Available online at www.sciencedirect.com
ScienceDirect
Computational Statistics & Data Analysis 52 (2008) 2066–2078
COMPUTATIONAL STATISTICS & DATA ANALYSIS
www.elsevier.com/locate/csda

On multistage ranked set sampling for distribution and median estimation

Mohammad Fraiwan Al-Saleh[a,*], Monjed Hisham Samuh[b]

[a]Department of Mathematics and Physics, Qatar University, Qatar
[b]Department of Mathematics and Computer Sciences, Palestine Polytechnic University, Hebron, Palestine

Received 1 January 2007; received in revised form 30 June 2007; accepted 1 July 2007
Available online 10 July 2007

Abstract

A variation of ranked set sampling (RSS), multistage RSS (MSRSS), is investigated for the estimation of the distribution function and some of its quantiles, in particular the median. It is shown that this method is significantly more efficient than simple random sampling (SRS). The method becomes more and more effective as the number of stages r increases. Two estimators of the median based on MSRSS are proposed and compared to the sample median obtained by SRS.
© 2007 Elsevier B.V. All rights reserved.

Keywords: Empirical distribution function; Multistage ranked set sampling; Ranked set sampling; Simple random sampling

1. Introduction

Ranked set sampling (RSS) was introduced by McIntyre (1952), (republished in McIntyre, 2005), for estimating pasture yields. It was first used to obtain a more efficient estimator of the population mean, as compared to simple random sampling (SRS). RSS is a method of sampling that can be advantageous when quantification of a large number of sampling units is costly but small sets of units can be easily ranked according to the characteristic under investigation, without actual quantification. RSS scheme can be described as follows:

(i) Randomly select m sets, each of size m elements from the population of interest.
(ii) The elements of each set in Step (i) are ranked visually or by any negligible cost method that does not require actual measurements.
(iii) Identify by judgment the ith minimum from the ith set, $i = 1, 2, \ldots, m$.

The set of the m elements obtained is called a ranked set sample.

(iv) Repeat Steps (i)–(iii) h times (cycles), if necessary, to obtain an RSS of size $n = mh$.

* Corresponding author.
E-mail addresses: malsaleh@qu.edu.qa (M.F. Al-Saleh), mhstat@ppu.edu (M.H. Samuh).

0167-9473/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.csda.2007.07.002

Takahasi and Wakimoto (1968) established a very important mathematical theory of RSS. They showed that the mean of the RSS is an unbiased estimator of the population mean, and has smaller variance than the mean of a SRS:

$$1 \leqslant \mathit{eff}(\hat{\mu}_{\text{RSS}}; \hat{\mu}_{\text{SRS}}) = \frac{V(\hat{\mu}_{\text{SRS}})}{V(\hat{\mu}_{\text{RSS}})} \leqslant \frac{m+1}{2}.$$

Stokes and Sager (1988) used RSS to estimate distribution functions. They showed that the empirical distribution function (edf) of a RSS is an unbiased estimator of the distribution function and has a smaller variance than that from a SRS. Al-Saleh and Al-Kadiri (2000) considered double RSS (DRSS), as a procedure that increases the efficiency of the RSS estimator without increasing the set size m. It was shown that the DRSS estimator of the mean is more efficient than that using RSS. Furthermore, ranking in the second stage is in some sense, easier than ranking in the first stage. Al-Saleh and Al-Omari (2002) generalized DRSS to MSRSS. The MSRSS scheme can be described as follows:

1. Randomly selected m^{r+1} sample units from the target population, where r is the number of stages, and m is the set size.
2. Allocate the m^{r+1} selected units randomly into m^{r-1} sets, each of size m^2.
3. For each set in Step (2), apply (i)–(iii) of the RSS procedure described above, to obtain a (*judgment*) ranked set of size m. This step yields m^{r-1} (*judgment*) ranked sets, of size m each.
4. Without doing any actual quantification on these ranked sets, repeat Step (3) on the m^{r-1} ranked sets to obtain m^{r-2} second stage (*judgment*) ranked sets, of size m each.
5. The process is continued using Step (3), without doing any actual quantification, until we end up with one rth stage ranked set of size m.
6. Repeat Steps (1)–(5) h times, if necessary, to obtain a rth stage RSS of size$n = mh$.

Since judgment ranking of a large size sample is prone to ranking errors, in practice, m should be small (2, 3 or 4). As given in the literature (see Takahasi and Wakimoto, 1968), for some distributions, such as highly skewed ones, small m does not lead to significant efficiency gain. One advantage of the multistage RSS (MSRSS) procedure is that efficiency gain can be achieved through the increase of the number of cycles r rather than the set size m.

In many practical situations, the variable of interest say X is fairly correlated with another concomitant variable Y, which can be easily ranked or measurements on it are available. In this case, the MSRSS can operate easily on the concomitant variable Y with a suitable number of stages r and set size m. Then at the end, the values of the corresponding variable X are obtained on the identified elements of MSRSS. For extremely large r, the efficiency is about m^2 when estimating the mean of uniform distribution while it is only $(m+1)/2$ for $r=1$. Clearly here, the work done on the Y-variable should not be of concern given the modern technology. Of course the gain in efficiency over SRS depends on how strong is the relation between the two variables.

Sometimes, we may have a massive data set. Though in these days and age, it is not a problem to store such data in the computer, as pointed out by a referee, it is still not easy to summarize the data using several carefully chosen data points that have high information content; it could be the case that not all the data are available at once, and data may come sequentially. Furthermore, some statistical packages may not be capable to handle massive data sets at once. A suitable data selection procedure should select data with high information content. The MSRSS is essentially a data selection procedure that has this property.

For more work on RSS and its variations see Chen et al. (2003), Al-Saleh (2004, 2007), Zheng and Al-Saleh (2002), Al-Saleh and Samawi (2007), and Al-Saleh and Ananbeh (2007).

2. Estimation of distribution functions using MSRSS

Assume that the variable of interest X has a probability density function (pdf) f, with absolutely continuous cumulative distribution function (cdf) F; let $E(X)=\mu$ and $Var(X)=\sigma^2$. Let $Y_i^{(r)}$, $(i=1,2,\ldots,m)$, be the ith element of a MSRSS of size m at stage r, and let $f_i^{(r)}$ be its pdf and $F_i^{(r)}$ be its cdf. Under the assumption of no error in judgment ranking, $f_i^{(r)}$ is the density of the ith order statistic of a MSRSS, $Y_1^{(r-1)}, Y_2^{(r-1)}, \ldots, Y_m^{(r-1)}$, i.e. $Y_i^{(r)} \stackrel{d}{=} Y_{(i)}^{(r-1)}$; $Y_1^{(r)}, Y_2^{(r)}, \ldots, Y_m^{(r)}$ are independent. Let $X_1, X_2, \ldots, X_n$ denote a random sample of size $n = mh$ from F. The empirical distribution function (edf) is

$$\hat{F}(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leqslant x),$$

where $I(\cdot)$ is the indicator function. Clearly, being a sum of iid binomial random variables, $\hat{F}(x)$ is an unbiased estimator of $F(x)$ with variance

$$\frac{1}{n}F(x)(1-F(x)).$$

Note that, $\hat{F}(x)$ is symmetric in $X_1, X_2, \ldots, X_n$. Therefore, it is a function of the order statistics, $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, which is the complete sufficient statistic, and since $\hat{F}(x)$ is unbiased, it is the unique, minimum variance unbiased estimator, UMVUE, of $F(x)$ *based on a simple random sample* $X_1, X_2, \ldots, X_n$ from F. (See Lehmann and Casella, 1998, p. 109.)

Let $\{Y_{ji}^{(r)} : j = 1, 2, \ldots, h; i = 1, 2, \ldots, m\}$ be a MSRSS of size $n = hm$ at stage r, for each $j = 1, 2, \ldots, h$, $Y_{ji}^{(r)}$ has a cdf $F_i^{(r)}$, $i = 1, 2, \ldots, m$. Then the edf is

$$\hat{F}^{(r)}(x) = \frac{1}{mh}\sum_{j=1}^{h}\sum_{i=1}^{m} I(Y_{ji}^{(r)} \leqslant x).$$

The following lemma, which will be used frequently, is due to Al-Saleh and Al-Omari (2002).

Lemma.

(1) $F(x) = \frac{1}{m}\sum_{i=1}^{m} F_i^{(r)}(x)$; $f(x) = \frac{1}{m}\sum_{i=1}^{m} f_i^{(r)}(x)$.
(2) $\mu = \frac{1}{m}\sum_{i=1}^{m}\mu_i^{(r)}$.
(3) $\sigma^2 = \frac{1}{m}\sum_{i=1}^{m}\sigma_i^{2(r)} + \frac{1}{m}\sum_{i=1}^{m}(\mu_i^{(r)} - \mu)^2$.
(4) *For* $i = 1, 2, \ldots, m$, *we have*

$$\lim_{r\to\infty} F_i^{(r)}(x) = F_i^{(\infty)}(x) = \begin{cases} 0, & x < Q_{(i-1)/m}, \\ mF(x) - (i-1), & Q_{(i-1)/m} \leqslant x < Q_{i/m}, \\ 1, & x \geqslant Q_{i/m}, \end{cases}$$

where Q_q *is the number which satisfies* $\int_{-\infty}^{Q_q} f(x)\,\mathrm{d}x = q$. ($Q_q$ *is called a* 100q*-percentile*, $0 < q < 1$.)

Proposition 1.

(i) $E(\hat{F}^{(r)}(x)) = F(x)$.
(ii) $\mathrm{var}(\hat{F}^{(r)}(x)) = \frac{1}{m^2h}\sum_{i=1}^{m} F_i^{(r)}(x)(1 - F_i^{(r)}(x))$.
(iii) $\mathit{eff}^{(r)}(\hat{F}^{(r)}(x); \hat{F}(x)) = \frac{F(x)(1-F(x))}{F(x) - (1/m)\sum_{i=1}^{m}(F_i^{(r)}(x))^2}$.

Proof. Let

$$Z_{ji}^{(r)} = I(Y_{ji}^{(r)} \leqslant x) = \begin{cases} 1, & Y_{ji}^{(r)} \leqslant x, \\ 0, & o.w. \end{cases}$$

For each $i = 1, 2, \ldots, m$, $U_i^{(r)} = \sum_{j=1}^{h} Z_{ji}^{(r)}$ is $b(h, F_i^{(r)})$, and $U_i^{(r)}$ are independent.

Thus,

$$E(U_i^{(r)}) = hF_i^{(r)}, \quad \mathrm{var}(U_i^{(r)}) = hF_i^{(r)}(x)(1 - F_i^{(r)}(x)).$$

Therefore, using the above lemma

$$E(\hat{F}^{(r)}(x)) = \frac{1}{mh}\left(\sum_{i=1}^{m} E(U_i^{(r)})\right) = \frac{1}{m}\sum_{i=1}^{m} F_i^{(r)} = F(x).$$

Thus, $\hat{F}^{(r)}(x)$ is an unbiased estimator of $F(x)$.

Also,

$$Var(\hat{F}^{(r)}(x)) = \frac{1}{m^2h^2}\sum_{i=1}^{m} \mathrm{var}(U_i^{(r)}) = \frac{1}{m^2h}\sum_{i=1}^{m} F_i^{(r)}(x)(1 - F_i^{(r)}(x)).$$

The efficiency of $\hat{F}^{(r)}(x)$ w.r.t. $\hat{F}(x)$ is

$$\mathit{eff}^{(r)}(\hat{F}^{(r)}(x); \hat{F}(x)) = \frac{\mathrm{var}(\hat{F}(x))}{\mathrm{var}(\hat{F}^{(r)}(x))} = \frac{F(x)(1 - F(x))}{F(x) - (1/m)\sum_{i=1}^{m}(F_i^{(r)}(x))^2}.$$

Note that

$$F_k^{(r)}(x) = \sum_{t=k}^{m}\sum_{S_t} F_{i_1}^{(r-1)}(x)F_{i_2}^{(r-1)}(x)\cdot\cdots\cdot F_{i_t}^{(r-1)}(x)(1 - F_{i_{t+1}}^{(r-1)}(x)) \times (1 - F_{i_{t+2}}^{(r-1)}(x))\cdot\cdots\cdot(1 - F_{i_m}^{(r-1)}(x)),$$

where S_t is the set of all permutations $(i_1, i_2, i_3, \ldots, i_m)$, of the numbers $(1, 2, \ldots, m)$ for which $i_1 < i_2 < \cdots < i_t$, $i_{t+1} < i_{t+2} < \cdots < i_m$. (Al-Saleh and Al-Omari, 2002).

Proposition 2.

$$\mathit{eff}^{(r)}(\hat{F}^{(r)}(x); \hat{F}(x)) \geqslant 1.$$

Proof. Let $a_i = F_i^{(r)}(x)$. By Holder's inequality we have

$$\frac{1}{m}\left(\sum_{i=1}^{m} |a_i|\right)^2 \leqslant \sum_{i=1}^{m} a_i^2.$$

Thus,

$$\frac{1}{m}\left(\sum_{i=1}^{m} F_i^{(r)}(x)\right)^2 \leqslant \sum_{i=1}^{m}(F_i^{(r)}(x))^2.$$

Since

$$F(x) = \frac{1}{m}\sum_{i=1}^{m} F_i^{(r)}(x),$$

$$\frac{1}{m}(mF(x))^2 \leqslant \sum_{i=1}^{m}(F_i^{(r)}(x))^2 \quad\Leftrightarrow\quad \frac{1}{m}\left(\sum_{i=1}^{m}(F_i^{(r)}(x))^2\right) \geqslant (F(x))^2.$$

Therefore,

$$\mathit{eff}^{(r)}(\hat{F}^{(r)}(x); \hat{F}(x)) = \frac{F(x)(1 - F(x))}{F(x) - (1/m)\sum_{i=1}^{m}(F_i^{(r)}(x))^2} \geqslant \frac{F(x)(1 - F(x))}{F(x) - (F(x))^2} = 1. \qquad \square$$

Note: Based on the above proposition, $\hat{F}^{(r)}$ which is obtained based on the MSRSS is more efficient than $\hat{F}$, the MVUE of F obtained based on a SRS of equivalent size. Of course $\hat{F}$ dominates all other unbiased estimators that can

Table 1
The efficiency of $\hat{F}^{(r)}(x)$ w.r.t $\hat{F}(x)$; $m=2$, at different values of r

$q \rightarrow r_{\downarrow}$	0.10	0.20	0.25	0.30	0.50	0.70	0.75	0.80	0.90
1	1.0989	1.1905	1.2308	1.2658	1.3333	1.2658	1.2308	1.1905	1.0989
2	1.1197	1.2744	1.3594	1.4440	1.6410	1.4440	1.3594	1.2744	1.1197
3	1.1239	1.3094	1.4281	1.5600	1.9360	1.5600	1.4281	1.3094	1.1239
4	1.1248	1.3237	1.4636	1.6336	2.2230	1.6336	1.4636	1.3237	1.1248
5	1.1249	1.3295	1.4817	1.6792	2.5047	1.6792	1.4817	1.3295	1.1249
10	1.1250	1.3333	1.4994	1.7444	3.8683	1.7444	1.4994	1.3333	1.1250
15	1.1250	1.3333	1.5000	1.7496	5.1942	1.7496	1.5000	1.3333	1.1250
∞	1.1250	1.3333	1.5000	1.7500	∞	1.7500	1.5000	1.3333	1.1250

Table 2
The efficiency of $\hat{F}^{(r)}(x)$ w.r.t $\hat{F}(x)$; $m=3$, at different values of r

$q \rightarrow r_{\downarrow}$	0.10	0.20	0.25	0.30	0.50	0.70	0.75	0.80	0.90
1	1.1959	1.3676	1.4382	1.4966	1.6000	1.4966	1.4382	1.3676	1.1959
2	1.2581	1.5996	1.7677	1.9096	2.1223	1.9096	1.7677	1.5996	1.2581
3	1.2774	1.7504	2.0315	2.2771	2.4970	2.2771	2.0315	1.7504	1.2774
4	1.2832	1.8465	2.2451	2.6193	2.7286	2.6193	2.2451	1.8465	1.2832
5	1.2849	1.9065	2.4163	2.9420	2.8587	2.9420	2.4163	1.9065	1.2849
10	1.2857	1.9926	2.8506	4.3013	2.9954	4.3013	2.8506	1.9926	1.2857
15	1.2857	1.9994	2.9639	5.2737	2.9999	5.2737	2.9639	1.9994	1.2857
∞	1.2857	2.0000	3.0000	7.0000	3.0000	7.0000	3.0000	2.0000	1.2857

Table 3
The efficiency of $\hat{F}^{(r)}(x)$ w.r.t $\hat{F}(x)$, $m=4$, at different values of r

$q \rightarrow r_{\downarrow}$	0.10	0.20	0.25	0.30	0.50	0.70	0.75	0.80	0.90
1	1.2904	1.5311	1.6248	1.6998	1.8286	1.6998	1.6248	1.5311	1.2904
2	1.4130	1.9365	2.1530	2.3154	2.5966	2.3154	2.1530	1.9365	1.4130
3	1.4646	2.2729	2.6214	2.8244	3.2565	2.8244	2.6214	2.2729	1.4646
4	1.4857	2.5615	3.0623	3.2421	3.8580	3.2421	3.0623	2.5615	1.4857
5	1.4943	2.8089	3.4902	3.5902	4.4346	3.5902	3.4902	2.8089	1.4943
10	1.4999	3.5678	5.5476	4.6451	7.1888	4.6451	5.5476	3.5678	1.4999
15	1.5000	3.8531	7.5413	5.0439	9.8515	5.0439	7.5413	3.8531	1.5000
∞	1.5000	4.0000	∞	5.2500	∞	5.2500	∞	4.0000	1.5000

be obtained based on SRS, but not necessarily dominates unbiased estimators based on MSRSS. The two estimators are obtained based on two different sets of data, each set has different joint distribution.

Proposition 3. *$\mathit{eff}^{(r)}(\hat{F}^{(r)}(x);\,\hat{F}(x))$ is increasing in r for fixed m.*

Proof. Since

$$\hat{F}^{(r)}(x)=\frac{1}{mh}\sum_{j=1}^{h}\sum_{i=1}^{m}I(Y_{ji}^{(r)}\leqslant x),$$

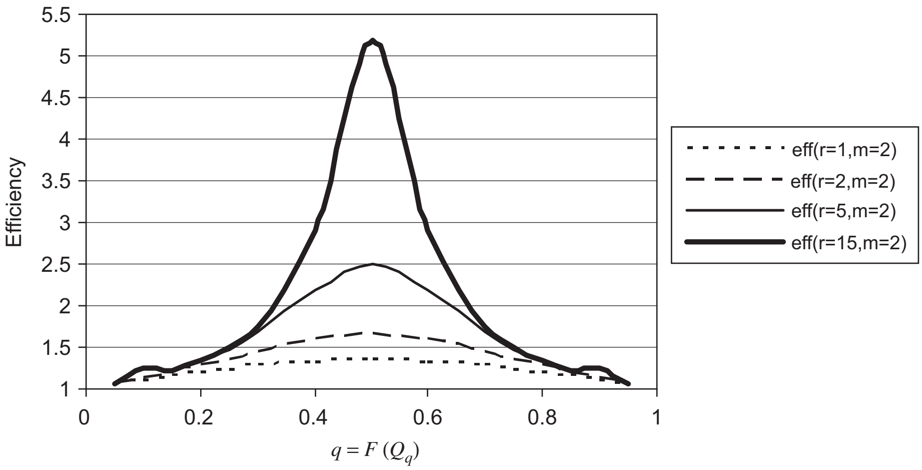

Fig. 1. The efficiency of $\hat{F}^{(r)}(x)$ w.r.t $\hat{F}(x)$; $m = 2$.

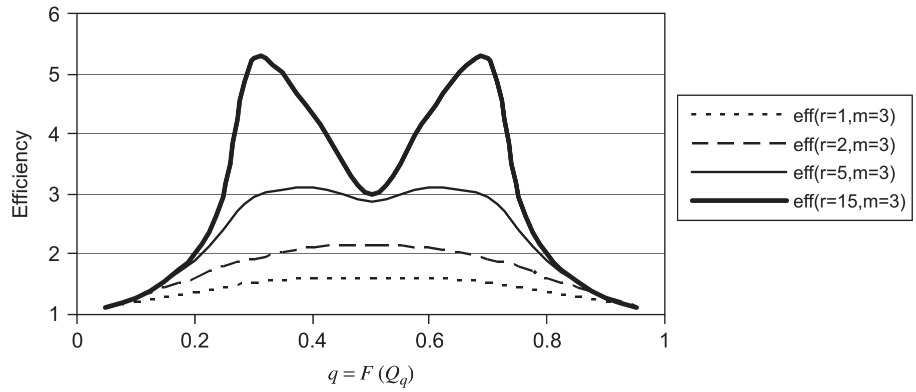

Fig. 2. The efficiency of $\hat{F}^{(r)}(x)$ w.r.t $\hat{F}(x)$; $m = 3$.

and for each i, $\{Y_{ji}^{(r)}, j = 1, \ldots, h\}$ are iid, we have

$$\begin{aligned}
\operatorname{var}(\hat{F}^{(r-1)}(x)) &= \frac{1}{m^2}\operatorname{var}\left(\sum_{i=1}^{m} I(Y_i^{(r-1)} \leqslant x)\right) = \frac{1}{m^2}\operatorname{var}\left(\sum_{i=1}^{m} I(Y_{(i)}^{(r-1)} \leqslant x)\right) \\
&= \frac{1}{m^2}\sum_{i=1}^{m}\operatorname{var}(I(Y_{(i)}^{(r-1)} \leqslant x)) + \frac{1}{m^2}\sum_{i\neq j}\operatorname{cov}(I(Y_{(i)}^{(r-1)} \leqslant x), I(Y_{(j)}^{(r-1)} \leqslant x)) \\
&= \frac{1}{m^2}\sum_{i=1}^{m}\operatorname{var}(I(Y_i^{(r)} \leqslant x)) + \frac{1}{m^2}\sum_{i\neq j} cov(I(Y_{(i)}^{(r-1)} \leqslant x), I(Y_{(j)}^{(r-1)} \leqslant x)) \\
&= \operatorname{var}(\hat{F}^{(r)}(x)) + \frac{1}{m^2}\sum_{i\neq j}\operatorname{cov}(I(Y_{(i)}^{(r-1)} \leqslant x), I(Y_{(j)}^{(r-1)} \leqslant x)).
\end{aligned}$$

Now,

$$\begin{aligned}
&\operatorname{cov}(I(Y_{(i)}^{(r-1)} \leqslant x), I(Y_{(j)}^{(r-1)} \leqslant x)) \\
&\quad = E(I(Y_{(i)}^{(r-1)} \leqslant x) I(Y_{(j)}^{(r-1)} \leqslant x)) - E(I(Y_{(i)}^{(r-1)} \leqslant x)) E(I(Y_{(j)}^{(r-1)} \leqslant x)).
\end{aligned}$$

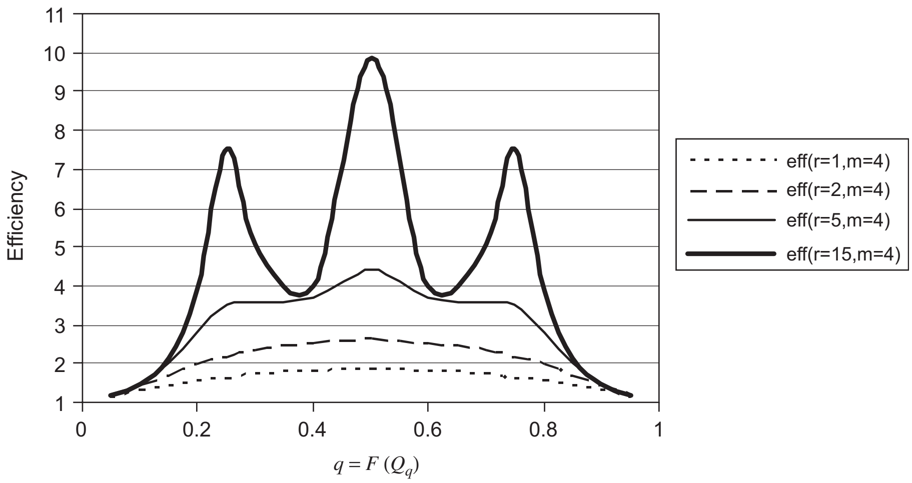

Fig. 3. The efficiency of $\hat{F}^{(r)}(x)$ w.r.t $\hat{F}(x)$; $m=4$.

But,

$$\begin{aligned} E(I(Y_{(i)}^{(r-1)} \leqslant x) I(Y_{(j)}^{(r-1)} \leqslant x)) &= P((Y_{(i)}^{(r-1)} \leqslant x), (Y_{(j)}^{(r-1)} \leqslant x)) \\ &\geqslant P((Y_{(i)}^{(r-1)} \leqslant x)) \times P((Y_{(j)}^{(r-1)} \leqslant x)) \\ &= E(I(Y_{(i)}^{(r-1)} \leqslant x)) E(I(Y_{(j)}^{(r-1)} \leqslant x)) \text{ (see Esary et al., 1967).} \end{aligned}$$

Thus,

$$\mathrm{cov}(I(Y_{(i)}^{(r-1)} \leqslant x), I(Y_{(j)}^{(r-1)} \leqslant x)) \geqslant 0.$$

Therefore,

$$\mathrm{var}(\hat{F}^{(r)}(x)) \leqslant \mathrm{var}(\hat{F}^{(r-1)}(x)),$$

which implies that $\mathit{eff}^{(r)}(\hat{F}^{(r)}(x); \hat{F}(x))$ is increasing in r for fixed m. □

Proposition 4. *For $i = 1, 2, \ldots, m$, we have,*

$$\begin{aligned} \lim_{r\to\infty} \mathit{eff}^{(r)}(\hat{F}^{(r)}(x); \hat{F}(x)) &= \mathit{eff}^{(\infty)}(\hat{F}^{(\infty)}(x); \hat{F}(x)) \\ &= \frac{mF(x)(1-F(x))}{(mF(x)-(i-1))(1-(mF(x)-(i-1)))} \quad \text{if } Q_{(i-1)/m} < x < Q_{i/m}. \end{aligned}$$

The proof is a direct application of the above lemma.

Tables 1–3 and Figs. 1–3 above report the efficiency for some special cases. It can be seen from the tables and figures that the efficiency of $\hat{F}^{(r)}(x)$ w.r.t $\hat{F}(x)$ is significantly larger than 1 especially at the interior values of the distribution. It is increasing in r and approaches a limit; it goes to infinity as r goes to infinity at $Q_{i/m}$, $i = 1, 2, \ldots, m-1$.

3. Estimation of the median using MSRSS

Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with pdf $f(x)$ and cdf $F(x)$, assumed to be differentiable, with $p(X_i \leqslant \theta) = \frac{1}{2}$, so that θ is the population median. Let $\hat{\theta}_{\mathrm{SRS}}$ be the sample median using a SRS. One way of defining $\hat{\theta}_{\mathrm{SRS}}$ is as follows:

$$\hat{\theta}_{\mathrm{SRS}} = \begin{cases} X_{((n+1)/2)}, & n \text{ is odd}, \\ \dfrac{X_{(n/2)} + X_{(n/2+1)}}{2}, & n \text{ is even}. \end{cases}$$

For the asymptotic properties of $\hat{\theta}_{\text{SRS}}$, we will use another similar definition of the median given in Serfling (1980), which is

$$\hat{\theta}_{\text{SRS}} = \begin{cases} X_{((n+1)/2)}, & n \text{ is odd}, \\ X_{(n/2)}, & n \text{ is even}. \end{cases}$$

Under some minor regularity conditions (see Serfling, 1980),

$$\hat{\theta}_{\text{SRS}} \text{ is } AN\left(\theta, \frac{1}{4nf^2(\theta)}\right).$$

The MSRSS of size $n = mh$ at stage r can be written as follows:

$$\begin{bmatrix} Y_{11}^{(r)}, & Y_{12}^{(r)}, & \cdots & Y_{1m}^{(r)} \\ Y_{21}^{(r)}, & Y_{22}^{(r)}, & \cdots & Y_{2m}^{(r)} \\ \vdots & \vdots & \cdots & \vdots \\ Y_{h1}^{(r)}, & Y_{h2}^{(r)}, & \cdots & Y_{hm}^{(r)} \end{bmatrix}.$$

For simplicity, the MSRSS at stage r is $\{Y_{ji}^{(r)} : \; j = 1, 2, \ldots, h; i = 1, 2, \ldots, m\}$. For each $j = 1, 2, \ldots, h$, $Y_{ji}^{(r)}$ has a pdf $f_i^{(r)}$ and cdf $F_i^{(r)}$, $i = 1, 2, \ldots, m$.

At stage r, for each j, let $\hat{\theta}_j^{(r)} = median\{Y_{ji}^{(r)} : i = 1, 2, \ldots, m\}$; that is, the median of each cycle in the MSRSS (the median of each row in the above matrix). Then, we have two possible estimators of the population median:

1. $\hat{\theta}_{\text{MSRSS1}}^{(r)} = median\{Y_{ji}^{(r)} : j = 1, 2, \ldots, h; i = 1, 2, \ldots, m\}$; that is, the overall median.
2. $\hat{\theta}_{\text{MSRSS2}}^{(r)} = median\{\hat{\theta}_j^{(r)} : j = 1, 2, \ldots, h\}$; that is, the median of the medians.

3.1. The efficiency of $\hat{\theta}_{\text{MSRSS1}}^{(r)}$ *w.r.t* $\hat{\theta}_{\text{SRS}}$

The following result is parallel to those on SRS and RSS (see Serfling, 1980; Chen, 2000; Samawi and Al-Sagheer, 2001).

Proposition 5. *Suppose that the density function f is positive in a neighborhood of* θ, *continuous at* θ *and that the ranking mechanism in MSRSS be such that*, $F(x) = (1/m)\sum_{i=1}^{m} F_i^{(r)}(x)$, *then for fixed m and r, and* $h \to \infty$,

$$\sqrt{mh}(\hat{\theta}_{\text{MSRSS1}} - \theta) \xrightarrow{\text{d}} N\left(0, \frac{\sigma^{2(r)}}{f^2(\theta)}\right),$$

where

$$\sigma^{2(r)} = \frac{1}{m}\sum_{i=1}^{m} F_i^{(r)}(\theta)(1 - F_i^{(r)}(\theta)).$$

Note that, $\sigma^{2(r)}$ depends only on $m\&r$, and it is free of F because $F_i^{(r)}(x)$ is eventually a function of F; $F_i^{(r)}(\theta)$ does not depend on F because $F(\theta) = 0.5$. The asymptotic efficiency of $\hat{\theta}_{\text{MSRSS1}}^{(r)}$ w.r.t $\hat{\theta}_{\text{SRS}}$ is

$$Aeff^{(r)}(\hat{\theta}_{\text{MSRSS1}}^{(r)}; \hat{\theta}_{\text{SRS}}) = \frac{A\text{var}(\hat{\theta}_{\text{SRS}})}{A\text{var}(\hat{\theta}_{\text{MSRSS1}}^{(r)})} = \frac{1}{(4/m)\sum_{i=1}^{m} F_i^{(r)}(\theta)(1 - F_i^{(r)}(\theta))}.$$

Numerical values of the efficiency and asymptotic efficiency are obtained by simulation. Four different distributions are considered, uniform distribution, $\mathrm{U}(0, \eta)$, normal distribution, $\mathrm{N}(\eta, 1)$, exponential distribution, $\mathrm{Exp}(\eta)$, and Cauchy distribution, $\mathrm{C}(\eta, 1)$. Without loss of generality, in the simulation, we assume that the unknown value of the parameter

Table 4
The efficiency of $\hat{\theta}^{(r)}_{\text{MSRSS1}}$ w.r.t $\hat{\theta}_{\text{SRS}}$, $m = 3$, at different values of $r$and h for $U(0, \theta)$

$h \rightarrow r_{\downarrow}$	1	2	3	4	5	10	30	$Aeff^{(r)}$
1	1.3982	1.8508	1.4952	1.6648	1.5450	1.6265	1.5985	1.6000
2	1.8828	2.3507	2.0743	2.1293	2.1086	2.1263	2.1252	2.1222
3	2.3828	2.5194	2.4355	2.5158	2.5037	2.5087	2.5004	2.4970
4	2.6678	2.8581	2.6964	2. 8140	2.7152	2.7397	2.7294	2.7286
5	2.7923	3.1873	2.8358	3.0603	2.8534	2.9475	2.8400	2.8587
6	3.4117	3.5049	3.1405	3.2326	3.0455	3.0486	3.0055	2.9279
7	3.6282	3.9560	3.2576	3.6629	3.1025	3.14769	3.0743	2.9636
8	3.7100	4.1390	3.5375	3.9635	3.3826	3.2063	3.1021	2.9819
9	3.8603	4.3962	3.8163	4.1263	3.4964	3.3109	3.1523	2.9908
10	3.9847	4.6772	3.9135	4.1718	3.5225	3.5964	3.1895	2.9954

Table 5
The efficiency of $\hat{\theta}^{(r)}_{\text{MSRSS1}}$ w.r.t $\hat{\theta}_{\text{SRS}}$, $m = 3$, at different values of $r$and h for $N(\theta, 1)$

$h \rightarrow r_{\downarrow}$	1	2	3	4	5	10	30	$Aeff^{(r)}$
1	1.6128	1.7421	1.5978	1.6898	1.5992	1.6105	1.6011	1.6000
2	2.2778	2.3995	2.1028	2.3394	2.1438	2.1263	2.1244	2.1222
3	2.7166	3.0493	2.4934	2.8322	2.5503	2.5147	2.5099	2.4970
4	3.2658	3.5244	2.9688	3.2791	2.8280	2.7936	2.7581	2.7286
5	3.8003	4.0097	3.1902	3.6596	3.5787	3.1499	3.0202	2.8587
6	4.4612	5.0101	3.7759	3.6203	3.5590	2.9908	2.9333	2.9279
7	4.6244	4.6839	3.8256	4.0614	3.7700	3.3345	3.1058	2.9636
8	4.8528	5.3807	3.9695	4.3850	3.8267	3.4662	3.1271	2.9819
9	5.1295	5.8570	4.0657	4.4258	3.8972	3.5383	3.1284	2.9908
10	6.3272	6.0578	4.2393	4.4558	3.9194	3.5937	3.1316	2.9954

Table 6
The efficiency of $\hat{\theta}^{(r)}_{\text{MSRSS1}}$ w.r.t $\hat{\theta}_{\text{SRS}}$ $m = 3$, at different values of $r$and h for $Exp(\theta)$

$h \rightarrow r_{\downarrow}$	1	2	3	4	5	10	30	$Aeff^{(r)}$
1	1.8134	1.7818	1.6767	1.7118	1.5901	1.6825	1.6122	1.6000
2	2.6100	2.5508	2.2446	2.4344	2.1207	2.2758	2.2207	2.1222
3	3.4524	3.3017	2.8316	2.9689	2.6643	2.7532	2.5552	2.4970
4	4.1373	3.9264	3.1670	3.4212	2.9662	3.0614	2.8397	2.7286
5	4.6432	4.3715	3.5318	3.7855	3.2714	3.2395	2.9889	2.8587
6	5.4756	5.2315	4.0593	4.2098	3.3937	3.3288	3.0337	2.9279
7	5.5304	5.5036	4.3116	4.2148	3.4835	3.4319	3.0950	2.9636
8	6.2324	6.0063	4.4745	4.2778	3.7514	3.5137	3.1202	2.9819
9	6.7976	6.1280	4.5034	4.2973	3.8629	3.6654	3.1941	2.9908
10	7.1865	6.3940	4.5941	4.5671	3.9163	3.6885	3.2494	2.9954

is 1 for the uniform and exponential distributions and 0 for the normal and Cauchy distributions. Tables 4–7 give the efficiency of $\hat{\theta}^{(r)}_{\text{MSRSS1}}$ with respect to $\hat{\theta}_{\text{SRS}}$ when the set size, $m = 3$, for the four distributions.

It can be seen from the tables that $eff^{(r)}(\hat{\theta}^{(r)}_{\text{MSRSS1}}; \hat{\theta}_{\text{SRS}}) > 1$. The efficiency is increasing in r for fixed h. This increase is slow for the skewed distribution and fast for the symmetric ones. The asymptotic efficiency is independent of the distribution function, $F(x)$; it is the same for all distributions.

Table 7
The efficiency of $\hat{\theta}^{(r)}_{\text{MSRSS1}}$ w.r.t $\hat{\theta}_{\text{SRS}}$ $m=3$, at different values of $r$and h for $C(\theta, 1)$

$h \to r_\downarrow$	2	3	4	5	6	10	30	$Aeff^{(r)}$
1	2.4174	1.9664	2.0132	1.7598	1.7526	1.7448	1.6276	1.6000
2	3.8360	2.7084	2.8461	2.4414	2.5224	2.3653	2.0988	2.1222
3	5.2023	3.5574	3.4316	3.0542	3.1198	2.8458	2.5467	2.4970
4	6.0235	4.0874	4.0116	3.3994	3.5448	3.2007	2.8570	2.7286
5	7.4283	4.1377	4.3114	3.6797	3.8088	3.4068	2.9646	2.8587
6	8.5046	4.9709	4.7244	3.9496	4.2472	3.6080	3.0514	2.9279
7	9.5654	5.2858	5.4363	4.4100	4.4371	3.6175	3.1465	2.9636
8	10.9265	5.4555	5.6064	4.5390	4.4016	3.6974	3.2583	2.9819
9	11.4655	5.7399	5.8746	4.6321	4.4591	3.7362	3.3627	2.9908
10	12.1943	6.0935	5.9757	4.7921	4.5023	3.8047	3.4035	2.9954

3.1.1. *The efficiency of* $\hat{\theta}^{(r)}_{\text{MSRSS1}}$ *w.r.t* $\hat{\theta}_{\text{SRS}}$ *when m and h are fixed but* $r \to \infty$

Consider the case of uniform distribution. It can be verified that

$$eff^{(\infty)}(\hat{\theta}^{(\infty)}_{\text{MSRSS1}}; \hat{\theta}_{\text{SRS}}) = \begin{cases} \dfrac{m^2(h+2)(4mh+7)}{(2mh+5)(2mh+3)}, & m \text{ and } h \text{ are odd,} \\ m^3\left(\dfrac{(h+1)(h+2)}{(mh+1)(mh+2)}\right), & m \text{ is odd but } h \text{ is even,} \\ m^3\left(\dfrac{(h+1)^2(h+2)}{2(mh+1)(mh+2)}\right), & m \text{ is even.} \end{cases}$$

Proposition 6. *The asymptotic efficiency of the overall median,* $\hat{\theta}^{(\infty)}_{\text{MSRSS1}}$, *obtained by a MSRSS* $\{Y^{(\infty)}_{ji} : j=1,2,\ldots,h; i=1,2,\ldots,m\}$, *w.r.t the median,* $\hat{\theta}_{\text{SRS}}$, *obtained by SRS, for the uniform distribution, is*

$$A\,eff^{(\infty)}(\hat{\theta}^{(\infty)}_{\text{MSRSS1}}; \hat{\theta}_{\text{SRS}}) = \begin{cases} m, & m \text{ is odd,} \\ \infty, & m \text{ is even.} \end{cases}$$

For other distributions, algebra gets more complicated. Table 8 gives the efficiency for the four distributions. It can be seen that $eff^{(\infty)}(\hat{\theta}^{(\infty)}_{\text{MSRSS1}}; \hat{\theta}_{\text{SRS}}) > 1$. The $eff^{(\infty)}(\hat{\theta}^{(\infty)}_{\text{MSRSS1}}; \hat{\theta}_{\text{SRS}})$ depends on m and h. When m is odd, the $eff^{(\infty)}(\hat{\theta}^{(\infty)}_{\text{MSRSS1}}; \hat{\theta}_{\text{SRS}})$ is decreasing in h and goes to m as $h \to \infty$. For even m, it goes to ∞ .

3.2. *The efficiency of* $\hat{\theta}^{(r)}_{\text{MSRSS2}}$ *w.r.t* $\hat{\theta}_{\text{SRS}}$

The second possible estimator of the median based on MSRSS is

$$\hat{\theta}^{(r)}_{\text{MSRSS2}} = median\{\hat{\theta}^{(r)}_j : j=1,2,\ldots,h\}.$$

Note that, $\hat{\theta}^{(r)}_1, \hat{\theta}^{(r)}_2, \ldots, \hat{\theta}^{(r)}_h$, are iid. Assume $\hat{\theta}^{(r)}_j$ has the density $g^{(r)}$.

The following result is parallel to those on SRS (see Serfling, 1980).

Under some minor regularity conditions, $\hat{\theta}^{(r)}_{\text{MSRSS2}}$ has an asymptotic normal distribution with mean θ and variance $1/4hg^{2(r)}(\theta)$. Thus, we have

$$\lim_{h\to\infty} eff^{(r)}(\hat{\theta}^{(r)}_{\text{MSRSS2}}; \hat{\theta}_{\text{SRS}}) = Aeff^{(r)}(\hat{\theta}^{(r)}_{\text{MSRSS2}}; \hat{\theta}_{\text{SRS}}) = \frac{g^{2(r)}(\theta)}{mf^2(\theta)}.$$

Tables 9–12 give the efficiency of $\hat{\theta}^{(r)}_{\text{MSRSS2}}$ with respect to $\hat{\theta}_{\text{SRS}}$ when the set size, $m=3$, the four distribution considered in the previous estimator. It can be seen that $eff^{(r)}(\hat{\theta}^{(r)}_{\text{MSRSS2}}; \hat{\theta}_{\text{SRS}}) > 1$ except for the exponential distribution. As $h \to \infty$, $\hat{\theta}^{(r)}_{\text{MSRSS1}}$ is better than $\hat{\theta}^{(r)}_{\text{MSRSS2}}$.

Table 8
The efficiency of $\hat{\theta}^{(r)}_{\text{MSRSS1}}$ w.r.t $\hat{\theta}_{\text{SRS}}$ for fixed h, as $r \to \infty$, for some distributions, when the set size, $m = 2, 3, 4, 5$

Distribution ↓	$h \to m_\downarrow$	1	2	3	4	5	10	30	$Aeff^{(\infty)}$
$U(0, 1)$	2	4.0123	4.7887	5.6818	6.6527	7.6762	12.6060	32.5771	∞
	3	5.1818	5.7681	4.0062	4.4473	3.6545	3.6080	3.2068	3.0000
	4	12.8660	12.9998	13.9145	15.6498	17.5334	26.4256	67.0058	∞
	5	11.1958	11.6628	7.2511	8.0882	6.4237	6.2217	5.3449	5.0000
$N(0, 1)$	2	2.7916	4.1705	5.3650	6.4037	7.6255	12.8784	32.7116	∞
	3	7.3795	7.0730	5.0027	4.6783	3.9484	3.7782	3.2019	3.0000
	4	15.8952	14.0888	15.2402	16.3372	18.7208	27.9786	66.1647	∞
	5	13.7335	12.8505	7.9890	8.7088	6.8876	6.3246	5.4799	5.0000
$Exp(1)$	2	1.6810	3.0653	4.4573	5.7849	7.1679	12.2657	34.7441	∞
	3	9.3919	8.7611	5.0200	5.1387	4.1876	3.8573	3.3093	3.0000
	4	17.1297	14.7899	15.4203	16.0437	18.0467	28.768	70.7205	∞
	5	16.6122	14.3481	8.9827	9.3702	6.9375	6.7811	5.3637	5.0000
$C(0, 1)$	2	–	–	5.0357	5.5407	7.8801	13.4214	33.6642	∞
	3	–	15.6655	7.1950	6.5106	5.0277	4.8172	3.6341	3.0000
	4	–	20.4072	18.2846	18.7258	20.6197	28.7759	68.1527	∞
	5	35.7114	19.2553	10.6330	10.1201	7.8922	6.8766	5.4725	5.0000

Table 9
The efficiency of $\hat{\theta}^{(r)}_{\text{MSRSS2}}$ w.r.t $\hat{\theta}_{\text{SRS}}$ when the set size, $m = 3$, at different stages, for different values of h for $U(0, \theta)$

$h \to r_\downarrow$	1	2	3	4	5	10	30	$Aeff^{(r)}$
1	1.3982	1.5413	1.2899	1.4498	1.2722	1.3320	1.2788	1.2312
2	1.8828	2.0456	1.7273	1.9368	1.7238	1.8305	1.7941	1.7059
3	2.3828	2.4622	2.1541	2.3587	2.1089	2.1996	2.1605	2.1121
4	2.6678	2.8914	2.5232	2.7283	2.4303	2.5842	2.5428	2.5576
5	2.7923	3.1780	2.7973	3.0503	2.7670	2.8627	2.7935	2.6723
6	3.4117	3.5156	3.0387	3.2495	2.8675	3.1715	2.8613	2.7024
7	3.6282	3.7791	3.1852	3.4691	2.9715	3.2490	3.0217	2.7638
8	3.7100	3.7949	3.2232	3.5449	3.2155	3.2913	3.0811	2.8261
9	3.8603	4.0510	3.7450	3.6311	3.4559	3.3809	3.1326	2.9411
10	3.9847	4.3140	3.9626	4.0677	3.4882	3.4071	3.1746	2.9523

Table 10
The efficiency of $\hat{\theta}^{(r)}_{\text{MSRSS2}}$ w.r.t $\hat{\theta}_{\text{SRS}}$ when the set size, $m = 3$, at different stages, for different values of h for $N(\theta, 1)$

$h \to r_\downarrow$	1	2	3	4	5	10	30	$Aeff^{(r)}$
1	1.6128	1.5571	1.3526	1.4425	1.2793	1.3162	1.2667	1.2401
2	2.2778	2.1024	1.8333	2.0295	1.8089	1.8358	1.7231	1.7678
3	2.7166	2.7292	2.2858	2.5152	2.2285	2.3189	2.1803	2.1361
4	3.2658	3.1992	2.7627	2.9882	2.6784	2.6334	2.4333	2.4056
5	3.8003	3.7118	3.0316	3.4106	2.9797	2.9257	2.7909	2.5030
6	4.4612	4.2985	3.5816	3.5838	3.2814	3.0461	2.9376	2.6735
7	4.6244	4.5019	3.6702	3.9072	3.3902	3.1310	3.0416	2.7054
8	4.8528	5.1964	3.8855	4.2655	3.5773	3.2519	3.1018	2.7725
9	5.1295	5.5907	4.0240	4.4089	3.7061	3.4015	3.2379	2.8792
10	6.3272	6.0017	4.2572	4.5925	3.8708	3.6851	3.3105	2.9523

Table 11
The efficiency of $\hat{\theta}^{(r)}_{\text{MSRSS2}}$ w.r.t $\hat{\theta}_{\text{SRS}}$ when the set size, $m=3$, at different stages, at different values of h for $Exp(\theta)$

$h \to r_\downarrow$	1	2	3	4	5	10	30	$Aeff^{(r)}$
1	1.8134	1.4822	1.4288	1.4404	1.2999	1.3864	1.2889	1.2500
2	2.6100	2.2255	1.9909	2.0877	1.8396	1.8805	1.8156	1.7355
3	3.4524	2.8845	2.5445	2.6038	2.3560	2.4062	2.2025	2.2027
4	4.1373	3.5207	2.9394	3.0965	2.7086	2.7689	2.5485	2.4845
5	4.6432	3.9975	3.3497	3.5029	3.0618	3.0152	2.7552	2.6790
6	5.4756	4.7667	3.5988	3.7000	3.2088	3.1410	2.8667	2.7026
7	5.5304	5.3987	4.0649	4.1492	3.3955	3.2634	2.9088	2.7534
8	6.2324	6.1909	4.3098	4.4262	3.5441	3.3097	2.9254	2.8034
9	6.7976	6.9823	4.5131	4.6890	3.8093	3.4522	2.9767	2.8725
10	7.1865	8.0430	4.7099	4.8934	3.9556	3.5856	3.0043	2.9302

Table 12
The efficiency of $\hat{\theta}^{(r)}_{\text{MSRSS2}}$ w.r.t $\hat{\theta}_{\text{SRS}}$ when the set size, $m=5$, at different stages, at different values of $h$for $C(\theta, 1)$

$h \to r_\downarrow$	1	2	3	4	5	10	30	$Aeff^{(r)}$
1	4.0317	2.2614	1.7999	1.8215	1.6324	1.6672	1.5285	1.5000
2	7.2506	4.0897	3.0302	3.1028	2.7391	2.6537	2.3754	2.0345
3	10.3300	5.9852	4.4884	4.2830	3.6681	3.4939	3.1960	2.9087
4	13.2477	7.5499	5.3385	5.2989	4.5814	4.3758	3.8490	3.6526
5	15.5618	9.9879	5.5033	5.4323	5.0176	5.0010	4.3975	4.4034
6	17.9813	11.6566	5.9676	5.7870	5.3545	5.1878	4.6565	4.5985
7	19.7657	13.0768	6.8332	6.4909	6.2988	5.7011	4.8912	4.6633
8	22.0900	14.9333	7.7099	7.6545	7.0121	5.9049	5.0093	4.7987
9	25.8663	15.7870	8.0982	7.9863	7.1099	6.0343	5.1090	4.8878
10	29.8823	16.9230	8.3649	8.1298	7.2545	6.2987	5.3323	4.9767

Table 13
The efficiency of $\hat{\theta}^{(r)}_{\text{MSRSS2}}$ w.r.t $\hat{\theta}_{\text{SRS}}$ for fixed h, as $r \to \infty$, for some distributions, when the set size, $m = 2, 3, 4, 5$

Distribution↓	$h \to m_\downarrow$	1	2	3	4	5	10	30	$Aeff^{(\infty)}$
$U(0,\theta)$	2	4.0123	6.2396	5.1192	6.3852	5.7605	6.7393	7.2317	7.6416
	3	5.1818	5.7857	4.4792	4.4473	3.7246	3.6080	3.2068	3.0000
	4	12.8660	17.2502	12.5478	15.2745	13.0559	14.5674	15.1003	15.0803
	5	11.1958	11.3640	7.5323	8.0882	6.3901	6.2217	5.3449	5.0000
$N(\theta,1)$	2	2.7916	3.2862	2.8728	3.3802	2.9837	3.4548	3.5740	3.7501
	3	7.3795	7.0730	5.0027	4.6783	3.9484	3.7782	3.3271	3.0000
	4	15.8952	18.1130	13.0361	14.6657	12.3544	13.3297	12.8811	13.0309
	5	13.7335	12.8505	7.9890	8.7088	6.8876	6.3246	5.4799	5.0000
$Exp(\theta)$	2	1.6810	1.1216	1.1173	1.1068	1.0464	0.8149	0.4376	0.0878
	3	9.3919	8.7611	5.0200	5.1387	4.1876	3.8573	3.3093	3.0000
	4	17.1297	14.7899	9.6008	9.3295	7.6815	6.4525	3.4444	0.7421
	5	16.6122	14.3481	8.9827	9.3702	6.9375	6.7811	5.3637	5.0000
$C(\theta,1)$	5	35.7114	19.2553	10.6330	10.1201	7.8922	6.8766	5.4725	5.0000

The efficiency of $\hat{\theta}^{(r)}_{\text{MSRSS2}}$ w.r.t $\hat{\theta}_{\text{SRS}}$ when m and h are fixed and $r \to \infty$ can be verified for uniform distribution to be

$$eff^{(\infty)}(\hat{\theta}^{(\infty)}_{\text{MSRSS2}}; \hat{\theta}_{\text{SRS}}) = \begin{cases} m^2\left(\dfrac{h+2}{mh+1}\right), & m \text{ and } h \text{ are odd,} \\ m^3\left(\dfrac{(h+1)(h+2)}{(mh+1)(mh+2)}\right), & m \text{ is odd but } h \text{ is even.} \end{cases}$$

The asymptotic efficiency of $\hat{\theta}^{(\infty)}_{\mathrm{MSRSS2}}$ obtained by a MSRSS $\{Y^{(\infty)}_{ji} : j=1,2,\ldots,h; i=1,2,\ldots,m\}$, w.r.t the median, $\hat{\theta}_{\mathrm{SRS}}$, obtained by SRS, for the uniform distribution, is

$$A\,\mathit{eff}^{(\infty)}(\hat{\theta}^{(\infty)}_{\mathrm{MSRSS2}}; \hat{\theta}_{\mathrm{SRS}}) = m \text{ when } m \text{ is odd.}$$

Based on Table 13, it can be seen that the $\mathit{eff}^{(\infty)}(\hat{\theta}^{(\infty)}_{\mathrm{MSRSS2}}; \hat{\theta}_{\mathrm{SRS}}) > 1$ except for the exponential distribution. The $\mathit{eff}^{(\infty)}(\hat{\theta}^{(\infty)}_{\mathrm{MSRSS2}}; \hat{\theta}_{\mathrm{SRS}})$ depends on the values of m and h. The asymptotic efficiency is independent of the distribution function, $F(x)$ when m is odd; in this case $\mathit{eff}^{(\infty)}(\hat{\theta}^{(\infty)}_{\mathrm{MSRSS2}}; \hat{\theta}_{\mathrm{SRS}})$ is decreasing in h and goes to m as $h \to \infty$. When $m=2$, the $\mathit{eff}^{(\infty)}(\hat{\theta}^{(\infty)}_{\mathrm{MSRSS2}}; \hat{\theta}_{\mathrm{SRS}})$ depends on the values of h, and it is decreasing in h. No trend when $m=4$ except for the exponential distribution, it is decreasing in h.

4. Concluding remarks

It has been seen from the results of the paper that multistage ranked set sampling, *when applicable*, can be very useful in the estimation of the distribution function and some quantiles of the underlying population. For example, with $m=3$, the empirical distribution based on MSRSS is more efficient than that based on SRS; on the average, the efficiency is at least 1.4 and it approaches 3 as the number of stages gets large. If the variable of interest is correlated with a concomitant variable that is easy to measure or measurements on it are available, then m can be taken larger and the judgment MSRSS on the variable of interest can be very efficient. Also, in this case we may sample values directly from the first, the middle and the upper third of the population of the concomitant variable (i.e. directly go to the case of $r=\infty$). Then make measurements on the corresponding judgment MSRSS for the variable of interest.

Acknowledgement

The authors are very thankful to the referee whose comments and suggestions were very useful in improving this revised version of the paper.

References

Al-Saleh, M.F., 2004. Steady state ranked set sampling and parametric estimation. J. Statist. Plann. Inference 123, 83–95.
Al-Saleh, M.F., 2007. On the similarity structure of order statistics. Commun. Statist. Theory Methods 36, 1433–1439.
Al-Saleh, M.F., Al-Kadiri, M., 2000. Double-ranked set sampling. Statist. Probab. Lett. 48, 205–212.
Al-Saleh, M.F., Al-Omari, A., 2002. Multistage ranked set sampling. J. Statist. Plann. Inference 102, 273–286.
Al-Saleh, M.F., Ananbeh, A., 2007. Estimating of the means of the bivariate normal using moving extreme ranked set sampling with concomitant variable. Statist. Papers 48, 179–195.
Al-Saleh, M.F., Samawi, H., 2007. Inclusion probability in ranked set sampling from finite population. TEST 16, 198–209.
Chen, Z., 2000. On ranked-set sample quantiles and their applications. J. Statist. Plann. Inference 83, 125–135.
Chen, Z., Bai, Z., Sinha, B.K., 2003. Ranked Set Sampling Theory and Applications. Springer, New York, USA.
Esary, J.D., Proschan, F., Walkup, D.W., 1967. Association of random variables, with application. Ann. Inst. Statist. Math. 38, 1466–1474.
Lehmann, E.L., Casella, G., 1998. Theory of Point Estimation. Springer, New York.
McIntyre, G.A., 1952. A method for unbiased selective sampling using ranked sets. Austral. J. Agricultural Res. 3, 385–390.
McIntyre, G.A., 2005. A method for unbiased selective sampling using ranked sets. Amer. Statist. 59, 230–232.
Samawi, H., Al-Sagheer, O., 2001. On the estimation of the distribution function using extreme and median ranked set sampling. Biometrical J. 43, 357–373.
Serfling, R.J., 1980. Approximation Theorems of Mathematical Statistics. Wiley, New York.
Stokes, S.L., Sager, T.W., 1988. Characterization of a RSS with application to estimating distribution functions. J. Amer. Statist. Assoc. 83, 374–381.
Takahasi, K., Wakimoto, K., 1968. On unbiased estimates of the population mean based on the sample stratified by means of ordering. Ann. Inst. Statist. Math. 20, 1–31.
Zheng, G., Al-Saleh, M.F., 2002. Modified maximum likelihood estimators based on RSS. Ann. Inst. Statist. Math. 54, 641–658.