveira and Morari (2000)). CNTMPC, which should be parameterized by the user, is based on the contraction property of the state feedback control system (Blanchini (1994); Malmgren and Nordstroem (1994)). With discrete formulation of the optimal control problem, a contractive state inequality constraint is introduced to impose system states at the end of each predicted sample instant to be contracted in norm with respect to the states at the beginning of the predicted sample instant. This inequality constraint is steered mainly by the contractive parameter which will be chosen to make the optimization problem feasible.

In this work we propose a new formulation of the optimal control problem to ensure the stability of the NMPC

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systems by introducing auxiliary state variables and corresponding linear state equations. We enforce system states to be contracted with respect to the auxiliary state variables by adding inequality constraints. Thus the stability properties of system states will conform to those of the auxiliary states, i.e. the system states will be stable, if the auxiliary states are stable. The eigenvalues of the linear state equations introduced will be determined to stabilize the auxiliary state variables and at the same time make the optimal control problem feasible. This is achieved by considering the eigenvalues as optimization variables in the optimal control problem. Therefore, the solution of the optimal control problem guarantees the feasibility, stability and optimality of the NMPC system.

The remaining of this paper is organized as follows, Section 2 presents the optimal control problem with a new setup to ensure the stability. Section 3 gives some notations and definitions regarding the open loop optimal control problem and its solution. Section 4 presents a stability analysis near the stationary point and in Section 5 two examples are given to examine the solution of optimal control problem using the setup of Section 2. The work will be concluded in Section 6.

2. OPTIMAL CONTROL PROBLEM

We consider the following optimal control problem with a quadratic performance index

$$\min_{x,z,u,a_z} J = \int_{t_0}^{t_f} (\|x(t)\|_Q^2 + \|u(t)\|_R^2 + \|z(t)\|_{\hat{Q}}^2)dt \quad (1a)$$

s.t.

$$x(t_0) = x_0, \quad (1b)$$
$$\dot{x}(t) = f(x(t), u(t), t), \quad \forall t \in [t_0, t_f], \quad (1c)$$
$$s(x(t), u(t), t) \geq 0, \quad \forall t \in [t_0, t_f], \quad (1d)$$
$$u_{\min} \leq u(t) \leq u_{\max}, \quad \forall t \in [t_0, t_f], \quad (1e)$$
$$\begin{aligned} &\dot{z}(t) = A_z z(t), \\ &z(t_0) = Z_0, \quad \forall t \in [t_0, t_f], \\ &A_z = \mathrm{diag}(a_z) \end{aligned} \quad (1f)$$
$$|x(t)| \leq |z(t)| \quad \forall t \in [t_0, t_f], \quad (1g)$$
$$a_z < 0, \quad (1h)$$

where J is a scalar function that represents the quadratic performance measure, ($J : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_{a_z}} \to \mathbb{R}$), Eqs. (1b) and (1c) represent initial condition and system dynamics, respectively, with system state vector $x(t) \in \mathbb{R}^{n_x}$, control vector $u(t) \in \mathbb{R}^{n_u}$ and $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$. A vector function $s : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_s}$ represents some path constraints on the system states.

This is in fact a new formulation of NMPC. Originally, only the vector $x(t) \in \mathbb{R}^{n_x}$ is supposed to be the system states in the optimal control problem (1). However, in the setup of (1), the system states are augmented from n_x to $2n_x$ by adding an auxiliary state vector $z(t) \in \mathbb{R}^{n_x}$ which is included in the setup of (1) using Eq. (1f). The auxiliary state is characterized mainly by the matrix $A_z \in \mathbb{R}^{n_x \times n_x}$ which has negative eigenvalues and thus is exponentially stable. This stability is obtained by satisfying the negativity of the eigenvalues $a_z \in \mathbb{R}^{n_x}$ and described through the inequality constraint (1h).

We will prove that the exponential decaying of the system state will be ensured by the inequality constraint (1g), at the terminal region, i.e. for $t \geq t_f$, we will prove that, under certain conditions, the equivalent solution of the optimal control problem is asymptotically stable.

The weighting matrices $Q \in \mathbb{R}^{n_x \times n_x}$, $R \in \mathbb{R}^{n_u \times n_u}$ and $\hat{Q} \in \mathbb{R}^{n_x \times n_x}$ are positive definite and can be chosen freely to make the optimal control problem (1) feasible where $\|x\|_Q^2$, $\|z\|_{\hat{Q}}^2$ and$\|u\|_R^2$ denote the Euclidean norm of the state vector x, z and u, respectively, e.g. $\|x\|_Q^2 = x^T Q x$. Note that the quadratic penalty term introduced with weighting penalty matrix $\hat{Q}$ is to penalize the auxiliary (augmented) state in the prediction horizon. A positive definite and symmetric weighting matrix $\hat{Q}$ can be determined off-line.

3. NOTATIONS AND DEFINITIONS

In this study we consider following notions and definitions regarding the open-loop optimal control problem (1).

Definition 1. (Equilibrium Point). A point $\eta = (\eta_x, \eta_u) \in (\mathbb{R}^{n_x}, \mathbb{R}^{n_u})$ is an equilibrium point of the system dynamics (1c) if $f(\eta, t) = 0$ and η is the point at which both the plant and model operate at a steady state (Murray et al. (1994)).

Definition 2. (Local Lyapunov Stability). The equilibrium point η is **uniformly (Lyapunov) stable** if all solutions which start near η. The initial conditions are in a neighborhood of η, remain near η for all time (Khalil (1992)).

Thus, for each $\varepsilon > 0$ and each $t_0 \in \mathbb{R}_+$, there exists a $\delta = \delta(\varepsilon)$ such that $\|x_0\| < \delta(\varepsilon)$ then $\|x(t)\| < \varepsilon$, $\forall t \geq t_0$.

Definition 3. (Asymptotical Stability).
The equilibrium point η is **uniformly asymptotical stable** if η is locally (uniformly) stable and solutions starting near η tend towards η as $t \to \infty$ (Khalil (1992)).

This means for each $t_0 \in \mathbb{R}_+$ there exists $\delta > 0$ such that $\|x_0\| < \delta$, then $\lim_{t \to \infty} x(t) = 0$, $\forall t \geq t_0$.

Definition 4. (Exponential Stability).
The equilibrium point η of the system is **exponentially stable** (Khalil (1992)) if it is asymptotically stable and if there exist constants $\alpha, \beta \, and \, \delta > 0$ such that, if $\|x_0\| < \delta$ then

$$\|x(t)\| \leq \alpha \|x_0\| e^{-\beta t} \qquad \forall t \geq t_0 \quad (2)$$

Definition 5. (Feasibility of the Optimal Control Problem). An optimal control problem (1) is feasible if there exist trajectory vectors $x^*(t) \in \mathbb{R}^{n_x}$, $u^*(t) \in \mathbb{R}^{n_u}$, $z^*(t) \in \mathbb{R}^{n_x}$ and eigenvalues $a_z^* \in \mathbb{R}^{n_x}$ that satisfy Eqs. (1b) to (1h) (Leineweber (1995))

Definition 6. (Optimality of the Optimal Control Problem). A trajectory vector $u^*(t) \in \mathbb{R}^{n_u}$ is a local optimal control (or local minimum solution) with corresponding trajectory state vectors $x^*(t), z^*(t) \in \mathbb{R}^{n_x}$ and corresponding eigenvalues $a_z^* \in \mathbb{R}^{n_x}$ if the optimal control problem (1) is feasible on $u^*(t)$, $x^*(t)$, $z^*(t)$ and a_z^* and the trajectory vectors $x^*(t) \in \mathbb{R}^{n_x}$ and $u^*(t) \in \mathbb{R}^{n_u}$ satisfy $J(x^*(t), z^*(t), u^*(t), a_z^*, t) \leq J(x(t), z(t), u(t), a_z, t)$ for all feasible neighborhood of x^*, u^*, z^* and a_z^* (Leineweber (1995)).

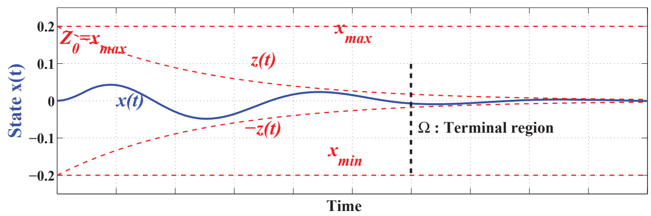

Fig. 1. State $x(t)$ (blue-solid), auxiliary states $\pm z(t)$ (red-dashed), $-0.2 \le x(t) \le 0.2 \Rightarrow -z(t) \le x(t) \le z(t)$, $Z_0 = x_{\max}$.

Definition 7. (Stabilizability). The dynamic system (1c) is a **stabilizable** system if a bounded input vector yields a bounded output (Nise (2004)).

Assumption 1. The system dynamics (1c) has a shifted equilibrium point at $\eta = (\eta_x, \eta_u) = (0, 0)$.

Assumption 2. The bounds of the system state vector are limited, i.e. $x_{\min} \ne -\infty$ and $x_{\max} \ne \infty$.

Since $|x(t)| \le |z(t)|$, $z(t) \in [0, x_{\max}]$, $-z(t) \in [x_{\min}, 0]$ and $Z_0 = \max\{z(t)\}$, we define the constant vector $Z_0 \in \mathbb{R}^{n_x}$ to represent the maximum and minimum values of the state $x(t)$ so that we can choose Z_0, e.g., as shown in Fig. 1, such that

$$Z_0 = \max\{|x_{\max}|, \quad |x_{\min}|\}. \tag{3}$$

Assumption 3. The linearization of the system dynamics (1c) around the origin is stabilizable (satisfies Definition 7 for $u_{\min} \le u(t) \le u_{\max}$), i.e., $\{(\partial f/\partial x)(0,0), (\partial f/\partial u)(0,0)\}$ is a stabilizable pair.

Assumption 4. The open-loop optimal control problem (1) has a local optimal control vector $u^*(t)$ with a corresponding optimal state vector and auxiliary state vector $x^*(t)$ and $z^*(t)$, respectively, that satisfy Definition 6. The state vector $x^*(t)$ is bounded by an optimal auxiliary state vector $z^*(t)$ which is characterized only by the optimal eigenvalues a^*.

Theorem 1. (Open-Loop Exponential Stability). Let the system dynamics (1c) has an equilibrium point at the origin with bounded and stabilizable states for $u \in [u_{\min}, u_{\max}]$ and Assumptions 1-4 are used for the open-loop optimal control problem (1) then the optimal state vector $x^*(t)$ is exponentially stable.

Proof. From Assumption 4, we have a point $(x^*, z^*, u^*, a^*) \in (\mathbb{R}^{n_x}, \mathbb{R}^{n_x}, \mathbb{R}^{n_u}, \mathbb{R}^{n_x})$ such that the optimal control problem (1) is feasible. Thus, it follows that

$$\begin{aligned}
&\dot{z}^*(t) = A_z z^*(t), \\
&z^*(t_0) = Z_0, && \forall t \in [t_0, t_f] \\
&A_z = \mathrm{diag}(a_z^*), \\
&-z^*(t) \le x^*(t) \le z^*(t), && \forall t \in [t_0, t_f]. \\
&a_z^* < 0.
\end{aligned}$$

Since the eigenvalues a_z^* is negative, the trajectory vector $z^*(t)$ is exponentially stable and $\lim_{t\to\infty} z^*(t) = 0$. That means, from $-z^*(t) \le x^*(t) \le z^*(t)$, and the time-varying trajectory vector $z(t)$ implies

$$z_l^*(t) = Z_{0,l} e^{-a_{z_l}^* t}$$

where the index l denotes the members in the vectors Z_0, and $a_z \in \mathbb{R}^{n_x}$, i.e. there exist α_z, β_z and $\delta_z > 0$ such that, if $\|z_0\| < \delta_z$ then

$$\|z^*(t)\| \le \alpha_z \|z_0\| e^{-\beta_z t}, \ \forall t \ge t_0.$$

Since $-z^*(t) \le x^*(t) \le z(t)$, there is $\|x_0\|$ such that

$$\|x_0\| < \|z_0\| < \delta_z,$$

and

$$\|x^*(t)\| \le \|z^*(t)\| \le \alpha_z \|z_0\| e^{-\beta_z t},$$

Let $\alpha_c = \frac{\|z_0\|}{\|x_0\|} > 1$,then

$$\|x^*(t)\| \le \|z^*(t)\| \le \alpha_c.\alpha_z \|x_0\| e^{-\beta_z t},$$

and

$$\alpha_c.\alpha_z > 0.$$

This means $x^*(t)$ is exponentially stable, too.

According to Theorem 1, the exponential stability of the system and auxiliary states can be obtained near the equilibrium point in the finite horizon. We solve this optimal control problem using a combined approach of multiple shooting method and collocation on finite elements (Tamimi and Li (2010)). Within this approach the multiple shooting method is used for discretizing the dynamic system, through which the optimal control problem is transformed to a nonlinear program (NLP). To solve this NLP problem state variables and their gradients are needed. Collocation on finite elements is used to carry out this task.

4. FEATURES AT THE EQUILIBRIUM POINT

To analyze the features of the proposed NMPC at the equilibrium point, we assume that the linearized system model (1c) around the origin

$$\dot{x} = A_x x + B_u u,$$

where $A_x = ({\partial f}/{\partial x})|_{(0,0)}$ and $B_u = ({\partial f}/{\partial u})|_{(0,0)}$, and dynamics A_x has negative eigenvalues, that means, all the poles of the linearized system are stable, then it can be concluded the system is stable at the equilibrium point, too. However, if A_x has some positive eigenvalues, that means the system has some poles on the right-hand side of the $s-$plane. The system states are augmented with an auxiliary state $z(t)$ and the open-loop optimal control problem (1) is solved. In addition, the prediction horizon is chosen such that the system and auxiliary states can reach a neighborhood of the equilibrium point. Then the optimal eigenvalues a_z and the optimal auxiliary state z will replace the instable poles of the linearized system model and act as a compensator.

To prove this issue, let us analyze the system states at the terminal point, the solution of the open-loop optimal control problem (1) implies the feasibility for all $t \in [t_0, t_f]$ and t_f is chosen such that the system states and auxiliary states reach a neighborhood of the equilibrium point. Then the infinite optimal control problem can be defined in the neighborhood of the equilibrium point with respect to the linearized system model

$$\min_{x,z,u} J = \int_{t_f}^{\infty} (\|x(t)\|_Q^2 + \|u(t)\|_R^2 + \|z(t)\|_{\hat{Q}}^2) dt \tag{4a}$$

s.t.

$$x(t_f) = x_f, \quad (4b)$$
$$\dot{x}(t) = A_x x(t) + B_u u(t), \qquad \forall t \geq t_f, \quad (4c)$$
$$\dot{z}(t) = A_z z(t),$$
$$z(t_f) = z_f, \qquad \forall t \in [t_0, t_f], \quad (4d)$$
$$A_z = \mathrm{diag}(a_z,)$$

where a diagonal matrix A_z is the same as in the finite-open-loop optimal control problem (1) that contains only negative eigenvalues. x_f and z_f are terminal values of the system and auxiliary state, respectively, that exist in the terminal region of the neighborhood of the origin which can be defined according to the following definition.

Definition 8. (Terminal Region). A terminal region Ω is a region in terminal time such that the system and auxiliary state reach a neighborhood of the equilibrium point, that means:

$$\left[x^T(t)\ z^T(t)\right]^T \in \Omega, \qquad t \geq t_f,$$
$$\left[x^T(t)\ z^T(t)\right]^T \approx \left[\eta_x^T\ \eta_z^T\right]^T = [0\ 0]^T.$$

Since the inequality constraint (1g) must be satisfied in the terminal region Ω, one of the following properties must be also satisfied, as shown in Fig. 1,

$$x(t) = z(t) - \delta = z^-(t), \qquad t \geq t_f \text{ or} \quad (5a)$$
$$x(t) = -z(t) + \delta = -z^+(t), \qquad t \geq t_f, \quad (5b)$$

where δ is a small value that represents the difference between the system state x and auxiliary state z in the terminal region which tends to zero.

Let us now rewrite the optimal control problem (4) with satisfying property (5a), i.e. $x(t) = z(t), \quad t \geq t_f$, then the open-loop optimal control problem will be

$$\min_{x,z,u} J = \int_{t_f}^{\infty} (\|[x^T(t)\ \ z^T(t)]^T\|_{Q_n}^2 + \|u(t)\|_R^2)dt \quad (6a)$$

s.t.

$$[x^T(t_f)\ \ z^T(t_f)]^T = [x_f^T\ \ z_f^T]^T, \quad (6b)$$
$$\begin{bmatrix}\dot{x}(t)\\ \dot{z}(t)\end{bmatrix} = A_n \begin{bmatrix}x(t)\\ z(t)\end{bmatrix} + B_n u(t), \qquad \forall t \geq t_f, \quad (6c)$$

where $A_n = \begin{bmatrix}A_x & 0\\ 0 & A_z\end{bmatrix}$, $B_n = \begin{bmatrix}B_u\\ 0\end{bmatrix}$ and $Q_n = \begin{bmatrix}Q & 0\\ 0 & \hat{Q}\end{bmatrix}$.

The solution of the open-loop optimal control problem (6) is equivalent to the problem solution by finding a linear feedback K_n, therefore

$$u = -K_n[x^T(t)\ \ z^T(t)]^T = [-K_x\ \ -K_z][x^T(t)\ \ z^T(t)]^T, \quad (7)$$

where $K_n \in \mathbb{R}^{n_u \times 2n_x}$ and $K_x,\ K_z \in \mathbb{R}^{n_u \times n_x}$ are designed to obtain the closed loop asymptotical stability of system (4c) such that

$$K_n = R^{-1}B^T P_n,$$

where P_n is a unique positive-definite solution and found by solving the following algebraic Riccati equation.

$$A_n^T P_n + P_n A_n - P_n B_n R^{-1} B_n^T P_n + Q_n = 0.$$

Combining Eq. (7) with Eq. (6c), we yield

$$\begin{bmatrix}\dot{x}(t)\\ \dot{z}(t)\end{bmatrix} = \begin{bmatrix}-B_u K_x & A_x - B_u K_z\\ 0 & A_z\end{bmatrix} \begin{bmatrix}x(t)\\ z(t)\end{bmatrix}. \quad (8)$$

On the other hand, if the property Eq. (5b) is satisfied, i.e. $x(t) = -z(t), \quad t \geq t_f$, then the solution of the open-loop optimal control problem is also equivalent to the closed

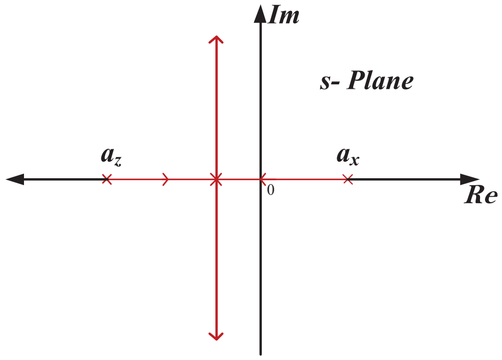

Fig. 2. Root locus of one dimensional system dynamics

loop optimal control problem by finding a linear control law $u = [-K_x\ \ -K_z][x^T(t)\ \ z^T(t)]^T, \quad t \geq t_f$ and Eq. (8) will be then

$$\begin{bmatrix}\dot{x}(t)\\ \dot{z}(t)\end{bmatrix} = \begin{bmatrix}-B_u K_x & -A_x - B_u K_z\\ 0 & A_z\end{bmatrix} \begin{bmatrix}x(t)\\ z(t)\end{bmatrix}. \quad (9)$$

The eigenvalues of the closed-loop system (8) and (9) are given by those of the state feedback regulator dynamics $-B_u K_x$ together with those of auxiliary state-system dynamics A_z. In case both matrices are asymptotically stable then so are the closed-loops (8) and (9).

Moreover, if the linearized system dynamics A_x has positive real part of the eigenvalues, i.e. the system has some poles in the right-hand side of the $s-$plane, the feedback gain K_x will substitute the state dynamics to a closed loop system with negative eigenvalues and brings the instable poles, in the open-loop case, to stable poles as shown in Fig. 2.

5. NUMERICAL EXAMPLES

We consider the following two optimal control problems to demonstrate the stability performance of the proposed NMPC formulation.

Example 1. (A demonstrative simple instable system). Consider the following optimal control problem

$$\min_{x,u} J = \int_0^5 (\|x(t)\|_q^2 + \|u(t)\|_r^2)dt \quad (10a)$$

s.t.

$$\dot{x}(t) = x(t) + u(t) \quad (10b)$$

where $x(0) = 1$, $u_{\min} = -1.5$, $u_{\max} = 1.5$, $q = 1$, $r = 1$. It is clear that exciting the plant model Eq.(10b) with a unit step control makes the state $x(t)$ instable, since the plant has an instable pole at $s = 1$.

Solving the problem using the proposed formulation described in Section 2 by adding an auxiliary state $\dot{z}(t) = A_z z(t), z(0) = 2$, an inequality constraint Eq. (11) and adding a new term, $(\int_0^5 \|z(t)\|_{\hat{q}}^2 dt)$, to the objective function with $\hat{q} = 0.05$. The solution leads to the optimized value $a_z = -2.0141$, which restricts and stabilizes the state in the first prediction horizon. In addition, it is seen that the stabilization inequality constraint Eq.(11) is more

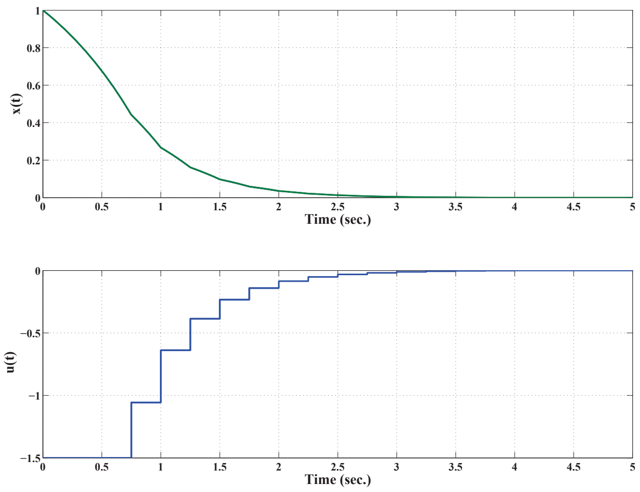

Fig. 3. State $x(t)$ and control $u(t)$ of Example 1

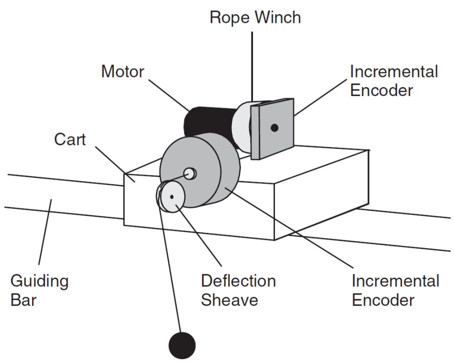

Fig. 4. Elements of the loading bridge

restrictive and the auxiliary state enforces the system state to drive to the equilibrium point, namely

$$-2e^{a_z t} \leq x(t) \leq 2e^{a_z t}. \tag{11}$$

Applying the equilibrium analysis in Section 4, at $t = t_f$, we put $x = z$ then the augmented system will be

$$\begin{bmatrix} \dot{x}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} -1 & -1.3318 \\ 0 & -2.0141 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}. \tag{12}$$

From the augmented system dynamics (12) it can be seen that all of the poles lay in the left hand side of s-plane, thus the system states are compensated.

Example 2. (Loading bridge). We consider a loading bridge with the mechanical setup as shown in Fig. 4. It consists of a cart which can be moved along a metal guiding bar by means of a transmission belt. A winch drive is mounted on top of the cart to change the length of a rope.

The control task is to move the cart by means of the transmission belt to the defined point at the metal guiding bar. In addition the movement should be carried out with a control effort. The optimal control problem is formulated as

$$\min_{x,z,u,a_z} J = \frac{1}{2}\int_0^5 (\|x(t)\|_Q^2 + \|u(t)\|_R^2 + \|z(t)\|_{\hat{Q}}^2)dt \tag{13a}$$

subject to system states where $x \in \mathbb{R}^6$ and $u \in \mathbb{R}^2$, for the model equations see (Amira-GmbH (2001)), with initial conditions $x(0) = [0.5\ \ 0\ \ 0\ \ 0.6\ \ 0]^T$, in addition we define

$$\dot{z} = A_z z, \quad A_z=\text{dig}([a_z]), \tag{13b}$$
$$z(0) = [0.6\ \ 1\ \ 0.5\ \ 1\ \ 1\ \ 1]^T, \tag{13c}$$
$$\eta_x - z(t) \leq x(t) \leq \eta_x + z(t), \tag{13d}$$
$$u_{\max} = [22.5\quad 3.75]^T,\ u_{\min} = [-22.5\ \ -3.75]^T, \tag{13e}$$

where the states x_1, x_2, x_3, x_4, x_5, and x_6 are the position of the cart , the velocity of the cart, the angle of the rope, the angular velocity of the rope, the length of the rope in m and the differentiation of the length of the rope, respectively. The controls u_1 and u_2 are the driving force of the cart and the winch, respectively. $z(t) \in \mathbb{R}^6$ are the auxiliary states where $A_z \in \mathbb{R}^{6\times 6}$ is a time-invariant diagonal matrix and $\eta_x \in \mathbb{R}^6$ is the operating point around which the system should be reached.

We solve the NMPC problem by defining Q=dig([200 0.5 0.5 0.5 1 1]), R=dig([0.05 0.05]), $\hat{Q}$=dig([200 150 100 50 50 50]), the desired operating point η_x=[0.4 0 0 0 1 0 $]^T$ and sampling time 0.25 sec. The optimal control solution leads to the objective function value of the main part (i.e. without the additional term) 1.8168. Fig. 5 shows that the systems state are forced to approach the operating point η_x.

Applying the equilibrium state analysis of Section 4, at $t = t_f$, we put $x = z$ then the augmented system of the form of Eq. (7) has a negative eigenvalues $\lambda = [0\ -3.6850\ 0\ 0\ -7.3197\ 0\ -2.6750\ -2.3070\ -3.0390]^T$ thus one can conclude that all poles of linearized system around the origin are stable poles.

6. CONCLUSIONS

In this paper, we proposed a new formulation of NMPC to ensure the stabilization of NMPC systems. In this formulation, the basic finite horizon optimization problem is extended. We introduce auxiliary linear states to the optimal control problem, which enforce the original states using inequality constraints. We use a hybrid multiple shooting combined with collocation method to solve this optimization problem efficiently. The optimization variables will be the discretized states and controls, the discretized auxiliary states and the eigenvalues of the auxiliary states. The optimal eigenvalues lead to the exponential stability of the linear states which restrict and thus stabilize the system states. To solve the dynamic optimization problem we use the framework of the numerical algorithm group (NAG) and the interior point optimizer (IPOPT) in C/C++ environment. In addition, we have shown the performance of the new approach is implementable by two examples. From these results it can be seen that this proposed approach can guarantee the stability of NMPC systems.

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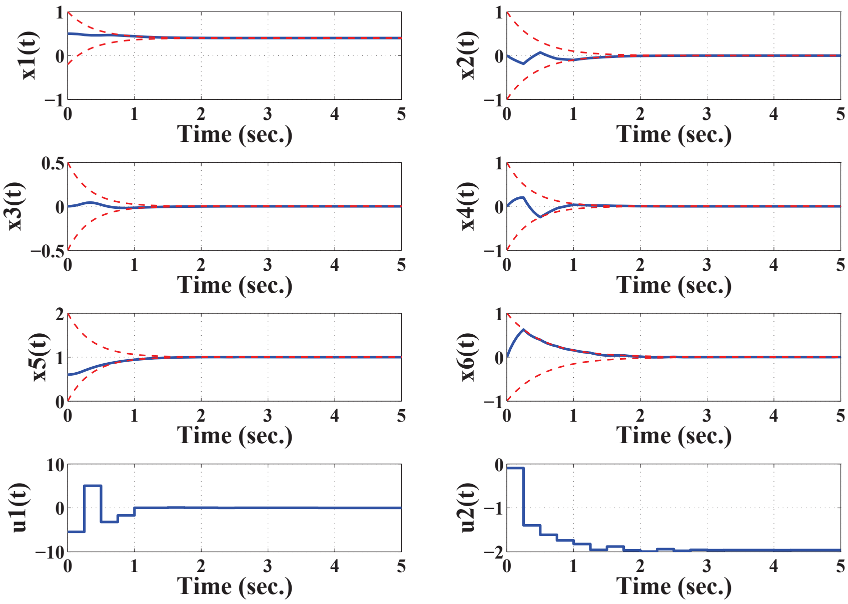

Fig. 5. Optimal control problem solution (states and controls) of example 2, $x_i(t)$, $i = 1, ..., 6$ and $u_1(t)$ and $u_2(t)$ (blue-solid), auxiliary states $\pm z_i(t)$, $i = 1, ..., 6$ (red-dashed)

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