Palestine Polytechnic University

Deanship of Graduate Studies and Scientific Research

Master Program of Mathematics

Winding Numbers and Solid Angles

Prepared by

Warda Farajallah

M.Sc. Thesis

Hebron - Palestine

2017

Winding Numbers and Solid Angles

Prepared by

Warda Farajallah

Supervisor

Dr. Ahmed Khamayseh

M.Sc. Thesis
Hebron - Palestine

Submitted to the Department of Mathematics at Palestine Polytechnic University as a partial fulfilment of the requirement for the degree of Master of Science.

Winding Numbers and Solid Angles

Prepared by

Warda Farajallah

Supervisor

Dr. Ahmed Khamayseh

Master thesis submitted an accepted, **Date September, 2017.**

The name and signature of the examining committee members

Dr. Ahmed Khamayseh	Head of committee	signature
Dr. Yousef Zahaykah	External Examiner	signature
Dr. Nureddin Rabie	Internal Examiner	signature

Palestine Polytechnic University

Declaration

I declare that the master thesis entitled "*Winding Numbers and Solid Angles* " is my own work, and hereby certify that unless stated, all work contained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgment is made in the text.

Warda Farajallah

Signature: _______________ Date: _______________

Statement of Permission to Use

In presenting this thesis in partial fulfillment of the requirements for the master degree in mathematics at Palestine Polytechnic University, I agree that the library shall make it available to borrowers under rules of library. Brief quotations from this thesis are allowable without special permission, provided that accurate acknowledgment of the source is made.

Permission for the extensive quotation from, reproduction, or publication of this thesis may be granted by my main supervisor, or in his absence, by the Dean of Graduate Studies and Scientific Research when, in the opinion either, the proposed use of the material is scholarly purpose. Any copying or use of the material in this thesis for financial gain shall not be allowed without may written permission.

Warda Farajallah

Signature: ______________ Date: ______________

Dedications

This work is dedicated to my family and my husband

Warda

Acknowledgements

Foremost, I would like to express my sincere gratitude to my supervisor Dr. Ahmed Khamayseh for his continuous support, encouragement and inspiring guidance throughout my thesis .

I also would like to express the deepest appreciation to the committee members Dr.Yousef Zahaykah and Dr. Nureddin Rabie for their valuable remarks.

Last but not the least, I would like to thank my family especially my parents and my husband for their continuous and endless support throughout all my life.

Abstract

Winding number is an important and familiar mathematical concept which is used in several medical and research areas such that in electro cardiology. In two dimensions, for any given polygon $\mathcal{P}$ we mean by winding number of $\mathcal{P}$ about a point $\mathbf{p}$ is how many time that this polygon travels around $\mathbf{p}$.

We established two methods for finding winding number for a given planar polygon $\mathcal{P}$ about an query point $\mathbf{p}$ in space, namely, direct method and projective method. In the first one we extend our works in two dimensions to three dimensions and in the second one base our approach on finding an orthogonal basis for a given plane in $\mathbb{R}^3$ and project each vector in the space to the xy-plane.

This thesis is mainly concerned with the winding number for a given polyhedron and solid angles. An analytical expression for solid angles including and explanation of triangular polyhedra with solid angles subtended by plane triangles are also introduced.

Contents

Chapter 1

Introduction

The concept of counting revolutions or winding is a dominant concept in most life, for example days are measured by revolutions of the earth, months are measured by revolutions of the moon around the earth, and our years by revolutions of the earth around the sun. Windings is an important and familiar notation in mathematics and sciences as well. To explain consider a closed, oriented curve in the xy-plane. We can imagine the curve as the path of motion of some object, with the orientation indicating the direction in which the object moves. Then need to the total number of counter clockwise turns that the object makes around the origin. The numbers that needed to be counted in the previous cases called the winding numbers which are the main concern for this thesis.

Winding numbers play a major role in science and its application on every day life, for example winding numbers appear in biomedical applications, namely, detection and counting of ovarian follicles and neuronal cells and estimation of cardiac motion from tagged MR images [6]. Another application is in robust inside-outside segmentation [13]. Winding number is also one of the most common algorithms for solving the point in polygon problem. In this problem we determine whether a query point lies inside a polygon or not [2].

Concept Winding number appears widely in the course of topology, physics, differential geometry and complex analysis. In topology, the winding number is an alternate term for the degree of a continuous mapping. In physics, winding numbers are frequently called topological quantum numbers. In differential geometry we can express the winding number of a differentiable curve γ with respect to the point $\mathbf{p} = (p_x, y_y)$ that does not lay on the curve as a line integral:

$$w(\gamma; \mathbf{p}) = \frac{1}{2\pi} \int_a^b \left(\frac{x(t) - p_x}{r^2(t)} dy - \frac{y(t) - p_y}{r^2(t)} dx \right) dt$$

where $r(t)$ is the distance between the point $\mathbf{p}$ and the point $(x(t), y(t))$ on the curve γ.

Winding numbers play a very important role throughout complex analysis with the residue theorem. In the context of complex analysis, the winding number can given by

$$w(\gamma; \mathbf{p}) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - \mathbf{p}}$$

This thesis has been in structured in five chapters, as follows

Chapter 1: A general overview materials that motivates the development and history of winding number.

Chapter 2: Introduces the basic definitions and concepts of geometry. It also includes a brief discussion for some main concepts and definitions of basic geometric subjects such that point, lines and planes. Furthermore definition of curve in complex plane with some property are given. Last section devoted to polyhedra and solid angles.

Chapter 3 : Sheds light on definitions of winding number in the plane, main properties and results. Examples to illustrate the concepts were given throughout this chapter.

Chapter 4: Winding numbers of planar polygon in the space is discussed. Implicit and explicit formula of Plane in 3D are also introduced. Direct and projective approach to find winding number of Planner Polygons were discussed.

Chapter 5 : Presents an analytical expression for the solid angle subtended by common object at some arbitrary point in space. It also introduced triangular polyhedra and expression of solid angle subtended by a plane triangle. Generalizing the 2D winding number is also mentioned in this chapter.

Chapter 2

The Mathematics of Polygons and Polyhedra

In this chapter, the basic definitions, concepts of geometry are introduced. So in the first section of this chapter we present a brief instruction of vectors. We also discuss the definition of some basic geometric subjects, namely; point, lines and planes, which are studied in details in [2]. Definition and some properties of the curve are also mentioned in this section. For the Definition and properties of the curves can be found in [11] and [21]. The second section presents the basic geometry of polygons and polyhedra. Finally the last section devoted to plane and solid angles.

2.1 Basic Geometry Definitions

In order to measure many physical quantities, such as force or velocity, we need to determine both a magnitude and a direction. Such quantities are conveniently represented as vectors. Also we cannot deal with any geometric objects without a background in vectors.

Definition 2.1. A vector is a directed line segment **ab** that can represents the displacement of a particle from its initial position **a** to a new position **b**.

Remark 2.1. *If the vector* $\boldsymbol{v}$ *starting from origin then it is called a position vector.*

We can represent vectors as geometric objects using arrows. The length of the arrow corresponds to the magnitude of the vector. The arrow points in the direction of the vector.

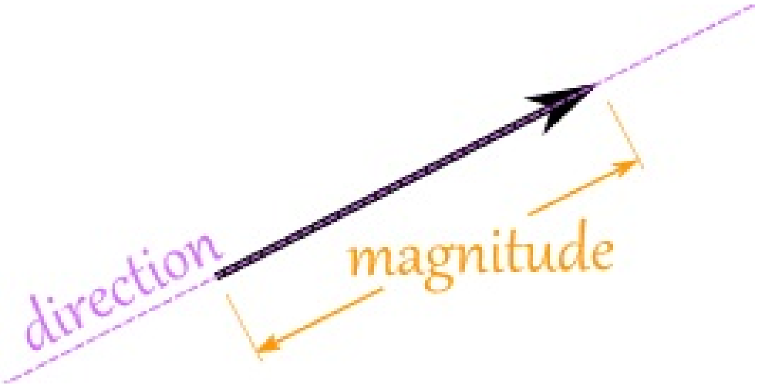

Figure 2.1: Vector Description.

Definition 2.2. The magnitude of a vector $\mathbf{v}$ with initial point $\mathbf{a} = (a_1, a_2, ..., a_n)$ and its terminal point $\mathbf{b} = (b_1, b_2, ..., b_n)$ (also called its length or norm), is given by

$$\|\mathbf{v}\| = \sqrt{\sum_{i=1}^{n} (b_i - a_i)^2}$$

Definition 2.3. Let $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{R}^n$, c is a real number, then:

1. $\mathbf{u}$ and $\mathbf{v}$ are equal if and only if they have the same magnitude and the same direction.

2. Multiplication of $\mathbf{u}$ by scalar c is a new vector which we denote by the symbol $c\ \mathbf{u}$. The magnitude of $c\ \mathbf{u}$ is $|\ c\ |\ \|\ \mathbf{u}\ \|$, and the direction of $c\ \mathbf{u}$ is the same as the direction of $\mathbf{u}$ if $c > 0$. However, the direction of $(-c\ \mathbf{u})$ is opposite of $\mathbf{u}$ provided $c > 0$.

3. To add or subtract vectors $\mathbf{u} = (u_1, u_2, ..., u_n)$ and $\mathbf{v} = (v_1, v_2, ..., v_n)$, add or subtract the corresponding coordinates:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, ..., u_n + v_n)$$
$$\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2, ..., u_n - v_n)$$

Geometrically, to add two or more vectors (drawn as arrows), join the tail of each succeeding vector to the head of the preceding one. The resultant vector is represented by an arrow from the tail of the first vector to the head of the last one, as shown in Figure 2.2 .

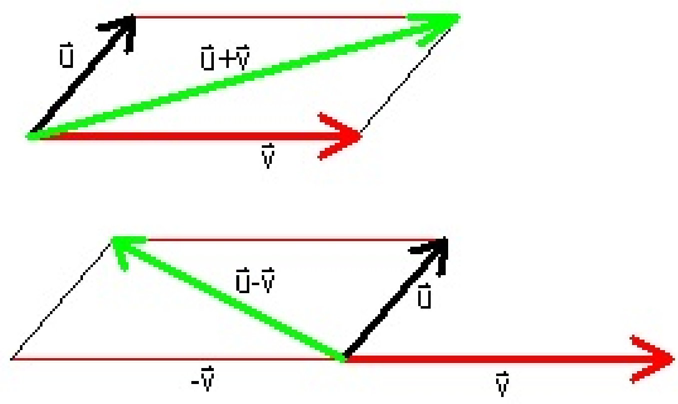

Figure 2.2: Vectors Addition and Subtraction.

4. The scalar product (also called dot product or Euclidean inner product) of two vectors **u** and **v** is defined in two distinct (though equivalent) ways:

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= u_1 v_1 + u_2 v_2 + ... + u_n v_n \\ &= ||\mathbf{u}||\ ||\mathbf{v}||\ cos\ \theta, \quad 0 \leq \theta \leq \pi\end{aligned}$$

5. If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are two vectors in $\mathbb{R}^3$, then:
   the vector product(cross product) of two vectors **u** and **v** is another vector **r** such that:

$$\text{r}=\begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Definition 2.4. If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are two vectors in $\mathbb{R}^3$, then

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta, \quad 0 \leq \theta \leq \pi$$

Remark 2.2. *The right-hand rule gives an intuitive sense of the vector* $\boldsymbol{r}$ *resulting from think of rotating* $\boldsymbol{u}$ *into* $\boldsymbol{v}$*, curling the fingers of your right hand in this angular direction. Then the extended thumb of your right hand will point in the direction of* $\boldsymbol{r}$*.*

Definition 2.5. A point is a precise location or place on a plane.

Definition 2.6. A **line** is a geometric object that is straight infinity long and infinity thin.

In space, a line is determined by a point and a vector giving the direction of the line. We have two formulas to determine the equation of line.

1. Point and direction formula: A vector equation for the line L through a point $\mathbf{p}_0$ and parallel to vector $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{p}_0 + t\mathbf{v}, \quad -\infty < t < \infty$$

   where $\mathbf{r}$ is a generic point on L.

2. Two point formula: If the line L passes through two points $\mathbf{p}_1$ and $\mathbf{p}_2$, then its vector equation is

$$\mathbf{r}(t) = \mathbf{p}_1 + t(\mathbf{p}_2 - \mathbf{p}_1), \quad -\infty < t < \infty$$

Definition 2.7. A **plane** is a flat surface extending infinitely in all direction.

Consider the plane containing the point $\mathbf{p_0}$ with the normal vector $\mathbf{n}$. A point $\mathbf{p}$ is in the plane if and only if the displacement vector, with initial point $\mathbf{p_0}$ and final point $\mathbf{p}$, is perpendicular to $\mathbf{n}$, that is:

$$\mathbf{n}.(\mathbf{p} - \mathbf{p_0}) = 0 \tag{2.1}$$

The previous equation 2.1 is the well known plane equation.

Definition 2.8. A **continuous curve** (or simply a curve) or path in the complex plane $\mathbb{C}$ is a continuous mapping γ from a closed interval $[a, b]$, into $\mathbb{C}$, where $\gamma(a), \gamma(b)$ are called initial and terminal points respectively.

Remark 2.3. *A curve γ is said to be closed when the initial and terminal points are the same ($\gamma(a) = \gamma(b)$), and simple closed if it has no self intersections except at the end points.*

Remark 2.4. *If $\gamma(t) = x(t) + iy(t), \;\; t \in [a, b]$, where $x(t), y(t)$ are real, then $x(t)$ and $y(t)$ are continuous real-valued function on $[a, b]$.*

Definition 2.9. The curve γ is said to be **smooth curve** if the function $\gamma(t) = x(t) + iy(t), \; t \in [a, b]$ is continuously differentiable on $[a, b]$ and $\gamma\prime(t) \neq 0$.

Definition 2.10. A complex-valued function f is said to be **continuous** on a continuously differentiable curve $\gamma : [a, b] \longrightarrow \mathbb{C}$ if

$$f(\gamma(t)) = u(t) + iv(t)$$

is continuous for $a \leq t \leq b$.

Definition 2.11. Suppose f is a continuous complex-valued function and $\gamma : [a, b] \longrightarrow \mathbb{C}$ is a smooth curve, then the expression

$$\begin{aligned}\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma\prime(t)dt \\ &= \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} f(\gamma(t))\gamma\prime(t)dt\end{aligned}$$

where $t_0 = a < t_1 < ... < t_j < t_{j+1} < ... < t_n = b$ and $(t_j, t_{j+1}), j = 0, 1, 2, ..., n-1$ being the interval in which γ is differentiable, is called the complex line integral of f along γ .

Definition 2.12. If γ_1, γ_2 are two curves with parametric interval $[a, b]$, we can define γ_1^* by

$$\gamma_1^* = \gamma_1(2t - a), \quad t \in [a, \frac{a+b}{2}]$$

and γ_2^* defined by

$$\gamma_2^* = \gamma_2(2t - b), \quad t \in [\frac{a+b}{2}, b]$$

are two curves with parametric intervals as indicated. Also,

$$\gamma_1^*(\frac{a+b}{2}) = \gamma_1(b) \quad \text{and} \quad \gamma_2^*(\frac{a+b}{2}) = \gamma_2(a)$$

If we assume that $\gamma_1(b) = \gamma_2(a)$, then we can define the sum or union of γ_1 and γ_2 as a continuous function on $[a, b]$ by

$$\gamma_1 + \gamma_2 \quad = \quad \begin{cases} \gamma_1^*(t), & if \quad t \in [a, \frac{a+b}{2}] \\ \gamma_2^*(t), & if \quad t \in [\frac{a+b}{2}, b] \end{cases}$$

Definition 2.13. If $\gamma : [a, b] \longrightarrow \mathbf{C}$ is a given curve, then the opposite curve $-\gamma$ of γ is defined by

$$(-\gamma)(t) = \gamma(a + b - t), \ a \le t \le b \ .$$

Therefore $-\gamma$ describe the same curve as γ, but in the reverse direction with the initial and terminal point interchanged.

2.2 Geometry of Polygons and Polyhedra

Polyhedra are beautiful three-dimensional geometrical figures that have fascinated philosophers, mathematicians and artists for millennia. The word polyhedron has slightly different meanings in geometry and algebraic geometry. In geometry, a polyhedron is simply a three-dimensional solid which consists of a collection of polygons, usually joined at their edges.

Definition 2.14. A polygon is a closed plane figure whose sides are line segments that intersect only at the endpoints.

Remark 2.5. *Polygon comes from Greek, (Poly) means "many" and (gon) means "angle".*

Definition 2.15.

1. Regular / Irregular Polygon: In a regular polygon all sides and interior angles are the same, otherwise it is irregular polygon.

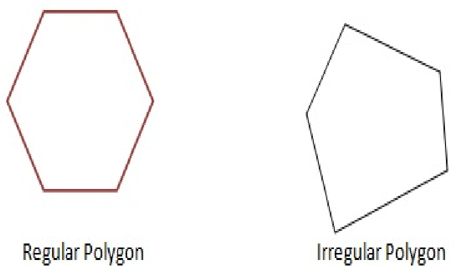

Figure 2.3: Regular and Irregular Polygon

2. Convex/Non Convex Polygon: A convex polygon has no internal angle can be more than 180°, otherwise it is non convex (concave) polygon.

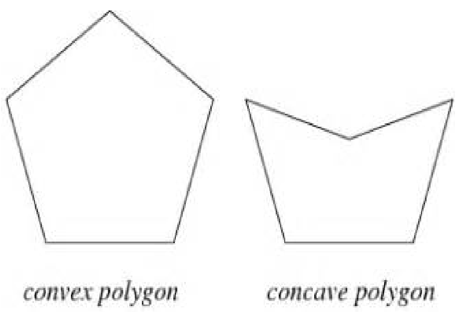

Figure 2.4: Convex and concave Polygon

3. Simple / Complex Polygon: In a complex polygon one or more sides crosses back over another side which creating multiple smaller polygons. A polygon that is not self-intersecting in this way is called a simple polygon.

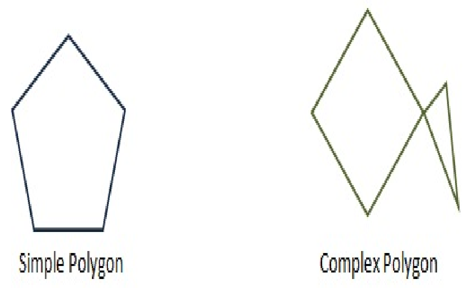

Figure 2.5: Simple and Complex Polygon

.

Definition 2.16. A polyhedron (plural polyhedrons or polyhedra) is a solid bounded by plane region. A polygons form the faces of the solid, and the segments common to these polygons are the edges of the polyhedron. Endpoints of the edges are the vertices of the polyhedron.

Remark 2.6. *Polyhedron comes from Greek, (poly) meaning "many" and (hedron) meaning "face".*

Polyhedra are often named according to the number of faces, the naming system is based on Greek prefixes for the number of faces and the root -hedron, perhaps these names are being asked to do too much; for example, to describe the inherent properties of a polyhedron and also certain of its relationships to other polyhedra.

Polyhedra	Number of faces
Tetrahedron	4
Hexahedron	6
Octahedron	8
Dodecahedron	12
Icosahedron	20

Table 2.1: Names for Some Polyhedra

Leonhard Euler ($Swiss, 1707 - 1763$) found that the number of vertices, edges, and faces of any polyhedron are related by Eulers Equation. This equation is given in the following theorem, which is stated without proof, for the proof refer to [9].

Theorem 2.1. (Eulers Equation): *The number of vertices V, the number of edges E, and the number of faces F of a polyhedron are related by the equation*

$$V + F = E + 2$$

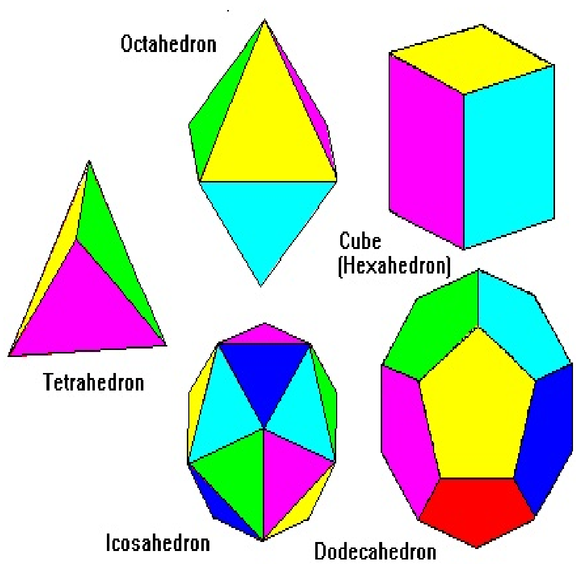

Figure 2.6: Some Polyhedra

2.3 Concept of Plane and Solid Angles

Given two intersecting lines or line segments, the amount of rotation about the point of intersection (the vertex) required to bring one into correspondence with the other is called the angle (plane angle) θ between them.

Definition 2.17.
A plane angle θ , measured in radian, is the measure of arc-length of a unit circle subtended by the angle.

The concept of an angle can be generalized from the circle to the sphere, in which case it is known as solid angle Ω subtended at a point. It is a measure of how large an object appears to an observer looking from that point.

Definition 2.18. The solid angle Ω subtended by a surface S is defined as the surface area of a unit sphere covered by the surface's projection onto the sphere which is measured in steradian.

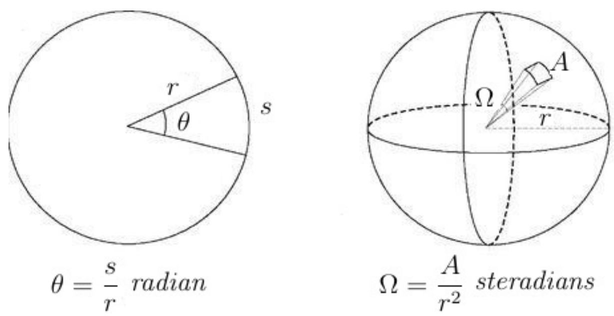

Figure 2.7: Plane and Solid Angle

The term "angle" can also be applied to the rotational offset between intersecting planes about their common line of intersection, in which case the angle is called the dihedral angle of the planes. The sides of the angle are perpendicular to the intersecting edge.

Chapter 3

Winding Number in the Plane

The concept of winding number appears in different fields as complex analysis, geometric topology, differential geometry, and physics. In this chapter alternative definitions, main properties and some of results of winding number are presented. The material of this chapter are mostly taken from [1], [8] and [21].

3.1 Definition of Winding Number

let $\mathbf{p}$ be a given point does not lie on a closed curve γ in $\mathbb{C}$, then there is a useful formula that measures how often γ winds around $\mathbf{p}$. The number of times the counting takes place, denoted by $w(\gamma; \mathbf{p})$, is called the winding number of γ about $\mathbf{p}$. Where Counter clockwise winding is assigned a positive winding number, while clockwise winding is assigned a negative winding number. In general the main concept (the formal definition) of winding number is given next in definition 3.1.

Definition 3.1. The position of an object moving in a plane relative to some fixed reference point $\mathbf{p} = (p_x, p_y)$ can be described by a vector-valued function $\mathbf{q}(t) = (x(t) - p_x, y(t) - p_y)$, where t is the time. This function $\mathbf{q}(t)$ is a parametrization for an oriented curve γ.

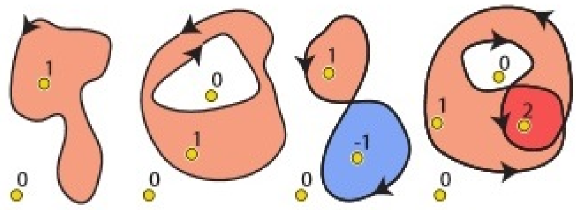

Figure 3.1: Description for Winding Numbers.

Assume that the origin does not lie on γ, so that $\mathbf{q}(t)$ is never the zero vector. For any particular value of t, the point $(x(t)-p_x, y(t)-p_y)$ on γ can be represented using polar coordinates $(r(t), \theta(t))$, where

$$r(t) = |\mathbf{q}(t)|,\ x(t)-p_x = r(t)\ cos\ \theta(t),\ y(t)-p_y = r(t)\ sin\ \theta(t) \qquad (3.1)$$

Clearly, the angle $\theta(t)$ is not uniquely defined by these equations. If $\theta_0(t)$ is one solution of the equations, then the general solution is $\theta(t) = \theta_0(t) + 2\pi s(t)$, where $s(t)$ is any integer-valued function of t. However, once we have selected an angle from among the various possible angles at any particular point on the curve, we can change this angle continuously as we move along the curve, never jumping by a multiple of 2π.

The resulting continuous angle function $\theta(t)$ is not uniquely determined by the parametrization $\mathbf{q}(t)$, but any two continuous angle functions differ by a constant, which is an integer multiple of 2π. As t varies from $t = t_1$ to $t = t_2$, the change $\theta(t_2) - \theta(t_1)$ does not depend on the particular angle function $\theta(t)$, since any two angle functions differ by a constant. This change can be interpreted as the total angle through which the position vector $\mathbf{q}$(t) turns as t varies from $t = t_1$ to $t = t_2$. In particular, $\theta(b) - \theta(a)$ represents the total signed angle through which $q(t)$ turns along the entire curve γ.

If γ is a closed curve ($\gamma(a) = \gamma(b)$), then the change $\theta(b) - \theta(a)$ must be an integer multiple of 2π. The integer $[\theta(b) - \theta(a)]/(2\pi)$ represents the number of times that

γ winds around the origin. This integer is called the index, or winding number of γ with respect to the origin.

Generally, if $\gamma : [a, b] \to \mathbb{C}$ be a closed curve and $\mathbf{p} \notin \gamma$, then the winding number of γ with respect to $\mathbf{p}$, denoted by $w(\gamma, \mathbf{p})$ is

$$w(\gamma; \mathbf{p}) = \frac{\theta_{\mathbf{p}}(b) - \theta_{\mathbf{p}}(a)}{2\pi} \tag{3.2}$$

where $\theta_{\mathbf{p}}$ is a continuous angle function of $\gamma \setminus \{\mathbf{p}\}$.

However, different definitions of winding number can be derived from the winding number formal definition. For example, in differential geometry which is a branch of mathematics that uses calculus to study the geometric properties of curves and surfaces, winding number can be defines as follow:

Definition 3.2. If we have a closed smooth curve $\gamma(t) = (x(t) - p_x, y(t) - p_y)$, then any angle function $\theta(t)$ has a derivative related to the polar coordinates θ and r by the equation:

$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{d}{dt}\left(tan^{-1}\left(\frac{y(t) - p_y}{x(t) - p_x}\right)\right) \\
&= \frac{1}{1 + \left(\frac{y(t) - p_y}{x(t) - p_x}\right)^2}\left(\frac{(x(t) - p_x)\ y'(t) - (y(t) - p_y)\ x'(t)}{(x - p_x)^2(t)}\right) \\
&= \left(\frac{(x(t) - p_x)\ y'(t) - (y(t) - p_y)\ x'(t)}{(x - p_x)^2(t) + (y - p_y)^2(t)}\right) \\
&= \left(\frac{(x(t) - p_x)\ y'(t) - (y(t) - p_y)\ x'(t)}{r^2(t)}\right)
\end{aligned}$$

Winding number $w(\gamma;\mathbf{p})$ along the curve γ with respect to the point $\mathbf{p}$ can be computed by line integral

$$\begin{aligned} w(\gamma;\mathbf{p}) &= \frac{1}{2\pi}\int_a^b d\theta(t) \\ &= \frac{1}{2\pi}\int_a^b \left(\frac{(x(t)-p_x)\ y'(t)-(y(t)-p_y)\ x'(t)}{r^2(t)}\right) dt \end{aligned} \tag{3.3}$$

Equation 3.3 can be written equivalently

$$w(\gamma;\mathbf{p}) = \frac{1}{2\pi}\int_a^b \left(\frac{x(t)-p_x}{r^2(t)}dy - \frac{y(t)-p_y}{r^2(t)}dx\right) dt$$

Our main interest is the winding number in complex analysis. We can illustrate the relation between formal definition of winding number and its formula in complex plane by next theorem.

Theorem 3.1. *let γ be a closed curve and $\mathbf{p}$ does not lie on γ, then the winding number $w(\gamma,\mathbf{p})$ of γ about $\mathbf{p}$ is given by the integral*

$$w(\gamma,\mathbf{p}) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\mathbf{p}}$$

Proof:

For any curve $\gamma : [a,b] \longrightarrow \mathbb{C}\setminus\{\mathbf{p}\}$ we can find a continuous polar coordinate expression about $\mathbf{p}$ such that

$$\gamma(t) = \mathbf{p} + r(t)\ e^{(i\theta(t))}$$

If the curve is differentiable, so that r and θ.

Using equation 3.2 and the fact that the curve is closed $r(b) = r(a)$ we get

$$\begin{aligned}\int_\gamma \frac{dz}{z-\mathbf{p}} &= \int_a^b \frac{\gamma'(t)}{\gamma(t)-\mathbf{p}}dt \\ &= \int_a^b \frac{r(t)\ i\ \theta'(t)\ e^{(i\theta(t))} + e^{(i\theta(t))}\ r'(t)}{r(t)\ e^{(i\theta(t))}}dt \\ &= \int_a^b i\theta'(t) + \frac{r'(t)}{r(t)}dt \\ &= i[\theta(b)-\theta(a)] + log(r(b)) - log(r(a)) \\ &= 2\pi i w(\gamma;\mathbf{p})\end{aligned}$$

Example 3.1. Let $\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$ where $\gamma(t) = z_0 + r\ e^{(ikt)},\ k \in \mathbb{Z}$, be a circle path with radius r around the center z_0 which is circulated k times. Then we have

$$w(\gamma,\mathbf{p}) = \begin{cases} k, & for\ \ all\ \ \mathbf{p}\ \ with\ \ |\mathbf{p}-z_0| < r \\ 0, & for\ \ all\ \ \mathbf{p}\ \ with\ \ |\mathbf{p}-z_0| > r \end{cases}$$

3.1.1 Winding Number of Polygon

Since the polygon is a piecewise smooth curve, then we can define the winding number of any point $\mathbf{p}$ does not lie on it. Winding number measures the number of times a polygon $\mathcal{P}$ encloses a point $\mathbf{p}$. To find $w(\mathcal{P};\mathbf{p})$, as an initial step and without loss of generality the geometry should be translated to that query point $\mathbf{p}$ is at the origin.

Let $\mathcal{P}$ be any closed convex polygon with vertices $(\mathbf{v}_0, \mathbf{v}_1, ..., \mathbf{v}_n)$ ($\mathbf{v}_0 = \mathbf{v}_n$) joined by a set of edges $(\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_n)$. Suppose that each edge $\mathbf{e}_i$ has an equation $\mathbf{e}_i = t\ \mathbf{v}_{i+1} + (1-t)\ \mathbf{v}_i,\ t \in [0,1]$, then by equation 3.3 we have

$$\begin{aligned}w(\mathcal{P};\mathbf{p}) &= \frac{1}{2\pi}\sum_{i=0}^{n-1}\int_0^1 \frac{x_i(t)\ y_i'(t) - y_i(t)\ x_i'(t)}{x_i^2(t)+y_i^2(t)}dt \\ &= \frac{1}{2\pi}\sum_{i=0}^{n-1}\theta_i \end{aligned} \tag{3.4}$$

where θ_i is the signed angle between the edges $\mathbf{pv}_i$ and $\mathbf{pv}_{i+1}$.

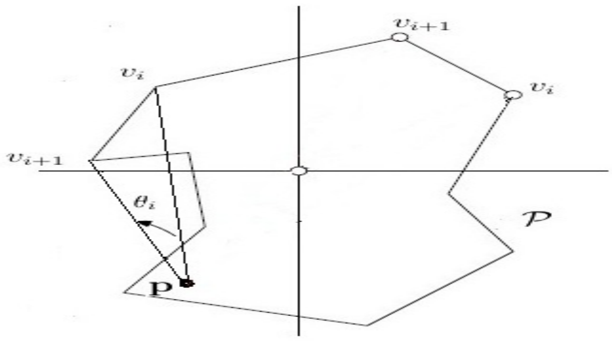

Figure 3.2: Winding Number of Non Convex Polygon

3.2 Properties of Winding Number

Their are some Properties of winding number presented in this section.

Theorem 3.2. *For any closed curve γ and $\mathbf{p} \notin \gamma$, then $w(\gamma; \mathbf{p})$ is an integer.*

Proof:

Consider a continuous function $g(t)$ on interval $[a, b]$ defined as follow

$$g(t) = \frac{1}{2\pi i} \int_a^t \frac{\gamma'(s)}{\gamma(s) - \mathbf{p}} dt, \; t \in [a, b] \tag{3.5}$$

Since $g(t)$ is continuous on $[a, b]$ and $g(a) = 0$, then

$$g'(t) = \frac{1}{2\pi i} \left\{ \frac{\gamma'(t)}{\gamma(t) - \mathbf{p}} \right\}, \; t \in [a, b] \tag{3.6}$$

Further

$$\begin{aligned} \frac{d}{dt}\{e^{(-2\pi i g(t))}\,(\gamma(t) - \mathbf{p})\} &= e^{(-2\pi i g(t))}\{\gamma'(t) - 2\pi i g'(t)(\gamma(t) - \mathbf{p})\} \\ &= e^{(-2\pi i \; g(t))}\,\{\gamma'(t) - \gamma'(t)\}, \; by \; 3.6 \\ &= 0, \; t \in [a, b] \end{aligned}$$

Consequently $e^{(-2\pi i g(t))}(\gamma(t) - \mathbf{p})$ must reduce to a constant, say M on $[a, b]$, that is

$$\gamma(t) - \mathbf{p} = M \; e^{(2\pi i g(t))} \tag{3.7}$$

Making $t = a$ in equation 3.7 , we find

$$\gamma(a) - \mathbf{p} = Me^{(2\pi i g(a))} = M$$

clearly, $M \neq 0$ since $(\gamma(t) \neq \mathbf{p})$. Substitute M in 3.7 , we have

$$e^{(2\pi i g(t))} = \frac{\gamma(t) - \mathbf{p}}{\gamma(a) - \mathbf{p}} \tag{3.8}$$

Letting $t = b$ in equation 3.8 and since γ is closed $(\gamma(a) = \gamma(b))$, therefore

$$e^{(2\pi i g(b))} = 1$$

Which shows that $g(b) = K$, for some integer K. Hence by 3.5

we have $w(\gamma; \mathbf{p}) = K$, an integer.

Theorem 3.3. *Considered a winding number as a function of* $\boldsymbol{p}$*, then the mapping* $\boldsymbol{p} \longmapsto w(\gamma; \boldsymbol{p})$ *is a continuous function of* $\boldsymbol{p}$ *for all* $\boldsymbol{p} \notin \gamma$*, where* γ *is a closed curve.*

Proof:

Let $\varepsilon > 0$ be given, want $\delta > 0$ such that if $|\mathbf{p} - \mathbf{p}_0| < \delta$ then $|w(\gamma, \mathbf{p}) - w(\gamma, \mathbf{p}_0)| < \varepsilon$.

Now

$$\begin{aligned} |w(\gamma, \mathbf{p}) - w(\gamma, \mathbf{p_0})| &= \frac{1}{2\pi} \left| \int_\gamma \left(\frac{1}{z - \mathbf{p}} - \frac{1}{z - \mathbf{p_0}} \right) dz \right| \\ &\leq \frac{1}{2\pi} \int_\gamma \left| \frac{\mathbf{p} - \mathbf{p}_0}{(z - \mathbf{p})(z - \mathbf{p}_0)} \right| |dz| \\ &= \frac{1}{2\pi} \int_\gamma \frac{|\mathbf{p} - \mathbf{p}_0|}{|z - \mathbf{p}||z - \mathbf{p}_0|} |dz| \end{aligned}$$

Suppose $\mathbf{p}, \mathbf{p}_0 \in \Delta(\mathbf{p}_0, d/2) = \{w \in \mathbb{C} : |w - \mathbf{p}_0| < d/2\}$, where $d = dist(\mathbf{p}_0, \gamma)$. Then $\forall z \in \gamma$ we have $|z - \mathbf{p}_0| \geq d$ and $|z - \mathbf{p}| \geq d/2, \quad \forall \mathbf{p} \in \Delta(\mathbf{p}_0, d/2), \mathbf{p} \neq \mathbf{p}_0$.

Hence

$$\begin{aligned} |w(\gamma, \mathbf{p}) - w(\gamma, \mathbf{p}_0)| &\leq \frac{|\mathbf{p} - \mathbf{p}_0|}{\pi d^2} \int_\gamma |dz| \\ &= \frac{|\mathbf{p} - \mathbf{p}_0|}{\pi d^2} L_\gamma \end{aligned}$$

For this to be $< \varepsilon$, we take $\delta < \dfrac{\pi d^2}{L_\gamma} \varepsilon$. Hence $w(\gamma; p)$ is continuous at $\mathbf{p}_0$.

Corollary 3.4. *Winding number $w(\gamma; \boldsymbol{p})$ is locally constant. This means that if $\boldsymbol{p} \notin \gamma$ then we can find small neighbourhood of $\boldsymbol{p}$ where $w(\gamma; \boldsymbol{p})$ is constant.*

This is clearly by continuity of winding number as a function of $\mathbf{p}$ and since it also takes integer values, so it must be constant in a neighbourhood of $\mathbf{p}$.

Theorem 3.5. *Let γ be a closed curve, then $w(\gamma, \boldsymbol{p}) = 0$ in the unbounded component of γ.[21]*

Proof:

Suppose that $\gamma \subset \Delta_R$, see Figure 3.3 . Then for any $\mathbf{p}$ in the unbounded component of $\mathbb{C} \setminus \overline{\Delta}_R$, we have $|z - \mathbf{p}| \geqslant |\mathbf{p}| - |z| > |\mathbf{p}| - R$

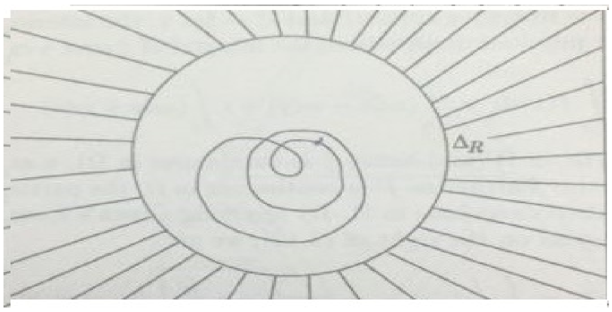

Figure 3.3:

Then

$$|w(\gamma; \mathbf{p})| = \left| \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - \mathbf{p}} \right| < \frac{L(\gamma)}{2\pi |\mathbf{p}| - R}$$

So $|w(\gamma; \mathbf{p})| < 1$, for $|\mathbf{p}|$ sufficiently large. But $|w(\gamma; \mathbf{p})|$ must be a non-negative integer and hence must be zero. Thus we must have $w(\gamma; \mathbf{p}) = 0$. By continuity of $w(\gamma; \mathbf{p})$, it follows that it vanishes in the unbounded component.

The concept of winding number is useful to characterize what is meant by the inside (interior) and the outside (exterior) of closed curve γ .

Definition 3.3. let γ be a closed curve in $\mathbb{C}$ then

$$Int(\gamma) = \{\mathbf{p} \notin \gamma : w(\gamma; \mathbf{p}) \neq 0\}$$
$$Ext(\gamma) = \{\mathbf{p} \notin \gamma : w(\gamma; \mathbf{p}) = 0\}$$

Theorem 3.6. *If γ is the sum of the two paths γ_1 and γ_2 in $\mathbb{C}$, then for every $\boldsymbol{p} \notin \gamma$*

1. $w(\gamma; \boldsymbol{p}) = w(\gamma_1; \boldsymbol{p}) + w(\gamma_2; \boldsymbol{p})$.
2. $w(-\gamma_1; \boldsymbol{p}) = -w(\gamma_1; \boldsymbol{p})$.

Proof.

The result follow directly from the definition.

3.2.1 Homotopy of Winding Number

Homotopy is a topological concept which identifies two geometric objects. In particular for two curves, if the first curve can be deformed continuously to another, these two curves are homotopic and are then considered equivalent.

Definition 3.4. Let $\gamma_0, \gamma_1 : [a, b] \longrightarrow \mathbb{C}$ are two paths with property

$$\gamma_0(a) = \gamma_1(a) = c \text{ and } \gamma_0(b) = \gamma_1(b) = d$$

We say that γ_0 and γ_1 are homotopic (or γ_0 is homotopic to γ_1), written $\gamma_0 \simeq \gamma_1$, if there is a continuous function $H : [a, b] \times [0, 1] \longrightarrow \mathbb{C}$ such that:

1. $H(s, t) \in \mathbb{C}$ for all $s \in [a, b], \quad t \in [0, 1]$
2. $H(s, 0) = \gamma_0(s)$ and $H(s, 1) = \gamma_1(s)$ for all $s \in [a, b]$.
3. $H(a, t) = \gamma_0(a) = \gamma_1(a) = c$ and $H(b, t) = \gamma_0(b) = \gamma_1(b) = d$ for all $t \in [0, 1]$.

Remark 3.1. *Such function H in definition 3.4 is called a homotopy between γ_0 and γ_1.*

Lemma 3.7. *Any two curves in a disk $D(x, r)$ with the same initial and terminal points are homotopic.*

Proof:

let $\gamma_0, \gamma_1 : [0, 1] \longrightarrow D(x, r)$ such that $\gamma_0(0) = \gamma_1(0) = c$ and $\gamma_0(1) = \gamma_1(1) = d$. Define

$$H(s, t) : [0, 1] \times [0, 1] \longrightarrow D(x, r)$$

such that $H(s, t) = (1 - t)\ \gamma_0(s) + t\ \gamma_1(s)$, then check that:

1. clearly $H(s, t) = (1 - t)\gamma_0(s) + t\gamma_1(s) \in D(x, r)$ for all $0 \leq s, t \leq 1$
2. $H(s, 0) = (1 - 0)\ \gamma_0(s) + 0\ \gamma_1(s) = \gamma_0(s)$ for all $0 \leq s \leq 1$.
3. $H(s, 1) = (1 - 1)\ \gamma_0(s) + 1\ \gamma_1(s) = \gamma_1(s)$ for all $0 \leq s \leq 1$.
4. $H(0, t) = (1 - t)\ \gamma_0(0) + t\ \gamma_1(0) = \gamma_0(0) = \gamma_1(0) = c$ for all $0 \leq t \leq 1$
5. $H(1, t) = (1 - t)\ \gamma_0(1) + t\ \gamma_1(1) = \gamma_0(1) = \gamma_1(1) = d$ for all $0 \leq t \leq 1$

so we have $\gamma_0 \simeq \gamma_1$ for all (γ_0, γ_1) in $D(x, r)$ with the same initial and terminal points.

Theorem 3.8. *Winding number is a homotopy invariant, which means that if two closed curves are homotopic in $\mathbb{C} \setminus \{\boldsymbol{p}\}$, then they have the same winding number.*

Proof.

Let $H : [a, b] \times [0, 1] \longrightarrow \mathbb{C} \setminus \{\mathbf{p}\}$ be a homotopy between γ_0 and γ_1. The function $\theta : [a, b] \times [0, 1] \longrightarrow \mathbb{R} \setminus \{2\pi\mathbb{Z}\}$ defined by $\theta(s, t) = arg(H(s, t) - \mathbf{p})$ is continuous in both variables, and can be lifted (by the lifting lemma) into a function $\overline{\theta}$.

Then you can define the continuous function

$$w(t) = \frac{\overline{\theta}(b, t) - \overline{\theta}(a, t)}{2\pi} \in \mathbb{Z}$$

Since $\mathbb{Z}$ is discrete, it has to be a constant map, so the winding numbers of $H(s, 0) = \gamma_0(s)$ and $H(s, 1) = \gamma_1(s)$ are the same.

3.3 Results of Winding Number

This section is intended to provide some results of winding number such that the two main integral theorems in complex analysis, Cauchy integral formula and Residue theorem.

Cauchy integral formula is a central theorem in complex analysis and an important result owing to Cauchy theorem, shows that the values of an analytic function f on the boundary of a closed curve γ determine the values of f interior to γ. Let we start with the one of the most important theorems in complex analysis which is Cauchy Theorem. The proof of Cauchy Theorem depends on green theorem [2].

Theorem 3.9. *Let f be analytic on and inside a simple closed curve γ and $f'(z)$ is also continuous on and inside γ, then*

$$\int_{\gamma} f(z)dz = 0$$

Theorem 3.10. (Cauchy Integral Formula): *Suppose Ω is an open subset of the complex plane $\mathbb{C}$, $f : \Omega \longrightarrow \mathbb{C}$ be a analytic function on and inside a positively oriented simple closed curve γ. Then at any interior point $\boldsymbol{p}$*

$$f(\boldsymbol{p}) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - \boldsymbol{p}} dz \tag{3.9}$$

Proof:

Inside the curve γ, drawing a small circle γ_r with radius r around the point $\mathbf{p}$ small enough to be completely inside γ, see Figure 3.4 .

From Cauchy's Theorem we can deform the curve γ into γ_r.

$$\int_{\gamma} \frac{f(z)}{z - \mathbf{p}} dz = \int_{\gamma_r} \frac{f(z)}{z - \mathbf{p}} dz$$

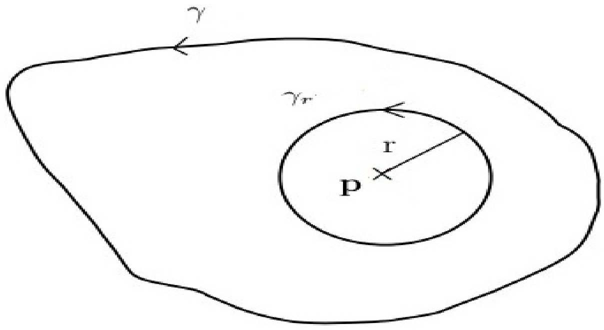

Figure 3.4: Circle γ_r inscribed in curve γ

Rewrite the second integral as

$$\int_{\gamma_r} \frac{f(z)}{z-\mathbf{p}} dz) = f(\mathbf{p}) \int_{\gamma_r} \frac{dz}{z-\mathbf{p}} + \int_{\gamma_r} \frac{f(z)-f(\mathbf{p})}{z-\mathbf{p}} dz \tag{3.10}$$

Using polar coordinates, $z = \mathbf{p} + r\ e^{(i\theta)},\ 0 \leq \theta \leq 2\pi,$ the first integral on the right in equation 3.10 is computed to be

$$f(\mathbf{p}) \int_{\gamma_r} \frac{dz}{z-\mathbf{p}} = f(\mathbf{p}) \int_0^{2\pi} \pi \frac{i\ r\ e^{(i\ \theta)}}{r\ e^{(i\ \theta)}} d\theta = 2\pi\ i\ f(\mathbf{p}) \tag{3.11}$$

Because f is continuous, for all $\varepsilon > 0$, there exist $r > 0$ such that $|f(z) - f(\mathbf{p})| < \varepsilon$ for small enough $|z - \mathbf{p}| = r$.
Then

$$\begin{aligned} \left| \int_{\gamma_r} \frac{f(z)-f(\mathbf{p})}{z-\mathbf{p}} dz \right| &\leq \int_{\gamma_r} \left| \frac{f(z)-f(\mathbf{p})}{z-\mathbf{p}} \right| |dz| \\ &= \int_{\gamma_r} \frac{|f(z)-f(\mathbf{p})|}{|z-\mathbf{p}|} |dz| \\ &< \frac{\varepsilon}{r} \int_0^{2\pi} r\ d\theta \\ &= 2\pi\varepsilon \end{aligned}$$

Thus as $\varepsilon \longrightarrow 0$, the second integral in equation 3.10 vanishes. Hence equation 3.11 and 3.10 yield Cauchy's Integral Formula.

Theorem 3.11. *If γ winds multiple loops around the point $\boldsymbol{p}$, then equation 3.9 becomes*

$$w(\gamma;\boldsymbol{p})f(\boldsymbol{p}) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-\boldsymbol{p}}dz \tag{3.12}$$

Proof:

For $\mathbf{p}, z \in \Omega$, consider

$$F(z) = \begin{cases} \dfrac{f(z)-f(\mathbf{p})}{z-\mathbf{p}}, & z \neq \mathbf{p} \\ f'(\mathbf{p}) , & z = \mathbf{p} \end{cases}$$

Clearly

$$\lim_{z\to\mathbf{p}} F(z) = f'(\mathbf{p}) = F(\mathbf{p})$$

and so $F(z)$ is continuous at $\mathbf{p}$. Also $f(z) - f(\mathbf{p})$ and $(z-\mathbf{p})^{-1}$ are both analytic functions on $\Omega \setminus \{\mathbf{p}\}$. Thus F is analytic on $\Omega \setminus \{\mathbf{p}\}$. By continuity of F at $\mathbf{p}$, we have that $F(z)$ is analytic on Ω. Also by Cauchy's Theorem

$$\int_\gamma F(z)dz = \int_\gamma \frac{f(z)-f(\mathbf{p})}{z-\mathbf{p}}\, dz = 0 \tag{3.13}$$

Further, by definition of winding number, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-\mathbf{p}}dz - f(\mathbf{p})w(\gamma;\mathbf{p}) &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-\mathbf{p}}dz - f(\mathbf{p})\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-\mathbf{p}} \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(\mathbf{p})}{z-\mathbf{p}}dz \qquad (3.14)\end{aligned}$$

Hence by equations 3.13 and 3.14 we have

$$w(\gamma;\mathbf{p})f(\mathbf{p}) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-\mathbf{p}}dz$$

Example 3.2. Let γ be a closed curve define as a circle $|z-i|=3$, evaluate

$$\int_\gamma \frac{sin\ \pi z^2 + cos\ \pi z^2}{(z-1)(z-2)}dz$$

Solution:

By partial fraction

$$\frac{sin\ \pi z^2 + cos\ \pi z^2}{(z-1)(z-2)} = \frac{sin\ \pi z^2 + cos\ \pi z^2}{(z-2)} - \frac{sin\ \pi z^2 + cos\ \pi z^2}{(z-1)}$$

Therefor

$$\int_\gamma \frac{sin\ \pi z^2 + cos\ \pi z^2}{(z-1)(z-2)}dz = \int_\gamma \frac{sin\ \pi z^2 + cos\ \pi z^2}{(z-2)}dz - \int_\gamma \frac{sin\ \pi z^2 + cos\ \pi z^2}{(z-1)}dz$$

Apply Cauchy's Integral Formula

$$\begin{aligned} \int_\gamma \frac{sin\ \pi z^2 + cos\ \pi z^2}{(z-2)}dz &= 2\pi i f(2) \\ &= 2\pi i\ (sin\ \pi(2)^2 + cos\ \pi(2)^2) \\ &= 2\pi i \end{aligned}$$

Also

$$\begin{aligned} \int_\gamma \frac{sin\ \pi z^2 + cos\ \pi z^2}{(z-1)}dz &= 2\pi i f(1) \\ &= 2\pi i\ (sin\ \pi(1)^2 + cos\ \pi(1)^2) \\ &= -2\pi i \end{aligned}$$

Hence

$$\int_\gamma \frac{sin\ \pi z^2 + cos\ \pi z^2}{(z-1)(z-2)}\ dz = (2\pi i) - (-2\pi i) = 4\pi i$$

We shall now formulate and prove Cauchy Residue Theorem. It is one of the most important theorems in complex analysis. We note that it includes the integral theorem and the integral formula of Cauchy. By Cauchy Residue Theorem we integrate functions with isolated singularities over closed curves. It is very useful in applications such as evaluation of definite integrals of various types. We introduce definition of singularity point of the function.

Definition 3.5. Let $\mathbf{p}$ be a point in $\mathbb{C}$ and f be a function analytic in $D(\mathbf{p};\varepsilon)\setminus\{\mathbf{p}\}$, for some $\varepsilon > 0$. Then the function f is said to have an isolated singularity at $\mathbf{p}$. There are three types of singular points:

1. A function f has a removable singularity at $\mathbf{p}$ if it extends to a function analytic on $D(\mathbf{p};\varepsilon)$.
2. A function f has a pole at $\mathbf{p}$ if $(z-\mathbf{p})^n f(z)$ extends to a function analytic on $D(\mathbf{p};\varepsilon)$ (for some n; the least n that will do is called the order of the pole.)
3. A function f has an essential singularity otherwise.

Definition 3.6. If the complex function f has an isolated singularity at the point $\mathbf{p}$, then f has a Laurent series representation

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-\mathbf{p})^k =$$
$$\dots + \frac{a_{-2}}{(z-\mathbf{p})^2} + \frac{a_{-1}}{(z-\mathbf{p})} + a_0 + a_1(z-\mathbf{p}) + a_2(z-\mathbf{p})^2 + \dots \; z \in D(\mathbf{p};\varepsilon)\setminus\{\mathbf{p}\}$$

with

$$a_k = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-\mathbf{p})^{k+1}}dz$$

where C is any circle centred at $\mathbf{p}$ and lying inside D. The negative part of series $\sum_{k=-\infty}^{-1} a_k(z-\mathbf{p})^k$ is referred to as the principal part of the series.

Definition 3.7. Let $\mathbf{p}$ be an isolated singularity of the complex valued function $f(z)$, we define the residue of $f(z)$ at $\mathbf{p}$ written $Res(f;\mathbf{p})$ as follow

$$Res(f;\mathbf{p}) = a_{-1} = \frac{1}{2\pi i}\int_C f(z)dz$$

where $C = C(\varepsilon;\mathbf{p})$ be a circle about $\mathbf{p}$ that contains no other singularity of f and a_{-1} is the coefficient of $\dfrac{1}{(z-\mathbf{p})}$ in the Laurent series given in definition 3.6 .

In calculation we need other way to find the residue number of a function with respect to its singularity.

Theorem 3.12. *If the function f has an isolating singularity of order one at $\boldsymbol{p}$, then*

$$Res(f;\boldsymbol{p}) = \lim_{z\to\boldsymbol{p}} f(z)\ (z-\boldsymbol{p})$$

and if $\boldsymbol{p}$ is an isolating singularity of order n then

$$Res(f;\boldsymbol{p}) = \frac{1}{(n-1)!}\ \lim_{z\to\boldsymbol{p}} \frac{d^{n-1}}{dz^{n-1}} f(z)\ (z-\boldsymbol{p})^n$$

Proof:

First part of theorem. Since $f(z)$ has a simple pole at $\mathbf{p}$, then the Laurent expansion of f about $\mathbf{p}$ has the form

$$f(z) = \frac{a_{-1}}{(z-\mathbf{p})} + a_0 + a_1(z-\mathbf{p}) + ...$$

Multiplying both sides by the term $(z-\mathbf{p})$, we get

$$(z-\mathbf{p})f(z) = a_{-1} + (z-\mathbf{p})a_0 + a_1(z-\mathbf{p})^2 + ...$$

Taking the limit as $z\to\mathbf{p}$, we obtain

$$\begin{aligned}\lim_{z\to\mathbf{p}}(z-\mathbf{p})f(z) &= \lim_{z\to\mathbf{p}} a_{-1} + 0 + 0 + ...\\ &= a_{-1}\\ &= Res(f;\mathbf{p})\end{aligned}$$

Proof the second part of theorem. Since $f(z)$ has a simple pole of order n at $\mathbf{p}$, then the Laurent expansion of f about $\mathbf{p}$ has the form

$$f(z) = \frac{a_{-n}}{(z-\mathbf{p})^n} + ... + \frac{a_{-2}}{(z-\mathbf{p})^2} + \frac{a_{-1}}{(z-\mathbf{p})} + a_0 + a_1(z-\mathbf{p}) + ... \tag{3.15}$$

Multiply 3.15 by $(z-\mathbf{p})^n$

$$(z-\mathbf{p})^n f(z) = a_{-n} + ... + a_{-2}(z-\mathbf{p})^{n-2} + a_{-1}(z-\mathbf{p})^{n-1} + a_0(z-\mathbf{p})^n + a_1(z-\mathbf{p})^{n+1} + ...$$

Differentiate both sides $(n-1)$ times:

$$\frac{d^{n-1}}{dz^{n-1}}(z-\mathbf{p})^n f(z) = (n-1)!\ \ a_{-1} + n!\ \ a_0(z-\mathbf{p}) + ... \tag{3.16}$$

Taking limit of equation 3.16 as $z \to \mathbf{p}$

$$\begin{aligned}\lim_{z\to\mathbf{p}} \frac{d^{n-1}}{dz^{n-1}}(z-\mathbf{p})^n f(z) &= \lim_{z\to\mathbf{p}}(n-1)!a_{-1} + 0 + ... \\ &= (n-1)!\ a_{-1} \\ &= (n-1)!\ Res(f;\mathbf{p})\end{aligned}$$

Therefore

$$Res(f;\mathbf{p}) = \frac{1}{n-1!}\ \lim_{z\to\mathbf{p}}\ \frac{d^{n-1}}{dz^{n-1}} f(z)(z-\mathbf{p})^n$$

Theorem 3.13. (Cauchy Residue Theorem): *Let Ω be an open subset of complex plane and $S = (\boldsymbol{p}_1, \boldsymbol{p}_2, ..., \boldsymbol{p}_n)$ be the finite set of isolated singularity points of f. If the function $f : \Omega \setminus \{S\} \longrightarrow \mathbb{C}$ be a holomorphic function, then*

$$\int_\gamma f(z)dz = (2\pi i)\ \sum_{k=1}^{n}\ w(\gamma;\boldsymbol{p}_k)\ Res(f,\boldsymbol{p}_k)$$

Where γ be a closed curve in $\Omega \setminus \{S\}$.

Proof:

As a first step, drawing closed paths $(\gamma_1, \gamma_2, ..., \gamma_n)$ around each singularity $(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n)$ such that γ_j are mutually disjoint and all lie inside γ, for all $j = 1, 2, ...n$.

Since $\mathbf{p}_j$ $(j = 1, 2, ..., n)$ is an isolated singularity of f, then f has Laurent expansion of the form

$$f(z) = \sum_{k=-\infty}^{\infty} a_{kj}(z-\mathbf{p}_j)^k \tag{3.17}$$

We Consider the principal part

$$g_j(z) = \sum_{k=-\infty}^{-1} a_{kj}(z-\mathbf{p}_j)^k$$

Since each $\mathbf{p}_j$ lies inside γ, we have

$$\begin{aligned}\int_\gamma g_j(z)dz &= a_{-1j}\int_\gamma \frac{dz}{z-\mathbf{p}_j}\\ &= 2\pi i\ a_{-1j}\ w(\gamma;\mathbf{p}_k)\\ &= 2\pi i\ w(\gamma;\mathbf{p}_k)\ Res(f;\mathbf{p}_k)\end{aligned}$$

Let $F(z)$ be a function defined as follow

$$F(z) = f(z) - \sum_{k=1}^{n} g_k(z) \tag{3.18}$$

Note that $F(z)$ is analytic on $\Omega \setminus \{(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n)\}$, since $f(z)$ is analytic on $\Omega \setminus \{(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n)\}$, and each $g_j(z)$ is analytic on $\mathbb{C} \setminus \{(\mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n)\}$. Thus all $\mathbf{p}_j, (j = 1, 2, ..., n)$ are removable singularities of F and hence, by 3.17 , we write

$$F(z) = \sum_{k=0}^{\infty} a_{kj}(z-\mathbf{p}_j)^k - \sum_{k=1,k\neq j}^{n} g_k(z)$$

Since the function $g_k, k \neq j$ is analytic on $\mathbb{C} \setminus \{\mathbf{p}_i\}$, $\lim_{z\to\mathbf{p}_j} F(z)$ exists and equals

$$a_{0j} - \sum_{k=1,k\neq j}^{n} g_k(z)$$

If the value of F is corrected at the points $\mathbf{p}_j$, then F becomes analytic on all of Ω. Hence by Cauchy Theorem

$$\int_\gamma F(z)dz = 0$$

and so by equation 3.18

$$\int_\gamma f(z)dz = \sum_{k=1}^{n}\int_\gamma g_k(z)dz = 2\pi i \sum_{k=1}^{n} (w(\gamma;\mathbf{p}_k)\ Res(f,\mathbf{p}_k)$$

[21]

Example 3.3. Use Cauchy Residue Theorem to evaluate

$$I = \int_\gamma \frac{dz}{z^2+a^2}$$

Where $a > 0$ and γ is the non simple contour of Figure 3.5

The residue of $\frac{1}{z^2+a^2}$ is

$$Res(f(z); ia) = \lim_{z \to ia} \frac{1}{z^2+a^2}(z - ia) = \frac{1}{2ai}$$

and

$$Res(f(z); -ia) = \lim_{z \to -ia} \frac{1}{z^2+a^2}(z + ia) = \frac{1}{-2ai}$$

By Figure 3.5 $w(ia) = 2$ and $w(-ai) = 1$. Thus

$$I = 2\pi i \left[2\left(\frac{1}{2ai}\right) - \frac{1}{2ai} \right] = \frac{\pi}{a}$$

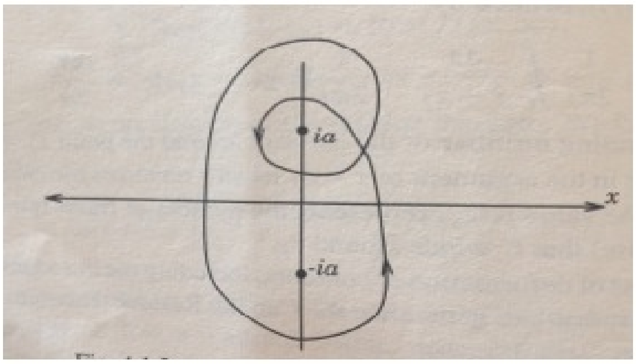

Figure 3.5: Non simple curve for example 3.3

In most applications, γ will be a simple closed curve and it is frequently the case that each $w(\gamma; \mathbf{p}_i)$ is either 0 or 1 . Then by this observation, we can simplify theorem to be Residue Theorem for simple closed curves.

Theorem 3.14. (Residue Theorem for Simple Closed Curves): *Let γ be a simple close curve in $\mathbb{C}$ and under hypothesis of theorem 3.13 we have.*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f, z_k)$$

Proof:

We enclose each of the points $\mathbf{p}_k$ by small non-intersecting closed curves, each of which lies within $\gamma : \gamma_1, \gamma_2, ..., \gamma_n$ and is connected to the main closed crve by cross cuts.

By Cauchy Theorem we have

$$\int_{\gamma} f(z)dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)dz \tag{3.19}$$

Substituting

$$2\pi i \; Res(f; \mathbf{p}_k) = \int_{\gamma_k} f(z)dz$$

in equation 3.19 we get

$$\int_{\gamma} f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f; \mathbf{p}_k)$$

Chapter 4

Winding Number in Planar Space

The previous chapters was devoted for the main concept of winding number in the plane and some of related theorems, in this chapter winding number in the space will be discussed.
Implicit and explicit formula of plane in 3D are introduced . Winding Number of planar in space were found using direct and projective approach.

4.1 Implicit Formula of the Plane in 3D

Geometrically, a plane is just a linear object in more than two dimensions, and linear means that it has constant slope (in each direction). A plane is determined by a point $\mathbf{p}_0 = (x_0, y_0, z_0)$ in the plane and a non zero normal vector $\mathbf{n} = (a, b, c)$. If $\mathbf{p} = (x, y, z)$ is any other point in the plane and $\mathbf{r_0}$,$\mathbf{r}$ are the position vectors of the points $\mathbf{p}_0$ and $\mathbf{p}$, respectively, then an equation of the plane is

$$\mathbf{n}.(\mathbf{r} - \mathbf{r_0}) = 0$$

or equivalently the familiar equation of the plane

$$ax + by + cz = d \tag{4.1}$$

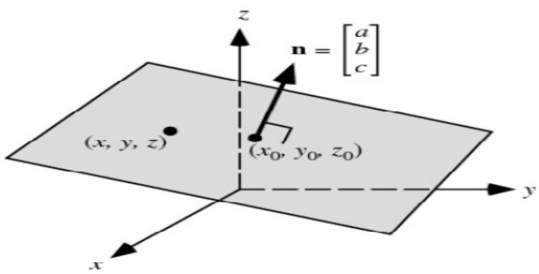

Figure 4.1: Plane Description

4.2 Parametric Formula of the Plane

Alternatively, a plane may be described parametrically by a point and two vectors lying on it, and the vector equation given by

$$\mathbf{r} = \mathbf{r_0} + s\mathbf{u} + t\ \mathbf{v},\ \ s, t \in \mathbb{R} \tag{4.2}$$

Where $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are two vectors lying on the plane and parallel to the plane but not parallel between them. So the parametric equations of a plane is given by

$$x = x_0 + su_1 + tv_1$$
$$y = y_0 + su_2 + tv_2$$
$$z = z_0 + su_3 + tv_3$$

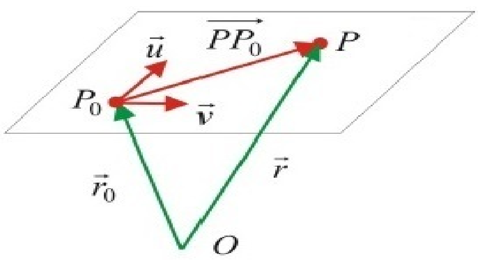

Figure 4.2: Point Two vectors Plane

4.3 Winding Number of Planar Polygons in Space

For any polygon $\mathcal{P}$ in the space or any high dimensional space, a winding number could be found by two approach. The first one is geometric approach which is a direct ways as it is a generalization for a 2D space. The second one is a linear algebra approach.

4.3.1 Direct Approach

This method finds the winding number of a given polygon in space by using the equation 3.4 that was introduced in the previous chapter. Connect each vertex v_i of the given polygon with the query point $\mathbf{p_0}$, then the winding number will equal the sum of the signed angles θ_i that formed between each line segment $\mathbf{pv_i}$ and $\mathbf{pv_{i+1}}$, see Figure 4.3 .

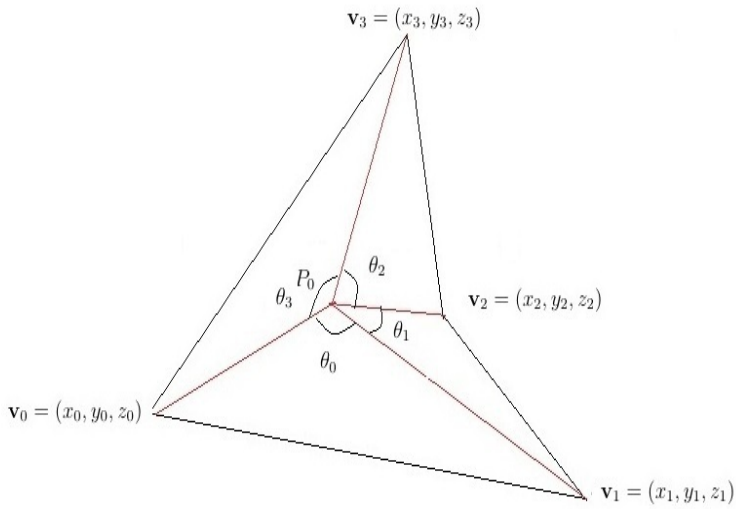

Figure 4.3: Non convex Polygon in the Space

Given a polygon $\mathcal{P}$ with n vertices $\mathbf{v}_i = (x_i, y_i, z_i)$, for $i = (1, 2, ..., n)$ given in a counter-clockwise order and with plane normal vector $\mathbf{n}$.

The winding number $w(\mathcal{P};\mathbf{p_0})$ is given by

$$\begin{aligned} w(\mathcal{P};\mathbf{p}_0) &= \frac{1}{2\pi}\sum_{i=0}^{n-1}\theta_i \\ &= \frac{1}{2\pi}\sum_{i=0}^{n-1}\cos^{-1}\left(\frac{\overline{\mathbf{p}_0\mathbf{v}_{i+1}}\cdot\overline{\mathbf{p}_0\mathbf{v}_i}}{||\mathbf{p}_0\mathbf{v}_{i+1}||\;\;||\mathbf{p}_0\mathbf{v}_i||}\right) sgn\left[(\overline{\mathbf{p}_0\mathbf{v}_i}\times\overline{\mathbf{p}_0\mathbf{v}_{i+1}})\cdot\mathbf{n}\right] \\ &= \frac{1}{2\pi}\sum_{i=0}^{n-1}\cos^{-1}\left(\frac{\overline{\mathbf{C}_{i+1}}\cdot\overline{\mathbf{C}_i}}{||\mathbf{C}_{i+1}||\;\;||\mathbf{C}_i||}\right) sgn\left[(\overline{\mathbf{C}_i}\times\overline{\mathbf{C}_{i+1}})\cdot\mathbf{n}\right] \end{aligned}$$

4.3.2 Projective Approach

In this subsection another approach for finding the winding number of a planner polygon in the space is present. First, we take any planner polygon in space and find the winding number for it's projected polygon on the xy-plane . This particular approach is an isomorphism, it preserve the shape, orientation and angle.

We construct an orthogonal basis of a given plane, and getting formula for a map takes one plane to another or planar polygon to the xy-plane and vies versa.
Let a plane with a polygon $\mathcal{P}$ laying on it has an equation

$$\mathbf{r} = \mathbf{r_0} + s\mathbf{u} + t\ \mathbf{v},\ for\ s,t \in \mathbb{R}$$

Consider for this plane the normal $\mathbf{n}$=(a,b,c), and the polygon $\mathcal{P}$ with vertices $\mathbf{v}_i = (x_i, y_i, z_i)$, for $i = (1, 2, ..., n)$. Where are $\mathbf{e}_1 = \mathbf{v}_{i+1} - \mathbf{v}_i$ and then $\mathbf{e}_2$= $\mathbf{n}$ $\times\mathbf{e}_1$. Clearly, $(\mathbf{e}_1, \mathbf{e}_2)$ is an orthogonal basis of the plane in $\mathbb{R}^3$.

Now Since $(\mathbf{p} - \mathbf{p}_0)\cdot$ $\mathbf{n} = 0$ and $(\mathbf{e}_1, \mathbf{e}_2), \mathbf{n}$ is our right-handed orthogonal basis, then any point $\mathbf{p}$ in $\mathbb{R}^3$ can be written as a linear combination of element of basis, i.e.,

$$\begin{aligned} \mathbf{p} &= (\mathbf{p}\cdot\mathbf{n})\mathbf{n} + (\mathbf{p}\cdot\mathbf{e}_1)\mathbf{e}_1 + (\mathbf{p}\cdot\mathbf{e}_2)\mathbf{e}_2 \\ &= (\mathbf{p}_0\cdot\mathbf{n})\mathbf{n} + (\mathbf{p}\cdot\mathbf{e}_1)\mathbf{e}_1 + (\mathbf{p}\cdot\mathbf{e}_2)\mathbf{e}_2 \end{aligned}$$

Clearly, a mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$ is given by

$$(s,t) = (s(x,y,z), t(x,y,z)) = (\mathbf{p}\cdot\mathbf{e}_1, \mathbf{p}\cdot\mathbf{e}_2) \tag{4.3}$$

and the inverse mapping is given by

$$(x,y,z) = (x(s,t), y(s,t), z(s,t)) = (\mathbf{p}_0\cdot\mathbf{n})\mathbf{n} + s\mathbf{e}_1 + t\mathbf{e}_2 \tag{4.4}$$

As we mentioned this mapping is an isomorphism map, which mean is angle, shape and orientation preserve. By equations 4.3 , any polygon in the space can be reflected to the xy-plane with new component of any point on this polygon. If we have a polygon $\mathcal{P}$ in the space with vertices $\mathbf{v}_i = (x_i, y_i, z_i)$, for $i = (1, 2, ..., n)$, and assuming that the reflected polygon of $\mathcal{P}$ in the xy-plane is $\mathcal{Q}$ with new vertices $\mathbf{u}_i = (s_i, t_i)$, for $i = (1, 2, ..., n)$, then the winding number of a polygon $\mathcal{Q}$ around the point $\mathbf{q}_0$ is given by

$$\begin{aligned} w(\mathcal{Q};\mathbf{q}_0) &= \frac{1}{2\pi}\sum_{i=0}^{n-1}\theta_i \\ &= \frac{1}{2\pi}\sum_{i=0}^{n-1}\cos^{-1}\left(\frac{\overline{\mathbf{q}_0\mathbf{u}_{i+1}}\cdot\overline{\mathbf{q}_0\mathbf{u}_i}}{||\mathbf{q}_0\mathbf{u}_{i+1}||\ \ ||\mathbf{q}_0\mathbf{u}_i||}\right) sgn\left[(\overline{\mathbf{q}_0\mathbf{u}_i}\times\overline{\mathbf{q}_0\mathbf{u}_{i+1}})\cdot\hat{\mathbf{k}}\right] \\ &= \frac{1}{2\pi}\sum_{i=0}^{n-1}\cos^{-1}\left(\frac{\overline{\mathbf{C}_{i+1}}\cdot\overline{\mathbf{C}_i}}{||\mathbf{C}_{i+1}||\ \ ||\mathbf{C}_i||}\right) sgn\left[(\overline{\mathbf{C}_i}\times\overline{\mathbf{C}_{i+1}})\cdot\hat{\mathbf{k}}\right] \end{aligned}$$

To verify that this winding number is equal to that in the space, it is sufficient to prove that

$$\mathbf{p}_0\mathbf{v}_{i+1}\cdot\mathbf{p}_0\mathbf{v}_i = \mathbf{q}_0\mathbf{u}_{i+1}\cdot\mathbf{q}_0\mathbf{u}_i$$

Starting with left hand side and using equation 4.3

$$\begin{aligned} \mathbf{q}_0\mathbf{u}_{i+1} &= \mathbf{u}_{i+1} - \mathbf{q}_0 \\ &= (\mathbf{v}_{i+1}\cdot\mathbf{e}_1, \mathbf{v}_{i+1}\cdot\mathbf{e}_2) - (\mathbf{p}_0\cdot\mathbf{e}_1, \mathbf{v}_{i+1}\cdot\mathbf{e}_2) \\ &= ((\mathbf{v}_{i+1}-\mathbf{p}_0)\cdot\mathbf{e}_1, (\mathbf{v}_{i+1}-\mathbf{p}_0)\cdot\mathbf{e}_2) \\ &= ((\mathbf{p}_0\mathbf{v}_{i+1})\cdot\mathbf{e}_1, (\mathbf{p}_0\mathbf{v}_{i+1})\cdot\mathbf{e}_2) \end{aligned}$$

Using equation 4.4

$$((\mathbf{p}_0\mathbf{v}_{i+1}) \cdot \mathbf{e}_1, (\mathbf{p}_0\mathbf{v}_{i+1}) \cdot \mathbf{e}_2) = (\mathbf{p}_0 \cdot \mathbf{n}) \cdot \mathbf{n} + ((\mathbf{p}_0\mathbf{v}_{i+1}) \cdot \mathbf{e}_1)\,\mathbf{e}_1 + ((\mathbf{p}_0\mathbf{v}_{i+1}) \cdot \mathbf{e}_2)\,\mathbf{e}_2$$

By the same way we get that

$$\mathbf{q}_0\mathbf{u}_i = (\mathbf{p}_0 \cdot \mathbf{n}) \cdot \mathbf{n} + ((\mathbf{p}_0\mathbf{v}_i) \cdot \mathbf{e}_1)\,\mathbf{e}_1 + ((\mathbf{p}_0\mathbf{v}_i) \cdot \mathbf{e}_2)\,\mathbf{e}_2$$

clearly $\mathbf{q}_0\mathbf{u}_{i+1} = \mathbf{p}_0\mathbf{v}_{i+1}$ and $\mathbf{q}_0\mathbf{u}_i = \mathbf{p}_0\mathbf{v}_i$, so we have

$$\mathbf{p}_0\mathbf{v}_{i+1} \cdot \mathbf{p}_0\mathbf{v}_i = \mathbf{q}_0\mathbf{u}_{i+1} \cdot \mathbf{q}_0\mathbf{u}_i$$

Chapter 5

Winding Number in 3D

An analytical expression is presented for the solid angle subtended by common object at some arbitrary point in the space. Triangular polyhedra and expression of solid angle subtended by a plane triangle are also introduced. The key idea is to use an analytic expression for the solid angle subtended by a plane triangle for generalizing the 2D winding number. For the solid angle subtended by a plane triangle the material was almost taken from [10], [13], and [31].

5.1 Formulation of Solid Angle

The problem of calculating solid angles appears in many areas of science and applied mathematics, so it's important that we know the true meaning of this mathematical measure. Definition and the method of calculate solid angle of spherical cap, sphere, hemisphere, cone, cube and faceted polyhedra are presented in this section.

A solid angle is a three dimensional angular volume that is defined analogously to the definition of a plane angle (ordinary angle) in two dimensions as we mentioned in chapter two. In particular, for solving Neuman and Dirichlet boundary-value problem of potential theory a formulation in which the potential ϕ at some point inside the volume conductor can be found as a solution of the following integral equation

$$\phi(\mathbf{Y}) = G(\mathbf{Y}) - \frac{1}{4\pi}\int_S \phi(\mathbf{X})\frac{\mathbf{R}\cdot d\mathbf{S}}{R^3} \tag{5.1}$$

Where $G(\mathbf{Y})$ is the potential at field point $\mathbf{Y}$, $\mathbf{R} = \mathbf{R(Y,X)}$ (norm of R) is a vector pointing from surface point $\mathbf{X}$ to field point $\mathbf{Y}$, and $d\mathbf{S} = d\mathbf{S}(\mathbf{X})$ is an element of the bounding surface S, directed along the local normal.

In the case when the shape of the surface S is more complicated, then surface S can be triangulate into a large number of small plane triangles Δ_j with constant potential σ_j for each triangle. Assuming that the triangles are taken sufficiently small, then integral equation 5.1 can be approximated by

$$\int_S \phi\frac{\mathbf{R}\cdot d\mathbf{S}}{R^3} = \sum_{j=1}^{N}\sigma_j\int_{\Delta_j}\frac{\mathbf{R}\cdot d\mathbf{S}}{R^3} \tag{5.2}$$

The integral on the right-hand side of 5.2 is the solid angle ω_{ij} subtended by triangle Δ_j at field point $\mathbf{Y}_i$.

Without loss of generality, assume that the origin is the observation point for the plane triangle of vertices $(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k)$, and $\mathbf{R} = \mathbf{n}\ R$, $d\mathbf{S} = \mathbf{n}\ dS$, then the solid angle Ω subtended by this triangle at $\mathbf{Y} = 0$ given by

$$\begin{aligned}
\Omega &= \int_S \frac{\mathbf{R}\cdot d\mathbf{S}}{R^3} \\
&= \int_S \frac{\mathbf{R}\cdot\mathbf{n}\ dS}{R^3} \\
&= \frac{\mathbf{R}\cdot\mathbf{n}}{R^3}\int_S dS \\
&= \frac{R\ \mathbf{n}\cdot\mathbf{n}}{R^3}\int_S dS \\
&= \frac{S}{R^2}
\end{aligned}$$

Where S is the area of the spherical triangle $(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k)$ of the sphere with radius R. In the case of unit sphere $\Omega = 4\pi$.

In particular, we will derive the differential solid angle by using spherical coordinates. Taking any surface S with two dimensions (θ, φ) and $\mathbf{q} = (x, y, z)$ be any point on the sphere with parametric representation of the sphere

$$\mathbf{q}(\theta, \varphi) = r\ sin\varphi\ cos\theta\ \mathbf{i} + r\ sin\varphi\ sin\theta\ \mathbf{j} + r\ cos\varphi\ \mathbf{k}$$

Where θ is the colatitude (polar angle) $(0 \leq \theta \leq \pi)$, φ is the longitude (azimuth angle) $(0 \leq \varphi \leq 2\pi)$, and $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are the normal basis of $\mathbb{R}^3$. We then consider formulating the solid angle of the hemisphere with parameters $(0 \leq \theta \leq 2\pi)$ and $(0 \leq \varphi \leq \frac{\pi}{2})$.

Note that the element area vector $\mathbf{dA}$ is defined as $\mathbf{dA} = \mathbf{q}_\theta \times \mathbf{q}_\varphi$. Where partial derivatives of $\mathbf{q}$ with respect to θ are given by

$$\begin{aligned} x_\theta &= -r\ sin\ \varphi\ sin\ \theta\ d\theta \\ y_\theta &= r\ sin\ \varphi\ cos\ \theta\ d\theta \\ z_\theta &= 0\ d\theta, \end{aligned}$$

and the partial derivatives of $\mathbf{q}$ with respect to φ are given by

$$\begin{aligned} x_\varphi &= r\ cos\ \varphi\ cos\ \theta\ d\varphi \\ y_\varphi &= r\ cos\ \varphi\ sin\ \theta\ d\varphi \\ z_\varphi &= -r\ sin\ \varphi\ d\varphi \end{aligned}$$

Next, we need to determine scalar area element $|\mathbf{dA}| = \|\mathbf{q}_\theta \times \mathbf{q}_\varphi\|$

$$\begin{aligned} \mathbf{q}_\theta \times \mathbf{q}_\varphi &= -r^2\ sin^2\varphi\ cos\ \theta\ \mathbf{i} - r^2 sin\ \varphi\ cos\ \varphi\ sin^2\theta\ \mathbf{k} - r^2 sin^2\varphi\ sin\ \theta\ \mathbf{j} - r^2 sin\ \varphi\ cos\ \varphi\ cos^2\theta\ \mathbf{k} \\ &= -r^2\ sin^2\varphi\ cos\ \theta\ \mathbf{i} - r^2\ sin^2\varphi\ sin\ \theta\ \mathbf{j} - r^2\ sin\ \varphi\ cos\ \theta\ (sin^2\theta + cos^2\theta)\ \mathbf{k} \\ &= -r^2\ sin^2\varphi\ cos\theta\ \mathbf{i} - r^2\ sin^2\varphi\ sin\theta\ \mathbf{j} - r^2\ sin\ \varphi\ cos\ \varphi\ \mathbf{k} \end{aligned}$$

Hence,

$$\|\mathbf{q}_\theta \times \mathbf{q}_\varphi\| = r^2\ sin\ \varphi\ d\theta\ d\varphi$$

Therefore differential solid angle is given by the following equation

$$\begin{aligned} d\Omega &= \frac{dA}{r^2} \\ &= \frac{(r\ sin\ \theta\ d\varphi)(rd\theta)}{r^2} \\ &= sin\ \theta\ d\theta\ d\varphi \end{aligned}$$

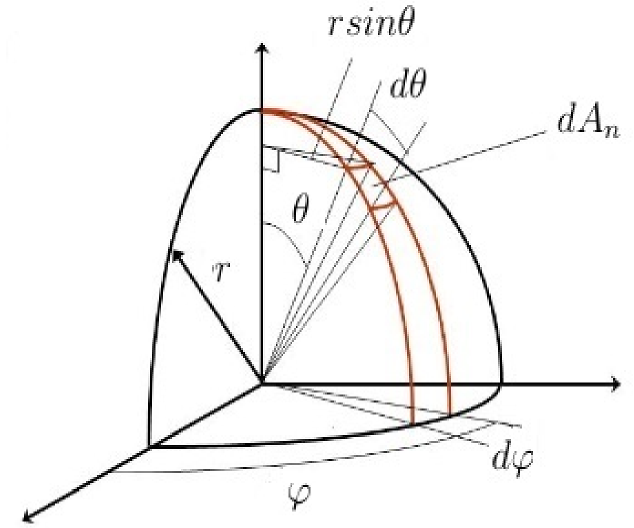

Figure 5.1: Differential Solid Angle

5.1.1 Solid Angles for Common Objects

This subsection discus some of the common objects solid angles. Next example consider the solid angel of spherical cap, sphere, hemisphere and cone.

Example 5.1. To find the solid angel of a spherical cap of a unit sphere, just compute the following double integral using the unit surface element in spherical coordinates

$$\begin{aligned} \int_0^{2\pi}\int_0^{\theta} sin\theta' d\theta' d\varphi &= (2\pi)\int_0^{\theta} sin\theta' d\theta' \\ &= 2\pi(1 - cos\theta) \end{aligned} \quad (5.3)$$

For theta $\theta = \pi$ (entire sphere) equation 5.3 becomes 4π, and when $\theta = \frac{\pi}{2}$ the spherical cap becomes a hemisphere having a solid angle 2π. The solid angle subtended by a cone whose apex has angle 2θ is the area of a spherical cap on a unit sphere

$$\Omega = 2\pi(1 - cos\theta)$$

For small angle θ we can approximate $cos(\theta)$ as $cos\theta \approx 1 - \frac{\theta^2}{2}$, which giving an approximate value $\pi\theta^2$ for the solid angle subtended by the cone.

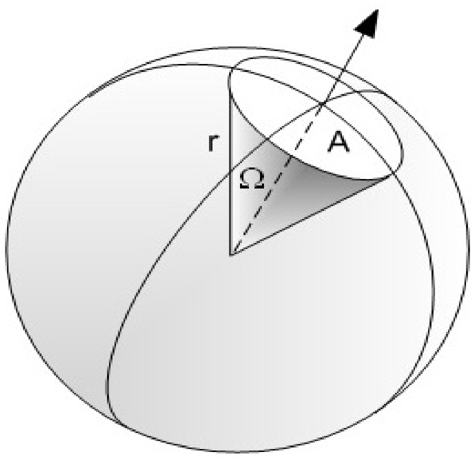

Figure 5.2: Solid Angle of Cone

Next example handle the solid angle for a face of a cube.

Example 5.2. Let's calculate the solid angle that is subtended by the face of a cube. Now since the cube has 6 sides, and all of them are equal in size. So if we project the cube onto the sphere, the surface area covered by one of the faces will be 1/6 of the total surface area. Then the solid angle subtended by one face of the cube will be 1/6 of the solid angle subtended by the entire sphere. So the solid angle subtended by one face of the cube is $\frac{2\pi}{3}$.

5.2 Winding Number of Triangular Polyhedra

Winding number immediately generalizes to $\mathbb{R}^3$ by replacing angle with solid angle with the same classification properties apply as in $\mathbb{R}^2$. Let $\mathbf{p}$ be any point in $\mathbb{R}^3$ and S an arbitrary closed surface, then the winding number $w(S;\mathbf{p})$ of the surface S with respect to the reference point $\mathbf{p}$ can be defined as

$$w(S;\mathbf{p}) = \frac{\Omega(\mathbf{p})}{4\pi} \tag{5.4}$$

Without loss of generality, any analytic surface can be approximated as triangular polyhedra. So 5.4 becomes

$$w(S;\mathbf{p}) = \sum_{f=1}^{m} \frac{\Omega_f(\mathbf{p})}{4\pi}$$

Where $\Omega_f(\mathbf{p})$ is the solid angle of the oriented triangle with vertices $(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k)$ with respect to the point $\mathbf{p}$. To compute each $\Omega_f(\mathbf{p})$ we will make use of an equality obtained by Van Oosterom and Strackee [31]. Consider a tetrahedron based at the origin of $\mathbb{R}^3$ and spanned by three non-zero vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c} \in \mathbb{R}^3$. Then the equality expresses the solid angle $\Omega(\mathbf{a}, \mathbf{b}, \mathbf{c})$ of the tetrahedron in terms of these vectors

$$tan(\frac{\Omega(\mathbf{p})}{2}) = \frac{|[\mathbf{a}, \mathbf{b}, \mathbf{c}]|}{(\| \mathbf{a} \| \| \mathbf{b} \| \| \mathbf{c} \|) + (\mathbf{a} \cdot \mathbf{b}) \| \mathbf{c} \| + (\mathbf{b} \cdot \mathbf{c}) \| \mathbf{a} \| + (\mathbf{c} \cdot \mathbf{a}) \| \mathbf{b} \|} \tag{5.5}$$

Where $\mathbf{a} = \mathbf{v}_i - \mathbf{p}$, $\mathbf{b} = \mathbf{v}_j - \mathbf{p}$, $\mathbf{c} = \mathbf{v}_k - \mathbf{p}$.

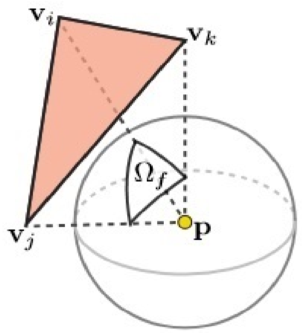

Figure 5.3: The Solid Angle of the Spherical Triangle Defined by **a**,**b**,**c**

Without loss of generality, assume that our spherical triangle has vertices (**A**, **B**, **C**) on the unit sphere, with the center at the vertex **O**. We denote the unit vectors (**OA**, **OB**, and **OC**) by $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$, respectively. Also we will use the standard notation a, b, c for the sides of our spherical triangle(**T**=**ABC**), and **A**, **B**, **C** for the angles. So equation 5.5 become

$$tan(\frac{\Omega}{2}) = \frac{|[\mathbf{a}, \mathbf{b}, \mathbf{c}]|}{1 + (\mathbf{a} \cdot \mathbf{b}) + (\mathbf{b} \cdot \mathbf{c}) + (\mathbf{c} \cdot \mathbf{a})} \tag{5.6}$$

Using terms of dot products and the triple product $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ and according to Figure 5.4 , we have

$$tan(\frac{\Omega)}{2}) = \frac{(1 - cos^2 a - cos^2 b - cos^2 c + 2\ cos\ a\ cos\ b\ cos\ c)^{\frac{1}{2}}}{1 + cos\ c + cos\ a + cos\ b} \tag{5.7}$$

The numerator of 5.6 and 5.7 are the same and equal the volume of parallelepiped with edges $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Let us prove this fact, starting from expression for that volume in terms of components of the unit vectors

$$V = |\ [\mathbf{a}, \mathbf{b}, \mathbf{c}]\ | = \pm \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then we have by the product theorem for determinants

$$V^2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} = \begin{vmatrix} 1 & cos\ c & cos\ b \\ cos\ c & 1 & cos\ a \\ cos\ b & cos\ a & 1 \end{vmatrix}$$

$$= 1 - cos^2 a - cos^2 b - cos^2 c + 2\ cos\ a\ cos\ b\ cos\ c$$

According to Figure 5.4 the altitude h from the point **A** of the parallelepiped with edges **a**,**b**,**c** is

$$h = sin\ b\ sin\ C$$

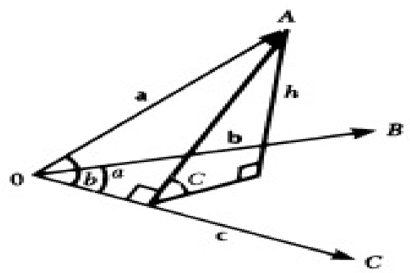

Figure 5.4:

Now since the volume equal the multiplication of area of base and altitude h we have

$$V = sin\ a\ sin\ b\ sin\ C \tag{5.8}$$

To continue our prove we will use the following construction. let L and M be the midpoints of BC and AC, respectively, in spherical triangle ABC. The great circles AB and LM intersect in antipodal points P and Q. Draw arcs AA', BB' and CC' perpendicular to LM. Then triangles AMA' and CMC' are congruent, as are BLB' and CLC'. Thus the angle $(A, MA') = (C, MC') = u$, say, and the angle $(B, B'L) = (C, C'L) = v$. Furthermore $AA' = CC' = BB'$. Thus triangles $AA'P$ and $BB'Q$ are congruent, because the angles at P and Q are equal and the angles at A' and B' are right angles. Hence angle $(A, A'P) = (B, QB')$. Call this angle t. We have now fig 5.5

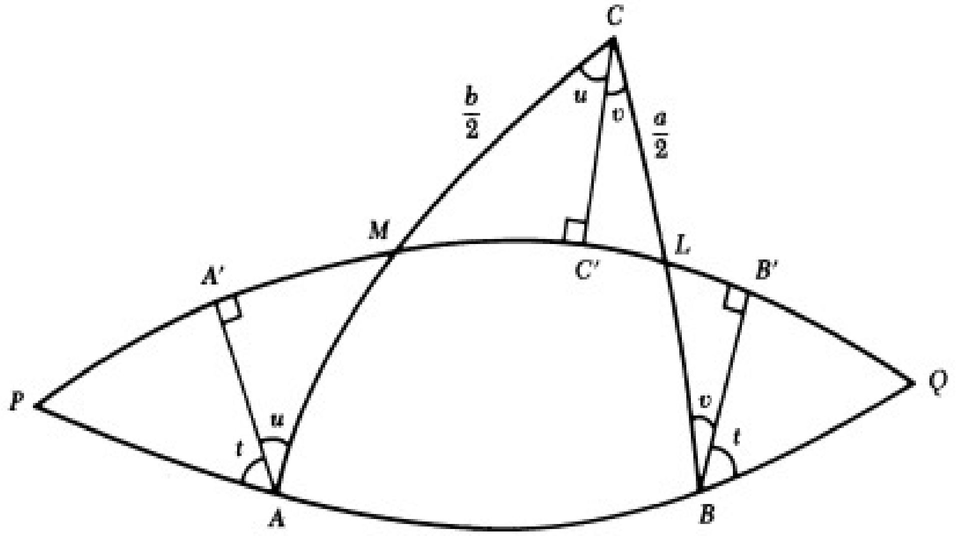

Figure 5.5:

$$A + u + t = \pi,\ B + v + t = \pi,\ C = u + v$$

and by addition

$$A + B + C + 2t = 2\pi$$

or

$$2t = 2\pi - (A + B + C) = \pi - E$$

Thus

$$\frac{1}{2}(\pi - E) = t = (A, A'P)$$

Now by elementary spherical trigonometry, we get the following expressions for tan E/2

$$\begin{aligned} tan\ \frac{E}{2} &= \frac{1}{tan\ t} \\ &= \frac{sin\ AA'}{tan\ PA'} \\ &= \frac{sin\ AA'}{cot\ ML} \\ &= \frac{sin\ CC' sin\ ML}{cos\ ML} \\ &= \frac{sin\ \frac{b}{2}\ sin\ M\ sin\ ML}{cos\ ML} \\ &= \frac{sin\ \frac{b}{2}\ sin\ C\ sin\ \frac{a}{2}}{cos\ ML} \end{aligned} \tag{5.9}$$

(In second equality $tan\ t$ expressed by a well-known formula for the right angled triangle $AA'P$. The third equality comes from $PA' + ML = (\frac{1}{2})PQ = \frac{\pi}{2}$, and the last one from the law of sines $(\frac{sin\ b}{sin\ B} = \frac{sin\ c}{sin\ C} = \frac{sin\ a}{sin\ A})$ for triangle CML.)
By the law of cosines for triangle CML, the denominator cos ML expressed as

$$\begin{aligned} cos\ ML &= cos\ \frac{a}{2}\ cos\ \frac{b}{2} + cos\ C\ sin\ \frac{a}{2}\ sin\ \frac{b}{2} \\ &= \frac{4\ cos^2\ \frac{a}{2}\ cos^2\ \frac{b}{2} + cos\ C\ sin\ a\ sin\ b}{4\ cos\ \frac{a}{2}\ cos\ \frac{b}{2}} \\ &= \frac{(1 + cos\ a)(1 + cos\ b) + cos\ c - cos\ a\ cos\ b}{4\ cos\ \frac{a}{2}\ cos\ \frac{b}{2}} \\ &= \frac{1 + cos\ a + cos\ b + cos\ c}{4\ cos\ \frac{a}{2}\ cos\ \frac{b}{2}} \end{aligned}$$

Inserting this in 5.9 we get

$$tan \frac{E}{2} = \frac{sin\ a\ sin\ b\ sinC}{1 + cos\ a + cos\ b + cos\ c}$$

Which is 5.7 with numerator 5.8 .[10]

Bibliography

[1] Ablowitz, M. J., & Fokas, A. S. (2003). *Complex variables: introduction and applications* . Cambridge University Press.

[2] Abu-Munshar, E.(2013). *Efficient computational geometry algorithms for spatial search and query(un published master thesis).* Palestine Polytechnic University, Hebron, Palestine.

[3] Ahlfors, L. V.(1953). *Complex analysis: an introduction to the theory of analytic functions of one complex variable.* McGraw-Hill.

[4] Arai, Z.(2013). *A rigorous numerical algorithm for computing the linking number of links.* Nonlinear Theory and Its Applications, IEICE, 4(1), 104-110.

[5] Arnold, D. N. (1997). *Complex analysis* . Dept. of Mathematics, Penn State Univ.

[6] Becciu, A., Fuster, A., Pottek, M., Van den Heuvel, B., Romeny, B.T.H.,& Van Assen, H.(2011). *3D winding number: Theory and application to medical imaging.* Journal of Biomedical Imaging.

[7] Chernov, V. V., & Rudyak, Y. B.(2008). *On generalized winding numbers.* St. Petersburg Math,20(5), 217-233.

[8] Crofoot,R.B. (2002). *Running with rover* . Mathematics Magazine, 75(4),311-316.

[9] Eppstein, D.(2013). *Twenty proofs of euler's formula* . Retrieved from http://www.math.caltech.edu/ 2014-15/2term/ma006b/.

[10] Eriksson, F.(1990). *On the measure of solid angles.* Mathematics Magazine, 63(3), 184-187.

[11] Freitag, E.,& Busam, R.(2005). *Complex analysis.* Universitext.

[12] Huggett, S., & Jordan, D.(2009). *A topological aperitif.* Springer Science & Business Media.

[13] Jacobson, A., Kavan, L., & Sorkine-Hornung, O.(2013). *Robust inside-outside segmentation using generalized winding numbers.* ACM Transactions on Graphics (TOG), 32(4), 33.

[14] Jeff Erickson, J.(2013). *Computational topology.* Retrieved from http://jeffe.cs.illinois.edu/teaching/comptop/.

[15] Koch, R.(2016). *Complex variable outline.* Retrieved from http://pages.uoregon.edu/koch/.

[16] Kravitz, B.(2009). *Lecture on solid angle.* Retrieved from https://marine.rutgers.edu/dmcs/ms552/2009/.

[17] Lehman, E.(2012). *Spherical geometry.* Retrieved from http://cs.uef.fi/matematiikka/kurssit/SphericalGeometry/.

[18] Mathar, R. J.(2015). *Solid angle of a rectangular plate.* Max-Planck Institute of Astronomy, Knigstuhl, 17, 69117.

[19] Mazonka, O.(2012). *Solid angle of conical surfaces, polyhedral cones, and intersecting spherical caps.* arXiv preprint arXiv:1205.1396.

[20] Narasimhan, R.(1985). *The Winding number and the residue theorem. In complex analysis in one variable.* Birkhauser Boston,(pp. 70-88).

[21] Ponnusamy, S.(1995). *Foundations of complex analysis* . Narosa Publishing House.

[22] Quincey, P.(2016). *The range of options for handling plane angle and solid angle within a system of units* . Metrologia, 53(2), 840.

[23] Rao, M., & Stetkaer, H.(1991). *Complex analysis: an invitation.* World Scientific Publishing Co Inc.

[24] Reid, M., & Szendroi, B.(2005). *Geometry and topology.* Cambridge University Press.

[25] Ribando, J. M.(2006). *Measuring solid angles beyond dimension three.* Discrete & Computational Geometry, 36(3), 479-487.

[26] Roe, J.(2015). *Winding around: The winding number in topology, geometry, and analysis* American Mathematical Society,(Vol. 76).

[27] Sarason, D.(2007). *Complex function theory* . Providence, RI: American Mathematical Society,(Vol. 49).

[28] Tatum, J.(2016). *Plane and spherical trigonometry.* Retrieved from http://astrowww.phys.uvic.ca/ tatum/celmechs/.

[29] Todhunter, I.(1863). *Spherical trigonometry, for the use of colleges and schools: with numerous examples.* Macmillan.

[30] Urea, C., Fajardo, M., & King, A.(2013, July). *An Areapreserving parametrization for spherical rectangles.* In Computer Graphics Forum (Vol. 4, No. 32, pp. 59-66).

[31] Van Oosterom, A., & Strackee, J. (1983). *The solid angle of a plane triangle* . IEEE transactions on Biomedical Engineering, (2), 125-126.

[32] Wilkins, D.R. (2008). *Functions of a complex variable.* Retrieved from http://www.maths.tcd.ie/ dwilkins/Courses/214/.

[33] Wischmann, HA., Drenckhahn, R., Wagner, M., & Fuchs, M.(1996). *Systematic distribution of the auto sotid angle and related integrals onto the adjacent triangles for the node based boundary element method* . Medical and Biological Engineering and Computing,34(Supplement 1, Part 2):245-246.

[34] Zhao, L., & Zaderman, V.,FM.(2016). *Numerically stable algorithms for winding number computation and polynomial root-finding* . FWCG2016 ,CUNY Graduate Center, October 28, 2016,New York, NY.