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Ideals, Congruences and Derivations in Distributive Lattices

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Dedications

This work is dedicated to
my father Hisham Alkurd
my mother Nidaa Alsaheb
my husband Hazem Alsaheb
my grandmother Um Mahmoud Alsaheb

Alaa

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Abstract

This thesis aims to develop a better understanding of ideals, congruence relations and derivations in distributive lattices. We present the definition of a partially ordered set, a lattice and a distributive lattice, and we furnish the relation between these concepts. We introduce the concept of ideal, filter, prime ideal and maximal ideal in lattices . Also we discuss some results related to ideals and their relationship with distributivity. We introduce the concepts of congruence relations, quotient lattices and kernels. In addition we characterize the distributive lattices by kernels. Furthermore we discuss the notion of derivation in lattices and its properties. We compare between derivations in lattices and lattice homomorphisms. Also we present the concepts of d-ideal, injective ideal and discuss two types of congruences on a distributive lattice with respect to derivations . Finally the Stone's result for ideals of a distributive lattice is extended to the case of injective ideals and d-prime ideals.

Contents

Introduction

Formalization of propositional Logic that led to Boolean Algebra was initially developed by George Boole in the first half of the nineteenth century, Pierce and Schroder continued to investigate the Boolean Algebra at the end of the nineteenth century and introduced the lattice concept while independent Richard Dedekind's researches led to the same discovery. In fact Dedekind wrote two papers on the subject of lattice theory, but then the subject lay relatively dormant till the thirties of the twentieth century that the lattice theory became a subject for systematic studies.

Lattice Theory was further developed by Garrett Birkhoffs in the 1930s , Birkhoff demonstrated that lattice theory provides a unifying framework for unrelated developments in many mathematical disciplines. Since then, many mathematicians have contributed to the subject, including Birkhoff himself, Richard Dedekind, George Gratzer, Aleksandr Kurosh, Anatoly Malcev, Oystein Ore, Gian-Carlo Rota, Alfred Tarski and Johnny von Neumann.

In parallel with the development of lattice theory; studies of distributive lattices were also developed, distributive lattices have received more attention than other types of lattices and played a central role in providing many results in general lattice theory. Many conditions on lattices are weakened forms of distributivity. Moreover, in many applications the condition of distributivity is imposed on lattices arising in various areas of mathematics, especially algebra.

In the 19th century, important results due to Minkowshi motivated the use of lattice theory in the theory and geometry numbers. The evolution of computer science in the 20th century led to lattice applications in various theoretical areas such as information theory, information access controls and cryptanalysis.

The notion of lattice ideals played an important role in lattice theoretical researches. On the other hand, the study of congruence relations on lattices had became a special interest to many authors . In the paper [14], Gratzer and Schmidt also studied an inter-relation between ideals and congruence relations in a lattice. Birkhoff [6], Gratzer, Szasz and many authors have studied various types of ideals and congruences all intimated to some extent the behavior of ideals in a distributive lattice.

In the past several years, there has been an ongoing interest in derivations of rings. Several authors like Bell H.E., Kappe L.C. [5] and Kaya K. [17] have studied derivations in rings and prime rings after Posner [21] had given the definition of the derivation in ring theory. Szasz have introduced and developed the theory of derivations in lattice structure. In a series of papers [25] and [26] he established the main properties of derivations of lattices. Ferrari L. [10] extended these concepts to lattices and he embedded any lattice having some additional properties into the lattice of its derivations.

The material of this thesis lies in three chapters, each one contains basic definitions, examples and important theorems.

Chapter one: In this chapter we begin with basic definitions needed in this work. We introduce the concept of partially ordered set, totally ordered set, duality, a lattice, a sublattice and a distributive lattice. Also we study some basic properties of these concepts. We learn how to represent any finite partially ordered set graphically. In addition special elements within a partial order such as least and greatest, minimal and maximal, upper and lower bound will be given. Furthermore we introduce several maps between partially ordered sets and between lattices.

Chapter two: In this chapter we discuss ideals and congruence relations on lattices. We introduce the concept of ideal, filter, principal ideal, prime ideal and maximal ideal. Furthermore we discuss various theorems about ideals and their relationship with distributivity, including the famous theorem of Stone which is meant for ideals, filters and prime ideals of a distributive lattice. In addition this chapter is devoted to congruence relations on lattices, quotient lattices and kernels. We characterize the distributive lattice by using kernel of congruence relations.

Chapter three: Includes the concept of derivation on lattices. Examples are presented to illustrate this concept. Also we will mention some properties of derivation and compare between derivations and lattice homomorphisms. Furthermore the concepts of d-ideal, d-prime ideal, injective ideal and maximal injective ideal are introduced in a distributive lattice with respect to derivations. Also the Stone's theorem for ideals of a distributive lattice is extended to the case of injective ideals and d-prime ideals. In addition two types of congruences are presented in a distributive lattice with respect to derivation.

Chapter 1

Preliminaries

In this chapter, the basic definitions that include the definition of partially ordered sets, lattices and distributive lattices are introduced. The relation between these concepts is investigated. In addition; examples are presented to illustrate these concepts.

1.1 Partially Ordered Sets

This section describes the basic theory of partially ordered sets. We introduce the definition of a partially ordered set and totally ordered set. We learn how to represent any finite partially ordered set graphically. Also we discuss special elements of partially ordered set. Duality which is very important will be introduced. Finally, we give several maps between partially ordered sets.

1.1.1 Definitions and Diagrams

Definition 1.1.1. A **Partial Order** on a nonempty set P is a binary relation $\preccurlyeq$ on P that is **reflexive**, **antisymmetric** and **transitive**, specifically, for all $x, y, z \in P$, we have that

(1) $x \preccurlyeq x$. (reflexive)

(2) if $x \preccurlyeq y$ and $y \preccurlyeq x$ then $x = y$. (antisymmetric)

(3) if $x \preccurlyeq y$ and $y \preccurlyeq z$ then $x \preccurlyeq z$. (transitive)

The pair $(P, \preccurlyeq)$ is called a **partially ordered set** or a **poset**, although it is often said that P is a poset, when the order relation is understood.

Remark 1.1.1.

1. *The symbol $\preccurlyeq$ is read* **related to** *or* **contained in**. *If $x \preccurlyeq y$ but $x \neq y$, we write $x \prec y$ or $y \succ x$.*

2. *If $x \preccurlyeq y$ or $y \preccurlyeq x$, then x and y are said to be* **comparable**. *Otherwise, x and y are* **incomparable**, *denoted by $x \parallel y$.*

3. *Any subset S of a poset P is it self a poset under the same relation.*

Definition 1.1.2. [7] A poset $(P, \preccurlyeq)$ is **totally ordered** if every $x, y \in P$ are comparable, that is $x \preccurlyeq y$ or $y \preccurlyeq x$.

The alternative name for totally ordered set is a **chain**. At the opposite extreme from a chain is an antichain. A poset $(P, \preccurlyeq)$ is **antichain** if every $x, y \in P$ are incomparable.

Example 1.1.1. The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ of natural numbers, integers, rationals, and real numbers form chains under the usual order $\leq$ (less than or equal).

Example 1.1.2. On the set $\mathbb{N} = \{1, 2, ...\}$ of natural numbers the relation $|$ defined by $x \mid y$ if and only if $y = kx$ for some $k \in \mathbb{N}$, is a partial order and $(\mathbb{N}, |)$ is a poset. On the other hand, $(\mathbb{N}, |)$ is not a chain, since not every $x, y \in \mathbb{N}$ are comparable. For example, $2 \parallel 3$.

One of the most useful and alternative features of posets is that, in the finite case at least, they can be "drawn". To describe how to represent posets diagrammatically, we need the idea of covering.

Definition 1.1.3. [22] Let $(P, \preccurlyeq)$ be a poset. Then y **cover** x in P, denoted by $x \sqsubset y$, if $x \prec y$ and no element in P lies strictly between x and y, that is,

$$x \preccurlyeq z \preccurlyeq y \Longrightarrow z = x \text{ or } z = y.$$

If $a \sqsubset b$ or $a = b$, then we write $a \sqsubseteq b$.

For a finite poset P, the covering relation uniquely determines the partial order on P, since $a \preccurlyeq b$ if and only if there is a finite sequence of elements of P such that

$$a \sqsubset p_1 \sqsubset p_2 \sqsubset \ldots \sqsubset p_n \sqsubset b.$$

Using the covering relation, one can obtain a graphical representation of any finite poset P as follows:
Draw a small circle to represent each element of P, placing a higher than b whenever $a \succ b$ and then draw a straight segment from a to b whenever $a \sqsupset b$. The resulting figure is called a **diagram** or a **Hasse diagram** of P. This is illustrated in the following examples.

Example 1.1.3. Figure 1.1 shows the Hasse diagram of the poset $\{1, 2, 3, 4, 6, 12\}$ ordered by divisibility.

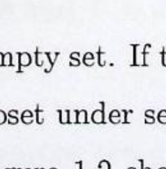

Figure 1.1:

Example 1.1.4. Let X be a nonempty set. If the power set $\mathbb{P}(X)$ of X is the set of all subsets of X, then $\mathbb{P}(X)$ is a poset under set inclusion: for $A, B \in P$, we define $A \preccurlyeq B$ if and only if $A \subseteq B$. Figure 1.2 shows the Hasse diagram of the poset $\mathbb{P}(\{a, b, c\})$ under set inclusion.

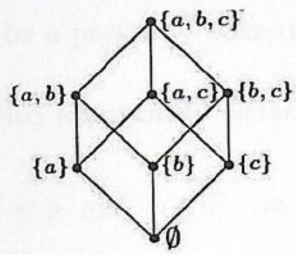

Figure 1.2:

1.1.2 Extremal Elements of Posets

Suppose that $\preccurlyeq$ is a partial order on a set P. Various types of extremal elements of P play important roles.

Definition 1.1.4. Let $(P, \preccurlyeq)$ be a partially ordered set.

(1) An element $m \in P$ is called a **greatest** element of P if $p \preccurlyeq m$ for all $p \in P$.

(2) An element $n \in P$ is called a **least** element of P if $n \preccurlyeq p$ for all $p \in P$.

The least and greatest elements of a partially ordered set are also called bottom and top, or $\perp$ and $\top$, respectively. A poset $(P, \preccurlyeq)$ can have at most one least element and at most one greatest element. That is, least and greatest elements are unique, if they exist. (The uniqueness comes from the antisymmetry).

Definition 1.1.5. A partially ordered set is **bounded** if it has both a bottom and a top.

Example 1.1.5.

(1) In $(\mathbb{P}(X), \subseteq)$, we have $\perp = \emptyset$ and $\top = X$. Therefore, the poset $(\mathbb{P}(X), \subseteq)$ is bounded.

(2) A finite chain always has a bottom and a top elements. So, it is bounded. But an infinite chain need not have. For example, the chain $(\mathbb{N}, \leq)$ has bottom element 1, but no top.

Definition 1.1.6. Let $(P, \preccurlyeq)$ be a partially ordered set.

(1) An element $m \in P$ is called a **maximal** element of P if

$$m \preccurlyeq p \text{ and } p \in P \quad \Rightarrow \quad m = p.$$

(2) An element $n \in P$ is called a **minimal** element of P if

$$p \preccurlyeq n \text{ and } p \in P \quad \Rightarrow \quad p = n.$$

Clearly a least element must be minimal and a greatest element must be maximal, but the converse is not true.

Example 1.1.6. Figure 1.3 shows the Hasse diagram of the poset $\{2, 3, 4, 5, 6, 8, 12, 18\}$ ordered by divisibility.

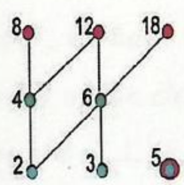

Figure 1.3:

It is clear that 5, 8, 12, 18 are maximal elements and 2, 3, 5 are minimal elements. But this poset possesses neither bottom nor top .

Definition 1.1.7. Let $(P, \preccurlyeq)$ be a partially ordered set and let $S \subseteq P$.

1. An **upper bound** for S is an element $x \in P$ such that

$$s \preccurlyeq x \qquad \forall s \in S.$$

The set of all upper bounds for S is denoted by S^u. If S^u has a least element, it is called the **join** or **least upper bound** or **supremum** of S and is denoted by $\bigvee S$. The join of a finite set $S = \{a_1, ..., a_n\}$ is denoted by

$$a_1 \vee ... \vee a_n$$

2. A **lower bound** for S is an element $x \in P$ such that

$$x \preccurlyeq s \qquad \forall s \in S.$$

The set of all lower bounds for S is denoted by S^ℓ. If S^ℓ has a greatest element, it is called the **meet** or **greatest lower bound** or **infimum** of S and is denoted by $\bigwedge S$. The join of a finite set $S = \{a_1, ..., a_n\}$ is denoted by

$$a_1 \wedge ... \wedge a_n$$

Example 1.1.7. Consider the poset $(\mathbb{N}, |)$. For any two numbers $a, b \in \mathbb{N}$, the least upper bound of a and b or $a \vee b$ is their least common multiple. The greatest lower bound of a and b or $a \wedge b$ is their greatest common divisor.

Remark 1.1.2. *[22] For any poset P, we have $\emptyset \subseteq P$. The join of the empty set $\emptyset$ is, by definition, the least upper bound of the elements of $\emptyset$. Since every element of P is an upper bound for $\emptyset$, we have $\bigvee \emptyset = \bot$ if P has a least element $\bot$; otherwise $\emptyset$ has no join. Similarly, $\bigwedge \emptyset = \top$ if $\top$ exist; otherwise $\bigwedge \emptyset$ does not exist.*

1.1.3 Duality

Definition 1.1.8. Let $(P, \preccurlyeq)$ be a poset. The **dual poset** $(P^\partial, \succcurlyeq)$ is the poset with the same underlying set but whose order relation is the opposite of $\preccurlyeq$, that is, $x \preccurlyeq y$ in P^∂ if and only if $y \preccurlyeq x$ in P.

Of course, $(P^\partial)^\partial = P$.

Each statement about the poset P corresponds to a statement about P^∂. For example, x is an upper bound for the set S in the poset $(P, \preccurlyeq)$ if and only if x is a lower bound for S in P^∂. Similarly, $u = \bigvee S$ in the poset $(P, \preccurlyeq)$ if and only if $u = \bigwedge S$

in $(P^\partial, \succcurlyeq)$. In general, given any statement Φ about posets, we obtain the *dual statement* Φ^∂ by replacing each occurrence of $\preccurlyeq$ by $\succcurlyeq$ and vice versa. This allows us to formulate the following principle.

The Duality Principle of the posets: Given a statement Φ about posets which is true in all posets, then the dual statement Φ^∂ is also true for all posets.

The validity of this principle follows from the fact that any poset can be regarded as the dual of some other poset. The duality principle allows us to simplify proofs of certain statements that concern posets. For statements involving both a concept and its dual we need to prove only half of the statement; the other half follows by applying the duality principle. For instance, once we prove the statement "any subset of a poset can have at most one least upper bound", the dual statement "any subset of a poset can have at most one greatest lower bound" follows. Two statements Φ and Ψ are equivalent if they are true in exactly the same posets. If a statement or definition is equivalent to its dual then it is said to be **self-dual**.

For a finite poset P, we obtain a diagram for P^∂ simply by turning upside down a diagram for P. Figure 1.4 provides a simple illustration.

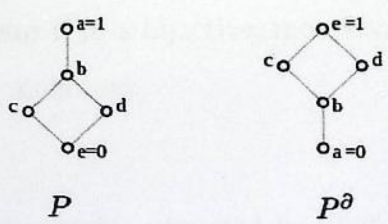

Figure 1.4:

1.1.4 Monotone Maps

It is reasonable to consider functions between partially ordered sets having certain additional properties that are related to the ordering relations of the two sets. The most fundamental condition that occurs in this context is monotonicity. Monotone functions are central in order theory. They appear in most articles on the subject and examples from special applications are found in these places. Some notable special monotone functions are order embeddings and order isomorphisms.

Definition 1.1.9. Let P and Q be posets. A function $f : P \to Q$ is said to be

(1) **order-preserving** or **monotone** or **isotone** if

$$x \preccurlyeq y \quad \Rightarrow \quad f(x) \preccurlyeq f(y)$$

and **strictly monotone** if

$$x \prec y \quad \Rightarrow \quad f(x) \prec f(y);$$

(2) an **order embedding** if

$$x \preccurlyeq y \quad \Leftrightarrow \quad f(x) \preccurlyeq f(y);$$

(3) an **order isomorphism** if it is bijective, monotone and whose inverse $(f^{-1} : Q \to P)$ is also monotone.

Two posets P and Q are isomorphic, denoted by $P \cong Q$, if there exists an order isomorphism between them.
We shall say that P and Q are dually isomorphic if $P \cong Q^{\partial}$ or equivalently, $Q \cong P^{\partial}$. In particular, if $P \cong P^{\partial}$, then we say that P is self dual.

Remark 1.1.3. *An order embedding is automatically a one-to-one map. By using reflexivity and antisymmetry of $\preccurlyeq$, first in Q and then in P, we can get*

$$\begin{aligned} f(x) = f(y) &\Leftrightarrow f(x) \preccurlyeq f(y) \text{ and } f(y) \preccurlyeq f(x) \\ &\Leftrightarrow x \preccurlyeq y \text{ and } y \preccurlyeq x \\ &\Leftrightarrow x = y. \end{aligned}$$

Remark 1.1.4. *If f is a monotone bijection, then f^{-1} need not be monotone, that is, f need not be an order isomorphism. For example, map two incomparable elements to two comparable elements.*

A useful criterion for an isomorphism of posets is the following.

Theorem 1.1.1. *[7] Posets P and Q are isomorphic if and only if there is a surjective mapping $f : P \to Q$ such that*

$$x \preccurlyeq y \Leftrightarrow f(x) \preccurlyeq f(y).$$

Example 1.1.8. Let E be the set of all positive even integers. Define the map $f : (\mathbb{N}, \leq) \to (E, \leq)$ by

$$f(n) = 2n \quad \text{for all } n \in \mathbb{N}.$$

f is one-to-one and onto. Also

$$n_1 \leq n_2 \quad \Leftrightarrow \quad 2n_1 \leq 2n_2.$$

Hence, f is an order isomorphism and $\mathbb{N} \cong E$.

For finite isomorphic posets the Hasse diagrams look the same except for the names of the elements. Let $f : P \to Q$ be an order isomorphism, and let H be any Hasse diagram of a poset P. If each label x of H is replaced by $f(x)$, then H will become a Hasse diagram for the poset Q.

Example 1.1.9. Let $P = \{1, 2, 3, 6\}$ under $|$. Let $Q = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ under $\subseteq$. If $f(1) = \emptyset, f(2) = \{a\}, f(3) = \{b\}, f(6) = \{a, b\}$, then f is an order isomorphism.

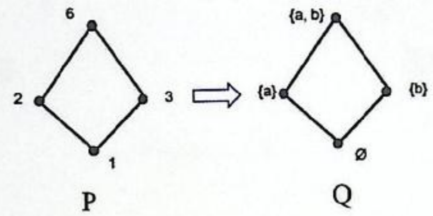

Figure 1.5:

1.2 Lattices

In this section we introduce the concept of lattice and sublattice in two ways. Examples are presented to illustrate this concepts. Furthermore we discuss lattice homomorphism and its properties. Finally we introduce a distributive lattice which is very important type of lattices.

1.2.1 Definition of Lattices

Many important properties of a poset P are expressed in terms of the existence of certain upper bounds or lower bounds of subsets of P. The most important class of posets defined in this way is lattices.

We can present a lattice in two ways: one based on the existence of an order relation satisfying certain properties and one based on the existence of binary operations satisfying certain algebraic properties.

In a partially ordered set P, the least upper bound $x \vee y$ of $\{x, y\}$ may fail to exist for two different reasons: (1) because x and y have no common upper bound, (2) or because they have no least upper bound. For example, in Figure 1.6, the element a and b in (i) are incomparable. Therefore, $\{a, b\}^u = \emptyset$ and hence $a \vee b$ does not exist. In (ii) we find that $\{a, b\}^u = \{c, d\}$ and thus $a \vee b$ does not exist as $\{a, b\}^u$ has no least element.

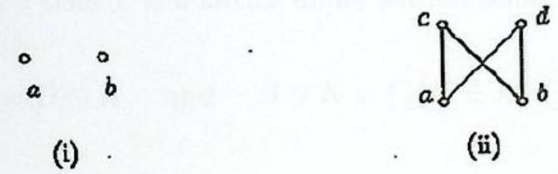

Figure 1.6:

Similar statements can be made for greateast lower bound.

A special structure arises when every pair of elements in a poset has a least upper bound and a greateast lower bound.

Definition 1.2.1. A **lattice** is a partailly ordered set $(L, \preccurlyeq)$ in which every pair of elements $x, y \in L$ has an infimum and a supremum (in L).

A lattice L is *trivial* if it has only one element; otherwise, it is *nontrivial.*

Example 1.2.1.

(1) Any totally ordered set is a lattice. For example, the set $\mathbb{Z}$ of integers under the ordinary order is a lattice, where $\forall a, b \in \mathbb{Z}$

$$a \wedge b = min\{a, b\} \quad \text{and} \quad a \vee b = max\{a, b\}.$$

(2) If X is a nonempty set, then the power set $\mathbb{P}(X)$ of X is lattice under set inclusion in which $\forall A, B \in \mathbb{P}(X)$

$$A \wedge B = A \cap B \quad \text{and} \quad A \vee B = A \cup B.$$

(3) The set $\mathbb{N}$ of natural numbers is a lattice under division, where $\forall a, b \in \mathbb{N}$

$$a \wedge b = gcd(a, b) \quad \text{and} \quad a \vee b = lcm(a, b).$$

(4) If X is a set, and L is a set of subsets of X that is closed under intersections and contains X , then L is a lattice under set inclusion. In which $\forall H, K \in L$

$$H \wedge K = H \cap K \quad \text{and} \quad H \vee K = \bigcap\{C \in L \mid (H \cup K) \subseteq C\}.$$

As special cases, the set $\mathcal{S}(G)$ of all subgroups of a group G is a lattice under set inclusion, where the meet of two subgroups is their intersection, and the join of two subgroups is the subgroup generated by their union. Normal subgroups are amenable. If $\mathcal{N}(G)$ is the set of all normal subgroups of G, then $\forall H, K \in \mathcal{N}(G)$, meet is again given by $\cap$ and $H \vee K = HK$, where

$$HK = \{hk \mid h \in H, k \in K\}.$$

Similar statements can be made for other algebraic objects, such as the submodules of a module, the subfields of a field, the subrings of a ring or the ideals of ring, in which the join of two ideals is their sum.

Note that not all posets are lattices. For example, any antichain with at least two elements is not a lattice. Another simple example, consider the set $\{0, 1, 2\}$ with the partial order $\{(0,0), (1,1), (2,2), (0,1), (0,2)\}$ which is indeed a poset (the verification is simple) but it is not a lattice because $1 \vee 2$ does not exist.

Remark 1.2.1. *[9] Let L be a lattice. Then for all $a, b, c, d \in L$:*

(1) $a \preccurlyeq b$ implies $a \vee c \preccurlyeq b \vee c$ and $a \wedge c \preccurlyeq b \wedge c$.

(2) $a \preccurlyeq b$ and $c \preccurlyeq d$ implies $a \vee c \preccurlyeq b \vee d$ and $a \wedge c \preccurlyeq b \wedge d$.

Theorem 1.2.1. *[9] Let L be a lattice. Then the following properties hold for all $a, b, c \in L$:*

(L1) idempotency

$$a \vee a = a, \quad a \wedge a = a$$

*(L2) **commutativity***

$$a \vee b = b \vee a, \quad a \wedge b = b \wedge a$$

*(L3) **associativity***

$$(a \vee b) \vee c = a \vee (b \vee c), \quad (a \wedge b) \wedge c = a \wedge (b \wedge c)$$

*(L4) **absorption***

$$a \vee (a \wedge b) = a, \quad a \wedge (a \vee b) = a.$$

Proof. Let L be a lattice. Then the idempotency and commutativity are evident. the associativity is also evident since $(a \vee b) \vee c$ and $a \vee (b \vee c)$ are both equal to the least upper bound of $\{a, b, c\}$. For $(L4)$, since $a \vee b$ is the least upper bound for $\{a, b\}$, so $a \preccurlyeq a \vee b$. Therefore, $a \wedge (a \vee b) = a$. By duality, $a \vee (a \wedge b) = a$. ■

We introduced lattices as partially ordered sets. However, we may adopt an alternative viewpoint. In mathematics, and more specifically in abstract algebra, the term algebraic structure generally refers to a set with one or more operations defined on it. Lattices can also be characterized as algebraic structures satisfying certain axiomatic identities.

Theorem 1.2.2. *[22] If $(L, \vee, \wedge)$ is a nonempty set L with two binary operations $\vee$ and $\wedge$ satisfying (L1)-(L4), then L is a lattice where join is $\vee$, meet is $\wedge$ and the order relation is given by*

$$a \preccurlyeq b \quad \textit{if} \quad a \vee b = b$$

or equivalently,

$$a \preccurlyeq b \quad \textit{if} \quad a \wedge b = a.$$

Moreover, since the set of axioms (L1)-(L4) is self-dual, it follows that if a statement holds in every lattice, then any dual statement holds in every lattice.

Proof. Let L be a nonempty set with two binary operations $\vee$ and $\wedge$ satisfying (L1)-(L4). As to the equivalence of the statements $a \vee b = b$ and $a \wedge b = a$, the absorption laws imply that

$$a \wedge b = a \quad \Rightarrow \quad a \vee b = (a \wedge b) \vee b = b$$

and

$$a \vee b = b \quad \Rightarrow \quad a \wedge b = a \wedge (a \vee b) = a.$$

Next we show that $\preccurlyeq$ is a partial order. For (reflexivity), the idempotency law implies that $a \preccurlyeq a$. For (antisymmetry), if $a \preccurlyeq b$ and $b \preccurlyeq a$, then

$$b = a \vee b = a.$$

For (transitivity), if $a \preccurlyeq b$ and $b \preccurlyeq c$, then

$$a \vee c = a \vee (b \vee c) = (a \vee b) \vee c = b \vee c = c \Rightarrow a \vee c = c \Rightarrow a \preccurlyeq c.$$

Finally, we prove that $\wedge$ and $\vee$ are the meet and join in L. For the meet, $a \wedge b$ is the lower bound for a and b since

$$(a \wedge b) \wedge a = (b \wedge a) \wedge a = b \wedge (a \wedge a) = b \wedge a = a \wedge b$$

and so $a \wedge b \preccurlyeq a$, and

$$(a \wedge b) \wedge b = a \wedge (b \wedge b) = a \wedge b$$

and so $a \wedge b \preccurlyeq b$.

Also, if u is a lower bound of a and b, then $u \preccurlyeq a$ and $u \preccurlyeq b$. So

$$u \wedge a = u = u \wedge b$$

then we have

$$u \wedge (a \wedge b) = (u \wedge a) \wedge b = u \wedge b = u$$

whence $u \preccurlyeq a \wedge b$. Thus, $a \wedge b$ is the greatest lower bound of a and b. The proof of join is similar. ■

Let L be a lattice. It may happen that $(L, \preccurlyeq)$ has greatest and least elements as defined in Definition 1.1.4. When thinking of L as $(L, \vee, \wedge)$, it is appropriate to view these elements from a more algebric standpoint. We say L has a **one** if there exists $1 \in L$ such that $x \wedge 1 = x$ for all $x \in L$. Dually, L has a **zero** if there exists $0 \in L$ such that $x \vee 0 = x$ for all $x \in L$. Note that the lattice $(L, \vee, \wedge)$ has a one if and only if $(L, \preccurlyeq)$ has a greatest element. A dual statement holds for zero.

Definition 1.2.2. [9] A lattice $(L, \vee, \wedge)$ possessing 0 and 1 is called **bounded** lattice.

Note that a finite lattice L is automatically bounded, with $1 = \bigvee L$ and $0 = \bigwedge L$.

Example 1.2.2. Let $\mathbb{N}_0 = \{0, 1, 2, ...\}$. The lattice ($\mathbb{N}_0$, lcm, gcd) is bounded with $1 = 0$ and $0 = 1$, while ($\mathbb{N}$, lcm, gcd) is not bounded (1 does not exists).

1.2.2 Sublattices

Definition 1.2.3. [16] A **sublattice** of a lattice L is a nonempty subset M of L that is closed under infimums and supremums.
Equivalently, a **sublattice** $(M, \vee, \wedge)$ of the lattice $(L, \vee, \wedge)$ is defined on a nonempty subset M of L with the property that $a, b \in M$ implies that $a \vee b \in M$ and $a \wedge b \in M$ (the operations $\vee$ and $\wedge$ are formed in $(L, \vee, \wedge)$).

Example 1.2.3. Any one-element subset of a lattice is a sublattice. More generally, any nonempty chain in a lattice is a sublattice.

Definition 1.2.4. [6] Let L be a lattice and $a, b \in L$ such that $a \preccurlyeq b$. The **closed interval** $[a, b]$ is the set of all elements $x \in L$ which satisfy $a \preccurlyeq x \preccurlyeq b$.

$$u \wedge (a \wedge b) = (u \wedge a) \wedge b = u \wedge b = u$$

whence $u \preccurlyeq a \wedge b$. Thus, $a \wedge b$ is the greatest lower bound of a and b. The proof of join is similar. ■

Let L be a lattice. It may happen that $(L, \preccurlyeq)$ has greatest and least elements as defined in Definition 1.1.4. When thinking of L as $(L, \vee, \wedge)$, it is appropriate to view these elements from a more algebraic standpoint. We say L has a one if there exists $1 \in L$ such that $x \wedge 1 = x$ for all $x \in L$. Dually, L has a zero if there exists $0 \in L$ such that $x \vee 0 = x$ for all $x \in L$. Note that the lattice $(L, \vee, \wedge)$ has a one if and only if $(L, \preccurlyeq)$ has a greatest element. A dual statement holds for zero.

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(the operations $\vee$ and $\wedge$ are formed in [illegible]

Example 1.2.3. Any one-element [illegible] any nonempty [illegible] lattice [illegible]

Definition 1.2.4. [illegible] interval $[a, b]$ is the set of [illegible]

Example 1.2.4. Let L be a lattice and $a \prec b$, then the closed interval

$$[a, b] = \{x \in L \mid a \preccurlyeq x \preccurlyeq b\}$$

is a sublattice of L.

Of course, a sublattice of a lattice is again a lattice. But it is important to note that a subset S of a lattice L can be a lattice under the same order relation and yet not be a sublattice of L. We can see that in Example 1.2.5.

Example 1.2.5. Let $L = \{1, 2, 3, 6, 12\}$ under division (see Figure 1.7).

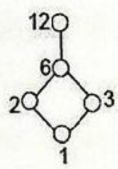


Figure 1.7:

The subset $S = \{1, 2, 3, 12\}$ is a lattice under division but not a sublattice of L. The subset $M = \{1, 2, 3, 6\}$ is a sublattice of L.

1.2.3 Lattice Homomorphisms

From the viewpoint of lattice as algebraic structures it is natural to regard as canonical those maps between lattices which preserve the operations join and meet. Since lattices are also ordered sets, order preserving maps are also available.

A mapping $f : P \to Q$ between two lattices is

- join preserving if $f(a \vee b) = f(a) \vee f(b)$ for all $a, b \in P$.
- meet preserving if $f(a \wedge b) = f(a) \wedge f(b)$ for all $a, b \in P$.

Remark 1.2.2. *A monotone map $f : P \to Q$ between lattices need not, in general, preserve meets and joins. Even an order embedding need not preserve meets and joins.*

Example 1.2.6. Consider the lattices of integers in Figures 1.8, where the order is division. The map $f : P \to Q$ defined by $f(n) = n$ is an order embedding, but

$$f(2 \vee 3) = f(12) = 12 \quad \text{and} \quad f(2) \vee f(3) = 2 \vee 3 = 6.$$

Hence, f does not preserve joins.

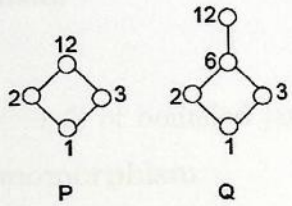


Figure 1.8:

Theorem 1.2.3. *[9] Let L and M be lattices and $f : L \to M$ a map. For all $a, b \in L$, the following are equivalent:*

(a) f is monotone;

(b) $f(a \vee b) \succcurlyeq f(a) \vee f(b)$;

(c) $f(a \wedge b) \preccurlyeq f(a) \wedge f(b)$.

Definition 1.2.5. [22] Let P and Q be lattices.

1. A function $f : P \to Q$ that preserves finite meets, that is,

$$f(a \wedge b) = f(a) \wedge f(b)$$

   is called a **meet-homomorphism**.

2. Dually, a **join-homomorphism** is a function $f : P \to Q$ that preserves finite joins.

Remark 1.2.3. *A meet-homomorphism is order preserving (or monotone), since*

$$a \preccurlyeq b \quad \Leftrightarrow \quad a \wedge b = a \quad \Rightarrow \quad f(a) \wedge f(b) = f(a) \quad \Leftrightarrow \quad f(a) \preccurlyeq f(b).$$

Dually, a join-homomorphism is order preserving as well.

Definition 1.2.6. [22] Let P and Q be lattices. A function $f : P \to Q$ that preserves finite meets and joins, that is, for which

$$f(a \wedge b) = f(a) \wedge f(b) \quad \text{and} \quad f(a \vee b) = f(a) \vee f(b)$$

is called a **lattice homomorphism** .

A lattice homomorphism $f : P \to Q$ of bounded lattices for which $f(0) = 0$ and $f(1) = 1$ is called a $\{0,1\}$ **-homomorphism.**

By induction, we see that the supremum and infimum of any finite set in a lattice are preserved under a lattice homomorphism. This is, however, not true for infinite sets in general. For instance, where $P = [0,1) \cup \{2\}$ and $Q = [0,1] \cup \{2\}$, the map $f : P \to Q$ defined by $f(x) = x$, is a lattice homomorphism from $(P, \leq)$ into $(Q, \leq)$, but $f(\bigvee[0,1)) = f(2) = 2$ while $\bigvee f([0,1)) = \bigvee[0,1) = 1$.

Example 1.2.7. [12] Figure 1.9 shows three maps of four-elements lattice into the three element chain.

The map in (i) is monotone but it is neither a meet- nor a join-homomorphism. The map in (ii) is a join-homomorphism but is not a meet-homomorphism, thus not a homomorphism. The map in (iii) is a lattice homomorphism.

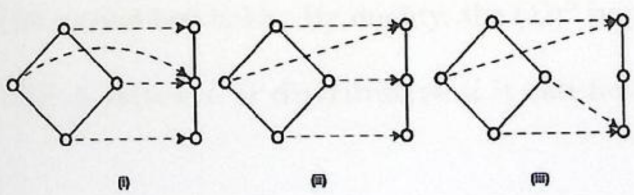


Figure 1.9:

Remark 1.2.4.

- *A **lattice endomorphism** is a lattice homomorphism of a lattice with itself.*
- *A **lattice isomorphism** is a bijective lattice homomorphism.*

1.2.4 Distributive Lattices

There are many special types of lattices like complete lattices, modular lattices, complemented lattices, etc. In this section we introduce one of the most important types, namely distributive lattices. Since lattices come with two binary operations $\wedge$ and $\vee$, it is natural to ask whether one of them distributes over the other, i.e. whether one or the other of the following **distributive laws** holds for every a, b, c in a lattice L:

$$(D) \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

$$(D)^{\partial} \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

Theorem 1.2.4. *[22] If either of the distributive laws holds for all elements of a lattice L, then so does the other.*

Proof. Suppose that the first distributive law (D) holds. Then applying it to the right side of the second distributive law $(D)^{\partial}$ and using absorption gives

$$\begin{aligned}(a \vee b) \wedge (a \vee c) &= [(a \vee b) \wedge a] \vee [(a \vee b) \wedge c] \\ &= a \vee [(a \vee b) \wedge c] \\ &= a \vee [(a \wedge c) \vee (b \wedge c)] \\ &= a \vee (b \wedge c),\end{aligned}$$

which shows that the second law holds. By duality, the $(D)^{\partial}$ implies (D) too. ■

Definition 1.2.7. [9] A lattice L is **distributive** if it satisfies the **distributive laws**.

Remark 1.2.5.

(1) *The distributive laws are dual to one another and so a lattice is distributive if and only if its dual lattice is distributive.*

(2) *The truth of D does not imply that of $(D)^{\partial}$ for individual elements of a lattice L. For example, in the lattice L with Hasse diagram given in Figure 1.10,*

$$b \wedge (a \vee c) = (b \wedge a) \vee (b \wedge c) = b \quad but \quad b \vee (a \wedge c) \neq (b \vee a) \wedge (b \vee c).$$

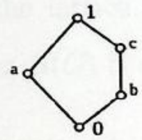


Figure 1.10:

(3) *For each distributive law, one side is related to the other in all lattices. For instance, we always have*

$$a \wedge (b \vee c) \succcurlyeq (a \wedge b) \vee (a \wedge c)$$

since each term on the right is related to each term on the left. So, we can define a distributive lattice as: a lattice is distributive if and only if either of the following inequalities holds:

$$a \wedge (b \vee c) \preccurlyeq (a \wedge b) \vee (a \wedge c)$$
$$a \vee (b \wedge c) \succcurlyeq (a \vee b) \wedge (a \vee c).$$

Let us consider some examples.

Example 1.2.8. Every sublattice of a distributive lattice is distributive.

Example 1.2.9. Any chain is a distributive lattice. In fact,

$$a \wedge (b \vee c) \text{ and } (a \wedge b) \vee (a \wedge c)$$

are both equal to a if $a \preccurlyeq b$ or $a \preccurlyeq c$; and both equal to $b \vee c$ in the alternative case that $a \succcurlyeq b$ and $a \succcurlyeq c$.

Example 1.2.10. [7] If L is a distributive lattice and $f : L \to M$ is a lattice homomorphism, then $f(L)$ the image of f is a distributive sublattice of M.

Example 1.2.11. The lattice $(\mathcal{S}(G), \subseteq)$ of subgroups of a group G is distributive if and only if G is locally cyclic (that is, any finite nonempty subset of G generates a cyclic subgroup). For instance, the lattice of subgroups of the group $(\mathbb{Z}, +)$ is distributive. Thus, if G is finite, then $\mathcal{S}(G)$ is distributive if and only if G is cyclic.

Definition 1.2.8. Let X be a nonempty set. A subset S of the power set $\mathbb{P}(X)$ is called a **ring of sets** if it is a sublattice of $\mathbb{P}(X)$, i.e. $\forall A, B \in S$

$$A \cap B \in S \quad \text{and} \quad A \cup B \in S.$$

Remark 1.2.6. *Any ring of sets is a distributive lattice, since*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Example 1.2.12. The lattices M_3 (the **diamond**) and N_5 (the **pentagon**) of Figure 1.11 are not distributive. In M_3 for example, we can see

$$x \vee (y \wedge z) = x \quad \text{but} \quad (x \vee y) \wedge (x \vee z) = 1.$$

Also, in N_5 we have

$$x \wedge (y \vee z) = x \quad \text{but} \quad (x \wedge y) \vee (x \wedge z) = z.$$

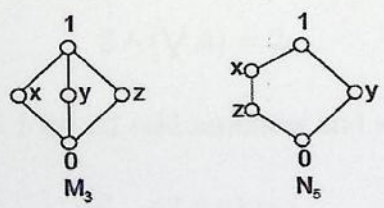


Figure 1.11:

We have as yet no way of showing that the distributive law is not satisfied except a random search for elements for which the law fails. Theorem 1.2.5 remedies this in a most satisfactory way. It implies that it is possible to determine whether or not a finite lattice is distributive from its diagram. This theorem is due to G. Birkhoff.

Theorem 1.2.5. *[16] A lattice is distributive if and only if it contains no sublattice that is isomorphic to* M_3*or to* N_5 *.*

Proof. see [16].

Theorem 1.2.6. *[6] In a distibutive lattice L, if* $a \vee b = c \vee b$ *and* $a \wedge b = c \wedge b$ *then* $a = c$ *for all* $a, b, c \in L$.

Proof. Let L be distributive. Let $a \vee b = c \vee b$ and $a \wedge b = c \wedge b$. By Using repeatedly (L4), (L2) and (D), we can get

$$\begin{aligned} a = a \wedge (b \vee a) &= a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \\ &= (b \wedge c) \vee (a \wedge c) = (b \vee a) \wedge c = (b \vee c) \wedge c = c. \end{aligned}$$ ■

Example 1.2.13. The lattice $(\mathbb{N}_0, \ |)$ is distributive. To show that use Theorem 1.2.6 and the identity $mn = \gcd\,(m, n)\mathrm{lcm}(m, n)$.

Remark 1.2.7. *Of course, distributivity extends to finite joins and meets, but not necessarily to infinite ones.*

Example 1.2.14. [22] The lattice $(\mathbb{N}_0, \ |)$ is distributive. However, let A be the set of odd natural numbers. Then $\bigvee A = 0$ and so

$$2 \wedge (\bigvee A) = 2.$$

On the other hand, $2 \wedge a = 1$ for all odd numbers and so

$$\bigvee_{a \in A} (2 \wedge a) = 1.$$

Chapter 2

Ideals And Congruence Relations on Lattices

In this chapter we introduce the concept of ideal and filter in lattices. We present some types of ideals like prime ideals, maximal ideals and principle ideals. Also we discuss the properties of ideals in lattices and in distributive lattices. In addition this chapter is devoted to congruence relations on lattices, quotient lattices and types of kernels. Furthermore we discuss some theorems and their relationship with distributivity.

2.1 Ideals and Filters

Ideals are fundamental importance in algebra. Filters, the order dual of lattice ideals, have a variety of application in logic and topology. In this section we introduce the concept of ideal and filter in a lattice. In addition we present the concept of principal ideal and principal filter. Also examples are presented on each concept. Furthermore we will mention some properties of ideals in lattices and in distributive lattices.

Definition 2.1.1. Let L be a lattice.

1. A nonempty subset I of L is called an **ideal** of L if

   (a) $x \in L$, $a \in I$ and $x \preccurlyeq a \Rightarrow x \in I$.

   (b) $a, b \in I \Rightarrow a \vee b \in I$.

   A **proper ideal**, that is, an ideal $I \neq L$. The set of all ideals of L is denoted by $\mathcal{I}(L)$.

2. Dually, a nonempty subset F of L is called a **filter** of L if

   (a) $x \in L$, $a \in F$ and $x \succcurlyeq a \Rightarrow x \in F$.

   (b) $a, b \in F \Rightarrow a \wedge b \in F$.

   A **proper filter**, that is, a filter $F \neq L$. The set of all filters of L is denoted by $\mathcal{F}(L)$.

Example 2.1.1. [9] Let L and M be bounded lattices and $f : L \to M$ a $\{0,1\}$-homomorphism. Then $f^{-1}(0)$ is an ideal and $f^{-1}(1)$ is a filter in L.

It is easy to show that an ideal I of a lattice with 1 is proper if and only if $1 \notin I$, and dually, a filter F of a lattice with 0 is proper if and only if $0 \notin F$.

Remark 2.1.1. *Every ideal I of lattice L is a sublattice, since*

$$a \wedge b \preccurlyeq a \quad \forall a, b \in L.$$

Hence, $a \wedge b \in I$. Dually, every filter F of L is a sublattice.

Let us say that two properties p and q of subsets of a lattice L are **complementary** if

$$S \text{ has property } p \quad \Leftrightarrow \quad L \setminus S \text{ has property } q.$$

Note that the properties of being an ideal and being a filter are not complementary.

Example 2.1.2. Let $L = \{0, a, b, c, 1\}$ be a lattice with Hasse diagram given in Figure 2.1. We can see that the subset $I = \{0, a\}$ is an ideal of L. On the other hand, $L \setminus I = \{b, c, 1\}$ is not a filter of L since

$$b, c \in L \setminus I \quad \text{but} \quad b \wedge c = a \notin L \setminus I.$$

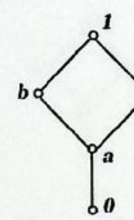


Figure 2.1:

Like other algebraic structures, we can also define the sublattice and the ideal generated by a subset of a lattice.

Definition 2.1.2. [22] Let H be a nonempty subset of a lattice L.

(1) The **ideal generated** by H, denoted by $(H]$, is the smallest ideal of L containing H.

(2) The **filter generated** by H, denoted by $[H)$, is the smallest filter of L containing H.

Remark 2.1.2. *[9] If $H = \{a\}$, then we write $(a]$ or $\downarrow a$ instead of $(\{a\}]$ such that*

$$(a] = \downarrow a = \{x \in L \mid x \preccurlyeq a\}$$

is known as the ***principal ideal*** *generated by a. Dually,*

$$[a) = \uparrow a = \{x \in L \mid x \succcurlyeq a\}$$

is known as the ***principal filter*** *generated by a.*

Example 2.1.3. Consider the lattice $L = \{1, 2, 4, 5, 10, 20\}$ with Hasse diagram given in Figure 2.2. The ideal generated by $\{2, 4\}$ is $(\{2, 4\}] = \{1, 2, 4\} = \downarrow 4$, and the ideal generated by $\{4, 5\}$ is $(\{4, 5\}] = \{1, 2, 4, 5, 10\} = \downarrow 10$. Of course, $\downarrow 20 = L$, since 20 is the top element in L.

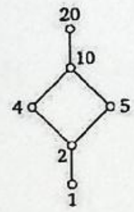


Figure 2.2:

Remark 2.1.3. *[9] In a finite lattice, every ideal or filter is principal: the ideal I equals $\downarrow \bigvee I$. Dually, the filter J of the finite lattice L equals $\uparrow \bigwedge J$.*

Note that the intersection $I \cap J$ of two ideals I and J of a lattice L is not empty, since if $i \in I$ and $j \in J$, then $i \wedge j \in I \cap J$.

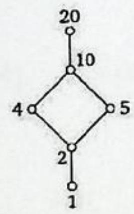
Theorem 2.1.1. *[7] If L is a lattice then, ordered by set inclusion, the set $\mathcal{I}(L)$ of ideals of L forms a lattice in which the lattice operations are given by*

$$\begin{aligned} J \wedge K &= J \cap K; \\ J \vee K &= \{x \in L \mid x \preccurlyeq j \vee k, \ for\ some\ j \in J,\ k \in K\}. \end{aligned}$$

Proof. We have to show that every pair of ideals of L has an infimum and a supremum in $\mathcal{I}(L)$. It is clear that if J and K are ideals of L, then so $J \cap K$, and that this is the biggest ideal of L that is contained in both J and K. Hence, $J \wedge k \in \mathcal{I}(L)$.

Now, any ideal that contains both J and K must clearly contain all the elements x such that $x \preccurlyeq j \vee k$ where $j \in J$ and $k \in K$. Conversely, the set of all such x clearly contains both J and K, and is contained in every ideal of L that contains both J and K.

Example 2.1.3. Consider the lattice $L = \{1, 2, 4, 5, 10, 20\}$ with Hasse diagram given in Figure 2.2. The ideal generated by $\{2,4\}$ is $(\{2,4\}] = \{1,2,4\} = \downarrow 4$, and the ideal generated by $\{4,5\}$ is $(\{4,5\}] = \{1,2,4,5,10\} = \downarrow 10$. Of course, $\downarrow 20 = L$, since 20 is the top element in L.



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Proof. We have to show that every pair of ideals of L has an infimum and a supremum in $\mathcal{I}(L)$. It is clear that if J and K are ideals of L, then so $J \cap K$, and that this is the biggest ideal of L that is contained in both J and K. Hence, $J \wedge k \in \mathcal{I}(L)$.

Now, any ideal that contains both J and K must clearly contain all the elements x such that $x \preccurlyeq j \vee k$ where $j \in J$ and $k \in K$. Conversely, the set of all such x clearly contains both J and K, and is contained in every ideal of L that contains both J and K.

Moreover, this set is also an ideal of L: if $x \in J \vee K$ and $r \in L$ such that $r \preccurlyeq x \preccurlyeq j \vee k$ for some $j \in J$ and $k \in K$, then (by transitivity)

$$r \preccurlyeq j \vee k \ \Rightarrow \ r \in J \vee K.$$

Also, if $x, y \in J \vee K$, then

$$x \preccurlyeq j \vee k \quad \text{and} \quad y \preccurlyeq j_1 \vee k_1 \quad \text{for some } j, j_1 \in J \text{ and } k, k_1 \in K.$$

Hence

$$x \vee y \preccurlyeq (j \vee k) \vee (j_1 \vee k_1) = (j \vee j_1) \vee (k \vee k_1),$$

where $(j \vee j_1) \in J$ and $(k \vee k_1) \in K$ since J and K are ideals. Therefore $x \vee y \in J \vee K$. Thus we see that $J \vee K$ exist in $\mathcal{I}(L)$. ■

Theorem 2.1.2. *[13] A lattice L is distributive if and only if for any two ideals I, J of L:*

$$I \vee J = \{i \vee j \mid i \in I, j \in J\}.$$

Proof. Let L be distributive. By Theorem 2.1.1, if $t \in I \vee J$, then $t \preccurlyeq i \vee j$, for some $i \in I$ and $j \in J$. Therefore,

$$t = t \wedge (i \vee j) = (t \wedge i) \vee (t \wedge j), \qquad t \wedge i \in I,\, t \wedge j \in J.$$

Conversely, if L is not distributive, then L contains elements a, b, c as in Figure 2.3.

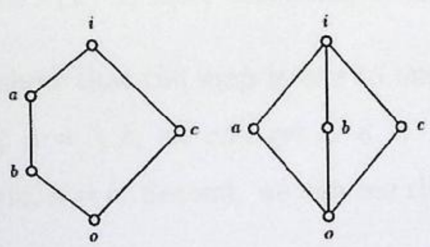

Figure 2.3:

Let $I = \downarrow b$ and $J = \downarrow c$: observe that $a \in I \vee J$, since $a \preccurlyeq b \vee c$. However, a has no representation as required by Theorem 2.1.2, since if $a = i \vee j$ for some $i \in I$, $j \in J$, then $j \preccurlyeq a$ and $j \preccurlyeq c$. Therefore, $j \preccurlyeq a \wedge c \prec b$. Hence $j \in I$ and so $a = i \vee j \in I$, a contradiction. ■

Theorem 2.1.3. *[22] A lattice L is distributive if and only if $\mathcal{I}(L)$ is distributive.*

Proof. Let L be distributive and $A, B, C \in \mathcal{I}(L)$. We must show that

$$A \cap (B \vee C) \subseteq (A \cap B) \vee (A \cap C).$$

If $x \in A \cap (B \vee C)$, then $x = a \in A$ and $x = b \vee c$ for some $b \in B$ and $c \in C$, so

$$x = a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \in (A \cap B) \vee (A \cap C)$$

for $a \in A$, $b \in B$ and $c \in C$. Hence $\mathcal{I}(L)$ is distributive. Conversely, Let $\mathcal{I}(L)$ be distributive. If L is not distributive lattice, then L contains elements a, b, c as in Figure 2.3. So

$$\downarrow a \cap \downarrow c = \downarrow b \cap \downarrow c \quad \text{and} \quad \downarrow a \vee \downarrow c = \downarrow b \vee \downarrow c \quad \text{but} \quad \downarrow a \neq \downarrow b.$$

Hence, by Theorem 1.2.6, $\mathcal{I}(L)$ is not distributive, a contradiction. ■

Theorem 2.1.4. *[22] The map $\varphi : L \to \mathcal{I}(L)$ such that $\varphi(a) = \downarrow a$ is an injective lattice homomorphism, that is, for all $a, b \in L$,*

$$\downarrow a \cap \downarrow b = \downarrow (a \wedge b) \quad \textit{and} \quad \downarrow a \vee \downarrow b = \downarrow (a \vee b)$$

The image of φ is sublattice $\mathcal{P}(L)$ of $\mathcal{I}(L)$ consisting of all principal ideals of L.

Proof. First, we have to show that the map is one to one. Let $a, b \in L$ such that $\varphi(a) = \varphi(b)$. Since $a \in \downarrow a = \downarrow b$, we can get $a \preccurlyeq b$. In the same way $b \preccurlyeq a$. Therefore, by antisymmetric, $a = b$. Second, we can see that

$$\begin{aligned} \downarrow a \cap \downarrow b &= \{x \in L \mid x \preccurlyeq a \text{ and } x \preccurlyeq b\} \\ &= \{x \in L \mid x \preccurlyeq a \wedge b\} \\ &= \downarrow (a \wedge b) \end{aligned}$$

then $j \preccurlyeq a$ and $j \preccurlyeq c$. Therefore, $j \preccurlyeq a \wedge c \prec b$. Hence $j \in I$ and so $a = i \vee j \in I$, a contradiction. ■

Theorem 2.1.3. *[22] A lattice L is distributive if and only if $\mathcal{I}(L)$ is distributive.*

Proof. Let L be distributive and $A, B, C \in \mathcal{I}(L)$. We must show that

$$A \cap (B \vee C) \subseteq (A \cap B) \vee (A \cap C).$$

If $x \in A \cap (B \vee C)$, then $x = a \in A$ and $x = b \vee c$ for some $b \in B$ and $c \in C$, so

$$x = a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \in (A \cap B) \vee (A \cap C)$$

for $a \in A$, $b \in B$ and $c \in C$. Hence $\mathcal{I}(L)$ is distributive. Conversely, Let $\mathcal{I}(L)$ be distributive. If L is not distributive lattice, then L contains elements a, b, c as in Figure 2.3. So

$$\downarrow a \cap \downarrow c = \downarrow b \cap \downarrow c \quad \text{and} \quad \downarrow a \vee \downarrow c = \downarrow b \vee \downarrow c \quad \text{but} \quad \downarrow a \neq \downarrow b.$$

Hence, by Theorem 1.2.6, $\mathcal{I}(L)$ is not distributive, a contradiction. ■

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and

$$\begin{aligned} \downarrow a \vee \downarrow b &= \{x \in L \mid x \preccurlyeq i \vee j \text{ } for \text{ } i \in \downarrow a \text{ } and \text{ } j \in \downarrow b\} \\ &= \{x \in L \mid x \preccurlyeq a \vee b\} \\ &= \downarrow (a \vee b). \end{aligned}$$

Hence, φ is a lattice homomorphism. The principal ideals thus form a sublattice of $\mathcal{I}(L)$, since for all $\downarrow a, \downarrow b \in \mathcal{P}(L)$

$$\downarrow a \cap \downarrow b = \downarrow (a \wedge b) \in \mathcal{P}(L) \quad \text{and} \quad \downarrow a \vee \downarrow b = \downarrow (a \vee b) \in \mathcal{P}(L).$$

2.2 Prime and Maximal Ideals

In this section we present special types of ideals which play a key role in lattice theory. Also we discuss the relation between these types in distributive lattices. In addition we prove the theorem of Stone which play an important role in Chapter 3.

Definition 2.2.1. [19] Let L be a lattice. If $a, b \in L$, then

(1) A proper ideal J is **prime** if

$$a \wedge b \in J \quad \Rightarrow \quad a \in J \quad \text{or } b \in J.$$

The set of prime ideals of L is denoted by $\mathcal{I}_{\mathcal{P}}(L)$.

(2) A proper filter F is **prime** if

$$a \vee b \in F \quad \Rightarrow \quad a \in F \quad \text{or } b \in F.$$

The set of prime filters of L is denoted by $\mathcal{F}_{\mathcal{P}}(L)$.

Example 2.2.1. Let L be the lattice given in Example 2.1.3. The ideal $\downarrow 10 = \{10, 4, 5, 2, 1\}$ is a prime ideal. The ideal $I = \{1, 2\}$ is not a prime ideal since $4 \wedge 5 = 2 \in I$, but $4 \notin I$ and $5 \notin I$.

and

$$\begin{aligned} \downarrow a \vee \downarrow b &= \{x \in L \mid x \preccurlyeq i \vee j \ for \ i \in \downarrow a \ and \ j \in \downarrow b\} \\ &= \{x \in L \mid x \preccurlyeq a \vee b\} \\ &= \downarrow (a \vee b). \end{aligned}$$

Hence, φ is a lattice homomorphism. The principal ideals thus form a sublattice of $\mathcal{I}(L)$, since for all $\downarrow a, \downarrow b \in \mathcal{P}(L)$

$$\downarrow a \cap \downarrow b = \downarrow (a \wedge b) \in \mathcal{P}(L) \quad \text{and} \quad \downarrow a \vee \downarrow b = \downarrow (a \vee b) \in \mathcal{P}(L).$$

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The set of prime filters of L is denoted by $\mathcal{F}_{\mathcal{P}}(L)$.

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Theorem 2.2.1. *[22] Let $f : L \to M$ be a lattice homomorphism.*

(1) If $I \in \mathcal{I}(M)$ and $f^{-1}(I) \neq \emptyset$, then $f^{-1}(I) \in \mathcal{I}(L)$.

(2) If $P \in \mathcal{I}_{\mathcal{P}}(M)$ and $f^{-1}(P) \neq \emptyset$, then $f^{-1}(P) \in \mathcal{I}_{\mathcal{P}}(L)$.

Proof. For(1), let I be an ideal of M and let $a, b \in f^{-1}(I)$. Then there exist $x, y \in I$ such that $f(a) = x$ and $f(b) = y$. So

$$\begin{aligned} f(a \vee b) &= f(a) \vee f(b) \\ &= x \vee y \in I. \end{aligned}$$

Therefore $a \vee b \in f^{-1}(I)$. Let $r \in L$ and $a \in f^{-1}(I)$ such that $r \preccurlyeq a$. Since f is a lattice homomorphism, f is order preserving. This implies $f(r) \preccurlyeq f(a) \in I$. But I is an ideal, so $f(r) \in I$. Hence $r \in f^{-1}(I)$. For (2), let P be a prime ideal of M. We have to show that $f^{-1}(P)$ is a prime ideal of L. From(1), $f^{-1}(P)$ is an ideal. Let $a \wedge b \in f^{-1}(P)$. Then

$$f(a \wedge b) = f(a) \wedge f(b) \in P.$$

Since P is prime, we can get

$$f(a) \in P \text{ or } f(b) \in P \quad \Rightarrow \quad a \in f^{-1}(P) \text{ or } b \in f^{-1}(P).$$

Hence $f^{-1}(P) \in \mathcal{I}_{\mathcal{P}}(L)$. ■

Remark 2.2.1. *[22] The properties of being a prime ideal and being a prime filter are complementary, that is, a subset J of a lattice L is a prime ideal if and only if $L \setminus J$ is a prime filter. Thus it is easy to switch between $\mathcal{I}_{\mathcal{P}}(L)$ and $\mathcal{F}_{\mathcal{P}}(L)$.*

Definition 2.2.2. Let L be a lattice.

(1) A proper ideal J of L is **maximal** if for any ideal I,

$$J \subseteq I \subseteq L \quad \Rightarrow \quad I = J \quad \text{or} \quad I = L.$$

(2) A proper filter F of L is **maximal** if for any filter X,

$$F \subseteq X \subseteq L \quad \Rightarrow \quad X = F \quad \text{or} \quad X = L.$$

In ring theory, maximal ideals are prime. This is not true in general for lattices.

Example 2.2.2. In the lattice M_3 of Figure 2.4, the ideal $I = \{0, x\}$ is maximal. However, $y \wedge z \in I$ but $y \notin I$ and $z \notin I$ and so I is not prime.

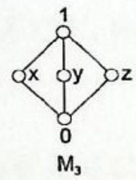

Figure 2.4:

Theorem 2.2.2. *[7] In a distributive lattice L, all maximal ideals are prime.*

Proof. Let M be a maximal ideal of L. Suppose that $a \wedge b \in M$ but $a \notin M$. Then $\downarrow a \vee M = L$ and so $b \preccurlyeq a \vee m$ for some $m \in M$. Hence,

$$b = b \wedge (a \vee m) = (b \wedge a) \vee (b \wedge m) \in M$$

which shows that M is prime. ■

One of the main results of prime ideals in distributive lattices is the following result due to Stone.

Theorem 2.2.3. *(Stone 1936) Let L be a distributive lattice, let I be an ideal of L, let F be a filter of L, and let $I \cap F = \emptyset$. Then there exists a prime ideal P of L such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Proof. Some form of the Axiom of Choice is needed to prove this statement. The most convenient form for this proof is:

Zorn's Lemma: Let A be a set and let $\mathcal{X}$ be a nonempty subset of $\mathbb{P}(A)$. Let us assume that $\mathcal{X}$ has the following property: If C is a chain in $(\mathcal{X}, \subseteq)$, then $\bigcup C \in \mathcal{X}$. Then $\mathcal{X}$ has a maximal element.

Let $\mathcal{X}$ be the set of all ideals of L that contain I and are disjoint from F. We verify that $\mathcal{X}$ satisfies the hypothesis of Zorn's Lemma. The set $\mathcal{X}$ is nonempty, since $I \in \mathcal{X}$. Let C be a chain in $\mathcal{X}$ and let $M = \bigcup C$. We have to show that M is an ideal:

1. if $a, b \in M$, then $a \in X$ and $b \in Y$, for some $X, Y \in C$; since C is a chain, either $X \subseteq Y$ or $Y \subseteq X$; if say $X \subseteq Y$, then $a, b \in Y$, and so $a \vee b \in Y \subseteq M$, since Y is an ideal.

2. also, if $a \in M$ and $b \preccurlyeq a$, then $a \in X \in C$, since X is an ideal, $b \in X \subseteq M$.

Thus M is an ideal. It is obvious that $I \subseteq M$ and $M \cap F = \emptyset$, verifying that $M \in \mathcal{X}$. Therefore, by Zorn's Lemma, $\mathcal{X}$ has a maximal element P.
We claim that P is a prime ideal. Indeed, if P is not prime, then there exist $a, b \in L$ such that $a, b \notin P$ but $a \wedge b \in P$. The maximality of P yields that $(P \vee \downarrow a) \cap F \neq \emptyset$ and $(P \vee \downarrow b) \cap F \neq \emptyset$. Thus there are $p, q \in P$ such that $p \vee a \in F$ and $q \vee b \in F$. Then $x = (p \vee a) \wedge (q \vee b) \in F$, since F is filter. Expanding by distributivity,

$$x = (p \wedge q) \vee (a \wedge q) \vee (p \wedge b) \vee (a \wedge b) \in P$$

thus $P \cap F \neq \emptyset$, a contradiction. ■

Corollary 2.2.4. *Let L be a distributive lattice, let I be an ideal of L, and let $a \in L$ and $a \notin I$. Then there is a prime ideal P such that $I \subseteq P$ and $a \notin P$.*

Proof. Let I be an ideal such that $a \notin I$. If $F = \uparrow a$, then $I \cap F = \emptyset$. Apply Theorem 2.2.3 to I and F. ■

Corollary 2.2.5. *Let L be a distributive lattice, $a, b \in L$ and $a \neq b$. Then there is a prime ideal containing exactly one of a and b.*

Proof. Either $\downarrow a \cap \uparrow b = \emptyset$ or $\uparrow a \cap \downarrow b = \emptyset$, so we can apply Corollary 2.2.4. ■

Corollary 2.2.6. *Every ideal I of a distributive lattice is the intersection of all prime ideals containing it.*

Let $\mathcal{X}$ be the set of all ideals of L that contain I and are disjoint from F. We verify that $\mathcal{X}$ satisfies the hypothesis of Zorn's Lemma. The set $\mathcal{X}$ is nonempty, since $I \in \mathcal{X}$. Let C be a chain in $\mathcal{X}$ and let $M = \bigcup C$. We have to show that M is an ideal:

1. if $a, b \in M$, then $a \in X$ and $b \in Y$, for some $X, Y \in C$; since C is a chain, either $X \subseteq Y$ or $Y \subseteq X$; if say $X \subseteq Y$, then $a, b \in Y$, and so $a \vee b \in Y \subseteq M$, since Y is an ideal.

2. also, if $a \in M$ and $b \preccurlyeq a$, then $a \in X \in C$, since X is an ideal, $b \in X \subseteq M$.

Thus M is an ideal. It is obvious that $I \subseteq M$ and $M \cap F = \emptyset$, verifying that $M \in \mathcal{X}$. Therefore, by Zorn's Lemma, $\mathcal{X}$ has a maximal element P.
We claim that P is a prime ideal. Indeed, if P is not prime, then there exist $a, b \in L$ such that $a, b \notin P$ but $a \wedge b \in P$. The maximality of P yields that $(P \vee \downarrow a) \cap F \neq \emptyset$ and $(P \vee \downarrow b) \cap F \neq \emptyset$. Thus there are $p, q \in P$ such that $p \vee a \in F$ and $q \vee b \in F$. Then $x = (p \vee a) \wedge (q \vee b) \in F$, since F is filter. Expanding by distributivity,

$$x = (p \wedge q) \vee (a \wedge q) \vee (p \wedge b) \vee (a \wedge b) \in P$$

thus $P \cap F \neq \emptyset$, a contradiction. ■

Corollary 2.2.4. *Let L be a distributive lattice, let I be an ideal of L, and let $a \in L$ and $a \notin I$. Then there is a prime ideal P such that $I \subseteq P$ and $a \notin P$.*

Proof. Let I be an ideal such that $a \notin I$. If $F = \uparrow a$, then $I \cap F = \emptyset$. Apply Theorem 2.2.3 to I and F. ■

Corollary 2.2.5. *Let L be a distributive lattice, $a, b \in L$ and $a \neq b$. Then there is a prime ideal containing exactly one of a and b.*

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Corollary 2.2.6. *Every ideal I of a distributive lattice is the intersection of all prime ideals containing it.*

Proof. Let I be an ideal of a distributive lattice L and let

$$I_1 = \bigcap(P \mid I \subseteq P, P \text{ is a prime ideal of } L).$$

If $I \neq I_1$, then there is an $a \in I_1 \setminus I$, and so, by Corollary 2.2.4, there is a prime ideal P, with $I \subseteq P$ and $a \notin P$. But then $a \notin I_1$ is a contradiction. ■

Theorem 2.2.7. *[9] Let L be a lattice and let $X = \mathcal{I}_{\mathcal{P}}(L)$. Then the map $\eta : L \to \mathbb{P}(X)$ defined by*

$$\eta : a \to X_a := \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid a \notin I\}$$

is a lattice homomorphism.

Proof. We have to show that $X_{a\vee b} = X_a \cup X_b$ and $X_{a\wedge b} = X_a \cap X_b$, for all $a, b \in L$. Take $I \in \mathcal{I}_{\mathcal{P}}(L)$. Since I is an ideal,

$$a \vee b \in I \text{ if and only if } a \in I \text{ and } b \in I$$

and, since I is prime,

$$a \wedge b \in I \text{ if and only if } a \in I \text{ or } b \in I.$$

Thus we have

$$\begin{aligned} X_{a\vee b} &= \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid a \vee b \notin I\} \\ &= \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid a \notin I \text{ or } b \notin I\} \\ &= X_a \cup X_b. \end{aligned}$$

Similarly,

$$\begin{aligned} X_{a\wedge b} &= \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid a \wedge b \notin I\} \\ &= \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid a \notin I \text{ and } b \notin I\} \\ &= X_a \cap X_b. \end{aligned}$$ ■

Theorem 2.2.8. *[6] Every distributive lattice is isomorphic to a ring of sets.*

Proof. Let I be an ideal of a distributive lattice L and let

$$I_1 = \bigcap(P \mid I \subseteq P, P \text{ is a prime ideal of } L).$$

If $I \neq I_1$, then there is an $a \in I_1 \setminus I$, and so, by Corollary 2.2.4, there is a prime ideal P, with $I \subseteq P$ and $a \notin P$. But then $a \notin I_1$ is a contradiction. ■

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$$a \vee b \in I \text{ if and only if } a \in I \text{ and } b \in I$$

and, since I is prime,

$$a \wedge b \in I \text{ if and only if } a \in I \text{ or } b \in I.$$

Thus we have

$$\begin{aligned} X_{a\vee b} &= \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid a \vee b \notin I\} \\ &= \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid a \notin I \textit{ or } b \notin I\} \\ &= X_a \cup X_b. \end{aligned}$$

Similarly,

$$\begin{aligned} X_{a\wedge b} &= \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid a \wedge b \notin I\} \\ &= \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid a \notin I \textit{ and } b \notin I\} \\ &= X_a \cap X_b. \ ■ \end{aligned}$$

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If $I \neq I_1$, then there is an $a \in I_1 \setminus I$, and so, by Corollary 2.2.4, there is a prime ideal P, with $I \subseteq P$ and $a \notin P$. But then $a \notin I_1$ is a contradiction. ■

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Proof. We have to show that $X_{a\vee b} = X_a \cup X_b$ and $X_{a\wedge b} = X_a \cap X_b$, for all $a, b \in L$. Take $I \in \mathcal{I}_{\mathcal{P}}(L)$. Since I is an ideal,

$$a \vee b \in I \text{ if and only if } a \in I \text{ and } b \in I$$

and, since I is prime,

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Thus we have

$$\begin{aligned} X_{a\vee b} &= \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid a \vee b \notin I\} \\ &= \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid a \notin I \text{ or } b \notin I\} \\ &= X_a \cup X_b. \end{aligned}$$

Similarly,

$$\begin{aligned} X_{a\wedge b} &= \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid a \wedge b \notin I\} \\ &= \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid a \notin I \text{ and } b \notin I\} \\ &= X_a \cap X_b. \end{aligned}$$ ■

Theorem 2.2.8. *[6] Every distributive lattice is isomorphic to a ring of sets.*

Proof. Let L be a distributive lattice and let $X = \mathcal{I}_{\mathcal{P}}(L)$. Define the map $\eta : L \to \mathbb{P}(X)$ by

$$\eta : a \to X_a := \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid a \notin I\}.$$

By Theorem 2.2.7 η is a lattice homomorphism, so that the image of η is a sublattice of $\mathbb{P}(X)$, i.e. Im η = the image of η is a ring of sets. Now, we have to show that η is injective. If $a, b \in L$ and $a \neq b$, then Corollary 2.2.5 implies that there exist a prime ideal P such that $a \in P$ and $b \notin P$. Therefore, $P \in X_b$ and $P \notin X_a$. So, $\eta(a) \neq \eta(b)$. Hence $L \cong \text{Im } \eta$. ■

2.3 Congruence Relations on Lattices

Congruence relations play a central role in lattice theory. This section develops the rudiments of a theory which goes way beyond the scope of an introductory text such as this. In this section we introduce the congruence relation on the group. Then we discuss the concept of congruence relation on lattices. Some examples and properties are given to illustrate these concept.

Introducing Congruences

We recall that an equivalence relation on a set A is a binary operation on A which is reflexive, symmetric and transitive. We write $a \equiv b \pmod{\theta}$ or $a\ \theta\ b$ to indicate that a and b are related under the relation θ.

An equivalence relation θ on A gives rise to a partition of A into non empty disjoint subsets. These subsets are the **equivalence classes** or **blocks** of θ. Atypical block is of the form $[a]_\theta := \{x \in A \mid x \equiv a \pmod{\theta}\}$.

If A is an algebraic structure, such as a lattice, a group, a ring or a module, then an equivalence relation θ on A that also preserves the algebraic operations of A is

Proof. Let L be a distributive lattice and let $X = \mathcal{I}_{\mathcal{P}}(L)$. Define the map $\eta : L \to \mathbb{P}(X)$ by

$$\eta : a \to X_a := \{I \in \mathcal{I}_{\mathcal{P}}(L) \mid a \notin I\}.$$

By Theorem 2.2.7 η is a lattice homomorphism, so that the image of η is a sublattice of $\mathbb{P}(X)$, i.e. Im η = the image of η is a ring of sets. Now, we have to show that η is injective. If $a, b \in L$ and $a \neq b$, then Corollary 2.2.5 implies that there exist a prime ideal P such that $a \in P$ and $b \notin P$. Therefore, $P \in X_b$ and $P \notin X_a$. So, $\eta(a) \neq \eta(b)$. Hence $L \cong$ Im η. ■

2.3 Congruence Relations on Lattices

Congruence relations play a central role in lattice theory. This section develops the rudiments of a theory which goes way beyond the scope of an introductory text such as this. In this section we introduce the congruence relation on the group. Then we discuss the concept of congruence relation on lattices. Some examples and properties are given to illustrate these concept.

Introducing Congruences

We recall that an equivalence relation on a set A is a binary operation on A which is reflexive, symmetric and transitive. We write $a \equiv b \pmod{\theta}$ or $a\ \theta\ b$ to indicate that a and b are related under the relation θ.

An equivalence relation θ on A gives rise to a partition of A into non empty disjoint subsets. These subsets are the **equivalence classes** or **blocks** of θ. Atypical block is of the form $[a]_\theta := \{x \in A \mid x \equiv a \pmod{\theta}\}$.

If A is an algebraic structure, such as a lattice, a group, a ring or a module, then an equivalence relation θ on A that also preserves the algebraic operations of A is called a **congruence relation** on A. For example, a congruence relation θ on a group G is an equivalence relation that is preserve the group operation, i.e., for all $a, b, x, y \in G$, we have

1. $a \equiv a \pmod{\theta}$ reflexivity
2. if $a \equiv b \pmod{\theta}$ then $b \equiv a \pmod{\theta}$ symmetry
3. if $a \equiv b \pmod{\theta}$ and $b \equiv x \pmod{\theta}$, then $a \equiv x \pmod{\theta}$ transitivity
4. if $a \equiv b \pmod{\theta}$ and $x \equiv y \pmod{\theta}$, then $ax \equiv by \pmod{\theta}$
5. if $a \equiv b \pmod{\theta}$, then $a^{-1} \equiv b^{-1} \pmod{\theta}$. (This can actually be proven from the other four, so is strictly redundant).

Now, in elementary algebra, one teaches that there is a correspondence between certain special types of substructures and quotient structures. In the case of groups, for example, there is a correspondence between normal subgroups and quotient groups. A more complete story for groups must include the fact that a subgroup H of a group G is normal if and only if the equivalence relation modulo H defined by:

$$a \equiv b \pmod{H} \quad \Leftrightarrow \quad aH = bH$$

is a *congruence relation* on G.

Therefore, in a group, a congruence relation is the same thing as the coset decomposition for some normal subgroup. Similar statements can be made for rings or modules. However, in most treatments of elementary algebra, the role of the congruence relation is underplayed in favor of the role of the special substructure (normal subgroup, ideal, submodule).

Congruence relations on lattices

We begin with the definition of a congruence relation on a lattice.

Definition 2.3.1. [13] An equivalence relation θ on a lattice L is a **congruence relation** on L if for all $a, b, x, y \in L$,

$$a \equiv b \pmod{\theta} \quad \text{and} \quad x \equiv y \pmod{\theta}$$

imply

$$a \wedge x \equiv b \wedge y \pmod{\theta} \quad \text{and} \quad a \vee x \equiv b \vee y \pmod{\theta}.$$

The equivalence classes under a congruence relation θ are called **congruence classes** or **blocks**. The congruence class containing $a \in L$ is denoted by $[a]_\theta$. The set of all congruence classes for θ is denoted by L/θ. The set of all congruence relations on L is denoted by $\mathrm{Con}(L)$.

Example 2.3.1. In any lattice there are always two trivial congruence relations, the congruence relation θ_1 where each element is its own congruence class (block), and at the other extreme the congruence relation θ_2 with a single block. i.e.

$$x \equiv y \pmod{\theta_1} \quad \text{if and only if } x = y;$$
$$x \equiv y \pmod{\theta_2} \quad \text{for all } x, y \in L.$$

Example 2.3.2. Let L be a lattice with Hasse diagram given in Figure 2.5.

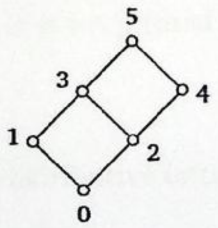

Figure 2.5:

The following are all congruences of L: $\theta_2 : \{\{0,1,2,3,4,5\}\}$,
θ_1 : $\{\{0\},\{1\},\{2\},\{3\},\{4\},\{5\}\}$, $\{\{0,1\},\{2,3\},\{4,5\}\}$, $\{\{0,2\},\{1,3\},\{4\},\{5\}\}$, $\{\{0,1,2,3\},\{4,5\}\}$, $\{\{0,2,4\},\{1,3,5\}\}$, $\{\{2,4\},\{3,5\},\{0\},\{1\}\}$, $\{\{0,1\},\{2,3,4,5\}\}$.

Example 2.3.3. In a finite chain C, a congruence relation is any decomposition of C into disjoint closed intervals, as in Figure 2.6.

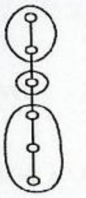

Figure 2.6:

Theorem 2.3.1. *[22] An equivalence relation θ on a lattice L is a congruence relation if and only if for all $a, b, x \in L$,*

$$a \equiv b \ (mod\ \theta) \quad \Rightarrow \quad a \vee x \equiv b \vee x \ (mod\ \theta) \text{ and } a \wedge x \equiv b \wedge x \ (mod\ \theta).$$

Proof. Assume that θ is a congruence on a lattice L. If $a \equiv b \pmod{\theta}$, then, since $x \equiv x \pmod{\theta}$, we have

$$a \vee x \equiv b \vee x \pmod{\theta} \quad \text{and} \quad a \wedge x \equiv b \wedge x \pmod{\theta}.$$

Conversely, let the stated property holds, if $a \equiv b \pmod{\theta}$, $x \equiv y \pmod{\theta}$, then

$$a \wedge x \equiv b \wedge x \pmod{\theta} \quad \text{and} \quad b \wedge x \equiv b \wedge y \pmod{\theta}.$$

By transitivity, we have

$$a \wedge x \equiv b \wedge y \pmod{\theta}.$$

Similarly for joins. ■

Example 2.3.4. [22] Let L be a distributive lattice and let $t \in L$. Then the binary relations δ, μ defined by

$$a \equiv b \pmod{\delta} \quad \text{if} \quad a \vee t = b \vee t$$

and

$$a \equiv b \pmod{\mu} \quad \text{if} \quad a \wedge t = b \wedge t$$

are both congruence relations on L.

It is easy to see that these relations are equivalence relations. For δ, it is clear that for all $a, b, x \in L$

$$a \equiv b \ (\mathrm{mod}\ \delta) \Rightarrow \ (a \vee x) \equiv (b \vee x) \ (\mathrm{mod}\ \delta)$$

and for meet, the distributivity of L gives

$$\begin{aligned} a \equiv b \ (mod\ \delta) \ &\Rightarrow \ a \vee t = b \vee t \\ &\Rightarrow \ (a \vee t) \wedge (x \vee t) = (b \vee t) \wedge (x \vee t) \\ &\Rightarrow \ (a \wedge x) \vee t = (b \wedge x) \vee t \\ &\Rightarrow \ (a \wedge x) \equiv (b \wedge x) \ (mod\ \delta). \end{aligned}$$

A similar argument can be made for μ. ■

Remark 2.3.1. *Let L be a lattice. If $a, b \in L$, $a \preccurlyeq b$ and $a \equiv b$ (mod θ), then all elements in the interval $[a, b]$ are congruent, for if $x \in [a, b]$, then*

$$x = b \wedge x \equiv a \wedge x = a \ (mod\ \theta)$$

and so every element of $[a, b]$ is congruent to a.

Definition 2.3.2. [7] A nonempty subset S of a lattice L is convex if whenever $a, b \in S$, then $[a, b] \subseteq S$.

Theorem 2.3.2. *[13] Let θ be a congruence relation on a lattice L. Then for all $a \in L$, $[a]_\theta$ is a convex sublattice.*

Proof. Let $x, y \in [a]_\theta$; then $x \equiv a$ (mod θ) and $y \equiv a$ (mod θ). Therefore,

$$x \wedge y \equiv a \wedge a = a \ (\mathrm{mod}\ \theta) \quad \text{and} \quad x \vee y \equiv a \vee a = a \ (\mathrm{mod}\ \theta),$$

proving that $[a]_\theta$ is a sublattice. Now, we need to prove that $[a]_\theta$ is convex subset. If $x, y \in [a]_\theta$ and $t \in [x, y]$, then $x \equiv a$ (mod θ) and $y \equiv a$ (mod θ). Therefore,

$$t = t \wedge y \equiv t \wedge a \ (\mathrm{mod}\ \theta),$$

and so

$$t = t \vee x \equiv (t \wedge a) \vee x \equiv (t \wedge a) \vee a = a \ (\mathrm{mod}\ \theta).$$

$$a \equiv b \text{ (mod } \delta) \Rightarrow (a \vee x) \equiv (b \vee x) \text{ (mod } \delta)$$

and for meet, the distributivity of L gives

$$\begin{aligned} a \equiv b \ (mod\ \delta) &\Rightarrow a \vee t = b \vee t \\ &\Rightarrow (a \vee t) \wedge (x \vee t) = (b \vee t) \wedge (x \vee t) \\ &\Rightarrow (a \wedge x) \vee t = (b \wedge x) \vee t \\ &\Rightarrow (a \wedge x) \equiv (b \wedge x) \ (mod\ \delta). \end{aligned}$$

A similar argument can be made for μ. ■

Remark 2.3.1. *Let L be a lattice. If $a, b \in L$, $a \preccurlyeq b$ and $a \equiv b$ (mod θ), then all elements in the interval $[a, b]$ are congruent, for if $x \in [a, b]$, then*

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Proof. Let $x, y \in [a]_\theta$; then $x \equiv a$ (mod θ) and $y \equiv a$ (mod θ). Therefore,

$$x \wedge y \equiv a \wedge a = a \text{ (mod } \theta) \quad \text{and} \quad x \vee y \equiv a \vee a = a \text{ (mod } \theta),$$

proving that $[a]_\theta$ is a sublattice. Now, we need to prove that $[a]_\theta$ is convex subset. If $x, y \in [a]_\theta$ and $t \in [x, y]$, then $x \equiv a$ (mod θ) and $y \equiv a$ (mod θ). Therefore,

$$t = t \wedge y \equiv t \wedge a \text{ (mod } \theta),$$

and so

$$t = t \vee x \equiv (t \wedge a) \vee x \equiv (t \wedge a) \vee a = a \text{ (mod } \theta).$$

Hence, $t \in [a]_\theta$. So, $[x,y] \subseteq [a]_\theta$. ■

Theorem 2.3.3. *[9] Let θ be a congruence relation on a lattice L and let $a, b \in L$.*

$$a \equiv b \ (mod\ \theta) \quad \text{if and only if} \quad a \wedge b \equiv a \vee b \ (mod\ \theta).$$

Proof. If $a \equiv b \ (\text{mod}\ \theta)$, then

$$a = a \vee a \equiv a \vee b \ (\text{mod}\ \theta) \ \text{ and } \ a = a \wedge a \equiv a \wedge b \ (\text{mod}\ \theta).$$

Since θ is symmetric and transitive, we deduce $a \wedge b \equiv a \vee b \ (\text{mod}\ \theta)$. Conversely, assume $a \wedge b \equiv a \vee b \ (\text{mod}\ \theta)$. We have $a \wedge b \preccurlyeq a \preccurlyeq a \vee b$, so $a \wedge b \equiv a \ (\text{mod}\ \theta)$, and similarly, $a \wedge b \equiv b \ (\text{mod}\ \theta)$. Because θ is symmetric and transitive, it follows that $a \equiv b \ (\text{mod}\ \theta)$. ■

Sometimes a long computation is required to prove that a binary relation is a congruence relation. Such computations are often facilitated by the following theorem which assume only a reflexive binary relation.

Theorem 2.3.4. *[14] A reflexive binary relation θ on a lattice L is a congruence relation if and only if it satisfies the following three properties :*

(1) For all $a, b \in L$,

$$a \equiv b \ (mod\ \theta) \ \Leftrightarrow \ a \wedge b \equiv a \vee b \ (mod\ \theta).$$

(2) If $a \preccurlyeq b \preccurlyeq c$ in L, then

$$a \equiv b \ (mod\ \theta), \quad b \equiv c \ (mod\ \theta) \ \Rightarrow \ a \equiv c \ (mod\ \theta).$$

(3) If $a \preccurlyeq b$ and $a \equiv b \ (mod\ \theta)$, then

$$a \wedge x \equiv b \wedge x \ (mod\ \theta) \quad \text{and} \quad a \vee x \equiv b \vee x \ (mod\ \theta) \quad \text{for all } x \in L.$$

Hence, $t \in [a]_\theta$. So, $[x, y] \subseteq [a]_\theta$. ■

Theorem 2.3.3. *[9] Let θ be a congruence relation on a lattice L and let $a, b \in L$.*

$$a \equiv b \ (mod\ \theta) \quad \text{if and only if} \quad a \wedge b \equiv a \vee b \ (mod\ \theta).$$

Proof. If $a \equiv b$ (mod θ), then

$$a = a \vee a \equiv a \vee b \ (\text{mod } \theta) \ \text{ and } \ a = a \wedge a \equiv a \wedge b \ (\text{mod } \theta).$$

Since θ is symmetric and transitive, we deduce $a \wedge b \equiv a \vee b$ (mod θ). Conversely, assume $a \wedge b \equiv a \vee b$ (mod θ). We have $a \wedge b \preccurlyeq a \preccurlyeq a \vee b$, so $a \wedge b \equiv a$ (mod θ), and similarly, $a \wedge b \equiv b$ (mod θ). Because θ is symmetric and transitive, it follows that $a \equiv b$ (mod θ). ■

Sometimes a long computation is required to prove that a binary relation is a congruence relation. Such computations are often facilitated by the following theorem which assume only a reflexive binary relation.

Theorem 2.3.4. *[14] A reflexive binary relation θ on a lattice L is a congruence relation if and only if it satisfies the following three properties :*

(1) For all $a, b \in L$,

$$a \equiv b \ (mod\ \theta) \ \Leftrightarrow \ a \wedge b \equiv a \vee b \ (mod\ \theta).$$

(2) If $a \preccurlyeq b \preccurlyeq c$ in L, then

$$a \equiv b \ (mod\ \theta), \quad b \equiv c \ (mod\ \theta) \ \Rightarrow \ a \equiv c \ (mod\ \theta).$$

(3) If $a \preccurlyeq b$ and $a \equiv b$ (mod θ), then

$$a \wedge x \equiv b \wedge x \ (mod\ \theta) \quad \text{and} \quad a \vee x \equiv b \vee x \ (mod\ \theta) \quad \text{for all } x \in L.$$

Proof. The "only if" part being trivial. So, it is sufficient to prove that a reflexive relation θ satisfying conditions (1) - (3) is a congruence relation. First, note that θ is symmetric by (1). Next, we need to show that if $a \preccurlyeq b$ and $a \equiv b \pmod{\theta}$, then all elements in $[a, b]$ are congruent. It is easy to see that any $x \in [a, b]$ is congruent to both a and b, since by using (3) we have

$$a = a \wedge x \equiv b \wedge x = x \pmod{\theta}$$

and

$$x = a \vee x \equiv b \vee x = b \pmod{\theta}.$$

Hence, if $x, y \in [a, b]$, then $x \vee y \in [a, b]$. So $x \vee y \equiv a \pmod{\theta}$. But $x \wedge y \in [a, x \vee y]$ and so $x \wedge y \equiv x \vee y \pmod{\theta}$. By (1), $x \equiv y \pmod{\theta}$.

To prove that θ is transitive, let $a \equiv b \pmod{\theta}$ and $b \equiv c \pmod{\theta}$. Then by (1), $a \wedge b \equiv a \vee b \pmod{\theta}$. We know that $a \wedge b \preccurlyeq a \vee b$ and so by (3), we have

$$(a \wedge b) \vee b \equiv a \vee b \vee b \pmod{\theta} \quad \Rightarrow \quad b \equiv a \vee b \pmod{\theta} \quad \Rightarrow \quad b \vee c \equiv a \vee b \vee c \pmod{\theta}.$$

Similarly,

$$a \wedge b \equiv a \vee b \pmod{\theta} \quad \Rightarrow \quad a \wedge b \equiv b \pmod{\theta} \quad \Rightarrow \quad a \wedge b \wedge c \equiv b \wedge c \pmod{\theta}.$$

Hence,

$$a \wedge b \wedge c \equiv b \wedge c \equiv b \vee c \equiv a \vee b \vee c \pmod{\theta}$$

and

$$a \wedge b \wedge c \preccurlyeq b \wedge c \preccurlyeq b \vee c \preccurlyeq a \vee b \vee c.$$

Thus by applying (2) twice, we get $a \wedge b \wedge c \equiv a \vee b \vee c \pmod{\theta}$. Therefore, all elements in interval $[a \wedge b \wedge c,\ a \vee b \vee c]$, including a and c, are congruent. Thus, θ is an equivalence relation.

We complete the proof using Theorem 2.3.1. Suppose that $a \equiv b \pmod{\theta}$. If a and b are comparable, then (3) and symmetry of the relation imply that

$$a \wedge x \equiv b \wedge x \pmod{\theta} \quad \text{and} \quad a \vee x \equiv b \vee x \pmod{\theta}$$

for all $x \in L$. If $a \parallel b$, then by (1), $a \wedge b \equiv a \vee b \pmod{\theta}$ and by (3), we get

$$(a \wedge b) \wedge x \equiv (a \vee b) \wedge x \pmod{\theta} \quad \text{and} \quad (a \wedge b) \vee x \equiv (a \vee b) \vee x \pmod{\theta}$$

for all $x \in L$. But

$$a \wedge x, b \wedge x \in [(a \wedge b) \wedge x, (a \vee b) \wedge x]$$

and so $a \wedge x \equiv b \wedge x \pmod{\theta}$. Similarly,

$$a \vee x, b \vee x \in [(a \wedge b) \vee x, (a \vee b) \vee x]$$

and so $a \vee x \equiv b \vee x \pmod{\theta}$. ■

2.4 Quotient Lattices

Homomorphisms and congruence relations express two sides of the same phenomenon. To establish this fact, we first define quotient lattice and take closer look at the properties of this lattice.

The set $L/\theta = \{[a]_\theta \mid a \in L\}$ of congruence classes of a congruence relation θ is also a lattice. Indeed, it is precisely because θ is a congruence relation that the inherited lattice operations of θ are well defined on L/θ.

Theorem 2.4.1. *[22] Let θ be a congruence relation on a lattice L. The set L/θ of congruence classes is a lattice, called the quotient lattice of L modulo θ, under the operations*

$$[a]_\theta \wedge [b]_\theta = [a \wedge b]_\theta \quad \textit{and} \quad [a]_\theta \vee [b]_\theta = [a \vee b]_\theta$$

Proof. First, we must check that these operations are well defined. But if $a \equiv a' \pmod{\theta}$ and $b \equiv b' \pmod{\theta}$, then $a \wedge b \equiv a' \wedge b' \pmod{\theta}$ and so

$$[a]_\theta = [a']_\theta \text{ and } [b]_\theta = [b']_\theta \quad \Rightarrow \quad [a \wedge b]_\theta = [a' \wedge b']_\theta$$

which says that the meet operation is well defined. A similar argument applies to the join operation. Finally, it is routine to check the properties of a lattice. For instance, the associativity law

$$([a]_\theta \vee [b]_\theta) \vee [c]_\theta = [a]_\theta \vee ([b]_\theta \vee [c]_\theta)$$

is equivalent to

$$[(a \vee b) \vee c]_\theta = [a \vee (b \vee c)]_\theta$$

which follows from associativity in L. ■

In our discussion of congruences we have so far treated lattices as algebraic structures; the underlying partial order on a lattice has not been mentioned. Theorem 2.4.2 redress the balance and describe the partial order relation in the quotient lattice.

Theorem 2.4.2. *[22] Let L be a lattice and let $\theta \in Con(L)$.*

(1) $[a]_\theta \preccurlyeq [b]_\theta$ if and only if some element of $[a]_\theta$ is related to some element of $[b]_\theta$. In particular,

$$a \preccurlyeq b \quad \Rightarrow \quad [a]_\theta \preccurlyeq [b]_\theta.$$

(2) $[a]_\theta \parallel [b]_\theta$ if and only if $\alpha \parallel \beta$ for every $\alpha \in [a]_\theta$ and $\beta \in [b]_\theta$.

(3) $[a]_\theta \prec [b]_\theta$ if and only if

$$\alpha \prec \beta \quad or \quad \alpha \parallel \beta$$

for every $\alpha \in [a]_\theta$ and $\beta \in [b]_\theta$, with strict inequality for at least one pair (α, β).

Proof. For part (1),

$$a \preccurlyeq b \Leftrightarrow a \wedge b = a \Rightarrow [a \wedge b]_\theta = [a]_\theta \Leftrightarrow [a]_\theta \wedge [b]_\theta = [a]_\theta \Leftrightarrow [a]_\theta \preccurlyeq [b]_\theta.$$

Conversely,

$$[a]_\theta \preccurlyeq [b]_\theta \Rightarrow [a \wedge b]_\theta = [a]_\theta \Rightarrow a \wedge b \in [a]_\theta$$

and so $\alpha \preccurlyeq \beta$, where $\alpha = a \wedge b \in [a]_\theta$ and $\beta = b \in [b]_\theta$. Part(2) follows from part (1). For part (3) if $[a]_\theta \prec [b]_\theta$, then $[a]_\theta \preccurlyeq [b]_\theta$, $[a]_\theta \neq [b]_\theta$ and $[b]_\theta \not\preccurlyeq [a]_\theta$. Thus , part (1) implies that $\alpha \prec \beta$ for some $\alpha \in [a]_\theta$, $\beta \in [b]_\theta$ and that $\beta \preccurlyeq \alpha$ cannot happen for any $\alpha \in [a]_\theta$, $\beta \in [b]_\theta$. Conversely, suppose that $\alpha \prec \beta$ or $\alpha \parallel \beta$ for every $\alpha \in [a]_\theta$, $\beta \in [b]_\theta$, with strict inequality for at least one pair. Then part (1) implies that $[a]_\theta \preccurlyeq [b]_\theta$. However, $[a]_\theta = [b]_\theta$ implies that $a \in [a]_\theta$ and $a \in [b]_\theta$, which is false. Hence, $[a]_\theta \prec [b]_\theta$. ■

The quadrilateral argument : Let L be a lattice and suppose that $\{a, b, c, d\}$ is a 4-element subset of L. Then a, b and c, d are said to be the opposite sides of the **quadrilateral**$\langle a, b; c, d\rangle$ if $a \prec b$, $c \prec d$ and either

$$(a \vee d = b \text{ and } a \wedge d = c) \quad \text{or} \quad (b \vee c = d \text{ and } b \wedge c = a).$$

We say that the blocks of a partition of L are **quadrilateral-closed** if whenever a, b and c, d are opposite sides of a quadrilateral and $a, b \in A$ for some block A then $c, d \in B$ for some block B (see Figure 2.7).

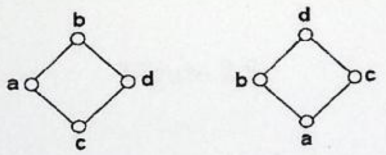

Figure 2.7:

Theorem 2.4.3. *[9] Let L be a lattice and let θ be an equivalence relation on L. Then θ is a congruence if and only if*

(1) each block of θ is a sublattice of L,

(2) each block of θ is convex,

(3) the blocks of θ are quadrilateral-closed.

Proof. Let θ be a congruence relation on L. Then by Theorem 2.3.2, each block of θ is a sublattice and convex. For (3), let a, b and c, d be opposite sides of quadrilateral, with $a \vee d = b$ and $a \wedge d = c$. We assume that $a, b \in [a]_\theta$ and $d \in [d]_\theta$ and seek to prove that $c \in [d]_\theta$. Since $a \equiv b \pmod{\theta}$, by Theorem 2.3.1, we can get

$$a \wedge d \equiv b \wedge d \ (mod\ \theta).$$
$$c \equiv d \ (mod\ \theta).$$

Conversely, assume that (1), (2) and (3) hold. We prove that $a \equiv b \pmod{\theta}$, implies that $a \wedge x \equiv b \wedge x \pmod{\theta}$ for all $x \in L$. Since (1), (2) and(3) are self dual, it follows that $a \vee x \equiv b \vee x \pmod{\theta}$ for all $x \in L$. First suppose that $u \equiv v \pmod{\theta}$ and $u \preccurlyeq v$. Then we can project x into the interval $[u, v]$ to get $u \vee (v \wedge x)$, as shown in Figure 2.8.

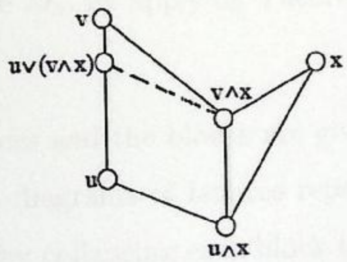

Figure 2.8:

The convexity of blocks implies that $u \equiv u \vee (v \wedge x) \pmod{\theta}$ and the (3) then implies that $u \wedge x \equiv v \wedge x \pmod{\theta}$.
For the general case, if $a \equiv b \pmod{\theta}$ then letting $u = a \wedge b$, and $v = a \vee b$, we have $a \wedge b \equiv a \vee b \pmod{\theta}$, and $a \wedge b \preccurlyeq a \vee b$. Hence, the previous remarks show that

$$(a \wedge b) \wedge x \equiv (a \vee b) \wedge x \pmod{\theta}$$

and since $a \wedge x$ and $b \wedge x$ lie in the interval between $(a \wedge b) \wedge x$ and $(a \vee b) \wedge x$, convexity implies that $a \wedge x \equiv b \wedge x \pmod{\theta}$, as desired. ■

Example 2.4.1. Consider the 4-element square lattice L with Hasse diagram given in Figure 2.9. Let θ be a congruence of L. Then the following cases hold:

$$0 \equiv a \ (\text{mod } \theta) \ \Leftrightarrow \ b \equiv 1 \ (\text{mod } \theta), 0 \equiv b \ (\text{mod } \theta) \ \Leftrightarrow \ a \equiv 1 \ (\text{mod } \theta),$$
$$a \equiv b \ (\text{mod } \theta) \ \Rightarrow \ 0 \equiv a \ (\text{mod } \theta) \text{ and so } \theta : \{0, 1, a, b\},$$
$$0 \equiv 1 \ (\text{mod } \theta) \ \Rightarrow \ \theta : \{0, 1, a, b\}.$$

Using these properties we deduce that there are four congruences in Con$L$: $\{\{0\}, \{1\}, \{a\}, \{b\}\}$, $\{0, 1, a, b\}$, $\{\{0, a\}, \{b, 1\}\}$ and $\{\{0, b\}, \{a, 1\}\}$.

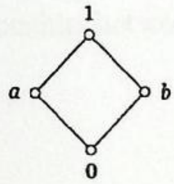

Figure 2.9:

Example 2.4.2. The lattice M_3, by applying Theorem 2.4.3, has only trivial congruences.

Some examples of congruences and the blocks are given in Figure 2.10. Note that the loops are drawn on the diagrams of lattices represents a partition of L. The quotient lattice is obtained by collapsing each block to a point.

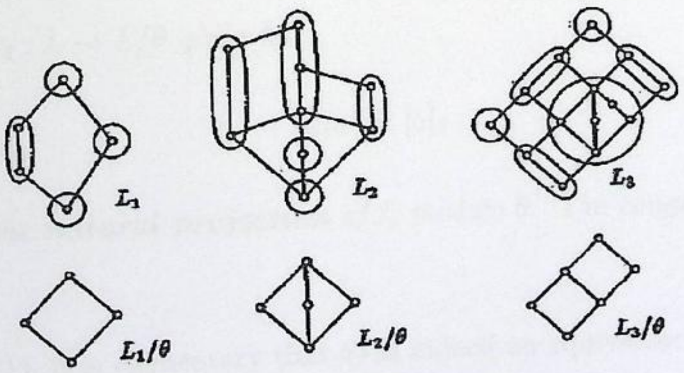

Figure 2.10:

Remark 2.4.1. *Let θ be a congruence relation on a lattice L. If L is a distributive, then L/θ is a distributive lattice. Since for all $[a]_\theta, [b]_\theta, [c]_\theta \in L/\theta$:*

$$[a]_\theta \wedge ([b]_\theta \vee [c]_\theta) = ([a]_\theta \wedge [b]_\theta) \vee ([a]_\theta \wedge [c]_\theta).$$

On the other hand, if L/θ is distributive lattice, then L it is not necessary to be distributive lattice. For instance, in Figure 2.10, L_1/θ is distributive lattice but L_1 is not distributive lattice.

Now we want to look at the relationship between congruence relations and lattice homomorphisms.

Theorem 2.4.4. *[22]*

(1) *Every lattice homomorphism $f : L \to M$ defines a congruence relation θ_f given by*

$$(\forall a, b \in L)\ a \equiv b\ (mod\ \theta_f) \quad \Leftrightarrow \quad f(a) = f(b)$$

*and called it is the **congruence kernel** of f. The congruence classes of θ_f are the sets $L/\theta_f = \{f^{-1}(x) \mid x \in image(f)\}$. Thus f is injective if and only if θ_f is equality.*

(2) *Every congruence relation θ on a lattice L defines a surjective lattice homomorphism $\pi_\theta : L \to L/\theta$ given by*

$$\pi_\theta(a) = [a]_\theta$$

*and called the **natural projection** of L modulo θ. The congruence kernel of π_θ is θ.*

Proof. For part(1), it is elementary that θ_f is indeed an equivalence relation. Now assume $a \equiv b$ (mod θ_f) and $c \equiv d$ (mod θ_f), so that $f(a) = f(b)$ and $f(d) = f(c)$, Hence since f preserves join,

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$$f(a \vee c) = f(a) \vee f(c) = f(b) \vee f(d) = f(b \vee d).$$

Therefore $a \vee c \equiv b \vee d \pmod{\theta_f}$. A similar argument can be made for meets. For part(2), it is clear that π_θ is surjective. Also,

$$\pi_\theta(a \wedge b) = [a \wedge b]_\theta = [a]_\theta \wedge [b]_\theta = \pi_\theta(a) \wedge \pi_\theta(b)$$

and similarly for join. Hence, π_θ is a surjective lattice homomorphism. Finally, the congruence kernel of π_θ is θ, since

$$a \equiv b \pmod{\theta} \quad \Leftrightarrow \quad [a]_\theta = [b]_\theta \quad \Leftrightarrow \quad \pi_\theta(a) = \pi_\theta(b). \quad \blacksquare$$

2.5 Kernels

In this section we define three types of kernels in lattices. One of them is related to homomorphism, and the other two kernels are related to congruence relation. Then we characterize the distributive lattices by using the kernel of congruence relations.

Definition 2.5.1. [13] Let $f : L \to M$ be a lattice homomorphism. If M has a least element 0 and if $f^{-1}(0)$ is nonempty, then the set

$$\ker(f) = f^{-1}(0) = \{a \in L \mid f(a) = 0\}$$

is called the **ideal kernel** of f.

Example 2.5.1. Let $f : L \to M$ be a lattice homomorphism with Hasse diagram given in Figure 2.11. The ideal kernel of f is $\{d, e\}$.

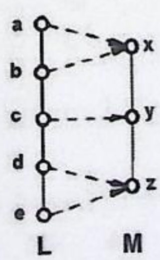

Figure 2.11:

Definition 2.5.2. [13] If $f : L \to M$ is a lattice homomorphism, then the congruence relation θ_f, defined in Theorem 2.4.4, is called the **congruence kernel** of f.

Example 2.5.2. In Figure 2.11 the congruence kernel of f is

$$\theta_f = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(b,a),(d,e),(e,d)\}.$$

Theorem 2.5.1. *[22] Let L be a lattice and $\theta \in \mathrm{Con}(L)$. A congruence class $[x]_\theta$ is an ideal of L if and only if it is the least element of L/θ, in which case it is called the* **ideal kernel** *of θ and is denoted by $\ker(\theta)$. If L has a 0, then $\ker(\theta) = [0]_\theta$.*

Proof. Let $[x]_\theta$ be an ideal of L. Then for any congruence classes $[a]_\theta$, we have $x \wedge a \in [x]_\theta$ and $x \wedge a \preccurlyeq a \in [a]_\theta$ and so, by Theorem 2.4.2, $[x]_\theta \preccurlyeq [a]_\theta$. Thus, $[x]_\theta$ is the least element of L/θ. Conversely, if $[x]_\theta$ is the least element of L/θ, then for any $a \preccurlyeq y \in [x]_\theta$ we have $[a]_\theta \preccurlyeq [y]_\theta = [x]_\theta$ and so $[a]_\theta = [x]_\theta$, that is, $a \in [x]_\theta$. Let $c, d \in [x]_\theta$. Since $[x]_\theta$ is a sublattice, we can get $c \vee d \in [x]_\theta$. Hence $[x]_\theta$ is an ideal . ■

Note that, not all ideals are ideal kernel of some congruence relation.

Example 2.5.3. Consider the lattice L with Hasse diagram given in Figure 2.12. The subset $I = \{0, a, b, c\}$ is an ideal, but is not an ideal kernel of any congruence

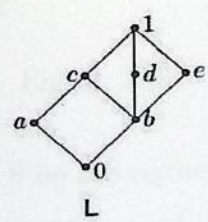

Figure 2.12:

relation. In fact, if I were the ideal kernel of a congruence θ, then we would have

$$d = d \vee 0 \equiv d \vee c = 1 \pmod{\theta} \quad \text{and} \quad e = e \vee 0 \equiv e \vee c = 1 \pmod{\theta}$$

whence the contradiction $0 \equiv b = d \wedge e \equiv 1 \pmod{\theta}$.

The existence in the Example 2.5.3 of a not distributive sublattice prompts a consideration of the distributive case. In so doing, we obtain the following characterisation of distributive lattices.

Theorem 2.5.2. *[7] A lattice L is distributive if and only if every ideal of L is an ideal kernel of some congruence relation.*

Proof. Suppose that L is distributive and let I be an ideal of L. Define the binary relation δ on L by

$$a \equiv b \,(\text{mod } \delta) \quad \Leftrightarrow \quad (\exists i \in I) \;\; a \vee i = b \vee i.$$

In Example 2.3.4, we proved that δ is a congruence relation. Now if $i \in I$, then $a \equiv i$ (mod δ) implies that, for some $j \in I$, $a \vee j = i \vee j \in I$ and hence $a \in I$. It follows that I is a δ class, and is indeed the bottom element of L/δ. Thus I is the $\ker(\delta)$. Conversely, suppose that every ideal of L is an ideal kernel of some congruence relation. If L were not distributive then it would contain as a sublattice one of the lattices in Figure 2.13.

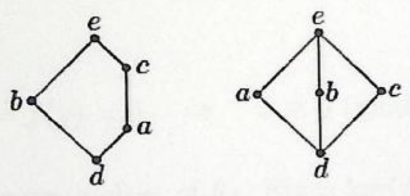

Figure 2.13:

In either case, let $I = \downarrow a$ and let θ be a congruence of which I is the $\ker(\theta)$. Then $e = a \vee b \equiv d \vee b = b$ (mod θ) from which we obtain $c = c \wedge e \equiv c \wedge b = d \equiv a$ (mod θ), which shows that $c \in I = \downarrow a$, a contradiction in each case. Thus L has no such sublattices and is therefore distributive. ■

The congruence δ considered in the proof of Theorem 2.5.2 has a more significant property.

Theorem 2.5.3. *[7] If I is an ideal of a distributive lattice then δ is the smallest congruence with ideal kernel I.*

Proof. Suppose that $x \equiv y \pmod{\delta}$, so that $x \vee i = y \vee i$ for some $i \in I$. Then for any congruence φ with ideal kernel I we have

$$[x]_\varphi = [x]_\varphi \vee [i]_\varphi = [x \vee i]_\varphi = [y \vee i]_\varphi = [y]_\varphi \vee [i]_\varphi = [y]_\varphi$$

so that $x \equiv y \pmod{\varphi}$ and consequently $\delta \subseteq \varphi$. ■

Theorem 2.5.4. *[22] Let M be a lattice with 0 and let $f : L \to M$ be a lattice homomorphism. Then the ideal kernel $\ker(f) = f^{-1}(0)$ is an ideal of L and*

$$\ker(\theta_f) = \ker(f).$$

Proof. If $x, y \in f^{-1}(0)$, then

$$f(x \vee y) = f(x) \vee f(y) = 0 \vee 0 = 0.$$

Hence $x \vee y \in f^{-1}(0)$. If $x \in f^{-1}(0)$, $r \in L$ and $r \preccurlyeq x$, then $f(r) \preccurlyeq f(x) = 0$. Therefore $f(r) = 0$ and $r \in f^{-1}(0)$. Hence $f^{-1}(0)$ is an ideal of L. Now let $x, y \in \ker(f)$. Then

$$f(x) = f(y) = 0 \quad \Rightarrow \quad x \equiv y \pmod{\theta_f}.$$

So $\ker(f) = [x]_{\theta_f}$ is a congruence class of θ_f. Since $\ker(f)$ is an ideal of L, $[x]_{\theta_f}$ is the least element of L/θ_f. Hence $\ker(f) = \ker(\theta_f)$. ■

Remark 2.5.1. *The lattice M is a* homomorphic image *of the lattice L if and only if there is a homomorphism of L onto M. Theorem 2.4.4 part (2) states that any quotient lattice is a homomorphic image.*

Theorem 2.5.5. *[12] (**The Homomorphism Theorem**). Let L be a lattice. Any homomorphic image of L is isomorphic to a suitable quotient lattice of L. In fact, if $f : L \to M$ is a homomorphism of L onto M and θ_f is the congruence kernel of f, then*

$$L/\theta_f \cong M;$$

an isomorphism(see Figure 2.14) is given by

$$\tau : L/\theta_f \to M \text{ such that } \tau([x]_{\theta_f}) = f(x)$$

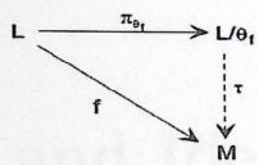

Figure 2.14:

Proof. To prove τ is an isomorphism we have to show check that τ: (1) is well defined, (2) is one to one, (3) is onto, and (4) preserves the operations.

(1) Let $[x]_{\theta_f} = [y]_{\theta_f}$. Then $x \equiv y \pmod{\theta_f}$; thus $f(x) = f(y)$, that is,

$$\tau([x]_{\theta_f}) = \tau([y]_{\theta_f}).$$

(2) Let $\tau([x]_{\theta_f}) = \tau([y]_{\theta_f})$. Then $x \equiv y \pmod{\theta_f}$; and so $[x]_{\theta_f} = [y]_{\theta_f}$.

(3) Let $a \in M$. Since f is onto, there is an $x \in L$ with $f(x) = a$. Thus

$$\tau([x]_{\theta_f}) = a.$$

(4) $\tau([x]_{\theta_f} \wedge [y]_{\theta_f}) = \tau([x \wedge y]_{\theta_f}) = f(x \wedge y) = f(x) \wedge f(y) = \tau([x]_{\theta_f}) \wedge \tau([y]_{\theta_f})$. ■

Chapter 3

Congruences and Ideals in Distributive Lattices with respect to A derivation

In this chapter we introduce the concept of derivation on lattices. Also we will mention some properties of derivation and compare between derivations and lattice homomorphisms. Furthermore the concepts of d-ideal, d-prime ideal, injective ideal and maximal injective ideal are introduced in a distributive lattice with respect to derivations. Also the Stone's theorem for ideals of a distributive lattice is extended to the case of injective ideals. In addition two types of congruences are introduced in a distributive lattice with respect to derivation.

3.1 Definition and Examples

In this section we introduce the concept of a derivation of a lattice. Also some examples are presented to illustrate this concept. Furthermore we will mention some properties of a derivation.

Given an algebraic structure $(A, +, *)$, where $+$ and $*$ denote two arbitrary binary operations, we call *derivation* of A any function $f : A \to A$ such that:

$$f(a+b) = f(a) + f(b),$$
$$f(a*b) = f(a)*b + a*f(b).$$

This definition clearly coincides with the usual (algebraic) notion of derivation when $(A, +, *)$ is a ring. However, it can be formally stated for every algebraic structure endowed with two binary operations. In this chapter, we will consider the special case in which $(A, +, *)$ is a lattice, so that $+$ and $*$ are, respectively, the join and the meet operations.

Definition 3.1.1. [26] Let $(L, \vee, \wedge)$ be a lattice; a **derivation** of L is a function $d : L \to L$ satisfying:

(1) $d(x \vee y) = d(x) \vee d(y)$,

(2) $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$, for all $x, y \in L$.

Example 3.1.1.

1. In every lattice L, the identity mapping i defined by $i(x) = x$ for each $x \in L$ is a derivation of L.
2. Let L be a lattice with least element 0. Then the mapping ψ defined by $\psi(x) = 0$ for each $x \in L$ is a derivation of L.
3. Take the lattice L in Figure 3.1.

Figure 3.1:

Define two functions d_1 and d_2 on L by

$$d_1(x) = \begin{cases} a_2, & x = 1 \\ x, & otherwise \end{cases} \qquad d_2(x) = \begin{cases} a_1, & x = 1 \\ x, & otherwise \end{cases},$$

d_1 is a derivation but d_2 is not a derivation. Since

$$d_2(a_2 \vee 1) = d_2(1) = a_1 \quad \text{but} \quad d_2(a_2) \vee d_2(1) = a_2 \vee a_1 = a_2.$$

Now we give some properties for the derivations of lattices.

Theorem 3.1.1. *[26] Let $d : L \to L$ be a derivation of a lattice L. Then for any $x, y \in L$, we have :*

(1) $d(x) \preccurlyeq x$.

(2) If $x \preccurlyeq y$ then $d(x) \preccurlyeq d(y)$. Hence, d is an isotone.

(3) If $x \preccurlyeq y$, then $d(x) = x \wedge d(y)$.

Proof. Let $d : L \to L$ be a lattice derivation. Given $x, y \in L$. For (1),

$$d(x) = d(x \wedge x) = (d(x) \wedge x) \vee (x \wedge d(x)) \quad \Rightarrow \quad d(x) = x \wedge d(x) \prec x \qquad \forall x \in L.$$

For (2), if $x \preccurlyeq y$, then $x \vee y = y$, and so

$$d(y) = d(x \vee y) = d(x) \vee d(y).$$

Hence,

$$d(x) \preccurlyeq d(y).$$

To prove (3), let $x \preccurlyeq y$ again. Then $d(x) \preccurlyeq x \preccurlyeq y$ by (1). Consequently

$$d(x) = d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)) = d(x) \vee (x \wedge d(y)),$$

i.e. $x \wedge d(y) \preccurlyeq d(x)$. On the other hand, $d(x) \preccurlyeq x$ by (1) and $d(x) \preccurlyeq d(y)$ by (2). Therefore,

$$x \wedge d(y) \succcurlyeq d(x).$$

So,

$$d(x) = x \wedge d(y). \blacksquare$$

Remark 3.1.1. *[10] let us have a look at the definition of derivation, and particularly to condition (2), i.e.*

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)).$$

We can prove that this condition is redundant, and we find a suitable way to simplify it. Indeed, since $x \wedge y \preccurlyeq x$, *we have* $d(x \wedge y) \preccurlyeq d(x)$, *and we also have* $d(x \wedge y) \preccurlyeq (x \wedge y) \preccurlyeq y$. *Therefore we have obtained* $d(x \wedge y) \preccurlyeq d(x) \wedge y$. *In the same way we also get* $d(x \wedge y) \preccurlyeq x \wedge d(y)$, *so we can conclude that condition (2) is equivalent to the following:*

$$(2') \ d(x \wedge y) = d(x) \wedge y = x \wedge d(y).$$

Definition 3.1.2. [10] An element a of a lattice L is said to be **distributive element** whenever, for every $x, y \in L$:

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y).$$

The set of all the distributive elements of a lattice is called the center of the lattice.

Note that all elements in a distributive lattice L are distributive elements. The next, fundamental Theorem states that for a very large class of lattices all the derivations are of the same form.

Theorem 3.1.2. *[10] Consider a lattice* L *with a greatest element* 1. *Then* $d : L \rightarrow L$ *is a derivation of* L *if and only if there exists a distributive element* $a \in L$ *such that* $d(x) = a \wedge x$, *for every* $x \in L$. *Obviously, in this case we have* $a = d(1)$.

Proof. If $a \in L$ is a distributive element, then every function of the form $d(x) = a \wedge x$ is a derivation (even if L does not have a greatest element) since $\forall x, y \in L$:

$$\begin{aligned} d(x \vee y) &= a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) = d(x) \vee d(y). \\ d(x \wedge y) &= a \wedge (x \wedge y) = (a \wedge x) \wedge y = d(x) \wedge y. \end{aligned}$$

Conversely, suppose $d : L \to L$ is a derivation; by definition we have:

$$d(x) = d(1 \wedge x) = d(1) \wedge x.$$

$d(1) \in L$ is a distributive element, since $\forall x, y \in L$

$$\begin{aligned} d(1) \wedge (x \vee y) &= d(1 \wedge (x \vee y)) \\ &= d((1 \wedge x) \vee (1 \wedge y)) \\ &= d(1 \wedge x) \vee d(1 \wedge y) \\ &= (d(1) \wedge x) \vee (d(1) \wedge y). \blacksquare \end{aligned}$$

Remark 3.1.2. *Theorem 3.1.2 is not necessarily true if a is not a distributive element.*

Example 3.1.2. Let L be the diamond lattice M_3 of Figure 3.2. Consider the function $f : L \to L$ such that $f(x) = b \wedge x$. Note that b is not a distributive element. Clearly, f is not a derivation on L:

$$f(a \vee c) = f(1) = b \quad \text{but} \quad f(a) \vee f(c) = 0.$$

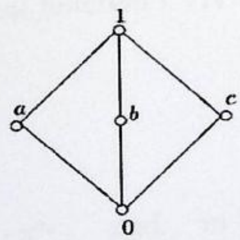

Figure 3.2:

Therefore, for a distributive lattice L with greatest element 1, the class of the derivations of L coincides with the class of the maps $d_a : L \to L$ denoted by $d_a(x) = a \wedge x$. More precisely, d_a is the only derivation of L such that $d(1) = a$. We will call these derivations **simple derivations** (even if the lattice L is not distributive), and we will say that d_a is the simple derivation associated with a.

Theorem 3.1.3. *[26] Every derivation d of a lattice L is of the form*

$$d(x) = c \wedge x$$

with a suitable chosen $c \in L$ if and only if L has a greatest element.

Proof. If 1 is the greatest element of L, then $d(x) = x \wedge d(1)$ for each $x \in L$. However, if L has no greatest element, then the identity mapping of L cannot be represented in the form $d(x) = c \wedge x$, because $c \wedge x \neq x$ for $x \succ c$. ■

The lattice $(\mathbb{N}, lcm, gcd)$ is a distributive lattice without greatest element. Theorem 3.1.4 defines and studies a particular function, whose appearences in algebra and geometry are plentiful. *"We call* ***radical*** *of a positive integer $n = p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$ (uniquely factorized into product of primes) the positive integer*

$$r(n) = r(p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}) = p_1 \cdot \ldots \cdot p_r$$

Therefore, the radical of a positive integer is the product of the primes of its factorization". [10]

Theorem 3.1.4. *[10] The radical function $r : \mathbb{N} \to \mathbb{N}$ is a derivation of the lattice* $(\mathbb{N}, lcm, gcd)$.

Proof. Let $n, m \in \mathbb{N}$ such that

$$n = p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r} \cdot p_{r+1}^{\alpha_{r+1}} \cdot \ldots \cdot p_s^{\alpha_s} \quad \text{and} \quad m = p_1^{\beta_1} \cdot \ldots \cdot p_r^{\beta_r} \cdot q_{r+1}^{\beta_{r+1}} \cdot \ldots \cdot q_t^{\beta_t}$$

(in this way we distinguish the common primes belonging to both n and m from the other primes occurring in the two factorizations). It is not difficult to show that the following equalities hold:

$$\begin{aligned} r(lcm(n,m)) &= r(p_1^{max(\alpha_1,\beta_1)} \cdot \ldots \cdot p_r^{max(\alpha_r,\beta_r)} \cdot p_{r+1}^{\alpha_{r+1}} \cdot \ldots \cdot p_s^{\alpha_s} \cdot q_{r+1}^{\beta_{r+1}} \cdot \ldots \cdot q_t^{\beta_t}) \\ &= p_1 \cdot \ldots \cdot p_r \cdot p_{r+1} \cdot \ldots \cdot p_s \cdot q_{r+1} \cdot \ldots \cdot q_t \\ &= lcm(p_1 \cdot \ldots \cdot p_s, p_1 \cdot \ldots \cdot p_r \cdot q_{r+1} \cdot \ldots \cdot q_t) \\ &= lcm(r(n), r(m)); \end{aligned}$$

$$\begin{aligned} r(gcd(n,m)) &= r(p_1^{min(\alpha_1,\beta_1)} \cdot \ldots \cdot p_r^{min(\alpha_r,\beta_r)}) \\ &= p_1 \cdot \ldots \cdot p_r \\ &= gcd(p_1^{\alpha_1} \cdot \ldots \cdot p_s^{\alpha_s}, p_1 \cdot \ldots \cdot p_r \cdot q_{r+1} \cdot \ldots \cdot q_t) \\ &= gcd(n, r(m)) \qquad (= gcd(r(n), m)). \blacksquare \end{aligned}$$

3.2 Derivations of Lattices vs lattice Homomorphisms

In this section we introduce some properties of derivations of lattices. In addition we compare between Derivations of lattices and lattices homomorphisms.

Theorem 3.2.1. *[10] A derivation d is a lattice endomorphism of L and also preserves the least element* 0.

Proof. If d is a derivation of a lattice L, then d is join preserving by the definition. So, we have to show that d is meet preserving. For $x, y \in L$, we get:

$$\begin{aligned} d(x \wedge y) &\preccurlyeq d(x) \wedge d(y), \qquad \textit{for } d \textit{ is isotone.} \\ d(x) \wedge d(y) &\preccurlyeq d(x) \wedge y = d(x \wedge y), \qquad \textit{from } (2'). \end{aligned}$$

Therefore, $d(x \wedge y) = d(x) \wedge d(y)$.

If L is a lattice with 0, then

$$d(0) = d(0 \wedge 0) = d(0) \wedge 0 = 0.$$

In fact, a lattice endomorphism need not be a lattice derivation.

Example 3.2.1. [26] Let L be a lattice. Given an element c different from the least element (eventually existing) of L, the mapping $\psi(x)$ defined by

$$\psi(x) = c \quad \text{for each} \quad x \in L$$

is an endomorphism of L which is not derivation because there exists at least one element $b \in L$ such that $b \prec c$ and thus $\psi(b) = c \succ b$. Hence, ψ is not a derivation.

Theorem 3.2.2. *[10] If $d : L \to L$ is a lattice derivation, then*

$$d(d(x)) = d(x) \ \ \forall x \in L.$$

Proof.

$$\begin{aligned} d(d(x)) &= d(d(x \wedge x)) = d(d(x) \wedge x) \\ &= d(x) \wedge d(x) = d(x). \ \blacksquare \end{aligned}$$

Remark 3.2.1. *If $f : L \to L$ is a lattice endomorphism, then it is not necessary that $f(f(x)) = f(x)$ for $x \in L$. For example, Let $L = \{0, a_1, a_2, 1\}$ be a lattice whose Hasse diagram is given in Figure 3.3. Define a lattice endomorphism $f : L \to L$ such that $f(1) = a_1, f(a_2) = f(a_1) = f(0) = 0$. Then $f(f(1)) \neq f(1)$.*

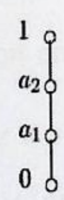

Figure 3.3:

Theorem 3.2.3. *[10] If d, f are derivations of the lattice L, then $d \circ f$ is a derivation too.*

Proof. Let d, f are derivations of the lattice L, then

$$\begin{aligned}(d \circ f)(x \vee y) &= d(f(x) \vee f(y)) = (d \circ f)(x) \vee (d \circ f)(y),\\ (d \circ f)(x \wedge y) &= d(f(x) \wedge y) = (d \circ f)(x) \wedge y.\end{aligned}$$

So, $d \circ f$ is a derivation. ■

Remark 3.2.2. *It is easy to show that if $f : L \to M$ and $d : M \to P$ are lattice homomorphisms, then $d \circ f$ is a lattice homomorphism too.*

Theorem 3.2.4. *[10] Let L be a distributive lattice and d, f be two derivations on L. Define*

$$(d \wedge f)(x) = d(x) \wedge f(x) \quad \textit{and} \quad (d \vee f)(x) = d(x) \vee f(x).$$

Then $d \vee f$ and $d \wedge f$ are derivations, and we also have $d \wedge f = d \circ f$.

Proof. First, we prove $d \vee f$ is a derivation on a distributive lattice L.

$$\begin{aligned}(d \vee f)(x \vee y) &= d(x \vee y) \vee f(x \vee y) = d(x) \vee d(y) \vee f(x) \vee f(y)\\ &= (d \vee f)(x) \vee (d \vee f)(y);\end{aligned}$$

$$\begin{aligned}(d \vee f)(x \wedge y) &= d(x \wedge y) \vee f(x \wedge y) = (d(x) \wedge y) \vee (f(x) \wedge y)\\ &= y \wedge (d(x) \vee f(x)) = y \wedge (d \vee f)(x).\end{aligned}$$

Similarly, we prove $d \wedge f$ is a derivation on L.

$$\begin{aligned}(d \wedge f)(x \vee y) &= d(x \vee y) \wedge f(x \vee y) = (d(x) \vee d(y)) \wedge (f(x) \vee f(y))\\ &= (d(x) \wedge f(y)) \vee (d(y) \wedge f(x)) \vee (d \wedge f)(x) \vee (d \wedge f)(y)\\ &= (d \wedge f)(x) \vee (d \wedge f)(y), \qquad (*)\end{aligned}$$

$$\begin{aligned}(d \wedge f)(x \wedge y) &= d(x \wedge y) \wedge f(x \wedge y) = d(x) \wedge y \wedge f(x) \wedge y\\ &= (d \wedge f)(x) \wedge y.\end{aligned}$$

$(*)$ can be explained as follows. We have

$$\begin{aligned}(d \wedge f)(x \wedge y) &= d(x \wedge y) \wedge f(x \wedge y)\\ &= d(x) \wedge y \wedge x \wedge f(y)\\ &= d(x) \wedge f(y),\end{aligned}$$

and so both $d(x) \wedge f(y) \preccurlyeq (d \wedge f)(x)$ and $d(x) \wedge f(y) \preccurlyeq (d \wedge f)(y)$. Finally, we immediately have:

$$(d \circ f)(x) = d(x \wedge f(x)) = d(x) \wedge f(x) = (d \wedge f)(x). \blacksquare$$

Remark 3.2.3. *[10] Note that the equality $d \circ f = d \wedge f$ does not depend on the distributivity of L; this means that the map $d \wedge f$ is a derivation even if the lattice L is not distributive.*

Example 3.2.2. Consider the non distributive lattice N_5 whose Hasse diagram is given in Figure 3.4. Since a and b are distributive elements, we can define the derivations d and f on N_5 by

$$d(x) = b \wedge x \qquad \text{and} \qquad f(x) = a \wedge x.$$

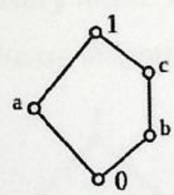

Figure 3.4:

Although N_5 is not a distributive lattice, but $(d \wedge f)(x) = (d \circ f)(x) = 0$ for all $x \in L$. Hence, $d \wedge f$ is a derivation on L. On the other hand, $d \vee f$ is not a derivation on N_5 since

$$(d \vee f)(c \wedge 1) = (d \vee f)(c) = b \quad \text{but} \quad c \wedge (d \vee f)(1) = c \wedge 1 = c.$$

Theorem 3.2.5. *[27] Let L be a distributive lattice and $\mathcal{D}(L)$ be a set of all derivations on L. Then $(\mathcal{D}(L), \vee, \wedge)$ is a distributive lattice.*

Proof. From Theorem 3.2.4, $\vee$ and $\wedge$ are binary operators on $\mathcal{D}(L)$. Define a binary relation $\preccurlyeq$ on $\mathcal{D}(L)$ by $d \preccurlyeq f$ if and only if $d \wedge f = d$. Then $\preccurlyeq$ is partial order relation on $\mathcal{D}(L)$ and infimum of $\{d, f\} = d \wedge f$, supremum of $\{d, f\} = d \vee f$. Therefore $(\mathcal{D}(L), \vee, \wedge)$ is a lattice.

In addition, for $d, f, g \in \mathcal{D}(L)$ and any $x \in L$, we have:

$$\begin{aligned}(d \wedge (f \vee g))(x) &= d(x) \wedge (f(x) \vee g(x)) \\ &= (d(x) \wedge f(x)) \vee (d(x) \wedge g(x)) \\ &= (d \wedge f)x) \vee (d \wedge g)x) \\ &= ((d \wedge f) \vee (d \wedge g))(x).\end{aligned}$$

Therefore, $d \wedge (f \vee g) = (d \wedge f) \vee (d \wedge g)$. Hence, $(\mathcal{D}(L), \vee, \wedge)$ is a distributive lattice. ■

Remark 3.2.4. *Let $Hom(L, M)$ be the set of all lattice homomorphism from a distributive lattice L to a distributive lattice M. Let $f, d \in Hom(L, M)$. For all $x \in L$, define*

$$(d \wedge f)(x) = d(x) \wedge f(x) \quad \text{and} \quad (d \vee f)(x) = d(x) \vee f(x).$$

Then $d \vee f$ and $d \wedge f$ are not necessary lattice homomorphisms. For instance, let L be a distributive lattice with Hasse diagram is given in Figure 3.5.

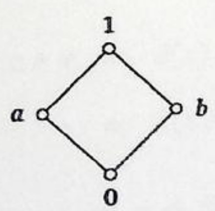

Figure 3.5:

Suppose $M = L$. Define two maps $d, f \in Hom(L, L)$ as follows:

$$d(x) = \begin{cases} 1, & x = 1 \text{ or } b, \\ 0, & \text{otherwise}, \end{cases} \qquad f(x) = \begin{cases} a, & x = 1 \text{ or } a, \\ 0, & \text{otherwise}. \end{cases}$$

Then $d \wedge f \notin Hom(L, L)$, since

$$(d \wedge f)(a \vee b) = (d \wedge f)(1) = a \quad but \quad (d \wedge f)(a) \vee (d \wedge f)(b) = 0 \vee 0 = 0.$$

Moreover, $d \vee f \notin Hom(L, L)$, since

$$(d \vee f)(a \wedge b) = (d \vee f)(0) = 0 \quad but \quad (d \vee f)(a) \wedge (d \vee f)(b) = a \wedge 1 = a.$$

Therefore $\wedge$ and $\vee$ are not binary operators. Hence we can not define a lattice on the set Hom(L, L).

Theorem 3.2.6. *[23] Let d be a derivation and I an ideal of a lattice L. Then we have*

(1) $d(I)$ is an ideal of L such that $d(I) \subseteq I$.

(2) $d^{-1}(I) = \{x \in L \mid d(x) \in I\}$ is an ideal of L.

Proof. For (1), $d(I) \neq \emptyset$. Let $a, b \in d(I)$. Then $a = d(x)$ and $b = d(y)$ for some $x, y \in I$. So

$$a \vee b = d(x) \vee d(y) = d(x \vee y) \in d(I).$$

Now, let $c \in d(I)$ and $x \in L$ such that $x \preccurlyeq c$. Then $c = d(z)$ for some $z \in I$. So

$$\begin{aligned} x = c \wedge x &= d(z) \wedge x \\ &= d(z \wedge x). \end{aligned}$$

Since I is an ideal and $z \wedge x \preccurlyeq z$, we get $x = d(z \wedge x) \in d(I)$. Hence, $d(I)$ is an ideal of L. To prove $d(I) \subseteq I$, let $x \in d(I)$. Then $x = d(y)$ for some $y \in I$. Since $d(y) \preccurlyeq y$, we get $d(y) = y \wedge d(y) \in I$. Hence, $x \in I$. For (2), $d^{-1}(I)$ is not empty sine $I \subseteq d^{-1}(I)$ and I is an ideal so I is not empty. Let $a, b \in d^{-1}(I)$. Then we have $d(a), d(b) \in I$. Since I is an ideal of L, we get

$$d(a \vee b) = d(a) \vee d(b) \in d(I) \quad \Rightarrow \quad a \vee b \in d^{-1}(I).$$

Again, let $c \in d^{-1}(I)$ and $x \in L$ such that $x \preccurlyeq c$. Then we get $d(c) \in I$. Since I is an ideal of L, we get $d(x) = d(c \wedge x) = d(c) \wedge x \in I$. Thus $x \in d^{-1}(I)$. Therefore, $d^{-1}(I)$ is an ideal of L.■

Remark 3.2.5. *Let $f : L \to L$ be a lattice endomorphism and I be an ideal of L. Then $f(I)$ may be not an ideal of L. For example, let L be a lattice of Figure 3.6. Define a lattice endomorphism $f : L \to L$ such that $f(x) = c$. Then $I = \{b, a, 0\}$ is*

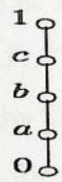

Figure 3.6:

an ideal of L but $f(I) = \{c\}$ is not an ideal of L. Moreover, $f(I) \nsubseteq I$.

In general, if a lattice homomorphism $f : L \to M$ is onto, then the image of an ideal I is again an ideal. To prove it, let I be an ideal of a lattice L: $f(I) \neq \emptyset$. Let $a, b \in f(I)$. Then $a = f(x)$ and $b = f(y)$ for some $x, y \in I$. So

$$a \vee b = f(x) \vee f(y) = f(x \vee y) \in f(I).$$

let $c \in f(I)$ and $x \in M$ such that $x \preccurlyeq c$. Then $c = f(z)$ for some $z \in I$. Since f is onto, there exist $w \in L$ such that $x = f(w)$. So

$$\begin{aligned} x = c \wedge x &= f(z) \wedge f(w) \\ &= f(z \wedge w). \end{aligned}$$

Since I is an ideal and $z \wedge w \preccurlyeq z$, we get $x = f(z \wedge w) \in f(I)$. Therefore, $f(I)$ is an ideal of L.

Hence The condition of onto which is necessary for getting the image of an ideal under a lattice homomorphism to become again an ideal is not required in case of a derivation.

Theorem 3.2.7. *[23] Let d be a derivation of a distributive lattice L and let I, J any two ideals of L. Then we have*

(1) $I \subseteq J$ implies that $d(I) \subseteq d(J)$.

(2) $d(I \cap J) = d(I) \cap d(J)$.

(3) $d(I \vee J) = d(I) \vee d(J)$.

Proof. For (1), suppose that $I \subseteq J$. Let $x \in d(I)$. Then we get that $x = d(y)$ for some $y \in I \subseteq J$. Hence, $x = d(y) \in d(J)$. Therefore, $d(I) \subseteq d(J)$.
For (2), clearly $d(I \cap J) \subseteq d(I) \cap d(J)$ by (1). Conversely, let $x \in d(I) \cap d(J)$. Then $x = d(a)$ for some $a \in I$ and $x = d(b)$ for some $b \in J$. Since $d(b) \preccurlyeq b$, we get that $d(b) \in J$ and hence $a \wedge d(b) \in I \cap J$. Thus

$$x = d(a) \wedge d(b) = d(a \wedge d(b)) \in d(I \cap J).$$

Therefore, $d(I) \cap d(J) \subseteq d(I \cap J)$. To prove (3), Clearly $d(I) \vee d(J) \subseteq d(I \vee J)$ by (1). Conversely, let $x \in d(I \vee J)$. Then $x = d(z)$ for some $z \in I \vee J$. Hence, $z = a \vee b$ for some $a \in I$ and $b \in J$. Thus

$$x = d(z) = d(a \vee b) = d(a) \vee d(b) \in d(I) \vee d(J).$$

Therefore, $d(I \vee J) \subseteq d(I) \vee d(J)$. ■

Remark 3.2.6. *Theorem 3.2.7 is true even if the lattice L is not distributive lattice. We can use the same proof for (1) and (2). To prove(3), $d(I) \vee d(J) \subseteq d(I \vee J)$ by (1). Conversely, let $x \in d(I \vee J)$. Then $x = d(z)$ for some $z \in I \vee J$. Hence, $z \preccurlyeq a \vee b$ for some $a \in I$ and $b \in J$. Thus*

$$x = d(z) \preccurlyeq d(a \vee b) = d(a) \vee d(b) \in d(I) \vee d(J).$$

Therefore, $d(I \vee J) \subseteq d(I) \vee d(J)$.

Remark 3.2.7. *Let $d : L \to M$ be a lattice homomorphism and let I, J any two ideals of L. Although $d(I)$ is not necessary to be an ideal of M, we can prove that d satisfy (1) and (2) in Theorem 3.2.7.*

Theorem 3.2.8. *A lattice L is distributive if and only if the map $d_a : L \to L$ such that $d_a(x) = a \wedge x$ is a lattice derivation for all $a \in L$.*

Proof. Let L be a distributive lattice. Then a is a distributive element $\forall a \in L$. So, $\forall x, y \in L$ we can get:

$$\begin{aligned} d_a(x \vee y) &= a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) = d_a(x) \vee d_a(y). \\ d_a(x \wedge y) &= a \wedge (x \wedge y) = (a \wedge x) \wedge y = d_a(x) \wedge y. \end{aligned}$$

Therefore, d_a is a lattice derivation for all $a \in L$. Conversely, let d_a be a derivation for all $a \in L$. Then $\forall a, x, y \in L$, we can get

$$\begin{aligned} a \wedge (x \vee y) &= d_a(x \vee y) \\ &= d_a(x) \vee d_a(y) \\ &= (a \wedge x) \vee (a \wedge y). \end{aligned}$$

Hence, L is a distributive lattice. ■

Theorem 3.2.9. *[22] A lattice L is distributive if and only if the map $m_a : L \to L$, defined by*

$$m_a(x) = a \wedge x$$

is a lattice homomorphism for all $a \in L$.

Proof. It is easy to see that m_a preserves joins and meets. As to preservation of joins, the equation

$$m_a(x \vee y) = m_a(x) \vee m_a(y)$$

is none other than the distributive law

$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y). \blacksquare$$

3.3 Ideals and Congruences in Distributive Lattices with respect to a Derivation

In this section we present many types of ideals in distributive lattice with respect to a derivation d such as d-ideal, injective ideal, d-prime ideal and maximal injective ideal. Also we furnish the relation between these types. Furthermore we introduce two types of congruences in a distributive lattice, one in terms of images of derivations and the other in terms of ideals generated by derivations. In addition we extend Stone's theorem to injective ideals and d-prime ideals.

Definition 3.3.1. The kernel of a derivation d of a lattice L with 0 is defined as the set

$$Ker\ d = \{x \in L \mid d(x) = 0\}.$$

Theorem 3.3.1. *[23] Let L be a lattice with 0. For any derivation d of a lattice L, $Ker\ d$ is an ideal of L.*

Proof. $Ker\ d$ is not empty since $0 = d(0) \in Ker\ d$. Let $x, y \in Ker\ d$. Then $d(x \vee y) = d(x) \vee d(y) = 0$. Hence $x \vee y \in Ker\ d$. Again, let $x \in Ker\ d$ and $r \in L$ such that $r \preccurlyeq x$. Since $d(r) \preccurlyeq r \preccurlyeq x$, we get

$$\begin{aligned} d(r) &= d(r) \wedge x \\ &= d(r \wedge x) \\ &= r \wedge d(x) \\ &= r \wedge 0 = 0. \end{aligned}$$

Hence $r \in Ker\ d$. Thus $Ker\ d$ is an ideal of L. ■

Remark 3.3.1. *Throughout this section a distributive lattice L stands for a distributive lattice with 0, unless otherwise mentioned.*

In the following, we introduce a congruence relation in terms of a derivation of a distributive lattice.

Definition 3.3.2. [24] Let d be a derivation of a distributive lattice L. Then define a relation θ^d with respect to d on L by

$$x \equiv y \pmod{\theta^d} \quad \Leftrightarrow \quad d(x) = d(y)$$

for $x, y \in L$.

Recall that in Chapter 2, for a congruence relation θ, we defined the **ideal kernel** of θ as the least element of L/θ. Denoted by $\ker(\theta)$.

Theorem 3.3.2. *[24] For any derivation d of a distributive lattice L, we have the following:*

(1) θ^d is a congruence relation on L.

(2) $\mathrm{Ker}(\theta^d) = Ker\ d$.

Proof. For(1), clearly θ^d is an equivalence relation on L. Now let $x \equiv y \pmod{\theta^d}$. Then we have $d(x) = d(y)$. Let c be an arbitrary element of L. Then

$$\begin{aligned} d(x \wedge c) &= d(x) \wedge c \\ &= d(y) \wedge c \\ &= d(y \wedge c). \end{aligned}$$

Hence $x \wedge c \equiv y \wedge c \pmod{\theta^d}$. Again

$$\begin{aligned} d(x \vee c) &= d(x) \vee d(c) \\ &= d(y) \vee d(c) \\ &= d(y \vee c). \end{aligned}$$

Hence $x \vee c \equiv y \vee c \pmod{\theta^d}$. Therefore θ^d is a congruence relation on L. For (2), if d any derivation of L, then

$$\begin{aligned} Ker(\theta^d) &= \{x \in L \mid x \equiv 0 \ (mod\ \theta^d)\} \\ &= \{x \in L \mid d(x) = d(0) = 0\} = Ker\ d. \blacksquare \end{aligned}$$

Theorem 3.3.3. *[24] For any derivation d of a distributive lattice L, we have the following:*

(1) $d(x) = x$ for all $x \in d(L)$.

(2) If $x \equiv y$ $(mod\ \theta^d)$ and $x, y \in d(L)$, then $x = y$.

Proof. For(1), let $x \in d(L)$. Then $x = d(a)$ for some $a \in L$. Since d is a derivation, we get that

$$d(x) = d(d(a)) = d(a) = x.$$

For(2), let $x \equiv y \pmod{\theta^d}$. Then $d(x) = d(y)$. Since d is a derivation and $x, y \in d(L)$, we get $x = y$. $\blacksquare$

Definition 3.3.3. [23] Let $d : L \to L$ be a derivation of a distributive lattice L. An ideal I of L is called a d-**ideal** if $I = d(I)$.
A **proper** d-**ideal** is a d-ideal $I \neq L$. For any derivation d of L, the set of all d-ideals of L is denoted by $\mathcal{I}_d(L)$.

Example 3.3.1.

(1) The zero ideal $\{0\}$ is a d-ideal of a distributive lattice L since $d(0) = 0$.

(2) If a derivation d is onto, then $d(L) = L$ and hence L is a d-ideal.

Example 3.3.2. Consider the distributive lattice $L = \{1, 2, 3, 6, 12\}$ ordered by divisibility. (see Figure 3.7).
Define a self map $d : L \to L$ such that $d(x) = x \wedge 2$. Then $d(1) = d(3) = 1$ and $d(2) = d(6) = d(12) = 2$. Clearly d is a derivation on L. Consider the ideal $I = \{1, 2\}$. We can see that $d(I) = I$. Therefore, I is a d-ideal of L.

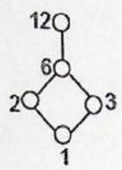

Figure 3.7:

Example 3.3.3. Let L be a distributive lattice with Hasse diagram given in Figure 3.8. Define a self map $d : L \to L$ such that $d(x) = x \wedge 3$. Then $d(0) = 0, d(1) =$

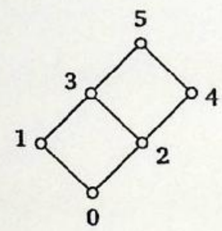

Figure 3.8:

$1, d(2) = d(4) = 2$ and $d(3) = d(5) = 3$. So, d is a derivation and the ideal $I = \{0, 1, 2, 3\}$ is a d-ideal.

Remark 3.3.2. *Let L be a distributive lattice. Define a derivation $d_a(x) = a \wedge x$ on L. Then for all $c \preccurlyeq a$, the ideal $J = \downarrow c$ is a d-ideal.*

Proof. *If $x \in J$, then $x \preccurlyeq c \preccurlyeq a$. so we can get $d_a(x) = a \wedge x = x$. Hence $d_a(x) = x$ for all $x \in J$. Therefore $d_a(J) = J$.* ■

Theorem 3.3.4. *[23] Let d be a derivation of a distributive lattice L. Then $\mathcal{I}_d(L)$ is a distributive lattice with respect to set inclusion.*

Proof. $\{0\} \in \mathcal{I}_d(L)$, so $\mathcal{I}_d(L)$ is not empty. We have to show that every pair of d-ideals of L has an infimum and a supremum in $\mathcal{I}_d(L)$. If J and K are d-ideals of L, then by using Theorem 3.2.7 we have :

$$\begin{aligned} d(J \cap K) &= d(J) \cap d(K) = J \cap K \\ d(J \vee K) &= d(J) \vee d(K) = J \vee K. \end{aligned}$$

So $J \cap K$ and $J \vee K$ are d-ideals. Hence $(\mathcal{I}_d(L), \cap, \vee)$ is a sublattice of a lattice $(\mathcal{I}(L), \cap, \vee)$ of all ideals of L. Since L is a distributive lattice, by Theorem 2.1.3, we get that the lattice $(\mathcal{I}(L), \cap, \vee)$ is a distributive. Therefore, $(\mathcal{I}_d(L), \cap, \vee)$ is a distributive lattice. ■

Definition 3.3.4. [23] Let d be a derivation of a distributive lattice L. An ideal I of L is called an **injective ideal** with respect to d if for $x, y \in L, d(x) = d(y)$ and $x \in I$ implies that $y \in I$. The set of all injective ideals of L is denoted by $\mathcal{I}^d(L)$.

Example 3.3.4. Consider the distributive lattice $\{1, 2, 3, 6, 12\}$ ordered by divisibility. Let d be the derivation given in Example 3.3.2. Then $I = \{1, 3\}$ is an injective ideal with respect to d.

Example 3.3.5. Let d be a derivation of a distributive lattice L. Evidently

$$Ker\ d = \{x \in L \mid d(x) = 0\}$$

is an injective ideal with respect to d.

Though the zero ideal $\{0\}$ is a d-ideal, there is no guarantee that it is an injective ideal. However, a set of equivalent conditions are established for $\{0\}$ to become an injective ideal.

Theorem 3.3.5. *[23] Let d be a derivation of a distributive lattice L. Then the following conditions are equivalent.*

(1) $\{0\}$ is injective ideal with respect to d.

(2) $Ker\ d = \{0\}$.

(3) $d(x) = 0$ implies that $x = 0$ for all $x \in L$.

Proof. (1) $\Rightarrow$ (2): Assume that $\{0\}$ is injective with respect to d. Let $x \in Ker\ d$. Then $d(x) = 0 = d(0)$. Since $\{0\}$ is an injective ideal, we can get that $x \in \{0\}$. Therefore $Ker\ d = \{0\}$. (2) $\Rightarrow$ (3): Trivial. (3) $\Rightarrow$ (1): Assume the condition (3). Let $d(x) = d(y)$ and $x \in \{0\}$. Hence, $d(y) = d(x) = d(0) = 0$. By (3), $y = 0 \in \{0\}$.

■

Theorem 3.3.6. *[23] Let d be a derivation of a distributive lattice L. Then the following conditions are equivalent.*

(1) d is injective.

(2) Every ideal is injective with respect to d.

(3) Every prime ideal is injective with respect to d.

Proof. (1) $\Rightarrow$ (2): Assume I is an ideal of L. Let $x, y \in I$, $d(x) = d(y)$ and $x \in I$. Since d is injective, $y = x \in I$. (2) $\Rightarrow$ (3): Obvious. (3) $\Rightarrow$ (1): Assume that every prime ideal of L is an injective ideal. Let $x, y \in L$ be such that $d(x) = d(y)$. Suppose that $x \neq y$. Without loss of generality, we can assume that $\downarrow x \cap \uparrow y = \emptyset$. Then by Corollary 2.2.5, there exists a prime ideal P such that $x \in P$ and $y \notin P$, which is a contradiction to P is an injective ideal. ■

Let I be an ideal of a distributive lattice L. If I is a d-ideal, then I is not necessary to be an injective ideal with respect to a derivation d and vice versa.

Example 3.3.6. Consider a distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given in Figure 3.9

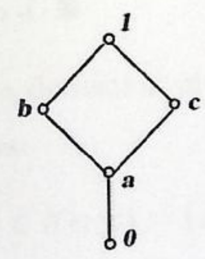

Figure 3.9:

Define a derivation $d : L \to L$ as follows:

$$d(x) = \begin{cases} 0, & if x = 0 \\ a, & if x = a, b \\ c, & if x = c, 1 \end{cases}$$

The ideal $J = \{0, a, c\}$ is a d-ideal. Now $d(a) = d(b)$ and $a \in J$ but $b \notin J$. Therefore, J is not an injective ideal with respect to d. The ideal $I = \{0, a, b\}$ is an injective ideal with respect to d but not a d-ideal since $d(I) = \{0, a\}$ and hence $d(I) \neq I$.

Theorem 3.3.7. *[23] Let d be a derivation of a distributive lattice L. Then a d-ideal I of L is injective with respect to d if and only if for any* $x \in L$, $d(x) \in I$ *implies* $x \in I$.

Proof. Let I be a d-ideal of L. Assume that I is an injective ideal with respect to d. Let $x \in L$. Suppose $d(x) \in I = d(I)$. Then $d(x) = d(a)$ for some $a \in I$. Since I is injective and $a \in I$, we get that $x \in I$. Conversely, let $x, y \in L$, $d(x) = d(y)$ and $x \in I$. Since $x \in I = d(I)$, we get $x = d(a)$ for some $a \in I$. Hence,

$$d(y) = d(x) = d(d(a)).$$

But $d(d(a)) = d(a)$ and $d(a) \preccurlyeq a \in I$ which implies that $y \in I$. Therefore, I is an injective ideal of L with respect to d. ■

Definition 3.3.5. [24] Let d be a derivation of a distributive lattice L. For any $a \in L$, define the set $(a)^d$ as follows:

$$(a)^d = \{x \in L \mid a \wedge x \in Ker\ d\} = \{x \in L \mid a \wedge d(x) = 0\}.$$

If $a \in Ker\ d$, then for all $x \in L$

$$d(a \wedge x) = d(a) \wedge x = 0 \quad \Rightarrow \quad a \wedge x \in Ker\ d.$$

Therefore $(a)^d = L$. Otherwise, let $a \notin Ker\ d$. Suppose $a \in (a)^d$. Hence $d(a) = 0$, which is contradiction. Hence $a \notin (a)^d$.

Theorem 3.3.8. *[24] Let L be a distributive lattice with 0 and let d be a derivation of L. Then $(a)^d$ is an ideal in L.*

Proof. $0 \in (a)^d$. Let $x, y \in (a)^d$. Then we get

$$\begin{aligned} a \wedge d(x \vee y) &= a \wedge (d(x) \vee d(y)) \\ &= (a \wedge d(x)) \vee (a \wedge d(y)) \\ &= 0 \vee 0 = 0. \end{aligned}$$

Hence $x \vee y \in (a)^d$. Let $r \in L$ and $x \in (a)^d$ such that $r \preccurlyeq x$. Then $x \wedge r = r$. So

$$\begin{aligned} a \wedge d(r) &= a \wedge d(x \wedge r) \\ &= a \wedge d(x) \wedge r \\ &= 0 \wedge r = 0. \end{aligned}$$

Hence $r \in (a)^d$. Therefore $(a)^d$ is an ideal of L. ■

Remark 3.3.3. *Let d be a derivation of a distributive lattice L. For any $a \in L$, the ideal $(a)^d$ is an injective ideal with respect to d.*

Proof. Let $a \in L$. By Theorem 3.3.8, $(a)^d$ is ideal. Let $x, y \in L$, $d(x) = d(y)$ and $x \in (a)^d$. So

$$a \wedge d(x) = 0 \quad \Rightarrow \quad a \wedge d(y) = 0.$$

Hence $y \in (a)^d$. ■

Lemma 3.3.9. *[24] Let d be a derivation of a distributive lattice L. Then for any $a, b \in L$, we have the following:*

(1) $a \preccurlyeq b$ implies $(b)^d \subseteq (a)^d$.

(2) $(a \vee b)^d = (a)^d \cap (b)^d$.

Proof. For (1), Suppose $a \preccurlyeq b$. This implies $a \wedge b = a$. Let $x \in (b)^d$. Then $b \wedge d(x) = 0$. Thus

$$\begin{aligned} a \wedge d(x) &= a \wedge b \wedge d(x) \\ &= a \wedge 0 = 0. \end{aligned}$$

Therefore $x \in (a)^d$. For (2), Let $a, b \in L$. Then by (1) we have $(a \vee b)^d \subseteq (a)^d \cap (b)^d$. Conversely, Let $x \in (a)^d \cap (b)^d$. So $a \wedge d(x) = 0$ and $b \wedge d(x) = 0$. Then

$$\begin{aligned} (a \vee b) \wedge d(x) &= (a \wedge d(x)) \vee (b \wedge d(x)) \\ &= 0 \vee 0 = 0. \end{aligned}$$

Hence, $x \in (a \vee b)^d$. ■

Lemma 3.3.10. *[24] Let d be a derivation of a distributive lattice L. For any $a, b, c \in L$, we have*

(1) $(a)^d = (b)^d$ implies $(a \wedge c)^d = (b \wedge c)^d$.

(2) $(a)^d = (b)^d$ implies $(a \vee c)^d = (b \vee c)^d$.

Proof. For(1), assume that $(a)^d = (b)^d$. Let $x \in (a \wedge c)^d$. Then

$$a \wedge c \wedge d(x) = 0 \;\Rightarrow\; a \wedge d(x \wedge c) = a \wedge d(x) \wedge c = 0.$$

Thus $x \wedge c \in (a)^d = (b)^d$. Therefore

$$b \wedge d(x \wedge c) = b \wedge d(x) \wedge c = 0 \;\Rightarrow\; b \wedge c \wedge d(x) = 0.$$

Thus $x \in (b \wedge c)^d$. Hence $(a \wedge c)^d \subseteq (b \wedge c)^d$. Similarly, we can get $(b \wedge c)^d \subseteq (a \wedge c)^d$. For(2), let $(a)^d = (b)^d$. By Lemma 3.3.9,

$$\begin{aligned} (a \vee c)^d &= (a)^d \cap (c)^d \\ &= (b)^d \cap (c)^d \\ &= (b \vee c)^d. \blacksquare \end{aligned}$$

Definition 3.3.6. [24] Let d be a derivation of a distributive lattice L. Then for any $x, y \in L$, define a relation θ_d on L with respect to d, as

$$x \equiv y \ (\text{mod } \theta_d) \quad \Leftrightarrow \quad (x)^d = (y)^d.$$

Theorem 3.3.11. *[24] For any derivation d of a distributive lattice L, the binary relation θ_d defined on L is a congruence relation on L.*

Proof. It is easy to show that θ_d is an equivalence relation. Let $x, y \in L$ such that $x \equiv y \ (\text{mod } \theta_d)$. Then by Lemma 3.3.10, for any $c \in L$, we get

$$x \wedge c \equiv y \wedge c \ (\text{mod } \theta_d) \quad \text{and} \quad x \vee c \equiv y \vee c \ (\text{mod } \theta_d). \blacksquare$$

The concept of kernel elements is introduced in the following.

Definition 3.3.7. [24] Let d be a derivation of a distributive lattice L. An element $x \in L$ is called a **kernel element** if $(x)^d = Ker\ d$. The set of all kernel elements of L is denoted by $\mathcal{K}_d$.

Example 3.3.7. Consider the distributive lattice $\{1, 2, 3, 6, 12\}$ under division (see Figure 3.7). Define a derivation $d : L \to L$ such that $d(x) = x \wedge 2$. Then

$$Ker\ d = \{1, 3\} \text{ and } \mathcal{K}_d = \{2, 6, 12\}.$$

Theorem 3.3.12. *[24] For any derivation d of a distributive lattice L, we have the following:*

(1) $\mathcal{K}_d$ is a congruence class with respect to θ_d.

(2) $Ker\ d \subseteq (x)^d$ for all $x \in L$.

(3) $\mathcal{K}_d$ is closed under $\wedge$ and $\vee$ of L.

(4) $\mathcal{K}_d$ is a filter of L, whenever $\mathcal{K}_d \neq \emptyset$.

Proof. For(1), we know that for any $x \in L$, the congruence class with respect to θ_d is the set $[x]_{\theta_d} = \{t \in L \mid x \equiv t \ (\text{mod } \theta_d)\}$. So, its clear that $\mathcal{K}_d$ is a congruence class with respect to θ_d. For(2), Let $a \in Ker\ d$. Then

$$d(a) = 0 \quad \Rightarrow \quad x \wedge d(a) = 0 \text{ for all } x \in L.$$

Thus $a \in (x)^d$ for all $x \in L$. Therefore $Ker\ d \subseteq (x)^d$ for all $x \in L$. For (3), let $a, b \in \mathcal{K}_d$. Then we get $(a)^d = (b)^d = Ker\ d$. Then

$$\begin{aligned} (a \vee b)^d &= (a)^d \cap (b)^d \\ &= Ker\ d \cap Ker\ d = Ker\ d. \end{aligned}$$

Hence $a \vee b \in \mathcal{K}_d$. To prove that $\mathcal{K}_d$ is closed under $\wedge$. By (2), we get $Ker\ d \subseteq (a \wedge b)^d$. Conversely, let $x \in (a \wedge b)^d$. Then

$$\begin{aligned} a \wedge b \wedge d(x) = 0 &\Rightarrow d(x) \wedge a \wedge b = 0 \\ &\Rightarrow d(x \wedge a) \wedge b = 0 \\ &\Rightarrow x \wedge a \in (b)^d = Ker\ d \\ &\Rightarrow d(x \wedge a) = 0 \\ &\Rightarrow d(x) \wedge a = 0 \\ &\Rightarrow a \wedge d(x) = 0 \\ &\Rightarrow x \in (a)^d = Ker\ d. \end{aligned}$$

Hence $(a \wedge b)^d \subseteq Ker\ d$. Thus $(a \wedge b)^d = Ker\ d$. Therefore $a \wedge b \in \mathcal{K}_d$. For (4), let $a, b \in \mathcal{K}_d$. By (3), we get $a \wedge b \in \mathcal{K}_d$. Let $r \in L$ and $a \in \mathcal{K}_d$ such that $a \preccurlyeq r$. Then

$$\begin{aligned} (r)^d &= (r \vee a)^d \\ &= (r)^d \cap (a)^d \\ &= (r)^d \cap Ker\ d \\ &= Ker\ d. \end{aligned}$$

Hence $r \in \mathcal{K}_d$. Therefore $\mathcal{K}_d$ is a filter of L. ■

Definition 3.3.8. [23] Let d be a derivation of a distributive lattice L. Then for any ideal I of L, define an extension of I as

$$I' = \{x \in L \mid d(x) \in \downarrow d(a) \text{ for some } a \in I\}.$$

Example 3.3.8. Return to Example 3.3.2. Let $I = \{1, 2\}$. Then $I' = \{1, 2, 3, 6, 12\}$. If $J = \{1, 3\}$, then $J' = \{1, 3\} = J$.

Example 3.3.9. In Example 3.3.3. The extension of the ideal $J = \{0, 2\}$ is $J' = \{0, 2, 4\}$.

Theorem 3.3.13. *[23] Let d be a derivation of a distributive lattice L. Then for any ideal I of L, I' is an ideal of L.*

Proof. Let $x, y \in I'$. Then $d(x) \in \downarrow d(a)$ for some $a \in I$ and $d(y) \in \downarrow d(b)$ for some $b \in I$. So

$$d(x \vee y) = d(x) \vee d(y) \in \downarrow d(a) \vee \downarrow d(b) = \downarrow d(a \vee b).$$

Since I is an ideal, $a \vee b \in I$. Therefore, $x \vee y \in I'$. Now, let $r \in L$ and $x \in I'$ such that $r \preccurlyeq x$. Then $d(x) \in \downarrow d(a)$ for some $a \in I$. So

$$d(r) \preccurlyeq d(x) \in \downarrow d(a) \quad \Rightarrow \quad d(r) \in \downarrow d(a) \quad \Rightarrow \quad r \in I'.$$

Hence, I' is an ideal of L. ■

The following lemma is a routine verification.

Lemma 3.3.14. *[23] Let d be a derivation of a distributive lattice L. Then for any two ideals I, J of L, we have the following:*

(1) $I \subseteq I'$.

(2) $I \subseteq J$ *implies* $I' \subseteq J'$.

(3) $I' \cap J' = (I \cap J)'$.

Theorem 3.3.15. *[23] Let d be a derivation of a distributive lattice L . Then for any ideal I of L, I' is the smallest injective ideal with respect to d such that $I \subseteq I'$.*

Proof. By Theorem 3.3.13 and by Lemma 3.3.14, I' is an ideal containing I. We have to show that I' is an injective ideal with respect to d. Let $x, y \in L, d(x) = d(y)$ and $x \in I'$. Then $d(y) = d(x) \in \downarrow d(a)$ for some $a \in I$. Thus $y \in I'$. Let J be an injective ideal with respect to d such that $I \subseteq J$. Let $t \in I'$. Then $d(t) \in \downarrow d(b)$ for some $b \in I \subseteq J$. So

$$d(t) = d(b) \wedge d(t) = d(b \wedge d(t)).$$

Since J is injective ideal with respect to d and $(b \wedge d(t)) \in J$, we get $t \in J$. Thus $I' \subseteq J$. Hence, I' is the smallest injective ideal with respect to d such that $I \subseteq I'$.

■

Corollary 3.3.16. *[23] If I is an injective ideal, then $I = I'$.*

Theorem 3.3.17. *[23] The set $\mathcal{I}^d(L)$ of all injective ideals of a distributive lattice L, with respect to a given derivation of d of L, forms a distributive lattice on their own.*

Proof. For $I, J \in \mathcal{I}^d(L)$, define the operations $\wedge$ and $\sqcup$ such that

$$\begin{aligned} I \wedge J &= I \cap J, \\ I \sqcup J &= (I \vee J)'. \end{aligned}$$

We have to show that every pair of injective ideals of L with respect to d has an infimum and a supremum in $\mathcal{I}^d(L)$. It is clear that if I and J are injective ideals of L, then so $I \cap J$, and this is the biggest injective ideal of L that is contained in both I and J. Therefore, inf $\{I, J\} \in \mathcal{I}^d(L)$. By Theorem 3.3.15, we can get $(I \vee J)'$ is the smallest injective ideal with respect to d such that $I \vee J \subseteq (I \vee J)'$. Therefore, sup $\{I, J\} \in \mathcal{I}^d(L)$. Hence $(\mathcal{I}^d(L), \wedge, \sqcup)$ is a lattice. For $I, J, K \in \mathcal{I}^d(L)$ we have

$$\begin{aligned} I \sqcup (J \wedge K) &= (I \vee (J \cap K))' \\ &= ((I \vee J) \cap (I \vee K))' \\ &= (I \vee J)' \cap (I \vee K)' \\ &= (I \sqcup J) \cap (I \sqcup K). \end{aligned}$$

Therefore, $(\mathcal{I}^d(L), \wedge, \sqcup)$ is a distributive lattice. ■

Definition 3.3.9. [23] Let d be a derivation of a distributive lattice L. A proper ideal P of L is called a **d-prime ideal** if for any $x, y \in L$, $x \wedge y \in Ker\ d$ implies either $x \in P$ or $y \in P$.

Example 3.3.10. Let $L = \{0, 1, 2, 3, 4, 5\}$ be a distributive lattice with Hasse diagram given in Figure 3.10. Define a derivation $d : L \to L$ such that $d(x) = 1 \wedge x$. Then $d(0) = d(2) = d(4) = 0$ and $d(1) = d(3) = d(5) = 1$. Therefore, $Ker\ d = \{0, 2, 4\}$. The ideal $I = \{0, 2, 4\}$ is a d-prime ideal. But the ideal $J = \{0, 1\}$ is not a d-prime ideal since $3 \wedge 4 = 2 \in Ker\ d$ but $3 \notin J$ and $4 \notin J$.

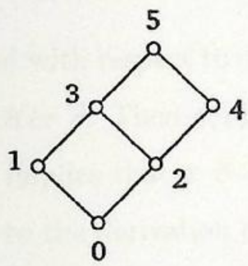

Figure 3.10:

Remark 3.3.4. *Let d be a derivation of a distributive lattice L.*

(1) If I is a prime ideal, then I is not necessarily to be a d-prime ideal. For instance, in Example 3.3.10, the ideal $I = \{0, 1\}$ is a prime ideal but not a d-prime ideal.

(2) If I is a d-prime ideal, then I is not necessarily to be a prime ideal. For instance, let L be a lattice given in Example 3.3.10. Define a derivation $d : L \to L$ such that $d(x) = 5 \wedge x$. Then $ker\ d = \{0\}$ and $I = \{0, 2\}$ is a d-prime ideal but not prime ideal since

$$3 \wedge 4 = 2 \in I \quad but \quad 3 \notin I \text{ and } 4 \notin I.$$

(3) If ker $d = \{0\}$, then the ideal $I = \{0\}$ is a d-prime ideal if and only if it is a prime ideal.

Definition 3.3.10. [23] Let d be a derivation of a distributive lattice L. A proper injective ideal J with respect to d is **maximal injective ideal** if for any injective ideal I:

$$J \subseteq I \subseteq L \quad \Rightarrow \quad I = J \quad \text{or} \quad I = L.$$

Lemma 3.3.18. *[23] Let d be a derivation of a distributive lattice L. Then for any injective ideal I of L with respect to d , $Ker\ d \subseteq I$. In other words, $Ker\ d$ is the smallest injective ideal of L.*

Proof. $Ker\ d$ is an injective ideal with respect to d of L. Let I be an injective ideal with respect to d. Suppose $x \in Ker\ d$. Then $d(x) = 0 = d(0)$. Since I is injective with respect to d and $0 \in I$, it implies that $x \in I$. Hence, $Ker\ d$ is the smallest injective ideal in L with respect to the derivation d. ■

In Chapter 2 we introduced Theorem 2.2.2: every maximal ideals in a distributive lattice are prime. Theorem 3.3.19 develop Theorem 2.2.2 to maximal injective ideal and a d-prime ideal.

Theorem 3.3.19. *[23] Let d be a derivation of a distributive lattice L. Then every maximal injective ideal with respect to the derivation d of L is a d-prime ideal.*

Proof. Let P be a maximal injective ideal of L. Choose $x, y \in L$ such that $x \notin P$ and $y \notin P$. Then

$$P \subset P \vee \downarrow x \subseteq (P \vee \downarrow x)' \quad \text{and} \quad P \subset P \vee \downarrow y \subseteq (P \vee \downarrow y)'.$$

By maximality of P, we can get

$$(P \vee \downarrow x)' = L \quad \text{and} \quad (P \vee \downarrow y)' = L.$$

Hence

$$\begin{aligned}(P \vee \downarrow (x \wedge y))' &= ((P \vee \downarrow x) \bigcap (P \vee \downarrow y))' \\ &= (P \vee \downarrow x)' \bigcap (P \vee \downarrow y)' \\ &= L.\end{aligned}$$

If $x \wedge y \in Ker\ d$, then $\downarrow (x \wedge y) \subseteq Ker\ d$. Therefore,

$$L = (P \vee \downarrow (x \wedge y)') \subseteq (P \vee Ker\ d)' \subseteq P' = P,$$

which is a contradiction to the fact that P is proper. Hence P is a d-prime ideal. ■

Theorem 3.3.20. *[23] Let d be a derivation of a distributive lattice L. Then every d-prime ideal P is an injective ideal with respect to d if for each $a \in P$ there exists $b \notin P$ such that $a \wedge b \in Ker\ d$.*

Proof. Let P be a d-prime ideal which satisfies the given property. Let $x, y \in L$. Suppose $d(x) = d(y)$ and $x \in P$. Then there exist $b \notin P$ such that $x \wedge b \in Ker\ d$. Hence

$$d(x) \wedge b = d(x \wedge b) = 0 \quad \Rightarrow \quad d(y \wedge b) = d(y) \wedge b = 0.$$

Therefore, $y \wedge b \in Ker\ d$. Since P is a d-prime ideal and $b \notin P$, we get $y \in P$. Hence, P is injective ideal with respect to d. ■

In Chapter 2, we introduced the celebrated result of Stone in Theorem 2.2.3: Let L be a distributive lattice, let I be an ideal, let F be a filter of L, and let $I \cap F = \emptyset$. Then there exists a prime ideal P of L such that $I \subseteq P$ and $P \cap F = \emptyset$. Theorem 3.3.21 generalise Theorem 2.2.3 which is meant for ideals, filters and prime ideals of a distributive lattice to the case of injective ideals, filters and d-prime ideals.

Theorem 3.3.21. *[23] Let d be a derivation of a distributive lattice L. Suppose I is an injective ideal with respect to d and F a filter of L such that $I \cap F = \emptyset$. Then there exists a d-prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Proof. Let I be an injective ideal with respect to d and F a filter of L such that $I \cap F = \emptyset$. Consider $\mathcal{X} = \{J \in I^d(L) \mid I \subseteq J \text{ and } J \cap F = \emptyset\}$. The set $\mathcal{X}$ is not empty, since $I \in \mathcal{X}$. Let C be a chain of elements of $\mathcal{X}$ and let $M = \bigcup C$. Then clearly M is an upper bound for C in $\mathcal{X}$. Hence by Zorn's lemma, $\mathcal{X}$ has a maximal element, say P. We have to show that P is d-prime. Let $x, y \in L$ such that $x \notin P$ and $y \notin P$. Then

$$P \subset P \vee \downarrow x \subseteq (P \vee \downarrow x)' \quad \text{and} \quad P \subset P \vee \downarrow y \subseteq (P \vee \downarrow y)'.$$

Since P is maximal injective ideal, we can get

$$(P \vee \downarrow x)' \cap F \neq \emptyset \quad \text{and} \quad (P \vee \downarrow y)' \cap F \neq \emptyset.$$

Choose $a \in (P \vee \downarrow x)' \cap F$ and $b \in (P \vee \downarrow y)' \cap F$. Then

$$\begin{aligned} a \wedge b &\in (P \vee \downarrow x)' \bigcap (P \vee \downarrow y)' \\ &= ((P \vee \downarrow x) \bigcap (P \vee \downarrow y))' \\ &= (P \vee \downarrow (x \wedge y))'. \end{aligned}$$

If $x \wedge y \in Ker\ d$, then $a \wedge b \in (P \vee \downarrow (x \wedge y))' \subseteq (P \ \vee \ Ker\ d)' = P' = P$. Thus, $a \wedge b \in P \cap F$, which is a contradiction to $P \cap F = \emptyset$. Hence, P is a d-prime ideal. ■

Corollary 3.3.22. *Let d be a derivation of a distributive lattice L, let I be an injective ideal with respect to d, and let $a \in L$ and $a \notin I$. Then there is a d-prime ideal P such that $I \subseteq P$ and $a \notin P$.*

Proof. Let I be an injective ideal with respect to d such that $a \notin I$. Define a filter F such that $F = \uparrow a = \{x \in L \mid x \succcurlyeq a\}$. Then $I \cap F = \emptyset$. By Theorem 2.1.1, there exists a d-prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$. Hence $a \notin P$. ■

Remark 3.3.5. *Let d be a derivation of a distributive lattice L. If $a, b \in L$ such that $a \neq b$, then it is not true that we can find a d-prime ideal containing exactly one of a and b. For instance, in Example 3.3.10 there is just one d-prime ideal that is $I = \{0, 2, 4\}$. For $1, 3 \in L$*

$$1 \neq 3 \quad \textit{but} \quad 1 \notin I \textit{ and } 3 \notin I.$$

Theorem 3.3.23. *Let d be a derivation of a distributive lattice L. Then every injective ideal I with respect to d is the intersection of all d-prime ideals containing it.*

Proof. Let I be an injective ideal of a distributive lattice L and let

$$I_1 = \bigcap(P \mid I \subseteq P,\ P \text{ is a } d\text{-prime ideal of } L).$$

If $I \neq I_1$, then there is an $a \in I_1 \setminus I$, and so $a \notin I$. By Corollary 3.3.22, there is a d- prime ideal P, with $I \subseteq P$ and $a \notin P$. But I_1 is the intersection of all d- prime ideal that contain I. Then $a \notin I_1$ is a contradiction. Hence $I_1 = I$. ■

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