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Master of Mathematics

Some Numerical Aspects of the Cauchon Algorithm

Submitted by:

Fatima Rasheed

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Graduate Advisory Committee:

Dr. Mohammad Adam (Supervisor), Palestine Polytechnic University.

Signature: Mohammad Adam Date: 7.10.2021

Dr. Ali Zain (Internal committee member), Palestine Polytechnic University.

Signature: Date: 7.10.2021

Prof. Dr. Jürgen Garloff (External committee member), Universität Konstanz.

Signature: Jürg Garloff Date: Oct 3, 2021

Thesis Approved

Dr. Nafez Nasreideen
Dean of Graduate Studies and Scientific Research
Palestine Polytechnic University

Signature: Date: 7.10.2021

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Abstract

A real matrix is called *totally nonnegative* or *totally positive* if all minors are nonnegative or positive, respectively. In this thesis, we derive the condensed form of the Restoration Algorithm which is a new result and it is the inverse of the condensed form of the Cauchon Algorithm where the latter provides an efficient tool to check a given matrix to be totally nonnegative or totally positive.

In this thesis we perform some elementary operations on a nonsingular totally nonnegative matrix and apply the condensed form of the Cauchon Algorithm on it, we determine how the entries of this matrix are changing after performing the elementary operations. Also, we present a new algorithm for computing the eigenvalues of a nonsingular totally nonnegative matrices with high accuracy using the results obtained in this thesis.

DEDICATION

This thesis is dedicated to:

The sake of Allah, my Creator and my Master,
My great supervisor Dr. Mohammad Adam, who encourage and support me,
My external committee member Prof. Dr. Jürgen Garloff,
My parents, the reason of what I become today.

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Contents

Chapter 1

Introduction

In this chapter, we introduce the totally nonnegative and totally positive matrices which play an important rule in mathematical applications.

This chapter consists of three sections. In Section 1.1, we introduce some notation, basic definitions, and various facts and identities that will be used throughout this thesis. In Section 1.2, we present the definition of totally nonnegative and totally positive matrices and review the operations that preserve total nonnegativity and total positivity. In Section 1.3, we recall various examples of totally positive and totally nonnegative matrices.

1.1 Preliminaries

The set of all $n \times m$ matrices with real elements will be denoted by $\mathbb{R}^{n\times m}$. If the number of rows of a matrix is equal to the number of columns, that is $n = m$, then the matrix is called a *square matrix of order* n. For $A \in \mathbb{R}^{n\times m}$, the notation $A = [a_{ij}]$ will indicate that the entries of A are $a_{ij} \in \mathbb{R}$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. The I_n denotes the *identity* matrix of order n, and for $1 \leq i, j \leq n$, we let E_{ij} denote the $n \times n$ *standard basic* matrix whose only nonzero entry is a 1 that occurs in the (i, j) position. The *transpose* of the matrix A will be denoted by A^T whose entry at the position (i, j) is a_{ji}.

For $A \in \mathbb{R}^{n\times m}$, $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_p\} \subseteq \{1, 2, \ldots, n\}$, and $\beta = \{\beta_1, \beta_2, \ldots, \beta_q\} \subseteq \{1, 2, \ldots, m\}$, the *submatrix* of A lying in the rows indexed by α and the columns indexed by β will be denoted by

$$A[\alpha|\beta] = A\left[\alpha_1, \alpha_2, \ldots, \alpha_p|\beta_1, \beta_2, \ldots, \beta_q\right] = A\begin{bmatrix} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_q \end{bmatrix}.$$

Similarly, $A(\alpha|\beta)$ is the submatrix obtained from A by deleting the rows indexed by α and columns indexed by β. We let α^c and β^c denote the *complements* of the index set α in $\{1, 2, \ldots, n\}$ and the index set β in $\{1, 2, \ldots, m\}$, respectively. So,

$A(\alpha|\beta) = A[\alpha^c|\beta^c]$. If $\alpha = \beta$, then the *principal submatrix* $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$. In the special case when $\alpha = \{1, 2, \ldots, k\}$ with $1 \leq k \leq \min\{n, m\}$, we refer to the principal submatrix $A[\alpha]$ as a *leading principal submatrix.*

For a square matrix A, $\det A$ denote the *determinant* of A. A *minor* in a given matrix A is the determinant of a square submatrix of $A \in \mathbb{R}^{n\times m}$, denoted by

$$\det A[\alpha|\beta] = \det A\left[\alpha_1, \alpha_2, \ldots, \alpha_p|\beta_1, \beta_2, \ldots, \beta_p\right] = \det A\begin{bmatrix} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_p \end{bmatrix},$$

and if $\alpha = \beta$, then the determinant of this principal submatrix is called *principal minor*, denoted by

$$\det A[\alpha] = \det A\begin{bmatrix} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \alpha_1, \alpha_2, \ldots, \alpha_p \end{bmatrix}.$$

The *dispersion* of α is the measure of the gaps in the index sequence α and is denoted by $d(\alpha)$ which is defined by $d(\alpha) := \alpha_p - \alpha_1 - p + 1$. If $d(\alpha) = 0$ we call α *contiguous*, if $d(\alpha) = d(\beta) = 0$ we call the submatrix $A[\alpha|\beta]$ *contiguous* and in the case $p = q$ we call the minor $\det A[\alpha|\beta]$ *contiguous.*

For integers p, n, we denote by $Q_{p,n}$ the set of all strictly increasing sequences of p integers chosen from $\{1, 2, \ldots, n\}$. We order the sequences from $Q_{p,n}$ with respect to the *lexicographical ordering*: If $\alpha, \beta \in Q_{p,n}$, we say that α *is less than or equal to* β *with respect to the lexicographical order*, denoted by

$$\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_p\} \leq \beta = \{\beta_1, \beta_2, \ldots, \beta_p\},$$

if and only if $\alpha = \beta$ or the first nonzero term in the following sequnce

$$\beta_1 - \alpha_1, \beta_2 - \alpha_2, \ldots, \beta_p - \alpha_p,$$

is positive.

In the following definition, we present some interesting families of matrices.

Definition 1.1. *[FJ11] An* $n \times n$ *matrix* $A = [a_{ij}]$ *is called*

1. diagonal *if* $a_{ij} = 0$ *whenever* $i \neq j$,
2. tridiagonal *if* $a_{ij} = 0$ *whenever* $|i - j| > 1$,
3. upper (lower) triangular *if* $a_{ij} = 0$ *whenever* $i > j$ $(i < j)$.

A tridiagonal matrix that is also upper (lower) triangular is called an upper (lower) bidiagonal matrix.

The following theorem introduces some important properties of determinants.

Theorem 1.1. *[KH05] Let A be a square matrix of order n. The determinant of A satisfies the following properties:*

1. *The determinants of a matrix and its transpose are equal, i.e.,* $\det A^T = \det A$.
2. *If matrix B results from matrix A by interchanging two rows (columns) of A, then* $\det B = -\det A$.
3. *If B obtained from A by multiplying a row (column) of A by a real constant c, then* $\det B = c \det A$.
4. *If $B = [b_{ij}]$ is obtained from $A = [a_{ij}]$ by adding to each element of the rth row (column) of A the corresponding elements in the sth row (column) multiplied by a constant c such that $r \neq s$, then* $\det B = \det A$.
5. *The determinant of upper (lower) triangular matrix is the product of the elements on the main diagonal, i.e.,* $\det A = \prod_{i=1}^{n} a_{ii}$ *if $A = [a_{ij}]$ is an upper (lower) triangular matrix.*

Definition 1.2. *[Pin10] The* p-th compound matrix *of an $n \times m$ matrix A is denoted by $A^{[p]}$ and is defined as the* $\begin{pmatrix} n \\ p \end{pmatrix} \times \begin{pmatrix} m \\ p \end{pmatrix}$ *matrix with entries*

$$\det A[\alpha|\beta],$$

where $\alpha \in Q_{p,n}$ and $\beta \in Q_{p,m}$ are arranged in lexicographical order, and $p = 1, 2, \ldots, \min\{n, m\}$.

The following theorem is a generalization of the multiplication formula of matrices which plays a significant rule in proving some fundamental results.

Theorem 1.2. *[FJ11](Cauchy-Binet Identity) Let $A \in \mathbb{R}^{n\times k}$ and $B \in \mathbb{R}^{k\times m}$. Then for $\alpha \in Q_{p,n}$ and $\beta \in Q_{p,m}$, where $1 \leq p \leq \min\{n, k, m\}$, we have*

$$\det AB[\alpha|\beta] = \sum_{\gamma} \det A[\alpha|\gamma] \det B[\gamma|\beta],$$

where the sum is taken over all sequences $\gamma \in Q_{p,k}$.

The following is an illustrative example of Theorem 1.2.

Example 1.1. *Let* $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 3 \\ 1 & 0 & -2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 4 & 2 & 0 & 1 \\ -1 & 0 & 3 & 2 \\ 1 & 1 & 2 & -1 \end{bmatrix}$. *Then* $\det AB[\alpha|\beta]$, *where* $\alpha = \{2, 3\}$, $\beta = \{3, 4\}$, *can be calculated by the application of the Cauchy-Binet identity:*

$$\begin{aligned}
\det AB\,[2,3|3,4] &= \det A\,[2,3|1,2]\det B\,[1,2|3,4] \\
&+ \det A\,[2,3|1,3]\det B\,[1,3|3,4] \\
&+ \det A\,[2,3|2,3]\det B\,[2,3|3,4] \\
&= \det A\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\det B\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \\
&+ \det A\begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}\det B\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \\
&+ \det A\begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix}\det B\begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} = 40.
\end{aligned}$$

The following determinant identity plays a rule in proving the results of this thesis.

Theorem 1.3. *[FJ11](Sylvester's Determinant Identity) Let* $A \in \mathbb{R}^{n\times m}$ *and suppose* $\alpha \in Q_{p,n}$, $\beta \in Q_{p,m}$ *be such that* $\det A[\alpha|\beta] \neq 0$. *Define the* $(n-p)\times(m-p)$ *matrix* $B = [b_{ij}]$ *whose entries are given by*

$$b_{ij} := \det A[\alpha \cup \{i\}|\beta \cup \{j\}],$$

with $i \in \alpha^c, j \in \beta^c$. *Then Sylvester's Identity states that for each* $\delta \subseteq \alpha^c, \gamma \subseteq \beta^c$, *with* $|\delta| = |\gamma| = r$,

$$\det B[\delta|\gamma] = (\det A[\alpha|\beta])^{r-1}\det A[\alpha\cup\delta|\beta\cup\gamma]. \tag{1.1}$$

Let A be a square matrix of order n be such that $n \geq 3$ and let $\alpha = \beta = \{2, \ldots, n-1\}$, so the matrix B in Theorem 1.3 is defined as follows:

$$B = \begin{bmatrix} b_{11} & b_{1n} \\ b_{n1} & b_{nn} \end{bmatrix},$$

with entries

$$\begin{aligned}
b_{11} &= \det A[\alpha \cup \{1\}|\beta \cup \{1\}] \\
&= \det A[1,2,\ldots,n-1|1,2,\ldots,n-1], \\
b_{1n} &= \det A[\alpha \cup \{1\}|\beta \cup \{n\}] \\
&= \det A[1,2,\ldots,n-1|2,\ldots,n-1,n], \\
b_{n1} &= \det A[\alpha \cup \{n\}|\beta \cup \{1\}] \\
&= \det A[2,\ldots,n-1,n|1,2,\ldots,n-1], \\
b_{nn} &= \det A[\alpha \cup \{n\}|\beta \cup \{n\}] \\
&= \det A[2,\ldots,n-1,n|2,\ldots,n-1,n].
\end{aligned}$$

For $\delta = \{1,n\}$ and $\gamma = \{1,n\}$ and by application of (1.1) we get

$$\det \begin{bmatrix} b_{11} & b_{1n} \\ b_{n1} & b_{nn} \end{bmatrix} = \det A[2,\ldots,n-1] \det A.$$

Hence,

$$\begin{aligned}
\det A &= \frac{b_{11}b_{nn} - b_{n1}b_{1n}}{\det A[2,\ldots,n-1]} \\
&= \frac{\det A[1,2,\ldots,n-1|1,2,\ldots,n-1] \det A[2,\ldots,n-1,n|2,\ldots,n-1,n]}{\det A[2,\ldots,n-1]} \\
&- \frac{\det A[2,\ldots,n-1,n|1,2,\ldots,n-1] \det A[1,2,\ldots,n-1|2,\ldots,n-1,n]}{\det A[2,\ldots,n-1]},
\end{aligned} \tag{1.2}$$

provided that $\det A[2,\ldots,n-1] \neq 0$.

The following example illustrates (1.2).

Example 1.2. *Let* $A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 2 \end{bmatrix}$. *By* (1.2), $\det A$ *can be found as follows:*

$$\begin{aligned}
\det A &= \frac{\det A \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix} \det A \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix} - \det A \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \det A \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}}{4} \\
&= \frac{-8(-10) - 0(2)}{4} = 20.
\end{aligned}$$

The following theorem gives the relationship between the determinant of a given square matrix and its minors.

Theorem 1.4. *[Kar68](Laplace Expansion by Minors) Let $A \in \mathbb{R}^{n\times n}$. Then for any $p = 1, \dots, n$ and $\alpha \in Q_{p,n}$ we have*

$$\det A = \sum_{\beta \in Q_{p,n}} (-1)^s \det A[\alpha|\beta] \det A[\alpha^c|\beta^c],$$

where $s = \sum_{i\in\alpha} i + \sum_{j\in\beta} j$.

Example 1.3. *For $A = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 2 & 1 \\ 3 & 4 & -1 \end{bmatrix}$ and $\alpha = \{1, 2\}$. The $\det A$ can be calculated by using the Laplace expansion as follows:*

$$\begin{aligned} \det A &= \sum_{\beta \in Q_{p,n}} (-1)^s \det A[\alpha|\beta] \det A[\alpha^c|\beta^c] \\ &= (-1)^6 \det A\left[1,2|1,2\right] \det A\left[3|3\right] + (-1)^7 \det A\left[1,2|1,3\right] \det A\left[3|2\right] \\ &+ (-1)^8 \det A\left[1,2|2,3\right] \det A\left[3|1\right] \\ &= \det \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix} \det \begin{bmatrix} -1 \end{bmatrix} - \det \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \det \begin{bmatrix} 4 \end{bmatrix} \\ &+ \det \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix} \det \begin{bmatrix} 3 \end{bmatrix} \\ &= 18. \end{aligned}$$

The following lemma is a basic tool in getting some results of this thesis.

Lemma 1.1. *If $B = [b_{ij}]$ is a square matrix obtained from A by multiplying the ith row with a positive scalar x and add it to the next row multiplied by y, then for $p \geq 2$,*

$$\begin{aligned} \det B \begin{bmatrix} i+1, i+2, \dots, i+p \\ \beta_1, \beta_2, \dots, \beta_p \end{bmatrix} &= y \det A \begin{bmatrix} i+1, i+2, \dots, i+p \\ \beta_1, \beta_2, \dots, \beta_p \end{bmatrix} \\ &+ x \det A \begin{bmatrix} i, i+2, \dots, i+p \\ \beta_1, \beta_2, \dots, \beta_p \end{bmatrix}. \end{aligned} \tag{1.3}$$

Proof. The entries in the row $i+1$ of the matrix B are given by:

$$b_{i+1,j} = x\ a_{i,j} + y\ a_{i+1,j},\ j = 1, 2, \dots, n.$$

By the Laplace expansion along the first row of $B\begin{bmatrix} i+1, i+2, \ldots, i+p \\ \beta_1, \beta_2, \ldots, \beta_p \end{bmatrix}$, we get

$$\begin{aligned}
\det B\begin{bmatrix} i+1, i+2, \ldots, i+p \\ \beta_1, \beta_2, \ldots, \beta_p \end{bmatrix} &= \begin{vmatrix} x\, a_{i,\beta_1} + y\, a_{i+1,\beta_1} & x\, a_{i,\beta_2} + y\, a_{i+1,\beta_2} & \cdots & x\, a_{i,\beta_p} + y\, a_{i+1,\beta_p} \\ a_{i+2,\beta_1} & a_{i+2,\beta_2} & \cdots & a_{i+2,\beta_p} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i+p,\beta_1} & a_{i+p,\beta_2} & \cdots & a_{i+p,\beta_p} \end{vmatrix} \\
&= x \det A\begin{bmatrix} i, i+2, \ldots, i+p \\ \beta_1, \beta_2, \ldots, \beta_p \end{bmatrix} + y \det A\begin{bmatrix} i+1, i+2, \ldots, i+p \\ \beta_1, \beta_2, \ldots, \beta_p \end{bmatrix}.
\end{aligned}$$

□

1.2 Totally Nonnegative Matrices

In this section, we introduce the totally nonnegative and totally positive matrices, also we present some basic properties of these classes of matrices.

Definition 1.3. *[Pin10] An $n \times m$ matrix A is said to be* totally nonnegative (TN) *if all of its minors are nonnegative, i.e.,*

$$\det A\,[\alpha|\beta] = \det A\begin{bmatrix} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_p \end{bmatrix} \geq 0, \tag{1.4}$$

for all $\alpha \in Q_{p,n}$, $\beta \in Q_{p,m}$, and all $p = 1, 2, \ldots, \min\{n, m\}$. It is said to be totally positive (TP) *if strict inequalities hold in* (1.4).

Alternatively, by using the compound matrix terminology, we can restate the definition of TN (TP) in terms of the compound matrices. A matrix $A \in \mathbb{R}^{n \times m}$ is TN (TP) if all entries of the p-th compound matrices of A are nonnegative (positive), $p = 1, 2, \ldots, \min\{n, m\}$.

Example 1.4. *The matrix* $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$ *is TN.*

To check that A is totally nonnegative matrix we will show that all the p-th compound matrices of A are nonnegative, $p = 1, 2, 3$.

$$
\begin{aligned}
A^{[1]} &= A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}, \\
A^{[2]} &= \begin{bmatrix} 1 & 3 & 2 \\ 2 & 8 & 6 \\ 3 & 15 & 12 \\ 1 & 5 & 6 \\ 2 & 12 & 16 \\ 1 & 7 & 12 \end{bmatrix}, \\
A^{[3]} &= \begin{bmatrix} 2 \\ 6 \\ 6 \\ 2 \end{bmatrix}.
\end{aligned}
$$

Hence all entries in the p-th compound are nonnegative, $p = 1, 2, 3$. Therefore A is totally nonnegative matrix, also A is totally positive since all the entries are positive.

Proposition 1.1. *[Pin10] If A is an $n \times m$ TN (TP) matrix, then*

1. *The transpose of A is TN (TP).*
2. *If $B := [b_{ij}]$, where $b_{ij} = a_{n+1-i,m+1-j}, i = 1, \ldots, n, j = 1, 2, \ldots, m$, then B is TN (TP). That is, the matrix B which is obtained from A by reserving the order of both of its rows and columns is TN (TP).*

Proof. Let A be TN (TP).

1. By the definition, every submatrix of A is totally nonnegative, so all minors of A^T are nonnegative (positive). Hence A^T is TN (TP).
2. Let $\alpha \in Q_{p,n}$ and $\beta \in Q_{p,m}$. Then

$$
\begin{aligned}
\det B \begin{bmatrix} \alpha_1, \ldots, \alpha_{p-1}, \alpha_p \\ \beta_1, \ldots, \beta_{p-1}, \beta_p \end{bmatrix} &= \det A \begin{bmatrix} n+1-\alpha_1, \ldots, n+1-\alpha_{p-1}, n+1-\alpha_p \\ m+1-\beta_1, \ldots, m+1-\beta_{p-1}, m+1-\beta_p \end{bmatrix} \\
&= -\det A \begin{bmatrix} n+1-\alpha_1, \ldots, n+1-\alpha_p, n+1-\alpha_{p-1} \\ m+1-\beta_1, \ldots, m+1-\beta_{p-1}, m+1-\beta_p \end{bmatrix} \\
&= \det A \begin{bmatrix} n+1-\alpha_1, \ldots, n+1-\alpha_p, n+1-\alpha_{p-1} \\ m+1-\beta_1, \ldots, m+1-\beta_p, m+1-\beta_{p-1} \end{bmatrix} \\
&\;\;\vdots \\
&= \det A \begin{bmatrix} n+1-\alpha_p, \ldots, n+1-\alpha_2, n+1-\alpha_1 \\ m+1-\beta_p, \ldots, m+1-\beta_2, m+1-\beta_1 \end{bmatrix},
\end{aligned}
$$

which is nonnegative (positive). Therefore A is TN (TP).

□

Proposition 1.2. *[Pin10] If A is an $n \times m$ TN and B is an $m \times r$ TN, then AB is an $n \times r$ TN.*

Proof. Let A and B be TN. Then by Definition 1.3 we have

$$\det A\,[\alpha|\gamma] = \det A \begin{bmatrix} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \gamma_1, \gamma_2, \ldots, \gamma_p \end{bmatrix} \geq 0,$$

and

$$\det B\,[\gamma|\beta] = \det B \begin{bmatrix} \gamma_1, \gamma_2, \ldots, \gamma_p \\ \beta_1, \beta_2, \ldots, \beta_p \end{bmatrix} \geq 0,$$

for all $\alpha \in Q_{p,n}$, $\gamma \in Q_{p,m}$ and $\beta \in Q_{p,r}$, $p = 1, \ldots, \min\{n, r, m\}$.

By the application of the Cauchy-Binet Identity we obtain

$$\begin{aligned} \det AB\,[\alpha|\beta] &= \det AB \begin{bmatrix} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_p \end{bmatrix} \\ &= \sum_{\gamma} \det A \begin{bmatrix} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \gamma_1, \gamma_2, \ldots, \gamma_p \end{bmatrix} \det B \begin{bmatrix} \gamma_1, \gamma_2, \ldots, \gamma_p \\ \beta_1, \beta_2, \ldots, \beta_p \end{bmatrix} \geq 0. \end{aligned}$$

Therefore, AB is TN.

□

Proposition 1.3. *[Pin10] Assume A is an $n \times m$ TN (TP) matrix. The following operations preserve the resulting matrices as TN (TP).*

1. *Multiplying a row (column) by a positive scalar.*
2. *Adding a positive multiple of a row (column) to the previous or the next row (column).*

Proof. 1. Let A be an $n \times m$ TN matrix and B be the matrix obtained from A by multiplying the row (column) i with a positive constant c. For any $\alpha \in Q_{p,n}$ and $\beta \in Q_{p,m}$. If $i \notin \alpha$, then

$$\det B[\alpha|\beta] = \det A[\alpha|\beta] \geq 0.$$

If $i \in \alpha$, then by using Theorem 1.1, we have

$$\det B \begin{bmatrix} \alpha_1, \ldots, i, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_p \end{bmatrix} = c \det A \begin{bmatrix} \alpha_1, \ldots, i, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_p \end{bmatrix} \geq 0.$$

Since all the minors of A are nonnegative, the matrix B is TN.

2. Let B be the matrix that is obtained from the totally nonnegative matrix A by multiplying the row i with a positive constant c and add it to the next row i.e., to the row $i+1$. Let $\alpha \in Q_{p,n}$ and $\beta \in Q_{p,m}$. If $i+1 \notin \alpha$, then

$$\det B[\alpha|\beta] = \det A[\alpha|\beta] \geq 0.$$

Else if $i, i+1 \in \alpha$, then by using Theorem 1.1, we have

$$\begin{aligned}
\det B \begin{bmatrix} \alpha_1, \ldots, i, i+1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_p \end{bmatrix} &= \begin{vmatrix} a_{\alpha_1,\beta_1} & a_{\alpha_1,\beta_2} & \cdots & a_{\alpha_1,\beta_p} \\ \vdots & \vdots & \cdots & \vdots \\ a_{i,\beta_1} & a_{i,\beta_2} & \cdots & a_{i,\beta_p} \\ c\, a_{i,\beta_1} + a_{i+1,\beta_1} & c\, a_{i,\beta_2} + a_{i+1,\beta_2} & \cdots & c\, a_{i,\beta_p} + a_{i+1,\beta_p} \\ a_{i+2,\beta_1} & a_{i+2,\beta_2} & \cdots & a_{i+2,\beta_p} \\ \vdots & \vdots & \cdots & \vdots \\ a_{\alpha_p,\beta_1} & a_{\alpha_p,\beta_2} & \cdots & a_{\alpha_p,\beta_p} \end{vmatrix} \\
&= \det A \begin{bmatrix} \alpha_1, \ldots, i, i+1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_p \end{bmatrix} \\
&\geq 0.
\end{aligned}$$

Otherwise, if $i \notin \alpha$, then by Laplace expansion along the row $i+1$ in α, we have

$$\begin{aligned}
\det B\Big[\alpha|\beta\Big] &= c\ \det A\Big[(\alpha \cup \{i\}) \setminus \{i+1\}|\beta\Big] + \det A\Big[\alpha|\beta\Big] \\
&\geq 0.
\end{aligned}$$

Therefore, B is TN.

□

Definition 1.4. *[Pin10] Let $A \in \mathbb{R}^{n\times m}$, we define the* right shadow *of the submatrix*

$$A \begin{bmatrix} \alpha+1, \alpha+2, \ldots, \alpha+r \\ \beta+1, \beta+2, \ldots, \beta+r \end{bmatrix}$$

as the $(\alpha+r) \times (m-\beta)$ submatrix

$$A \begin{bmatrix} 1, 2, \ldots, \alpha+r \\ \beta+1, \beta+2, \ldots, m \end{bmatrix}$$

and the left shadow *of the submatrix is the $(n-\alpha) \times (\beta+r)$ submatrix*

$$A \begin{bmatrix} \alpha+1, \alpha+2, \ldots, n \\ 1, 2, \ldots, \beta+r \end{bmatrix}$$

Example 1.5. *Let*

$$A = \begin{bmatrix} 5 & 6 & 1 & 4 & 8 \\ 2 & 1 & 0 & 5 & 1 \\ 4 & 3 & 2 & 1 & 0 \\ 5 & 7 & 8 & 10 & 3 \end{bmatrix}.$$

For the submatrix

$$B = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}.$$

The right shadow of the submatrix B is

$$\begin{bmatrix} 6 & 1 & 4 & 8 \\ 1 & 0 & 5 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix},$$

and the left shadow of the submatrix B is

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 2 \\ 5 & 7 & 8 \end{bmatrix}.$$

Proposition 1.4. *[Pin10] If A is an $n \times m$ TN and*

$$\operatorname{rank} A \begin{bmatrix} \alpha+1, \alpha+2, \ldots, \alpha+r \\ \beta+1, \beta+2, \ldots, \beta+r \end{bmatrix} = r-1,$$

then at least one of the following holds. Either the rows $\alpha+1, \alpha+2, \ldots, \alpha+r$ or the columns $\beta+1, \beta+2, \ldots, \beta+r$ of A are linearly dependent, or the right or the left shadow of

$$A \begin{bmatrix} \alpha+1, \alpha+2, \ldots, \alpha+r \\ \beta+1, \beta+2, \ldots, \beta+r \end{bmatrix}$$

has rank $r-1$.

By permuting some rows and columns of a given matrix A and the fact that A is singular if and only if its rank is less than the number or rows and columns of A, we have the following lemma.

Lemma 1.2. *Let A be an $n \times n$ matrix and suppose A has a zero submatrix of size $p \times q$ where $p + q \geq n + 1$. Then* $\det A = 0$.

Theorem 1.5. *[Kar68] Let A be an $n \times n$ nonsingular TN. Then all principal minors of A are positive.*

Proof. We will proceed by induction on the size of the principal submatrices. Firstly, we will prove that the principal minors of order 1 are positive, i.e.,

$$a_{rr} > 0, \forall r \in \{1, \ldots, n\}. \tag{1.5}$$

On the contrary, assume that $a_{rr} = 0$ for some $r \in \{1, \ldots, n\}$. By Proposition 1.4 either the r-th row or r-th column of A is zero, or the right or left shadow of a_{rr} is zero. Since A is nonsingular, i.e., $\det A \neq 0$, we cannot have that the r-th row or r-th column is zero. Thus either the left or right shadow of a_{rr} is zero. Assume without less of generality that its right shadow is zero, so $a_{ij} = 0, \forall i \leq r$ and $\forall j \geq r$. Since the right shadow of a_{rr} is a zero submatrix of size $r \times (n - r + 1)$ such that $r + n - r + 1 = n + 1$, by Lemma 1.2 we have $\det A = 0$ but A is nonsingular, therefore $a_{rr} > 0$. So when the size of the principal minors is 1, these principal minors are positive.

Assume that all principal minors of order $p - 1$ of A are positive. We want to prove that all principal minors of order p are positive. We do this again by contradiction. Let $\alpha \in Q_{p,n}$ be such that $\det A[\alpha] = 0$, whereas $\det A[\alpha'] > 0$, where $\alpha = \alpha' \cup \{\alpha_s\}$ for some $s \in \{1, \ldots, p\}$. Let $B = [b_{ij}]$ be defined as follows

$$b_{ij} = \det A \left[\ \alpha' \bigcup \{i\} | \alpha' \bigcup \{j\}\ \right],$$

for all $i, j \in \alpha'^c$. For $i = j = \alpha_s$, we have

$$\begin{aligned} b_{\alpha_s,\alpha_s} &= \det A \left[\ \alpha' \bigcup \{\alpha_s\} | \alpha' \bigcup \{\alpha_s\}\ \right] \\ &= \det A[\alpha] = 0, \end{aligned} \tag{1.6}$$

But by Theorem 1.3 B is nonsingular TN. The diagonal entries of B must be positive by the induction hypothesis. Therefore,

$$b_{\alpha_s,\alpha_s} > 0,$$

which is a contradiction. Therefore, all principal minors of a nonsingular TN matrix are positive. □

1.3 Examples of Totally Nonnegative Matrices

In the previous section, we define the totally nonnegative matrices as the matrices with all of its minors are nonnegative. In this section, we introduce some fundamental examples of totally positive matrices and totally nonnegative matrices.

Definition 1.5. *[FJ11](Vandermonde matrix)*
For n real distinct positive numbers $0 < x_1 < x_2 < \cdots < x_n$, the Vandermonde

matrix $V(x_1, x_2, \dots, x_n)$ *is defined as follows:*

$$V(x_1, x_2, \dots, x_n) = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}.$$

By using the classical determinant formula for this matrix which is given by

$$\det V(x_1, x_2, \dots, x_n) = \prod_{i>j}(x_i - x_j),$$

we can easily see that $V(x_1, x_2, \dots, x_n)$ is TP.

Example 1.6. *The matrix A is a Vandermonde matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \end{bmatrix},$$

with $x_1 = 1, x_2 = 2, x_3 = 3$ and $x_4 = 5$.

Definition 1.6. *[FJ11](Pascal matrix)*
Consider the $n \times n$ matrix $P_n = [p_{ij}]$ which is defined as follows:

$$p_{ij} = \begin{cases} 1, & \text{for } \ i = 1 \text{ or } j = 1, \\ p_{i-1,j} + p_{i,j-1}, & \text{for } \ 2 \le i, j \le n. \end{cases}$$

P_n is called symmetric Pascal matrix.

We will prove that P_n is TP by showing the existence of a bidiagonal factorization of P_n with special form in Chapter 2.

Example 1.7. *The 4×4 symmetric Pascal matrix is given by*

$$P_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}.$$

Definition 1.7. *[Pin10](Cauchy matrix) The matrix $C = \left[\dfrac{1}{x_i + y_j}\right]$, $1 \le i, j \le n$, is called a* Cauchy matrix.

For $0 < x_1 < x_2 < \cdots < x_n$, $0 < y_1 < y_2 < \cdots < y_n$, C is a totally positive matrix, which can be shown by an induction argument and by using the following determinant formula

$$\det C = \frac{\prod_{l<k}(x_k - x_l)\prod_{l<k}(y_k - y_l)}{\prod_{i,j}(x_i + y_j)}, \ 1 \le l, k \le n.$$

Example 1.8. *The matrix*

$$A = \begin{bmatrix} 1/4 & 1/6 & 1/7 & 1/10 \\ 1/5 & 1/7 & 1/8 & 1/11 \\ 1/7 & 1/9 & 1/10 & 1/13 \\ 1/8 & 1/10 & 1/11 & 1/14 \end{bmatrix},$$

is a Cauchy matrix with $x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 5$ *and* $y_1 = 3, y_2 = 5, y_3 = 6, y_4 = 9$.

Definition 1.8. *[Pin10](Jacobi matrix) A* Jacobi matrix $A = [a_{ij}]$ *is a square tridiagonal matrix. In other words, it is an* $n \times n$ *matrix of the form*

$$A = \begin{bmatrix} a_1 & b_1 & 0 & \dots & 0 & 0 \\ c_1 & a_2 & b_2 & \dots & 0 & 0 \\ 0 & c_2 & a_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \dots & c_{n-1} & a_n \end{bmatrix},$$

where

$$\begin{cases} a_{ij} = 0, & for \ \ |i-j| \ge 2, \\ a_{ii} = a_i, & for \ \ i = 1, \dots, n, \\ a_{i,i+1} = b_i, & for \ \ i = 1, \dots, n-1, \\ a_{i+1,i} = c_i, & for \ \ i = 1, \dots, n-1. \end{cases}$$

Theorem 1.6. *[Pin10] A Jacobi matrix A is TN if and only if all of its off-diagonal elements* $\{b_i\}$, $\{c_i\}$, *and all its contiguous principal minors are nonnegative.*

Example 1.9. *The following matrix A is a Jacobi matrix,*

$$A = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 1 & 4 & 4 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 6 & 10 & 2 \\ 0 & 0 & 0 & 20 & 10 \end{bmatrix}.$$

By application of Theorem 1.6, it is easy to show that A is TN.

Chapter 2

The Cauchon Algorithm

In this chapter, we present the condensed form of the Cauchon algorithm. This algorithm provides an efficient method for checking total nonnegativity and total positivity. One of the applications of this algorithm is to find the bidiagonal factorization of a given nonsingular totally nonnegative or totally positive matrix.

This chapter consists of three sections. In Section 2.1, we present the Cauchon diagram, Cauchon matrix, condensed form of the Cauchon algorithm, our new algorithm the condensed form of the restoration algorithm, and we define the lacunary sequences with respect to a given Cauchon diagram. In Section 2.2, the bidiagonal factorization of a given nonsingular totally nonnegative matrix is introduced. In Section 2.3, the condensed form of the Cauchon algorithm is applied to nonsingular totally nonnegative matrices and a method to find the bidiagonal factorization from the resulting matrices is given.

2.1 Condensed Form of the Cauchon Algorithm and Totally Nonnegative Matrices

In this section, we present the Cauchon diagram, Cauchon matrix, and condensed form of the Cauchon algorithm. We will use this algorithm to check whether a given matrix is totally nonnegative (totally positive).

Definition 2.1. *[LL14] A* Cauchon diagram C *is an* $n \times m$ *grid consisting of* $n \cdot m$ *squares colored black and white, where each black square has the property that either every square to its left (in the same row) or every square above it (in the same column) is black.*

We define an $n \times m$ Cauchon diagram by identifying set of pairs of its black squares, i.e., we fix positions in a Cauchon diagram in the following way:
For a given Cauchon diagram C and $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$, we say that $(i, j) \in C$ if the square in row i and column j is black.

For example, for the Cauchon diagram C as shown in Figure 2.1, we have $(3,2) \notin C$, whereas $(3,1) \in C$.

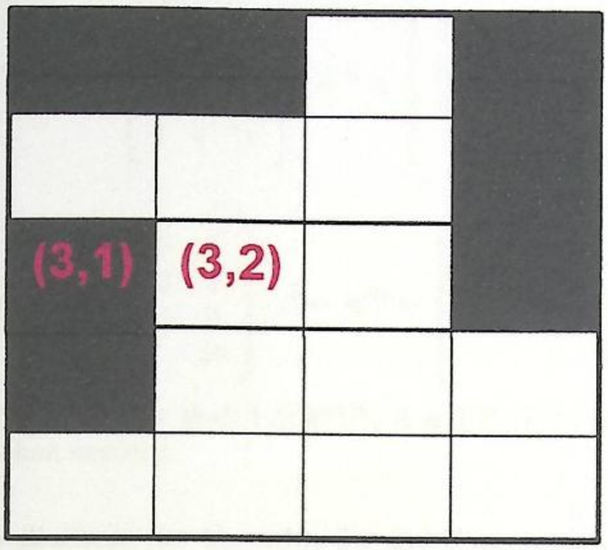

Figure 2.1: An example of a Cauchon diagram.

Definition 2.2. *[LL14] An $n \times m$ matrix $A = [a_{ij}]$ is called a* Cauchon matrix *associated with a given Cauchon diagram C if for all $(i,j), i \in \{1,\ldots,n\}, j \in \{1,\ldots,m\}$, we have $a_{ij} = 0$ if and only if $(i,j) \in C$. Equivalently, $a_{ij} = 0$ if and only if $a_{kj} = 0, \forall k = 1,\ldots,i-1$ or $a_{il} = 0, \forall l = 1,\ldots,j-1$ implies that A is Cauchon matrix.*

Algorithm 2.1. *[AG14](Condensed form of the Cauchon algorithm) Let $A = [a_{ij}] \in \mathbb{R}^{n \times m}$.*

- *Set $A^{(n)} := A$.*
- *For $k = n-1,\ldots,1$ define $A^{(k)} = [a_{ij}^{(k)}] \in \mathbb{R}^{n \times m}$ as follows;*
  For $i = 1,\ldots,k$, $j = 1,\ldots,m-1$,
  set $u_j := \min\left\{h \in \{j+1,\ldots,m\} \mid a_{k+1,h}^{(k+1)} \neq 0\right\}$ (we define that $u_j := \infty$ if this set is empty)

  $$a_{ij}^{(k)} = \begin{cases} a_{ij}^{(k+1)} - \frac{a_{k+1,j}^{(k+1)} a_{i,u_j}^{(k+1)}}{a_{k+1,u_j}^{(k+1)}} & \text{if } u_j < \infty, \\ a_{ij}^{(k+1)} & \text{if } u_j = \infty. \end{cases}$$

  For $i = k+1,\ldots,n, j = 1,\ldots,m$, and $i = 1,\ldots,k, j = m$,

  $$a_{ij}^{(k)} = a_{ij}^{(k+1)}.$$

- *Put $\tilde{A} = A^{(1)}$.*

Example 2.1. *Returning to Example 1.4, by application of the condensed form of the Cauchon algorithm to A we obtain*

$$A^{(4)} = A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}, A^{(3)} = \begin{bmatrix} 3/4 & 3/4 & 1 \\ 1/2 & 1 & 4 \\ 1/4 & 3/4 & 9 \\ 1 & 4 & 16 \end{bmatrix},$$

$$A^{(2)} = \begin{bmatrix} 1/2 & 2/3 & 1 \\ 1/6 & 2/3 & 4 \\ 1/4 & 3/4 & 9 \\ 1 & 4 & 16 \end{bmatrix}, \tilde{A} = A^{(1)} = \begin{bmatrix} 1/3 & 1/2 & 1 \\ 1/6 & 2/3 & 4 \\ 1/4 & 3/4 & 9 \\ 1 & 4 & 16 \end{bmatrix}.$$

Theorem 2.1. *[LL14] Assume that $A \in \mathbb{R}^{n\times m}$, A is TP (TN) if and only if $\tilde{A} > 0$ ($\tilde{A} \geq 0$ and a Cauchon matrix).*

In the following we illustrate the Algorithm 2.1 and Theorem 2.1.

Example 2.2. *Let A be the following matrix.*

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & 4 & 16 \end{bmatrix}.$$

By the application of the condensed form of the Cauchon algorithm on A we have

$$A^{(3)} = A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \\ 1 & 4 & 16 \end{bmatrix}, A^{(2)} = \begin{bmatrix} 1/2 & 1 & 4 \\ 3/4 & 3/4 & 1 \\ 1 & 4 & 16 \end{bmatrix},$$

$$\tilde{A} = A^{(1)} = \begin{bmatrix} -1/2 & -2 & 4 \\ 3/4 & 3/4 & 1 \\ 1 & 4 & 16 \end{bmatrix}.$$

Since $\tilde{A}$ is not entrywise positive, A is not TP.

Example 2.3. *Let A be the following 3×3 Vandermonde matrix with $x_1 = 3, x_2 = 7, x_3 = 11$. By the application of the condensed form of the Cauchon algorithm we will show that A is TP.*

$$A^{(3)} = A = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 7 & 49 \\ 1 & 11 & 121 \end{bmatrix}, A^{(2)} = \begin{bmatrix} 8/11 & 24/11 & 9 \\ 4/11 & 28/11 & 49 \\ 1 & 11 & 121 \end{bmatrix},$$

$$\tilde{A} = A^{(1)} = \begin{bmatrix} 32/77 & 132/77 & 9 \\ 4/11 & 28/11 & 49 \\ 1 & 11 & 121 \end{bmatrix}.$$

Since $\tilde{A} > 0$, A is TP.

Example 2.4. *The following matrix A is a Jacobi matrix. We will use the condensed form of the Cauchon algorithm to show that A is TN.*

$$A = \begin{bmatrix} 6 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 3 & 4 & 2 \\ 0 & 0 & 6 & 12 \end{bmatrix}.$$

By the application of the condensed form of the Cauchon algorithm we have

$$A^{(4)} = A = \begin{bmatrix} 6 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 3 & 4 & 2 \\ 0 & 0 & 6 & 12 \end{bmatrix}, A^{(3)} = \begin{bmatrix} 6 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 3 & 3 & 2 \\ 0 & 0 & 6 & 12 \end{bmatrix},$$

$$A^{(2)} = \begin{bmatrix} 6 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 3 & 3 & 2 \\ 0 & 0 & 6 & 12 \end{bmatrix}, \tilde{A} = A^{(1)} = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 3 & 3 & 2 \\ 0 & 0 & 6 & 12 \end{bmatrix}.$$

Since $\tilde{A}$ is nonnegative and a Cauchon matrix, A is TN.

In [Lau04], an algorithm called restoration algorithm is introduced as the inverse of the Cauchon algorithm. In the following, we present the condensed form of this algorithm.

Algorithm 2.2. *(The condensed form of the restoration algorithm)*
Let $A = [a_{ij}] \in \mathbb{R}^{n \times m}$.

- *Set $A^{(1)} := A$.*
- *For $k = 2, \ldots, n$ define $A^{(k)} = [a_{ij}^{(k)}] \in \mathbb{R}^{n \times m}$ as follows*
  For $i = 1, \ldots, k-1$, For $j = m-1, \ldots, 1$,
  if $j = m-1$,

$$a_{i,m-1}^{(k)} = \begin{cases} a_{i,m-1}^{(k-1)} + \frac{a_{k,m-1}^{(k-1)} a_{i,m}^{(k-1)}}{a_{k,m}^{(k-1)}} & \text{if } a_{k,m} \neq 0, \\ a_{i,m-1}^{(k-1)} & \text{if } a_{k,m} = 0. \end{cases}$$

else, for $j = m-2, \ldots, 1$,
set $u_j := \min\left\{ h \in \{j+1, \ldots, m-1\} \mid a_{k,h}^{(k-1)} \neq 0 \right\}$ *(we set* $u_j := \infty$ *if this set is empty)*

$$a_{ij}^{(k)} = \begin{cases} a_{ij}^{(k-1)} + \frac{a_{k,j}^{(k-1)} a_{i,u_j}^{(k)}}{a_{k,u_j}^{(k-1)}} & \text{if } u_j < \infty, \\ a_{ij}^{(k-1)} & \text{if } u_j = \infty. \end{cases}$$

For $i = k, \ldots, n, j = 1, \ldots, m,$ *and* $i = 1, \ldots, k-1, j = m,$

$$a_{ij}^{(k)} = a_{ij}^{(k-1)}.$$

- *Put* $\bar{A} = A^{(n)}$, $\bar{A}$ *is called the matrix obtained from* A *by the restoration algorithm.*

Example 2.5. *For the following matrix*

$$A = \begin{bmatrix} 1/3 & 1/2 & 1 \\ 1/6 & 2/3 & 4 \\ 1/4 & 3/4 & 9 \\ 1 & 4 & 16 \end{bmatrix},$$

when we apply the condensed form of the restoration algorithm, we have

$$A^{(1)} = A = \begin{bmatrix} 1/3 & 1/2 & 1 \\ 1/6 & 2/3 & 4 \\ 1/4 & 3/4 & 9 \\ 1 & 4 & 16 \end{bmatrix}, A^{(2)} = \begin{bmatrix} 1/2 & 2/3 & 1 \\ 1/6 & 2/3 & 4 \\ 1/4 & 3/4 & 9 \\ 1 & 4 & 16 \end{bmatrix},$$

$$A^{(3)} = \begin{bmatrix} 3/4 & 3/4 & 1 \\ 1/2 & 1 & 4 \\ 1/4 & 3/4 & 9 \\ 1 & 4 & 16 \end{bmatrix}, \bar{A} = A^{(4)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}.$$

Observe that A is a positive matrix and when we apply the condensed form of the restoration algorithm on A we get the matrix $\bar{A}$ which is TP according to Theorem 2.1.

Example 2.6. *For the following matrix*

$$A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 3 & 3 & 2 \\ 0 & 0 & 6 & 12 \end{bmatrix},$$

when we apply the condensed form of the restoration algorithm on A we have

$$A^{(1)} = A = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 3 & 3 & 2 \\ 0 & 0 & 6 & 12 \end{bmatrix}, A^{(2)} = \begin{bmatrix} 6 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 3 & 3 & 2 \\ 0 & 0 & 6 & 12 \end{bmatrix},$$

$$A^{(3)} = \begin{bmatrix} 6 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 3 & 3 & 2 \\ 0 & 0 & 6 & 12 \end{bmatrix}, \bar{A} = A^{(4)} = \begin{bmatrix} 6 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 3 & 4 & 2 \\ 0 & 0 & 6 & 12 \end{bmatrix}.$$

Observe that A is a nonnegative Cauchon matrix and when we apply the condensed form of the restoration algorithm on A we get the matrix $\bar{A}$ which is TN according to Theorem 2.1.

We can represent each entry of the matrix $\tilde{A}$ which is obtained by the application of the condensed form of the Cauchon Algorithm to a given TP matrix A as a ratio of two contiguous minors.

Theorem 2.2. *[Adm16] Let $A \in \mathbb{R}^{n \times n}$ be TP. Then the entries $\tilde{a}_{kj}$ of the matrix $\tilde{A}$ obtained from A by the condensed form of the Cauchon algorithm can be represented as*

$$\tilde{a}_{kj} = \frac{\det A\left[k, \ldots, k+w | j, \ldots, j+w\right]}{\det A\left[k+1, \ldots, k+w | j+1, \ldots, j+w\right]}, \quad k, j = 1, \ldots, n, \tag{2.1}$$

where $w = \min\{n-k, n-j\}$.

In the following, we introduce the definition of a lacunary sequence associated with Cauchon diagrams.

Definition 2.3. *[LL14] Let C be a Cauchon diagram. We say that a sequence*

$$\gamma = \{(\alpha_k, \beta_k), k = 0, 1, 2, \ldots, p\} \tag{2.2}$$

which is strictly increasing in both arguments is a lacunary sequence with respect to *C if the following conditions hold:*

1. *$(\alpha_k, \beta_k) \notin C, k = 1, 2, \ldots, p$.*
2. *$(\alpha, \beta) \in C$ for $\alpha_p < \alpha \leq n$ and $\beta_p < \beta \leq m$.*
3. *Let $s \in \{0, 1, \ldots, p-1\}$. Then $(\alpha, \beta) \in C$ if*

(a) *either for all* (α,β), $\alpha_s < \alpha < \alpha_{s+1}$ *and* $\beta_s < \beta$,
or for all (α,β), $\alpha_s < \alpha < \alpha_{s+1}$ *and* $\beta_0 \leq \beta < \beta_{s+1}$,
and

(b) *either for all* (α,β), $\alpha_s < \alpha$ *and* $\beta_s < \beta < \beta_{s+1}$,
or for all (α,β), $\alpha < \alpha_{s+1}$ *and* $\beta_s < \beta < \beta_{s+1}$.

Example 2.7. *In Figure 2.1, the sequence* $\{(1,1),(2,3),(4,4)\}$ *and* $\{(2,1),(5,2)\}$ *are a lacunary sequence with respect to* C *while the sequence* $\{(1,2),(3,3)\}$ *is not.*

Proposition 2.1. *[AMAFG18] Let* A *be an* $n \times m$ *matrix such that* $\tilde{A}$ *is a Cauchon matrix and let* $\gamma = \{(\alpha_0,\beta_0),(\alpha_1,\beta_1),\ldots,(\alpha_p,\beta_p)\}$ *be a lacunary sequence with respect to a given Cauchon diagram* C *associated with* $\tilde{A}$. *Then*

$$\det A[\alpha_0,\alpha_1,\ldots,\alpha_p|\beta_0,\beta_1,\ldots,\beta_p] = \tilde{a}_{\alpha_0,\beta_0}\,\tilde{a}_{\alpha_1,\beta_1}\cdots\tilde{a}_{\alpha_p,\beta_p}, \tag{2.3}$$

holds for all lacunary sequences γ *given by Definition 2.3.*

We can determine if a given TN matrix A is nonsingular or not by the value of the diagonal entries of $\tilde{A}$.

Proposition 2.2. *[AG13] Let* A *be an* $n \times n$ *TN matrix. Then* A *is nonsingular if and only if* $\tilde{a}_{ii} > 0, i = 1,2,\ldots,n$.

Proof. Let A be nonsingular TN and C be the Cauchon diagram associated with $\tilde{A}$. By Theorem 1.5, all the principal minors of A are positive and when we apply the condensed form of the Cauchon algorithm, we have

$$\tilde{a}_{nn} = a_{nn} > 0.$$

Assume there exist $1 \leq k < n$ such that $\tilde{a}_{kk} = 0$ and $\tilde{a}_{ii} > 0, i = k+1,\ldots,n$. Consider the lacunary sequence (with respect to C) $\{(i,i), i = k,k+1,\ldots,n\}$. Then by Proposition 2.1 it follows that if $(i,i) \in C$ then

$$\det A[k,\ldots,n] = \tilde{a}_{kk}\,\tilde{a}_{k+1,k+1}\cdots\tilde{a}_{nn} = 0,$$

contradicting Theorem 1.5. Conversely, assume that $\tilde{a}_{ii} > 0, i = 1,2,\ldots,n$. Then the sequence $\{(i,i), i = 1,2,\ldots,n\}$ is a lacunary sequence with respect to C and by Proposition 2.1 it follows that

$$\det A[1,2,\ldots,n] = \tilde{a}_{11}\,\tilde{a}_{22}\cdots\tilde{a}_{nn} > 0.$$

So, A is a nonsingular matrix. □

2.2 Bidiagonal Factorization of Totally Nonnegative Matrices

Factorization of matrices is one of the most important topics in matrix theory and it plays a central role in many related applied areas such as numerical analysis and statistics. In this section, our main focus will be on bidiagonal factorization for nonsingular totally nonnegative matrices which allows one to obtain algorithms with high relative accuracy for the computation of singular values, eigenvalues, and inverses of totally nonnegative matrices [Koe05], [Koe07].

Definition 2.4. *[FJ11](Elementary Bidiagonal Matrices)*
For any positive integer n and complex numbers s, t, we let

$$L_i(s) := I_n + sE_{i,i-1} \text{ and } U_j(t) := I_n + tE_{j-1,j}, \; 2 \le i, j \le n,$$

matrices of the form $L_i(s)$ or $U_j(t)$ above are called elementary bidiagonal matrices, *and the class of elementary bidiagonal matrices will be denoted by EB.*

The EB matrices $L_i(s)$ and $U_j(t)$ satisfy the following relations [FJ11]:

1. $L_i(s)L_j(t) = L_j(t)L_i(s)$, if $|i - j| > 1$,
2. $L_i(s)L_i(t) = L_i(s + t)$, for all i,
3. $L_{i+1}(r)L_i(s)L_{i+1}(t) = L_i(r')L_{i+1}(s')L_i(t')$, for each i,
4. $L_i(s)U_j(t) = U_j(t)L_i(s)$, if $i \neq j$.

In addition, relations from 1. to 3. are true for the upper EB matrices U_j.

Example 2.8. *The following two matrices are lower elementary bidiagonal matrices with $n = 3$*

$$L_2(-2) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; L_3(0.3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.3 & 1 \end{bmatrix}.$$

Example 2.9. *The following two matrices are upper elementary bidiagonal matrices with $n = 3$*

$$U_2(25) = \begin{bmatrix} 1 & 25 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; U_3(-6) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}.$$

Definition 2.5. *[FJ11](Generalized Elementary Bidiagonal Matrices)*
Matrices of the form

$$D + sE_{i,i-1} \text{ or } D + tE_{j-1,j}, \; 2 \le i, j \le n,$$

where D is an $n \times n$ diagonal matrix, n is a positive integer, and s, t are complex numbers are called generalized elementary bidiagonal matrices, *and the class will be denoted by GEB.*

It is easy to see that

$$\left(L_i(s)\right)^{-1} = L_i(-s).$$

It is obvious that any GEB matrix is TN whenever D is nonnegative and $s, t \geq 0$, and any EB matrix is TN whenever $s, t \geq 0$, see, e.g., [FJ11].

Theorem 2.3. *[Loe55] An $n \times n$ nonsingular TN(TP) matrix can be written as*

$$\begin{aligned}&\left(L_n(l_k)\ L_{n-1}(l_{k-1})\ \dots\ L_2(l_{k-n+2})\right)\left(L_n(l_{k-n+1})\dots L_3(l_{k-2n+4})\right)\dots\left(L_n(l_1)\right)\cdot\\ &\quad D\cdot\left(U_n(u_1)\right)\left(U_{n-1}(u_2)\ U_n(u_3)\right)\ \dots\ \left(U_2(u_{k-n+2})\dots U_{n-1}(u_{k-1})\ U_n(u_k)\right), \quad (2.4)\end{aligned}$$

where $k := \begin{pmatrix} n \\ 2 \end{pmatrix}, l_i, u_j \geq 0\ (l_i, u_j > 0)$ for all $i, j \in \{1, 2, \dots, k\}$, and $D =$ $\operatorname{diag}(d_1, d_2, \dots, d_n)$ is a diagonal matrix with all $d_i > 0$.

If we use the relation 1. for the EB matrices in Theorem 2.3, we can write (2.4) in another form as we present in the next theorem, see, e.g., [Koe05], [FJ11].

Theorem 2.4. *An $n \times n$ nonsingular $TN(TP)$ matrix can be written as*

$$A = L^{(1)} \dots L^{(n-1)} \cdot D \cdot U^{(n-1)} \dots U^{(1)}, \tag{2.5}$$

where $D = \operatorname{diag}(d_1, d_2, \dots, d_n)$, and

$$\begin{aligned} L^{(r)} &= L_{n-r+1}(l_1^{(r)})L_{n-r+2}(l_2^{(r)})\dots L_n(l_r^{(r)}), \; r = 1, \dots, n-1 \\ &= \begin{array}{r} \\ \\ n\text{-}r\text{+}1 \leftarrow \\ \\ \\ \\ \end{array} \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & l_1^{(r)} & 1 & & & \\ & & l_2^{(r)} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & l_r^{(r)} & 1 \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} U^{(r)} &= U_n(u_1^{(r)})U_{n-1}(u_2^{(r)})\ldots U_{r+1}(u_r^{(r)}),\ r=1,\ldots,n-1 \\ &= \begin{array}{r} \\ n\text{-}r \leftarrow \\ \\ \\ \\ \\ \end{array} \left[\begin{array}{cccccc} 1 & & & & & \\ & \ddots & u_r^{(r)} & & & \\ & & \ddots & \ddots & & \\ & & & 1 & u_2^{(r)} & \\ & & & & 1 & u_1^{(r)} \\ & & & & & 1 \end{array}\right]. \end{aligned}$$

For example, the symmetric Pascal matrix P_n can be written as [FJ11]

$$P_n = L_n(1)\cdots L_2(1)\begin{bmatrix} 1 & 0 \\ 0 & P_{n-1} \end{bmatrix} U_2(1)\cdots U_n(1).$$

By the induction, P_n has the factorization (2.4) in which the u_i's and l_i's involved are all equal to one. Consequently, the symmetric Pascal matrix P_n is TN for $n \geq 1$, since it is a product of TN matrices.

Theorem 2.5. *[Cry76] Any $n \times n$ TN matrix A can be written as*

$$A = \prod_{i=1}^{M} L^{(i)} \prod_{j=1}^{N} U^{(j)},$$

where $N, M < n$ and the matrices $L^{(i)}$ and $U^{(j)}$ are not required to be nonsingular, nor are they assumed to have constant main diagonal entries, as they are GEB matrices.

2.3 Cauchon Algorithm and Bidiagonal Factorization

In this section, we present how we can find the bidiagonal factorization of a given nonsingular TN matrix $A = [a_{ij}]$ by application of the condensed form of the Cauchon algorithm. The numbers l_i and u_j given in Theorem 2.3 can be obtained by running the condensed form of the Cauchon algorithm on $G = (A^{\#})^T$ and get in this way a bidiagonalization of A, where the entries of $G = [g_{ij}]$ are

$$g_{ij} = a_{n+1-j,n+1-i}, 1 \leq i, j \leq n.$$

Algorithm 2.3. *[Adm16](Bidiagonal Factorization)*
Let A be a TP matrix, we can find l_i's and u_j's given in Theorem 2.3 as follows:

- *Run the condensed form of the Cauchon algorithm on $G = (A^{\#})^T$.*

- $l_k = \frac{\tilde{g}_{n1}}{\tilde{g}_{n2}}$, $l_{k-1} = \frac{\tilde{g}_{n2}}{\tilde{g}_{n3}}$, $\dots$, $l_{k-n+2} = \frac{\tilde{g}_{n,n-1}}{\tilde{g}_{n,n}}$,
  $l_{k-n+1} = \frac{\tilde{g}_{n-1,1}}{\tilde{g}_{n-1,2}}$, $l_{k-n} = \frac{\tilde{g}_{n-1,2}}{\tilde{g}_{n-1,3}}$, $\dots$, $l_{k-2n+4} = \frac{\tilde{g}_{n-1,n-2}}{\tilde{g}_{n-1,n-1}}$, $\dots$,
  $l_1 = \frac{\tilde{g}_{21}}{\tilde{g}_{22}}$.
- $d_{ii} = \tilde{g}_{n+1-i,n+1-i}, i = 1, 2, \dots, n.$
- $u_1 = \frac{\tilde{g}_{12}}{\tilde{g}_{22}}$,
  $u_2 = \frac{\tilde{g}_{23}}{\tilde{g}_{33}}$, $u_3 = \frac{\tilde{g}_{13}}{\tilde{g}_{23}}$, $\dots$,
  $u_{k-n+2} = \frac{\tilde{g}_{n-1,n}}{\tilde{g}_{nn}}$, $\dots$, $u_{k-1} = \frac{\tilde{g}_{2n}}{\tilde{g}_{3n}}$, $u_k = \frac{\tilde{g}_{1n}}{\tilde{g}_{2n}}$.

Algorithm 2.4. *(Bidiagonal Factorization)*
Let A be a TP matrix, we can find $l_i^{(r)}$'s and $u_j^{(r)}$'s given in Theorem 2.4 as follows:

- *Run the condensed form of the Cauchon algorithm on $G = (A^{\#})^T$.*
- $l_1^{(1)} = \frac{\tilde{g}_{n1}}{\tilde{g}_{n2}}$,
  $l_1^{(2)} = \frac{\tilde{g}_{n2}}{\tilde{g}_{n3}}$, $l_2^{(2)} = \frac{\tilde{g}_{n-1,1}}{\tilde{g}_{n-1,2}}$, $\dots$,
  $l_1^{(r)} = \frac{\tilde{g}_{nr}}{\tilde{g}_{n,r+1}}$, $l_2^{(r)} = \frac{\tilde{g}_{n-1,r-1}}{\tilde{g}_{n-1,r}}$, $\dots$, $l_r^{(r)} = \frac{\tilde{g}_{n-r+1,1}}{\tilde{g}_{n-r+1,2}}$.
- $d_{ii} = \tilde{g}_{n+1-i,n+1-i}, i = 1, 2, \dots, n.$
- $u_1^{(1)} = \frac{\tilde{g}_{1n}}{\tilde{g}_{2n}}$,
  $u_1^{(2)} = \frac{\tilde{g}_{1,n-1}}{\tilde{g}_{2,n-1}}$,$u_2^{(2)} = \frac{\tilde{g}_{2n}}{\tilde{g}_{3n}}$, $\dots$,
  $u_1^{(r)} = \frac{\tilde{g}_{1,n-r+1}}{\tilde{g}_{2,n-r+1}}$, $u_2^{(r)} = \frac{\tilde{g}_{2,n-r+2}}{\tilde{g}_{3,n-r+2}}$,$\dots$, $u_r^{(r)} = \frac{\tilde{g}_{rn}}{\tilde{g}_{r+1,n}}$.

Example 2.10. *Let A be the following 4×4 Vandermonde matrix with $x_1 = 2, x_2 = 4, x_3 = 6$ and $x_4 = 8$:*

$$A = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \\ 1 & 6 & 36 & 216 \\ 1 & 8 & 64 & 512 \end{bmatrix}.$$

Firstly, we find the matrix G as follows:

$$G = (A^{\#})^T = \begin{bmatrix} 512 & 216 & 64 & 8 \\ 64 & 36 & 16 & 4 \\ 8 & 6 & 4 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

By the application of the condensed form of the Cauchon algorithm on G we get:

$$G^{(4)} = G = \begin{bmatrix} 512 & 216 & 64 & 8 \\ 64 & 36 & 16 & 4 \\ 8 & 6 & 4 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}, G^{(3)} = \begin{bmatrix} 296 & 152 & 56 & 8 \\ 28 & 20 & 12 & 4 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$G^{(2)} = \begin{bmatrix} 144 & 96 & 48 & 8 \\ 8 & 8 & 8 & 4 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \tilde{G} = G^{(1)} = \begin{bmatrix} 48 & 48 & 32 & 8 \\ 8 & 8 & 8 & 4 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The number k of l_i in (2.4) is $k = \begin{pmatrix} n \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6$, we find all of l_i, $i = 1, 2, \ldots, 6$ as follows:

$$l_6 = \frac{\tilde{g}_{41}}{\tilde{g}_{42}} = \frac{1}{1} = 1,\ l_5 = \frac{\tilde{g}_{42}}{\tilde{g}_{43}} = \frac{1}{1} = 1,\ l_4 = \frac{\tilde{g}_{43}}{\tilde{g}_{44}} = \frac{1}{1} = 1,$$

$$l_3 = \frac{\tilde{g}_{31}}{\tilde{g}_{32}} = \frac{2}{2} = 1,\ l_2 = \frac{\tilde{g}_{32}}{\tilde{g}_{33}} = \frac{2}{2} = 1,$$

$$l_1 = \frac{\tilde{g}_{21}}{\tilde{g}_{22}} = \frac{8}{8} = 1.$$

Also, the values of the entries on the diagonal matrix D are

$$d_{11} = \tilde{g}_{44} = 1,\ d_{22} = \tilde{g}_{33} = 2,\ d_{33} = \tilde{g}_{22} = 8,\ d_{44} = \tilde{g}_{11} = 48,$$

and the values of u_j, $j = 1, 2, \ldots, 6$ are

$$u_1 = \frac{\tilde{g}_{12}}{\tilde{g}_{22}} = \frac{48}{8} = 6,$$

$$u_2 = \frac{\tilde{g}_{23}}{\tilde{g}_{33}} = \frac{8}{2} = 4,\ u_3 = \frac{\tilde{g}_{13}}{\tilde{g}_{23}} = \frac{32}{8} = 4,$$

$$u_4 = \frac{\tilde{g}_{34}}{\tilde{g}_{44}} = \frac{2}{1} = 2,\ u_5 = \frac{\tilde{g}_{24}}{\tilde{g}_{34}} = \frac{4}{2} = 2,\ u_6 = \frac{\tilde{g}_{14}}{\tilde{g}_{24}} = \frac{8}{4} = 2.$$

So, the bidiagonal factorization of A is

$$A = \left(L_4(l_6)\ L_3(l_5)\ L_2(l_4)\right)\ \left(L_4(l_3)\ L_3(l_2)\right)\ \left(L_4(l_1)\right) \cdot D \\ \cdot \left(U_4(u_1)\right)\ \left(U_3(u_2)\ U_4(u_3)\right)\ \left(U_2(u_4)\ U_3(u_5)\ U_4(u_6)\right),$$

or

$$A=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&8&0\\0&0&0&48\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&6\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&4&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&4\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&2&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&2&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&4\\0&0&0&1\end{bmatrix},$$

$$A=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&8&0\\0&0&0&48\end{bmatrix}$$

$$\begin{bmatrix}1&2&0&0\\0&1&4&0\\0&0&1&6\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&2&0\\0&0&1&4\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&4\\0&0&0&1\end{bmatrix}.$$

Ramark 2.1. *[AG14] For any nonsingular TN matrix A, we can approximate A by a TP matrix A_ϵ as follows: Let A be a nonsingular TN. After the application of the condensed form of the Cauchon algorithm, we get $\tilde{A}$. Then by Theorem 2.1 and Proposition 2.2 $\tilde{A}$ is a nonnegative Cauchon matrix with positive diagonal entries. For a given zero entry in the upper part in $\tilde{A}$, all entries in the same column above it are zero. If the zero entry in the lower part, then all entries in the same row to the left of it are zero. We replace the zero entries from the bottom to the top in the upper part and from the right to the left in the lower part by increasing integer powers of a positive number ϵ. We call this matrix by $\tilde{A}_\epsilon$. We apply the condensed form of the Restoration Algorithm to $\tilde{A}_\epsilon$ and obtain by Theorem 2.1 the TP matrix A_ϵ since all entries in $\tilde{A}_\epsilon$ are positive. Since $\tilde{A}_\epsilon$ tends to $\tilde{A}$ as ϵ tend to 0, A_ϵ tends to A. So we can approximate the given nonsingular TN matrix A by the TP matrix A_ϵ.*

Example 2.11. *For the following nonsingular TN matrix*

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 7 & 6 & 4 & 1 \\ 4 & 4 & 4 & 1 \\ 3 & 3 & 3 & 3 \end{bmatrix},$$

when we apply the condensed form of the Cauchon algorithm for A, we get

$$\tilde{A} = \begin{bmatrix} 1/2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 3 & 3 & 3 & 3 \end{bmatrix}.$$

We can approximate A by a TP matrix A_ϵ. In the matrix $\tilde{A}$ we replace all zero entries in each row from the right to the left in the lower part and in each column from the bottom to the top in the upper part by increasing integer powers of a positive number ϵ, so we have

$$\tilde{A}_\epsilon = \begin{bmatrix} 1/2 & 1 & \epsilon^2 & \epsilon \\ 1 & 2 & \epsilon & 1 \\ \epsilon^2 & \epsilon & 3 & 1 \\ 3 & 3 & 3 & 3 \end{bmatrix}.$$

By application the condensed form of the restoration algorithm to $\tilde{A}_\epsilon$, we get

$$A_\epsilon = \begin{bmatrix} 2 + 10\epsilon + 7\epsilon^2 + 5/3\epsilon^3 + 2/3\epsilon^4 & 1 + 7\epsilon + 3\epsilon^2 + 2/3\epsilon^3 & 4\epsilon + 2\epsilon^2 & \epsilon \\ 7 + 4\epsilon + 4/3\epsilon^2 + 1/3\epsilon^3 & 6 + 2\epsilon + 1/3\epsilon^2 & 4 + \epsilon & 1 \\ 4 + \epsilon + \epsilon^2 & 4 + \epsilon & 4 & 1 \\ 3 & 3 & 3 & 3 \end{bmatrix},$$

which is a TP matrix since all entries in $\tilde{A}_\epsilon$ are positive. As ϵ tend to 0, $\tilde{A}_\epsilon$ tends to $\tilde{A}$, therefore A_ϵ tends to A. So we approximate the nonsingular TN matrix A by the TP matrix A_ϵ.

To extend Algorithm 2.3 to the nonsingular TN case, we approximate the given nonsingular TN matrix G by the TP matrix G_ϵ as described in the above remark. After cancellation of common powers of ϵ, we obtain that the denominators do not contain ϵ. Letting ϵ tend to 0, the extension of Algorithm 2.3 to the nonsingular TN case follows. So we set $\frac{0}{0} := 0$. Therefore, all quantities in Algorithm 2.3 are defined since in the upper part of $\tilde{G}$ if one entry is zero then the entry above it is zero, also if one entry is zero in the lower part then the entry to the left of it is zero.

Example 2.12. *Return to Example 2.4. We can find the bidiagonal factorization of* A.

$$A = \begin{bmatrix} 6 & 2 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 3 & 4 & 2 \\ 0 & 0 & 6 & 12 \end{bmatrix}.$$

A is a nonsingular TN matrix, by application Algorithm 2.4 to

$$G = (A^{\#})^T = \begin{bmatrix} 12 & 2 & 0 & 0 \\ 6 & 4 & 1 & 0 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 2 & 6 \end{bmatrix},$$

we get

$$G^{(4)} = G = \begin{bmatrix} 12 & 2 & 0 & 0 \\ 6 & 4 & 1 & 0 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 2 & 6 \end{bmatrix}, G^{(3)} = \begin{bmatrix} 12 & 2 & 0 & 0 \\ 6 & 4 & 1 & 0 \\ 0 & 3 & 4/3 & 2 \\ 0 & 0 & 2 & 6 \end{bmatrix},$$

$$G^{(2)} = \begin{bmatrix} 12 & 2 & 0 & 0 \\ 6 & 7/4 & 1 & 0 \\ 0 & 3 & 4/3 & 2 \\ 0 & 0 & 2 & 6 \end{bmatrix}, \tilde{G} = G^{(1)} = \begin{bmatrix} 36/7 & 2 & 0 & 0 \\ 6 & 7/4 & 1 & 0 \\ 0 & 3 & 4/3 & 2 \\ 0 & 0 & 2 & 6 \end{bmatrix}.$$

So, the bidiagonal factorization of A is

$$
\begin{aligned}
A &= L^{(1)}L^{(2)}L^{(3)} \cdot D \cdot U^{(3)}U^{(2)}U^{(1)} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 1 & 0 & 0 \\ 0 & 9/4 & 1 & 0 \\ 0 & 0 & 24/7 & 1 \end{bmatrix}
\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 4/3 & 0 & 0 \\ 0 & 0 & 7/4 & 0 \\ 0 & 0 & 0 & 36/7 \end{bmatrix} \\
&\quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 8/7 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 1 & 0 & 0 \\ 0 & 9/4 & 1 & 0 \\ 0 & 0 & 24/7 & 1 \end{bmatrix}
\begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 4/3 & 0 & 0 \\ 0 & 0 & 7/4 & 0 \\ 0 & 0 & 0 & 36/7 \end{bmatrix} \\
&\quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 8/7 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

Chapter 3

Elementary Operations and Cauchon Algorithm

In the previous chapters, we present the operations that preserve total nonnegativity and total positivity. Also, we apply the condensed form of the Cauchon algorithm on totally nonnegative and totally positive matrices and we introduce a method to calculate the bidiagonal factorization of nonsingular totally nonnegative matrices. In this chapter, we present how the elementary operations applied on A affect on $\tilde{A}$ and the bidiagonal factorization of A.

This chapter consists of two sections. In Section 3.1 we have a new results, we present how the entries of $\tilde{A}$ will be changed after performing the elementary operations on A. In Section 3.2, we introduce how the entries of the bidiagonal factorization of A will be changed after performing the elementary operations on A.

3.1 Elementary Operations and the Cauchon Algorithm

In this section, we perform on the matrix A the following elementary operations:

1. multiplying a row (column) by a positive scalar;
2. adding a positive multiple of a row (column) to the next row (column).

Each of these elementary operations preserves the total nonnegativity, see Proposition 1.3. In the following we present how the entries of $\tilde{A}$ will be changed after performing the above elementary operations on a TP matrix A.

Multiplying a row (column) by a positive scalar.
Let $A = [a_{kj}]$ be an $n \times n$ TP matrix and $B = [b_{kj}]$ be the matrix obtained from A by multiplying the row i by a positive scalar x. Then the entries of B are given as

follows:

$$b_{kj} = \begin{cases} x\, a_{kj}, & \text{for } k = i, \\ a_{kj}, & \text{for } k \neq i, \end{cases} \tag{3.1}$$

$k, j = 1, 2, \ldots, n$, and $x > 0$.

This elementary operation preserves the total positivity if A is a given TP matrix as we observe in Proposition 1.3. The following theorem displays how to compute $\tilde{B}$ which we get by running the condensed form of the Cauchon algorithm on B in terms of the entries of the matrix $\tilde{A}$.

Theorem 3.1. *Let A be an $n \times n$ TP matrix, $x > 0$, let $\tilde{A}$ be the matrix obtained from A by running the condensed form of the Cauchon algorithm on A, and let B defined as in* (3.1). *When we apply the condensed form of the Cauchon algorithm on B, we get the matrix $\tilde{B}$ with entries given as follows*

$$\tilde{b}_{kj} = \begin{cases} x\, \tilde{a}_{kj}, & \textit{for } k = i, \\ \tilde{a}_{kj}, & \textit{for } k \neq i, \end{cases}$$

$k, j = 1, 2, \ldots, n$, and $x > 0$. The matrix $\tilde{B}$ can be computed from $\tilde{A}$ in exactly n arithmetic operations without performing any subtractions.

Proof. Let $x > 0$ and A be a TP matrix. When we multiply the row i in the matrix A by x we get the matrix B which is defined as follows:

$$B = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1,n-1} & a_{1n} \\ \vdots & \vdots & & \vdots & \vdots \\ x\, a_{i1} & x\, a_{i2} & \ldots & x\, a_{i,n-1} & x\, a_{in} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{n.n-1} & a_{nn} \end{bmatrix}.$$

The matrix B is TP by Proposition 1.3. The entries of the matrix $\tilde{B}$ obtained from B by the condensed form of the Cauchon algorithm can be represented as

$$\tilde{b}_{kj} \;=\; \frac{\det B[k, \ldots, k+w | j, \ldots, j+w]}{\det B[k+1, \ldots, k+w | j+1, \ldots, j+w]}, \quad w = \min\{n-k, n-j\}.$$

To find all entries of $\tilde{B}$ in terms of the entries of $\tilde{A}$, we have three cases:
Case 1: $k < i$

$$\begin{aligned} \tilde{b}_{kj} &= \frac{\det B[k, \ldots, k+w | j, \ldots, j+w]}{\det B[k+1, \ldots, k+w | j+1, \ldots, j+w]} \\ &= \frac{\det A[k, \ldots, k+w | j, \ldots, j+w]}{\det A[k+1, \ldots, k+w | j+1, \ldots, j+w]} \\ &= \tilde{a}_{kj}, \end{aligned}$$

since the row i will be involved in the denominator and numerator of $\tilde{b}_{kj}$, or will not be involved.

Case 2: $k > i$

$$\begin{aligned}\tilde{b}_{kj} &= \frac{\det B[k,\ldots,k+w|j,\ldots,j+w]}{\det B[k+1,\ldots,k+w|j+1,\ldots,j+w]}\\ &= \frac{\det A[k,\ldots,k+w|j,\ldots,j+w]}{\det A[k+1,\ldots,k+w|j+1,\ldots,j+w]}\\ &= \tilde{a}_{kj},\end{aligned}$$

since the row i will not be involved in the denominator and numerator of $\tilde{b}_{kj}$.

Case 3: $k = i$

$$\begin{aligned}\tilde{b}_{kj} &= \frac{\det B[k,\ldots,k+w|j,\ldots,j+w]}{\det B[k+1,\ldots,k+w|j+1,\ldots,j+w]}\\ &= \frac{x\det A[k,\ldots,k+w|j,\ldots,j+w]}{\det A[k+1,\ldots,k+w|j+1,\ldots,j+w]}\\ &= x\,\tilde{a}_{kj}. \qquad (3.2)\end{aligned}$$

The expression in (3.2) involves no subtractions and requires n arithmetic operations, for $j = 1, 2, \ldots, n$. □

Similarly, when we multiply a column i in the matrix A by a positive scalar x and apply the condensed form of the Cauchon algorithm on the resulting matrix B. Then the entries of $\tilde{B}$ are given by:

$$\tilde{b}_{kj} = \begin{cases} x\,\tilde{a}_{kj}, & \text{for } j = i,\\ \tilde{a}_{kj}, & \text{for } j \neq i,\end{cases}$$

$k, j = 1, 2, \ldots, n.$

Example 3.1. *Let A be the following matrix*

$$A = \begin{bmatrix} 1 & 2 & 4\\ 1 & 3 & 9\\ 1 & 4 & 16\end{bmatrix}.$$

Then by the application the condensed form of the Cauchon algorithm to A we get

$$\tilde{A} = \begin{bmatrix} 1/6 & 2/3 & 4\\ 1/4 & 3/4 & 9\\ 1 & 4 & 16\end{bmatrix}.$$

Hence A is TP since all entries of $\tilde{A}$ are positive. Let B be the matrix obtained from A by multiplying the second row of A by 4. By application of Theorem 3.1, we have

$$\tilde{B} = \begin{bmatrix} 1/6 & 2/3 & 4 \\ 1 & 3 & 36 \\ 1 & 4 & 16 \end{bmatrix}.$$

Adding a positive multiple of a row to the next row.

Let A be an $n \times n$ matrix, we define the matrix $B = [b_{kj}]$ with entries given as follows:

$$b_{kj} = \begin{cases} \frac{1}{y}\, a_{kj}, & \text{for } \; k = i, \\ x\, a_{k-1,j} + y\, a_{kj}, & \text{for } \; k = i+1, \\ a_{kj}, & \text{for } \; k \neq i \text{ and } k \neq i+1, \end{cases} \tag{3.3}$$

$j = 1, 2, \ldots, n$ and $x, y > 0$. That is, B is obtained by multiplying the ith row in the matrix A by a positive scalar x and add it to the next row multiplied by y, and multiply row i by $1/y$. We want to find the entries of the matrix $\tilde{B}$ which we get by running the condensed form of the Cauchon algorithm on B in terms of the entries of the matrix $\tilde{A}$.

Theorem 3.2. *Let A be TP, $x, y > 0$, let $\tilde{A}$ be the matrix obtained by running the condensed form of the Cauchon algorithm on A, and let B be defined as in (3.3). When we apply the condensed form of the Cauchon algorithm on B, we get the matrix $\tilde{B}$ with entries as follows:*

$$\tilde{b}_{kj} = \begin{cases} \dfrac{\tilde{a}_{kj}}{y + x\left(\sum_{m=j+1}^{n} \frac{\tilde{a}_{km}}{\tilde{a}_{k+1,m}}\right)}, & \textit{for } \; k = i, 1 \leq j < n, \\ \frac{1}{y}\, \tilde{a}_{kj}, & \textit{for } \; k = i, j = n, \\ y\, \tilde{a}_{kj} + x\, \left(\tilde{a}_{k-1,j} + \tilde{a}_{kj}\left(\sum_{m=j+1}^{n} \frac{\tilde{a}_{k-1,m}}{\tilde{a}_{km}}\right)\right), & \textit{for } \; k = i+1, 1 \leq j < n, \\ y\, \tilde{a}_{kj} + x\, \tilde{a}_{k-1,j}, & \textit{for } \; k = i+1, j = n, \\ \tilde{a}_{kj}, & \textit{for } \; k \neq i \textit{ and } k \neq i+1, 1 \leq j \leq n. \end{cases} \tag{3.4}$$

Proof. The TP matrix B can be written as follows:

$$B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2,n-1} & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ 1/y \; a_{i1} & 1/y \; a_{i2} & \dots & 1/y \; a_{i,n-1} & 1/y \; a_{in} \\ ya_{i+1,1} + xa_{i,1} & ya_{i+1,2} + xa_{i,2} & \dots & ya_{i+1,n-1} + xa_{i,n-1} & ya_{i+1,n} + xa_{i,n} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & a_{n,2} & \dots & a_{n.n-1} & a_{nn} \end{bmatrix}.$$

To find the entries of $\tilde{B}$ in terms of the entries of $\tilde{A}$, we distinguish the following three cases:
Case 1: $k < i$ *or* $k > i + 1$.
Since B is TP, by Theorem 2.2 we have

$$\begin{aligned} \tilde{b}_{kj} &= \frac{\det B[k, \dots, k+w|j, \dots, j+w]}{\det B[k+1, \dots, k+w|j+1, \dots, j+w]} \\ &= \frac{\det A[k, \dots, k+w|j, \dots, j+w]}{\det A[k+1, \dots, k+w|j+1, \dots, j+w]} \\ &= \tilde{a}_{kj}. \end{aligned}$$

The second equality follows in the cases $k > i+1$ and $k+w < i$ since $B\left[k, \dots, k+w|j, \dots, j+w\right] = A\left[k, \dots, k+w|j, \dots, j+w\right]$. Otherwise, if $k+1 \leq i \leq k+w$ and $i+1 \leq k+w$, then by determinantal properties, the second equality also follows. If $i+1 > k+w$, then $i = k+w$ holds. Hence by determinantal properties, the second equality follows.

Case 2: $k = i + 1$.
Since $\tilde{a}_{i+1,n} = a_{i+1,n}$, $\tilde{a}_{in} = a_{in}$, and $\tilde{b}_{i+1,n} = b_{i+1,n}$, and from the definition of B it follows that

$$\tilde{b}_{i+1,n} = y \; \tilde{a}_{i+1,n} + x \; \tilde{a}_{in}.$$

In the following we will explain how the other entries in the row $i + 1$ of $\tilde{B}$ are changed. By Theorem 2.2 and determinantal properties, we obtain

$$\begin{aligned}
\tilde{b}_{i+1,n-1} &= \frac{\det B[i+1,i+2|n-1,n]}{\det B[i+2|n]} \\
&= \frac{y \det A[i+1,i+2|n-1,n] + x \det A[i,i+2|n-1,n]}{\det A[i+2|n]} \\
&= y\,\frac{\det A[i+1,i+2|n-1,n]}{\det A[i+2|n]} + x\,\frac{\det A[i,i+2|n-1,n]}{\det A[i+2|n]} \\
&= y\ \tilde{a}_{i+1,n-1} + x\,\frac{\det A[i,i+2|n-1,n]}{\det A[i+2|n]}. \qquad (3.5)
\end{aligned}$$

To find the value of the second term in (3.5) in terms of the entries of $\tilde{A}$, we add and subtract the following quantity to it:

$$\frac{\det A[i+1,i+2|n-1,n]}{\det A[i+2|n]} \cdot \frac{\det A[i|n]}{\det A[i+1|n]} = \tilde{a}_{i+1,n-1}\ \frac{\tilde{a}_{i,n}}{\tilde{a}_{i+1,n}}.$$

Hence,

$$\begin{aligned}
\frac{\det A[i,i+2|n-1,n]}{\det A[i+2|n]} &= \frac{\det A[i,i+2|n-1,n]}{\det A[i+2|n]} \\
&\pm \frac{\det A[i+1,i+2|n-1,n]}{\det A[i+2|n]} \cdot \frac{\det A[i|n]}{\det A[i+1|n]} \\
&= \frac{a_{i,n-1}a_{i+2,n} - a_{i,n}a_{i+2,n-1}}{a_{i+2,n}} \\
&+ \frac{\tilde{a}_{i,n}\ \tilde{a}_{i+1,n-1}}{\tilde{a}_{i+1,n}} - \frac{a_{i,n}}{a_{i+2,n}a_{i+1,n}}\left(a_{i+1,n-1}a_{i+2,n} - a_{i+1,n}a_{i+2,n-1}\right) \\
&= \frac{a_{i,n-1}a_{i+1,n} - a_{i,n}a_{i+1,n-1}}{a_{i+1,n}} + \frac{\tilde{a}_{i,n}\ \tilde{a}_{i+1,n-1}}{\tilde{a}_{i+1,n}} \\
&= \frac{\det A[i,i+1|n-1,n]}{\det A[i+1|n]} + \frac{\tilde{a}_{i,n}\ \tilde{a}_{i+1,n-1}}{\tilde{a}_{i+1,n}} \\
&= \tilde{a}_{i,n-1} + \tilde{a}_{i+1,n-1}\left(\frac{\tilde{a}_{i,n}}{\tilde{a}_{i+1,n}}\right). \qquad (3.6)
\end{aligned}$$

By substituting (3.6) into (3.5) we get

$$\tilde{b}_{i+1,n-1} = y\ \tilde{a}_{i+1,n-1} + x\ \left(\tilde{a}_{i,n-1} + \tilde{a}_{i+1,n-1}\left(\frac{\tilde{a}_{i,n}}{\tilde{a}_{i+1,n}}\right)\right). \qquad (3.7)$$

Now,

$$\begin{aligned}
\tilde{b}_{i+1,n-2} &= \frac{\det B[i+1,i+2,i+3|n-2,n-1,n]}{\det B[i+2,i+3|n-1,n]} \\
&= \frac{y \det A[i+1,i+2,i+3|n-2,n-1,n] + x \det A[i,i+2,i+3|n-2,n-1,n]}{\det A[i+2,i+3|n-1,n]} \\
&= y\, \frac{\det A[i+1,i+2,i+3|n-2,n-1,n]}{\det A[i+2,i+3|n-1,n]} + x\, \frac{\det A[i,i+2,i+3|n-2,n-1,n]}{\det A[i+2,i+3|n-1,n]} \\
&= y\, \tilde{a}_{i+1,n-2} + x\, \frac{\det A[i,i+2,i+3|n-2,n-1,n]}{\det A[i+2,i+3|n-1,n]}. \qquad (3.8)
\end{aligned}$$

To find the value of the second term in (3.8) in terms of the entries of $\tilde{A}$, we add and subtract the following quantity to it:

$$\frac{\det A[i+1,i+2,i+3|n-2,n-1,n]}{\det A[i+2,i+3|n-1,n]} \; \frac{\det A[i,i+2|n-1,n] / \det A[i+2|n]}{\det A[i+1,i+2|n-1,n] / \det A[i+2|n]}$$

$$= \tilde{a}_{i+1,n-2}\, \frac{\det A[i,i+2|n-1,n] / \det A[i+2|n]}{\tilde{a}_{i+1,n-1}}.$$

Hence, by using Sylvester's Determinant Identity, we obtain

$$\begin{aligned}
\frac{\det A[i,i+2,i+3|n-2,n-1,n]}{\det A[i+2,i+3|n-1,n]} &= \frac{\det A[i,i+2|n-2,n-1] \det A[i+2,i+3|n-1,n]}{\det A[i+2,i+3|n-1,n] \det A[i+2|n-1]} \\
&- \frac{\det A[i,i+2|n-1,n] \det A[i+2,i+3|n-2,n-1]}{\det A[i+2,i+3|n-1,n] \det A[i+2|n-1]} \\
&\pm \frac{\det A[i+1,i+2,i+3|n-2,n-1,n]}{\det A[i+2,i+3|n-1,n]} \; \frac{\det A[i,i+2|n-1,n] / \det A[i+2|n]}{\det A[i+1,i+2|n-1,n] / \det A[i+2|n]} \\
&= \frac{\det A[i,i+2|n-2,n-1]}{\det A[i+2|n-1]} - \frac{\det A[i,i+2|n-1,n] \det A[i+2,i+3|n-2,n-1]}{\det A[i+2,i+3|n-1,n] \det A[i+2|n-1]} \\
&- \frac{\det A[i+1,i+2,i+3|n-2,n-1,n]}{\det A[i+2,i+3|n-1,n]} \; \frac{\det A[i,i+2|n-1,n]}{\det A[i+1,i+2|n-1,n]} \qquad (3.9) \\
&+ \tilde{a}_{i+1,n-2} \frac{\det A[i,i+2|n-1,n] / \det A[i+2|n]}{\tilde{a}_{i+1,n-1}}.
\end{aligned}$$

Again, we use Sylvester's Determinant Identity, we have

$$\begin{aligned}
\frac{\det A[i+1,i+2,i+3|n-2,n-1,n]}{\det A[i+2,i+3|n-1,n]} &= \frac{\det A[i+1,i+2|n-2,n-1] \det A[i+2,i+3|n-1,n]}{\det A[i+2,i+3|n-1,n] \det A[i+2|n-1]} \\
&- \frac{\det A[i+1,i+2|n-1,n] \det A[i+2,i+3|n-2,n-1]}{\det A[i+2,i+3|n-1,n] \det A[i+2|n-1]} \\
&= \frac{\det A[i+1,i+2|n-2,n-1]}{\det A[i+2|n-1]} \\
&- \frac{\det A[i+2,i+3|n-2,n-1] \det A[i+1,i+2|n-1,n]}{\det A[i+2,i+3|n-1,n] \det A[i+2|n-1]}. \qquad (3.10)
\end{aligned}$$

By substituting (3.10) into (3.9) we obtain

$$\begin{aligned}
\frac{\det A[i,i+2,i+3|n-2,n-1,n]}{\det A[i+2,i+3|n-1,n]} &= \frac{\det A[i,i+2|n-2,n-1]}{\det A[i+2|n-1]} - \frac{\det A[i,i+2|n-1,n]\det A[i+2,i+3|n-2,n-1]}{\det A[i+2,i+3|n-1,n]\det A[i+2|n-1]} \\
&- \frac{\det A[i+1,i+2|n-2,n-1]}{\det A[i+2|n-1]}\ \frac{\det A[i,i+2|n-1,n]}{\det A[i+1,i+2|n-1,n]} \\
&+ \frac{\det A[i+2,i+3|n-2,n-1]\det A[i+1,i+2|n-1,n]}{\det A[i+2,i+3|n-1,n]\det A[i+2|n-1]}\ \frac{\det A[i,i+2|n-1,n]}{\det A[i+1,i+2|n-1,n]} \\
&+ \tilde{a}_{i+1,n-2}\frac{\det A[i,i+2|n-1,n]/\det A[i+2|n]}{\tilde{a}_{i+1,n-1}} \\
&= \frac{\det A[i,i+2|n-2,n-1]}{\det A[i+2|n-1]} - \frac{\det A[i+1,i+2|n-2,n-1]}{\det A[i+2|n-1]}\ \frac{\det A[i,i+2|n-1,n]}{\det A[i+1,i+2|n-1,n]} \\
&+ \tilde{a}_{i+1,n-2}\frac{\det A[i,i+2|n-1,n]/\det A[i+2|n]}{\tilde{a}_{i+1,n-1}} \\
&= \frac{\det A[i,i+1,i+2|n-2,n-1,n]}{\det A[i+1,i+2|n-1,n]} \\
&+ \tilde{a}_{i+1,n-2}\frac{\det A[i,i+2|n-1,n]/\det A[i+2|n]}{\tilde{a}_{i+1,n-1}}, \text{by Sylvester's Determinant Identity,} \\
&= \tilde{a}_{i,n-2} + \frac{\tilde{a}_{i+1,n-2}}{\tilde{a}_{i+1,n-1}}\left(\tilde{a}_{i,n-1} + \tilde{a}_{i+1,n-1}\frac{\tilde{a}_{i,n}}{\tilde{a}_{i+1,n}}\right), \text{by using (3.6).} \qquad (3.11)
\end{aligned}$$

By substituting (3.11) into (3.8), we get

$$\tilde{b}_{i+1,n-2} = y\ \tilde{a}_{i+1,n-2} + x\left(\tilde{a}_{i,n-2} + \tilde{a}_{i+1,n-2}\left(\frac{\tilde{a}_{i,n-1}}{\tilde{a}_{i+1,n-1}} + \frac{\tilde{a}_{i,n}}{\tilde{a}_{i+1,n}}\right)\right). \qquad (3.12)$$

We use the following decreasing induction on the column index of the entries in row $i+1$ of $\tilde{B}$, to get all the entries in the row $i+1$, which are given as follows:

$$\tilde{b}_{i+1,n-j} = y\ \tilde{a}_{i+1,n-j} + x\left(\tilde{a}_{i,n-j} + \tilde{a}_{i+1,n-j}\left(\sum_{m=n-j+1}^{n}\frac{\tilde{a}_{i,m}}{\tilde{a}_{i+1,m}}\right)\right),\ j = 1,2,\ldots,n-1. \qquad (3.13)$$

By using (3.7) and (3.12), (3.13) is true for $j = 1, 2$. Assume it is true when $j = \gamma$, i.e.,

$$\tilde{b}_{i+1,n-\gamma} = y\ \tilde{a}_{i+1,n-\gamma} + x\left(\tilde{a}_{i,n-\gamma} + \tilde{a}_{i+1,n-\gamma}\left(\sum_{m=n-\gamma+1}^{n}\frac{\tilde{a}_{i,m}}{\tilde{a}_{i+1,m}}\right)\right). \qquad (3.14)$$

From (2.1), we have

$$\begin{aligned}\tilde{b}_{i+1,n-\gamma} &= \frac{\det B[i+1,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]}{\det B[i+2,\ldots,i+1+w|n-\gamma+1,\ldots,n-\gamma+w]}\\ &= y\,\tilde{a}_{i+1,n-\gamma}+x\,\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma+1,\ldots,n-\gamma+w]}.\end{aligned} \tag{3.15}$$

From (3.14) and (3.15) we have

$$\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma+1,\ldots,n-\gamma+w]} = \tilde{a}_{i,n-\gamma}+\tilde{a}_{i+1,n-\gamma}\left(\sum_{m=n-\gamma+1}^{n}\frac{\tilde{a}_{i,m}}{\tilde{a}_{i+1,m}}\right). \tag{3.16}$$

If $n-\gamma > i+1$, then for $j = n-\gamma-1$ we are on the upper part or on the main diagonal of the matrix $\tilde{B}$. Now, we want to prove that (3.13) holds for $j = \gamma+1$.

$$\begin{aligned}\tilde{b}_{i+1,n-\gamma-1} &= \frac{\det B[i+1,\ldots,i+2+w|n-\gamma-1,\ldots,n-\gamma+w]}{\det B[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]}\\ &= \frac{y\det A[i+1,i+2,\ldots,i+2+w|n-\gamma-1,n-\gamma,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]}\\ &+ x\frac{\det A[i,i+2,\ldots,i+2+w|n-\gamma-1,n-\gamma,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]}\\ &= y\,\tilde{a}_{i+1,n-\gamma-1}+x\,\frac{\det A[i,i+2,\ldots,i+2+w|n-\gamma-1,n-\gamma,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]}.\end{aligned} \tag{3.17}$$

To find the value of the second term in (3.17), we apply Sylvester's Determinant Identity on its numerator to obtain

$$\frac{\det A[i,i+2,\ldots,i+2+w|n-\gamma-1,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]} = \frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma-1,\ldots,n-\gamma-1+w]\ \det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]\ \det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]}$$

$$-\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]\ \det A[i+2,\ldots,i+2+w|n-\gamma-1,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]\ \det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]}$$

$$=\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma-1,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]}-\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]\ \det A[i+2,\ldots,i+2+w|n-\gamma-1,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]\ \det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]}$$

$$\pm\frac{\det A\big[i+1,\ldots,i+w+2|n-\gamma-1,\ldots,n-\gamma+w\big]}{\det A\big[i+2,\ldots,i+w+2|n-\gamma,\ldots,n-\gamma+w\big]}\ \frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]/\det A[i+2,\ldots,i+1+w|n-\gamma+1,\ldots,n-\gamma+w]}{A\big\langle i+1,\ldots,i+w+1|n-\gamma,\ldots,n-\gamma+w\big\rangle}$$

$$=\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma-1,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]}-\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]\ \det A[i+2,\ldots,i+2+w|n-\gamma-1,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]\ \det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]}$$

$$-\frac{\det A[i+1,\ldots,i+2+w|n-\gamma-1,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]}\ \frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}{\det A[i+1,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}$$

$$+\tilde{a}_{i+1,n-\gamma-1}\ \frac{1}{\tilde{a}_{i+1,n-\gamma}}\left(\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma+1,\ldots,n-\gamma+w]}\right), \tag{3.18}$$

Now, again by Sylvester's Determinant Identity, we have

$$\frac{\det A[i+1,\ldots,i+2+w|n-\gamma-1,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]} = \frac{\det A[i+1,\ldots,i+1+w|n-\gamma-1,\ldots,n-\gamma-1+w]\ \det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]\ \det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]}$$

$$-\frac{\det A[i+1,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]\ \det A[i+2,\ldots,i+2+w|n-\gamma-1,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]\ \det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]}$$

$$= \frac{\det A[i+1,\ldots,i+1+w|n-\gamma-1,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]}$$

$$-\frac{\det A[i+1,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]\ \det A[i+2,\ldots,i+2+w|n-\gamma-1,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]\ \det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]}. \tag{3.19}$$

By substituting (3.19) into (3.18) and after simplifications, we get

$$\frac{\det A[i,i+2,\ldots,i+2+w|n-\gamma-1,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]} = \frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma-1,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]}$$

$$-\frac{\det A[i+1,\ldots,i+1+w|n-\gamma-1,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]}\ \frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}{\det A[i+1,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]}$$

$$+\frac{\tilde{a}_{i+1,n-\gamma-1}}{\tilde{a}_{i+1,n-\gamma}}\left(\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma+1,\ldots,n-\gamma+w]}\right)$$

$$\begin{aligned}
&= \frac{\det A[i,\ldots,i+1+w|n-\gamma-1,\ldots,n-\gamma+w]}{\det A[i+1,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]} \\
&\quad + \frac{\tilde{a}_{i+1,n-\gamma-1}}{\tilde{a}_{i+1,n-\gamma}}\left(\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma+1,\ldots,n-\gamma+w]}\right) \qquad (3.20) \\
&= \tilde{a}_{i,n-\gamma-1} + \frac{\tilde{a}_{i+1,n-\gamma-1}}{\tilde{a}_{i+1,n-\gamma}}\left(\tilde{a}_{i,n-\gamma} + \tilde{a}_{i+1,n-\gamma}\left(\sum_{m=n-\gamma+1}^{n}\frac{\tilde{a}_{i,m}}{\tilde{a}_{i+1,m}}\right)\right), \text{from (3.16)} \\
&= \tilde{a}_{i,n-\gamma-1} + \tilde{a}_{i+1,n-\gamma-1}\left(\sum_{m=n-\gamma}^{n}\frac{\tilde{a}_{i,m}}{\tilde{a}_{i+1,m}}\right). \qquad (3.21)
\end{aligned}$$

Hence by substituting (3.21) into (3.17) we conclude (3.13) is true when $j=\gamma+1$ on the upper part and on the main diagonal of $\tilde{B}$, i.e.,

$$\tilde{b}_{i+1,n-\gamma-1} = y\ \tilde{a}_{i+1,n-\gamma-1} + x\left(\tilde{a}_{i,n-\gamma-1} + \tilde{a}_{i+1,n-\gamma-1}\left(\sum_{m=n-\gamma}^{n}\frac{\tilde{a}_{i,m}}{\tilde{a}_{i+1,m}}\right)\right). \tag{3.22}$$

For $n-\gamma \le i+1$, then for $j = n-\gamma-1$ we are on the lower part of $\tilde{B}$ and we have

$$\begin{aligned}
\tilde{b}_{i+1,n-\gamma-1} &= \frac{\det B[i+1,\ldots,i+1+w|n-\gamma-1,\ldots,n-\gamma-1+w]}{\det B[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]} \\
&= y\,\frac{\det A[i+1,\ldots,i+1+w|n-\gamma-1,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]} \\
&+ x\,\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma-1,n-\gamma,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]} \\
&= y\,\tilde{a}_{i+1,n-\gamma-1} + x\,\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma-1,n-\gamma,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]}.
\end{aligned} \tag{3.23}$$

In (3.23), we can rewrite $\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma-1,n-\gamma,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]}$ by using the Sylvester's Determinant Identity and add and subtract the following quantity:

$$\frac{\det A\left[i,\ldots,i+w+1|n-\gamma-1,\ldots,n-\gamma+w\right]}{\det A\left[i+1,\ldots,i+w+1|n-\gamma,\ldots,n-\gamma+w\right]}.$$

Hence,

$$\begin{aligned}
&\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma-1,n-\gamma,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]} \pm \frac{\det A\left[i,\ldots,i+w+1|n-\gamma-1,\ldots,n-\gamma+w\right]}{\det A\left[i+1,\ldots,i+w+1|n-\gamma,\ldots,n-\gamma+w\right]} \\
&= \frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma-1,n-\gamma,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]} + \frac{\det A\left[i,\ldots,i+w+1|n-\gamma-1,\ldots,n-\gamma+w\right]}{\det A\left[i+1,\ldots,i+w+1|n-\gamma,\ldots,n-\gamma+w\right]} \\
&- \frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma-1,n-\gamma,\ldots,n-\gamma-1+w]}{\det A[i+1,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}\,\frac{\det A[i+1,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}{\det A[i+2,i+3,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma-1+w]} \\
&+ \frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}{\det A[i+1,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}\,\frac{\det A[i+1,i+2,\ldots,i+1+w|n-\gamma-1,n-\gamma,\ldots,n-\gamma-1+w]}{\det A[i+2,i+3,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma-1+w]} \\
&= \frac{\det A\left[i,\ldots,i+w+1|n-\gamma-1,\ldots,n-\gamma+w\right]}{\det A\left[i+1,\ldots,i+w+1|n-\gamma,\ldots,n-\gamma+w\right]} \\
&+ \frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}{\det A[i+1,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}\,\frac{\det A[i+1,i+2,\ldots,i+1+w|n-\gamma-1,n-\gamma,\ldots,n-\gamma-1+w]}{\det A[i+2,i+3,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma-1+w]} \\
&= \tilde{a}_{i,n-\gamma-1} + \tilde{a}_{i+1,n-\gamma-1}\left(\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]/\det A[i+2,\ldots,i+1+w|n-\gamma+1,\ldots,n-\gamma+w]}{\det A[i+1,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]/\det A[i+2,\ldots,i+1+w|n-\gamma+1,\ldots,n-\gamma+w]}\right)
\end{aligned}$$

$$= \tilde{a}_{i,n-\gamma-1} + \frac{\tilde{a}_{i+1,n-\gamma-1}}{\tilde{a}_{i+1,n-\gamma}}\left(\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma+1,\ldots,n-\gamma+w]}\right) \tag{3.24}$$

$$= \tilde{a}_{i,n-\gamma-1} + \frac{\tilde{a}_{i+1,n-\gamma-1}}{\tilde{a}_{i+1,n-\gamma}}\left(\tilde{a}_{i,n-\gamma} + \tilde{a}_{i+1,n-\gamma}\left(\sum_{m=n-\gamma+1}^{n}\frac{\tilde{a}_{i,m}}{\tilde{a}_{i+1,m}}\right)\right), \text{from (3.16).} \tag{3.25}$$

Hence by substituting (3.25) into (3.23) we conclude (3.13) is true when $j = \gamma + 1$

in the lower part of the matrix $\tilde{B}$, i.e.,

$$\tilde{b}_{i+1,n-\gamma-1} = y\ \tilde{a}_{i+1,n-\gamma-1} + x\left(\tilde{a}_{i,n-\gamma-1} + \tilde{a}_{i+1,n-\gamma-1}\left(\sum_{m=n-\gamma}^{n}\frac{\tilde{a}_{i,m}}{\tilde{a}_{i+1,m}}\right)\right).$$

Case 3: $k = i$

In this case we show that the entries in the row i of $\tilde{B}$ are given by

$$\tilde{b}_{i,n-j} = \begin{cases} \frac{1}{y}\ \tilde{a}_{i,n-j}, & \text{for } \ j = 0, \\ \frac{\tilde{a}_{i,n-j}}{y+x\left(\sum_{m=n-j+1}^{n}\frac{\tilde{a}_{i,m}}{\tilde{a}_{i+1,m}}\right)}, & \text{for } \ 1 \le j < n. \end{cases} \tag{3.26}$$

Because all the entries in the last column of B do not change when we apply the condensed form of the Cauchon algorithm on B. Hence for $j = 0$, (3.26) holds, i.e.,

$$\tilde{b}_{i,n} = \frac{1}{y}\ \tilde{a}_{i,n}. \tag{3.27}$$

To prove (3.26) for $j = 1, \ldots, n-1$ we use mathematical induction on j. For $j = 1$,

$$\begin{aligned} \tilde{b}_{i,n-1} &= \frac{\det B[i,i+1|n-1,n]}{\det B[i+1|n]} \\ &= \frac{\det A[i,i+1|n-1,n]}{y \det A[i+1|n] + x \det A[i|n]} \\ &= \frac{\det A[i,i+1|n-1,n] / \det A[i+1|n]}{\left(y \det A[i+1|n] + x \det A[i|n]\right) / \det A[i+1|n]} \\ &= \frac{\tilde{a}_{i,n-1}}{y + x\ \frac{\tilde{a}_{i,n}}{\tilde{a}_{i+1,n}}}. \end{aligned} \tag{3.28}$$

Assume (3.26) is true when $j = \gamma > 0$, that means

$$\tilde{b}_{i,n-\gamma} = \frac{\tilde{a}_{i,n-\gamma}}{y + x\left(\sum_{m=n-\gamma+1}^{n}\frac{\tilde{a}_{i,m}}{\tilde{a}_{i+1,m}}\right)}. \tag{3.29}$$

Assume $n-\gamma > i$, we want to prove that (3.26) is true when $j=\gamma+1$.

$$\begin{aligned}
\tilde{b}_{i,n-\gamma-1} &= \frac{\det B[i,\ldots,i+1+w|n-\gamma-1,\ldots,n-\gamma+w]}{\det B[i+1,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]}\\
&= \frac{\det A[i,\ldots,i+1+w|n-\gamma-1,\ldots,n-\gamma+w]}{y\det A[i+1,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]+x\det A[i,i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]}\\
&= \frac{\det A[i,\ldots,i+1+w|n-\gamma-1,\ldots,n-\gamma+w]/\det A[i+1,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]}{y+x\det A[i,i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]/\det A[i+1,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]}\\
&= \frac{\tilde{a}_{i,n-\gamma-1}}{y+x\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]/\det A[i+2,\ldots,i+1+w|n-\gamma+1,\ldots,n-\gamma+w]}{\det A[i+1,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]/\det A[i+2,\ldots,i+1+w|n-\gamma+1,\ldots,n-\gamma+w]}}\\
&= \frac{\tilde{a}_{i,n-\gamma-1}}{y+x\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma+w]/\det A[i+2,\ldots,i+1+w|n-\gamma+1,\ldots,n-\gamma+w]}{\tilde{a}_{i+1,n-\gamma}}}.
\end{aligned} \tag{3.30}$$

By using (3.16) we obtain

$$\begin{aligned}
\tilde{b}_{i,n-\gamma-1} &= \frac{\tilde{a}_{i,n-\gamma-1}}{y+x\frac{\tilde{a}_{i,n-\gamma}+\tilde{a}_{i+1,n-\gamma}\left(\sum_{m=n-\gamma+1}^{n}\frac{\tilde{a}_{i,m}}{\tilde{a}_{i+1,m}}\right)}{\tilde{a}_{i+1,n-\gamma}}}\\
&= \frac{\tilde{a}_{i,n-\gamma-1}}{y+x\left(\sum_{m=n-\gamma}^{n}\frac{\tilde{a}_{i,m}}{\tilde{a}_{i+1,m}}\right)}.
\end{aligned} \tag{3.31}$$

Hence by (3.31) we have that (3.26) is true when $j=\gamma+1$, and when $n-\gamma\le i$ we can proceed similarly to prove that (3.26) holds. □

Example 3.2. *Let A be the following Pascal matrix:*

$$A=\begin{bmatrix}1&1&1&1\\1&2&3&4\\1&3&6&10\\1&4&10&20\end{bmatrix}.$$

Then by application of the condensed form of the Cauchon algorithm to A we get

$$\tilde{A}=\begin{bmatrix}1/4&1/6&1/4&1\\1/6&2/10&6/10&4\\1/4&6/10&1&10\\1&4&10&20\end{bmatrix}.$$

Let B be the matrix obtained from A by multiplying the third row by 2 and add to it the second row multiplied by 10. Also multiply the second row by 1/2. Hence

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1/2 & 1 & 3/2 & 2 \\ 12 & 26 & 42 & 60 \\ 1 & 4 & 10 & 20 \end{bmatrix}.$$

By application of Theorem 3.2 with $x = 10$ and $y = 2$, the entries in the first and last rows don't change. The entries in the second row are changed as follows

$$\begin{aligned} \tilde{b}_{24} &= \frac{1}{y}\,\tilde{a}_{24} \\ &= \frac{1}{2}\,(4) = 2, \end{aligned}$$

the other entries are

$$\begin{aligned} \tilde{b}_{23} &= \frac{\tilde{a}_{23}}{y + x\left(\dfrac{\tilde{a}_{24}}{\tilde{a}_{34}}\right)} \\ &= \frac{6/10}{2 + 10\left(\dfrac{4}{10}\right)} \\ &= \frac{1}{10}, \\ \tilde{b}_{22} &= \frac{\tilde{a}_{22}}{y + x\left(\dfrac{\tilde{a}_{23}}{\tilde{a}_{33}} + \dfrac{\tilde{a}_{24}}{\tilde{a}_{34}}\right)} \\ &= \frac{2/10}{2 + 10\left(\dfrac{6}{10} + \dfrac{4}{10}\right)} \\ &= \frac{1}{60}, \\ \tilde{b}_{21} &= \frac{\tilde{a}_{21}}{y + x\left(\dfrac{\tilde{a}_{22}}{\tilde{a}_{32}} + \dfrac{\tilde{a}_{23}}{\tilde{a}_{33}} + \dfrac{\tilde{a}_{24}}{\tilde{a}_{34}}\right)} \\ &= \frac{1/6}{2 + 10\left(\dfrac{1}{3} + \dfrac{6}{10} + \dfrac{4}{10}\right)} \\ &= \frac{1}{92}. \end{aligned}$$

Also, the entries in the third row are given by

$$\begin{aligned}\tilde{b}_{34} &= y\,\tilde{a}_{34} + x\,\tilde{a}_{24}\\ &= 60,\end{aligned}$$

the other entries are

$$\begin{aligned}\tilde{b}_{33} &= y\,\tilde{a}_{33} + x\left(\tilde{a}_{23} + \tilde{a}_{33}\frac{\tilde{a}_{24}}{\tilde{a}_{34}}\right)\\ &= 2\,(1) + 10\left(\frac{6}{10} + 1\left(\frac{4}{10}\right)\right)\\ &= 12,\\ \tilde{b}_{32} &= y\,\tilde{a}_{32} + x\left(\tilde{a}_{22} + \tilde{a}_{32}\left(\frac{\tilde{a}_{23}}{\tilde{a}_{33}} + \frac{\tilde{a}_{24}}{\tilde{a}_{34}}\right)\right)\\ &= 2\left(\frac{6}{10}\right) + 10\left(\frac{2}{10} + \frac{6}{10}\left(\frac{6}{10} + \frac{4}{10}\right)\right)\\ &= \frac{46}{5},\\ \tilde{b}_{31} &= y\,\tilde{a}_{31} + x\left(\tilde{a}_{21} + \tilde{a}_{31}\left(\frac{\tilde{a}_{22}}{\tilde{a}_{32}} + \frac{\tilde{a}_{23}}{\tilde{a}_{33}} + \frac{\tilde{a}_{24}}{\tilde{a}_{34}}\right)\right)\\ &= 2\left(\frac{1}{4}\right) + 10\left(\frac{1}{6} + \frac{1}{4}\left(\frac{1}{3} + \frac{6}{10} + \frac{4}{10}\right)\right)\\ &= \frac{11}{2}.\end{aligned}$$

Therefore, the matrix $\tilde{B}$ is

$$\tilde{B} = \begin{bmatrix} 1/4 & 1/6 & 1/4 & 1\\ 1/92 & 1/60 & 1/10 & 2\\ 11/2 & 46/5 & 12 & 60\\ 1 & 4 & 10 & 20 \end{bmatrix}.$$

Ramark 3.1. *We extend Theorem 3.2 to the nonsingular TN case by the following. We replace zero entries from the bottom to the top in the upper part and from right to the left in the lower part of $\tilde{A}$ by increasing integral powers of a positive number ϵ, we get the positive matrix $\tilde{A}_\epsilon$. We apply the restoration algorithm to $\tilde{A}_\epsilon$ and obtain the TP matrix A_ϵ. Since $\tilde{A}_\epsilon$ tends to $\tilde{A}$ as ϵ tend to 0, A_ϵ tends to A. After cancellation of common powers of ϵ, we obtain that the denominators do not contain ϵ. Letting ϵ tend to 0, the extension of Theorem 3.2 to the nonsingular TN case follows. By setting all $\frac{0}{0} := 0$, all quantities in Theorem 3.2 are defined since in the upper part of $\tilde{A}$ if one entry is zero then the entry above it is zero, also if one entry is zero in* *the lower part then the entry to the left of it is zero. So, if A is a nonsingular TN, then we can write the entries of $\tilde{B}$ in Theorem 3.2 as follows*

$$\tilde{b}_{kj} = \begin{cases} \dfrac{\tilde{a}_{kj}}{y+x\left(\sum_{m=j+1}^{n}\frac{\tilde{a}_{km}}{\tilde{a}_{k+1,m}}\right)}, & \text{for } k=i, i\leq j<n, \\ \dfrac{\tilde{a}_{kj}}{y+x\left(\sum_{m=j+1}^{n}\frac{\tilde{a}_{km}}{\tilde{a}_{k+1,m}}\right)}, & \text{for } k=i, 1\leq j<i \text{ and } \tilde{a}_{k+1,j+1}\neq 0, \\ \dfrac{\tilde{a}_{kj}\,\tilde{a}_{k+1,j}}{y\,\tilde{a}_{k+1,j}+x\left(\sum_{m=j+1}^{n}\frac{\tilde{a}_{k+1,j}\,\tilde{a}_{km}}{\tilde{a}_{k+1,m}}\right)}=0, & \text{for } k=i, 1\leq j<i \text{ and } \tilde{a}_{k+1,j+1}=0, \\ \frac{1}{y}\,\tilde{a}_{kj}, & \text{for } k=i, j=n, \\ y\,\tilde{a}_{kj}+x\,\left(\tilde{a}_{k-1,j}+\tilde{a}_{kj}\left(\sum_{m=j+1}^{n}\frac{\tilde{a}_{k-1,m}}{\tilde{a}_{km}}\right)\right), & \text{for } k=i+1, i\leq j<n, \\ y\,\tilde{a}_{kj}+x\,\left(\tilde{a}_{k-1,j}+\tilde{a}_{kj}\left(\sum_{m=j+1}^{n}\frac{\tilde{a}_{k-1,m}}{\tilde{a}_{km}}\right)\right), & \text{for } k=i+1, 1\leq j<i \text{ and } \tilde{a}_{k,j+1}\neq 0, \\ y\,\tilde{a}_{kj}+x\,\left(\tilde{a}_{k-1,j}+\left(\sum_{m=j+1}^{n}\frac{\tilde{a}_{kj}\,\tilde{a}_{k-1,m}}{\tilde{a}_{km}}\right)\right)=x\,\tilde{a}_{k-1,j}, & \text{for } k=i+1, 1\leq j<i \text{ and } \tilde{a}_{k,j+1}=0, \\ y\,\tilde{a}_{kj}+x\,\tilde{a}_{k-1,j}, & \text{for } k=i+1, j=n, \\ \tilde{a}_{kj}, & \text{for } k\neq i \text{ and } k\neq i+1, 1\leq j\leq n. \end{cases} \tag{3.32}$$

In the following we illustrate Remark 3.1.

Example 3.3. *Let A be the following Jacobi matrix:*

$$A = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 1 & 4 & 4 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 6 & 10 & 2 \\ 0 & 0 & 0 & 20 & 10 \end{bmatrix}.$$

Then by application of the condensed form of the Cauchon algorithm to A we get

$$\tilde{A} = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 6 & 6 & 2 \\ 0 & 0 & 0 & 20 & 10 \end{bmatrix}.$$

Let B be the matrix obtained from A by multiplying the fourth row by 3 and add to it the third row multiplied by 4. Also multiply the third row by 1/3. Hence

$$B = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 1 & 4 & 4 & 0 & 0 \\ 0 & 1/3 & 1 & 1/3 & 0 \\ 0 & 4 & 30 & 34 & 6 \\ 0 & 0 & 0 & 20 & 10 \end{bmatrix}.$$

By application of (3.32) in Remark 3.1 with $x = 4$ and $y = 3$, the entries in the first, second, and fifth rows do not change. The entries in the third row are changed as follows

$$\begin{aligned} \tilde{b}_{35} &= \frac{1}{y}\tilde{a}_{35} \\ &= \frac{1}{3}(0) = 0, \end{aligned}$$

the other entries are

$$\begin{aligned} \tilde{b}_{34} &= \frac{\tilde{a}_{34}}{y + x\left(\dfrac{\tilde{a}_{35}}{\tilde{a}_{45}}\right)} \\ &= \frac{1}{3 + 4\left(\dfrac{0}{2}\right)} \\ &= \frac{1}{3}, \\ \tilde{b}_{33} &= \frac{\tilde{a}_{33}}{y + x\left(\dfrac{\tilde{a}_{34}}{\tilde{a}_{44}} + \dfrac{\tilde{a}_{35}}{\tilde{a}_{45}}\right)} \\ &= \frac{2}{3 + 4\left(\dfrac{1}{6} + \dfrac{0}{2}\right)} \\ &= \frac{6}{11}, \\ \tilde{b}_{32} &= \frac{\tilde{a}_{32}}{y + x\left(\dfrac{\tilde{a}_{33}}{\tilde{a}_{43}} + \dfrac{\tilde{a}_{34}}{\tilde{a}_{44}} + \dfrac{\tilde{a}_{35}}{\tilde{a}_{45}}\right)} \\ &= \frac{1}{3 + 4\left(\dfrac{2}{6} + \dfrac{1}{6} + \dfrac{0}{2}\right)} \\ &= \frac{1}{5}, \end{aligned}$$

$$\begin{aligned}
\tilde{b}_{31} &= \frac{\tilde{a}_{31}\tilde{a}_{41}}{y\,\tilde{a}_{41} + x\left(\dfrac{\tilde{a}_{32}\tilde{a}_{41}}{\tilde{a}_{42}} + \dfrac{\tilde{a}_{33}\tilde{a}_{41}}{\tilde{a}_{43}} + \dfrac{\tilde{a}_{34}\tilde{a}_{41}}{\tilde{a}_{44}} + \dfrac{\tilde{a}_{35}\tilde{a}_{41}}{\tilde{a}_{45}}\right)} \\
&= \frac{0(0)}{3(0) + 4\left(\dfrac{1(0)}{0} + \dfrac{2(0)}{6} + \dfrac{1(0)}{6} + \dfrac{0(0)}{2}\right)} \\
&= 0.
\end{aligned}$$

Also, the entries in the fourth row are given by

$$\begin{aligned}
\tilde{b}_{45} &= y\,\tilde{a}_{45} + x\,\tilde{a}_{35} \\
&= 3(2) + 4(0) = 6,
\end{aligned}$$

the other entries are

$$\begin{aligned}
\tilde{b}_{44} &= y\,\tilde{a}_{44} + x\left(\tilde{a}_{34} + \tilde{a}_{44}\frac{\tilde{a}_{35}}{\tilde{a}_{45}}\right) \\
&= 3\,(6) + 4\left(1 + 6\left(\frac{0}{2}\right)\right) \\
&= 22, \\
\tilde{b}_{43} &= y\,\tilde{a}_{43} + x\left(\tilde{a}_{33} + \tilde{a}_{43}\left(\frac{\tilde{a}_{34}}{\tilde{a}_{44}} + \frac{\tilde{a}_{35}}{\tilde{a}_{45}}\right)\right) \\
&= 3\,(6) + 4\left(2 + 6\left(\frac{1}{6} + \frac{0}{2}\right)\right) \\
&= 30, \\
\tilde{b}_{42} &= y\,\tilde{a}_{42} + x\left(\tilde{a}_{32} + \tilde{a}_{42}\left(\frac{\tilde{a}_{33}}{\tilde{a}_{43}} + \frac{\tilde{a}_{34}}{\tilde{a}_{44}} + \frac{\tilde{a}_{35}}{\tilde{a}_{45}}\right)\right) \\
&= 3\,(0) + 4\left(1 + 0\left(\frac{2}{6} + \frac{1}{6} + \frac{0}{2}\right)\right) \\
&= 4, \\
\tilde{b}_{41} &= y\,\tilde{a}_{41} + x\left(\tilde{a}_{31} + \left(\frac{\tilde{a}_{41}\tilde{a}_{32}}{\tilde{a}_{42}} + \frac{\tilde{a}_{41}\tilde{a}_{33}}{\tilde{a}_{43}} + \frac{\tilde{a}_{41}\tilde{a}_{34}}{\tilde{a}_{44}} + \frac{\tilde{a}_{41}\tilde{a}_{35}}{\tilde{a}_{45}}\right)\right) \\
&= 3\,(0) + 4\left(0 + \left(\frac{(0)1}{0} + \frac{(0)2}{6} + \frac{(0)1}{6} + \frac{(0)0}{2}\right)\right) \\
&= 0.
\end{aligned}$$

Therefore, the matrix $\tilde{B}$ is given by

$$\tilde{B} = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 \\ 0 & 1/5 & 6/11 & 1/3 & 0 \\ 0 & 4 & 30 & 22 & 6 \\ 0 & 0 & 0 & 20 & 10 \end{bmatrix}.$$

In the following theorem we rewrite the entries of the matrix $\tilde{B}$ that have been obtained in Theorem 3.2 so that we reduce the number of required arithmetic operations.

Theorem 3.3. *We can write the entries of the matrix $\tilde{B}$ in Theorem 3.2 as follows*

$$\tilde{b}_{kj} = \begin{cases} x\,\tilde{a}_{k-1,j} + \frac{\tilde{a}_{kj}\,\tilde{b}_{k,j+1}}{\tilde{a}_{k,j+1}}, & \textit{for} \quad k = i+1, 1 \le j < n, \\ y\,\tilde{a}_{kj} + x\,\tilde{a}_{k-1,j}, & \textit{for} \quad k = i+1, j = n, \\ \frac{\tilde{a}_{kj}\tilde{a}_{k+1,j+1}}{\tilde{b}_{k+1,j+1}}, & \textit{for} \quad k = i, 1 \le j < n, \\ \frac{1}{y}\,\tilde{a}_{kj}, & \textit{for} \quad k = i, j = n, \\ \tilde{a}_{kj}, & \textit{for} \quad k \ne i \textit{ and } k \ne i+1, 1 \le j \le n. \end{cases}$$

Therefore, the matrix $\tilde{B}$ can be computed from $\tilde{A}$ in at most $6n-2$ arithmetic operations without performing any subtractions.

Proof. From Theorem 3.2 for $k < i$ *or* $k > i+1$ and $1 \le j \le n$, we have

$$\tilde{b}_{kj} = \tilde{a}_{kj}.$$

In the following we investigate the cases $k = i+1$ and $k = i$.

Case 1: $k = i+1$.

For $k = i+1$ and $j = n$ or $k = i+1 = n$ and $1 \le j \le n$, we get

$$\tilde{b}_{i+1,j} = y\,\tilde{a}_{i+1,j} + x\,\tilde{a}_{ij}, \tag{3.33}$$

since the entries in the last row and columns do not change.

In the following assume that $k = i+1 \ne n$ and $1 \le j < n$. From (3.15), we have

$$\frac{\det A[i, i+2, \ldots, i+1+w | n-\gamma, n-\gamma+1, \ldots, n-\gamma+w]}{\det A[i+2, \ldots, i+1+w | n-\gamma+1, \ldots, n-\gamma+w]} = \frac{\tilde{b}_{i+1,n-\gamma} - y\,\tilde{a}_{i+1,n-\gamma}}{x}. \tag{3.34}$$

If $n-\gamma > i+1$, then for $j = n-\gamma-1$ we are on the upper part or on the diagonal of the matrix $\tilde{B}$. From (3.17), we obtain

$$\tilde{b}_{i+1,n-\gamma-1} = y\,\tilde{a}_{i+1,n-\gamma-1} + x\,\frac{\det A[i,i+2,\ldots,i+2+w|n-\gamma-1,n-\gamma,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]}. \tag{3.35}$$

Also, from (3.20) we get

$$\frac{\det A[i,i+2,\ldots,i+2+w|n-\gamma-1,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+2+w|n-\gamma,\ldots,n-\gamma+w]} = \tilde{a}_{i,n-\gamma-1} + \frac{\tilde{a}_{i+1,n-\gamma-1}}{\tilde{a}_{i+1,n-\gamma}}\left(\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma+1,\ldots,n-\gamma+w]}\right)$$

$$= \tilde{a}_{i,n-\gamma-1} + \frac{\tilde{a}_{i+1,n-\gamma-1}}{\tilde{a}_{i+1,n-\gamma}}\left(\frac{\tilde{b}_{i+1,n-\gamma} - y\,\tilde{a}_{i+1,n-\gamma}}{x}\right), \text{from (3.34).} \tag{3.36}$$

By substituting (3.36) into (3.35), we have

$$\begin{aligned}\tilde{b}_{i+1,n-\gamma-1} &= y\,\tilde{a}_{i+1,n-\gamma-1} + x\left(\tilde{a}_{i,n-\gamma-1} + \frac{\tilde{a}_{i+1,n-\gamma-1}}{\tilde{a}_{i+1,n-\gamma}}\left(\frac{\tilde{b}_{i+1,n-\gamma} - y\,\tilde{a}_{i+1,n-\gamma}}{x}\right)\right). \\ &= x\,\tilde{a}_{i,n-\gamma-1} + \frac{\tilde{a}_{i+1,n-\gamma-1}}{\tilde{a}_{i+1,n-\gamma}}\,\tilde{b}_{i+1,n-\gamma}. \end{aligned} \tag{3.37}$$

For $n-\gamma \leq i+1$, then for $j = n-\gamma-1$ we are on the lower part of the matrix $\tilde{B}$ and by (3.23) we obtain

$$\tilde{b}_{i+1,n-\gamma-1} = y\,\tilde{a}_{i+1,n-\gamma-1} + x\,\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma-1,n-\gamma,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]}. \tag{3.38}$$

From (3.24), we have

$$\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma-1,n-\gamma,\ldots,n-\gamma-1+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma,\ldots,n-\gamma-1+w]} = \tilde{a}_{i,n-\gamma-1} + \frac{\tilde{a}_{i+1,n-\gamma-1}}{\tilde{a}_{i+1,n-\gamma}}\left(\frac{\det A[i,i+2,\ldots,i+1+w|n-\gamma,n-\gamma+1,\ldots,n-\gamma+w]}{\det A[i+2,\ldots,i+1+w|n-\gamma+1,\ldots,n-\gamma+w]}\right)$$

$$= \tilde{a}_{i,n-\gamma-1} + \frac{\tilde{a}_{i+1,n-\gamma-1}}{\tilde{a}_{i+1,n-\gamma}}\left(\frac{\tilde{b}_{i+1,n-\gamma} - y\,\tilde{a}_{i+1,n-\gamma}}{x}\right), \text{from (3.34).} \tag{3.39}$$

Hence by substituting (3.39) into (3.38), we conclude that

$$\tilde{b}_{i+1,n-\gamma-1} = x\,\tilde{a}_{i,n-\gamma-1} + \frac{\tilde{a}_{i+1,n-\gamma-1}\,\tilde{b}_{i+1,n-\gamma}}{\tilde{a}_{i+1,n-\gamma}}.$$

Therefore, for $1 \leq j < n$, we get

$$\tilde{b}_{i+1,j} = x\ \tilde{a}_{i,j} + \frac{\tilde{a}_{i+1,j}\ \tilde{b}_{i+1,j+1}}{\tilde{a}_{i+1,j+1}},\ 1 \leq j < n, \tag{3.40}$$

and $\tilde{b}_{i+1,n} = y\ \tilde{a}_{i+1,n} + x\ \tilde{a}_{i,n}$. The expression in (3.40) involves no subtractions and requires three arithmetic operations for $j = n$, also requires $4(n-1)$ arithmetic operations for $1 \leq j < n$, so the total number of arithmetic operations to find all the entries in the row $i+1$ is $4n-1$.

Case 2: $k = i$.

For $k = i$ and $j = n$, we have

$$\tilde{b}_{in} = \frac{1}{y}\ \tilde{a}_{in}. \tag{3.41}$$

For $n - \gamma > i$, from (3.30), we get

$$\tilde{b}_{i,n-\gamma-1} = \frac{\tilde{a}_{i,n-\gamma-1}}{y + x\frac{\det A[i, i+2, \ldots, i+1+w|n-\gamma, \ldots, n-\gamma+w] / \det A[i+2, \ldots, i+1+w|n-\gamma+1, \ldots, n-\gamma+w]}{\tilde{a}_{i+1,n-\gamma}}},$$

by using (3.34) we obtain

$$\begin{aligned}\tilde{b}_{i,n-\gamma-1} &= \frac{\tilde{a}_{i,n-\gamma-1}}{y + x\frac{\left(\frac{\tilde{b}_{i+1,n-\gamma} - y\ \tilde{a}_{i+1,n-\gamma}}{x}\right)}{\tilde{a}_{i+1,n-\gamma}}} \\ &= \frac{\tilde{a}_{i,n-\gamma-1}\ \tilde{a}_{i+1,n-\gamma}}{\tilde{b}_{i+1,n-\gamma}}. \end{aligned} \tag{3.42}$$

For $n-\gamma \leq i$, we proceed similarly as in the case $n - \gamma > i$. Therefore, for $1 \leq j < n$, we have

$$\tilde{b}_{i,j} = \frac{\tilde{a}_{i,j}\ \tilde{a}_{i+1,j+1}}{\tilde{b}_{i+1,j+1}},\ 1 \leq j < n, \tag{3.43}$$

and $\tilde{b}_{i,n} = \frac{\tilde{a}_{i,n}}{y}$. The expression in (3.43) involves no subtractions and requires one arithmetic operation for $j = n$, also requires $2(n-1)$ arithmetic operations for $1 \leq j < n$, so the total number of arithmetic operations to find all the entries in the row i is $2n-1$. Therefore, the total number of arithmetic operations is $4n - 1 + 2n - 1 = 6n - 2$. □

Example 3.4. *Returning to Example 3.2, with the following Pascal matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix},$$

we have,

$$\tilde{A} = \begin{bmatrix} 1/4 & 1/6 & 1/4 & 1 \\ 1/6 & 2/10 & 6/10 & 4 \\ 1/4 & 6/10 & 1 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}.$$

Let B be the matrix obtained from A by multiplying the third row by 2 and add to it the second row multiplied by 10. Also multiply the second row by 1/2. Hence

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1/2 & 1 & 3/2 & 2 \\ 12 & 26 & 42 & 60 \\ 1 & 4 & 10 & 20 \end{bmatrix}.$$

By application of Theorem 3.3 with $x = 10$ and $y = 2$, the entries in the first and last rows do not change. Firstly, we find the entries in the third row which are given by

$$\begin{aligned} \tilde{b}_{34} &= y\,\tilde{a}_{34} + x\,\tilde{a}_{24} \\ &= 10(1) + 6(10) \\ &= 60, \end{aligned}$$

the other entries are

$$\begin{aligned} \tilde{b}_{33} &= x\,\tilde{a}_{23} + \frac{\tilde{a}_{33}\,\tilde{b}_{34}}{\tilde{a}_{34}} \\ &= 10\left(\frac{6}{10}\right) + \frac{1(60)}{10} \\ &= 12, \end{aligned}$$

$$\begin{aligned} \tilde{b}_{32} &= x\,\tilde{a}_{22} + \frac{\tilde{a}_{32}\,\tilde{b}_{33}}{\tilde{a}_{33}} \\ &= 10\left(\frac{2}{10}\right) + \frac{6(12)}{10} \\ &= \frac{46}{5}, \end{aligned}$$

$$\begin{aligned} \tilde{b}_{31} &= x\,\tilde{a}_{21} + \frac{\tilde{a}_{31}\,\tilde{b}_{32}}{\tilde{a}_{32}} \\ &= 10\left(\frac{1}{6}\right) + \frac{1}{4}\frac{10}{6}\left(\frac{46}{5}\right) \\ &= \frac{11}{2}. \end{aligned}$$

Also, the entries in the second row are changed as follows

$$\begin{aligned}\tilde{b}_{24} &= \frac{1}{2}\tilde{a}_{24} \\ &= \frac{1}{2}(4) = 2,\end{aligned}$$

the other entries are

$$\begin{aligned}\tilde{b}_{23} &= \frac{\tilde{a}_{23}\tilde{a}_{34}}{\tilde{b}_{34}} \\ &= \frac{\frac{6}{10}(10)}{60} \\ &= \frac{1}{10}, \\ \tilde{b}_{22} &= \frac{\tilde{a}_{22}\tilde{a}_{33}}{\tilde{b}_{33}} \\ &= \frac{\frac{2}{10}(1)}{12} \\ &= \frac{1}{60}, \\ \tilde{b}_{21} &= \frac{\tilde{a}_{21}\tilde{a}_{32}}{\tilde{b}_{32}} \\ &= \frac{\frac{1}{6}\left(\frac{6}{10}\right)}{\frac{46}{5}} \\ &= \frac{1}{92}.\end{aligned}$$

Therefore, the matrix $\tilde{B}$ is

$$\tilde{B} = \begin{bmatrix} 1/4 & 1/6 & 1/4 & 1 \\ 1/92 & 1/60 & 1/10 & 2 \\ 11/2 & 46/5 & 12 & 60 \\ 1 & 4 & 10 & 20 \end{bmatrix},$$

as in Example 3.2.

In the following algorithm we return to Remark 3.1 to extend Theorem 3.3 to the nonsingular TN case.

Algorithm 3.1. *For a given $\tilde{A}$ which is the matrix obtained by running the condensed form of the Cauchon algorithm on the nonsingular TN matrix A, x, y, and i also are given, the following algorithm implements the procedure from Theorem 3.3 and computes $\tilde{B}$ in at most $6n-2$ arithmetic operations.*

- $\tilde{b}_{kj} := \tilde{a}_{kj}$, for $k \neq i, i+1, j = 1, \ldots, n$.
- $\tilde{b}_{i+1,n} := y\, \tilde{a}_{i+1,n} + x\, \tilde{a}_{in}$ and $\tilde{b}_{in} = \frac{\tilde{a}_{in}}{y}$.
- For $j = n-1, \ldots, 1$,
    1. If $\tilde{a}_{i+1,j+1} \neq 0$, then

$$\tilde{b}_{kj} = \begin{cases} x\, \tilde{a}_{k-1,j} + \frac{\tilde{a}_{kj}\, \tilde{b}_{k,j+1}}{\tilde{a}_{k,j+1}}, & \text{if } \; k = i+1, \\ \frac{\tilde{a}_{kj}\tilde{a}_{k+1,j+1}}{\tilde{b}_{k+1,j+1}}, & \text{if } \; k = i. \end{cases}$$

    2. If $\tilde{a}_{i+1,j+1} = 0$ and $\tilde{a}_{i+1,j} = 0$, then

$$\tilde{b}_{kj} = \begin{cases} x\, \tilde{a}_{k-1,j}, & \text{if } \; k = i+1, \\ 0, & \text{if } \; k = i. \end{cases}$$

    3. If $\tilde{a}_{i+1,j+1} = 0$ and $\tilde{a}_{i+1,j} \neq 0$, then set $u_j := \min\{h \in \{j+2, \ldots, n\} | \tilde{a}_{i+1,h} \neq 0\}$ (we define that $u_j := \infty$ if the set is empty).
    If $u_j \neq \infty$, then

$$\tilde{b}_{kj} = \begin{cases} x\, \tilde{a}_{k-1,j} + \frac{\tilde{a}_{kj}\, \tilde{b}_{k,u_j}}{\tilde{a}_{k,u_j}}, & \text{if } \; k = i+1, \\ \frac{\tilde{a}_{kj}\tilde{a}_{k+1,u_j}}{\tilde{b}_{k+1,u_j}}, & \text{if } \; k = i. \end{cases}$$

    If $u_j = \infty$, then

$$\tilde{b}_{kj} = \begin{cases} y\, \tilde{a}_{kj} + x\, \tilde{a}_{k-1,j}, & \text{if } \; k = i+1, \\ \frac{\tilde{a}_{kj}}{y}, & \text{if } \; k = i. \end{cases}$$

3.2 Elementary Operations and the Bidiagonal Factorization

In this section, we present how the bidiagonal factorization of a given nonsingular totally nonnegative matrix A changes by adding a positive multiple of a column to the previous one. We define the matrix $J_i(x, y)$ as a lower bidiagonal matrix with entries

$$J_i(x,y) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & y & & \\ & & x & 1/y & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix},$$

where $x, y > 0$, and the entries y, x and $1/y$ are in the positions $(i-1, i-1)$, $(i, i-1)$, and (i, i), respectively.

In [Koe05], it was studied how the bidiagonal factorization of a nonsingular TN matrix A, denoted as $BD(A)$, changes when we multiply the ith column in the matrix A by a positive scalar x/y and add it to the column $i-1$, and scale column i and the resulting column $i-1$ by $1/y$ and y, respectively. In matrix language, this is equivalent to the product $AJ_i(x,y)$.

Theorem 3.4. *[Koe05] Let $x, y > 0$ and $BD(A)$ which is the bidiagonal factorization of a nonsingular TN matrix A are given. The $BD(AJ_i(x,y)$ can be computed in at most $4n + 2i + 12$ arithmetic operations without performing any subtractions.*

Let A be a nonsingular TN matrix, with a given bidiagonal factorization $BD(A)$. When we multiply the matrix A from the right by $J_i(x,y)$ in (3.44), we have

$$AJ_i(x,y) = L^{(1)}L^{(2)}L^{(3)}\dots L^{(n-2)}L^{(n-1)}DU^{(n-1)}\dots U^{(3)}U^{(2)}U^{(1)}J_i(x,y). \qquad (3.44)$$

This operation preserves the structure of the bidiagonal factorization as it was pointed out in [Koe05]:

$$\begin{aligned}
AJ_i &= L^{(1)}\dots L^{(i-2)}L^{(i-1)}L^{(i)}\dots L^{(n-2)}L^{(n-1)}DU^{(n-1)}\dots U^{(3)}U^{(2)}\underline{U^{(1)}J_i} \\
&= L^{(1)}\dots L^{(i-2)}L^{(i-1)}L^{(i)}\dots L^{(n-2)}L^{(n-1)}DU^{(n-1)}\dots U^{(3)}\underline{U^{(2)}J_i^{(1)}}\mathcal{U}^{(1)} \\
&= L^{(1)}\dots L^{(i-2)}L^{(i-1)}L^{(i)}\dots L^{(n-2)}L^{(n-1)}DU^{(n-1)}\dots \underline{U^{(3)}J_i^{(2)}}\mathcal{U}^{(2)}\mathcal{U}^{(1)} \\
&\vdots \\
&= L^{(1)}\dots L^{(i-2)}L^{(i-1)}L^{(i)}\dots L^{(n-2)}L^{(n-1)}\underline{DJ_i^{(n-1)}}\mathcal{U}^{(n-1)}\dots \mathcal{U}^{(2)}\mathcal{U}^{(1)} \\
&= L^{(1)}\dots L^{(i-2)}L^{(i-1)}L^{(i)}\dots L^{(n-2)}\underline{L^{(n-1)}J_i^{(n)}}\mathcal{D}\mathcal{U}^{(n-1)}\dots \mathcal{U}^{(2)}\mathcal{U}^{(1)} \\
&= L^{(1)}\dots L^{(i-2)}L^{(i-1)}L^{(i)}\dots L^{(n-3)}\underline{L^{(n-2)}J_{i+1}^{(n+1)}}\mathcal{L}^{(n-1)}\mathcal{D}\mathcal{U}^{(n-1)}\dots \mathcal{U}^{(2)}\mathcal{U}^{(1)} \\
&= L^{(1)}\dots L^{(i-2)}L^{(i-1)}L^{(i)}\dots \underline{L^{(n-3)}J_{i+2}^{(n+2)}}\mathcal{L}^{(n-2)}\mathcal{L}^{(n-1)}\mathcal{D}\mathcal{U}^{(n-1)}\dots \mathcal{U}^{(2)}\mathcal{U}^{(1)} \\
&\vdots \\
&= L^{(1)}\dots L^{(i-2)}\underline{L^{(i-1)}J_n^{(2n-i)}}\mathcal{L}^{(i)}\dots \mathcal{L}^{(n-2)}\mathcal{L}^{(n-1)}\mathcal{D}\mathcal{U}^{(n-1)}\dots \mathcal{U}^{(2)}\mathcal{U}^{(1)}, \\
&= L^{(1)}\dots L^{(i-2)}\mathcal{L}^{(i-1)}\mathcal{L}^{(i)}\dots \mathcal{L}^{(n-2)}\mathcal{L}^{(n-1)}\mathcal{D}\mathcal{U}^{(n-1)}\dots \mathcal{U}^{(2)}\mathcal{U}^{(1)}
\end{aligned}$$

where the matrix $J_m^{(k)} = J_m(x_k, y_k)$ for some x_k and y_k. The matrices $\mathcal{L}^{(k)}$, $\mathcal{D}$ and $\mathcal{U}^{(k)}$, $k = 1, \ldots, n-1$, are unit lower bidiagonal, diagonal, and unit upper bidiagonal, respectively.

For the upper bidiagonal matrices U we have the following transformation:

$$J_i(x', y') \cdot \mathcal{U} = U \cdot J_i(x, y), \tag{3.45}$$

where $x, y, x', y' > 0$ and

$$U = \begin{bmatrix} 1 & u_1 & & & \\ & 1 & u_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & u_{n-1} \\ & & & & 1 \end{bmatrix},$$

and

$$\mathcal{U} = \begin{bmatrix} 1 & u'_1 & & & \\ & 1 & u'_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & u'_{n-1} \\ & & & & 1 \end{bmatrix}.$$

The entries on both sides of (3.45) are given by

$$\begin{aligned} x' &= x, \\ y' &= y + u_{i-1}x, \\ u'_{i-2} &= u_{i-2}y, \\ u'_{i-1} &= u_{i-1}/(yy'), \\ u'_i &= u_i y', \\ u'_j &= u_j, \ for\ j \notin \{i-2, i-1, i\}. \end{aligned} \tag{3.46}$$

For the diagonal matrix D we have the following transformation:

$$J_i(x', 1) \cdot \mathcal{D} = D \cdot J_i(x, y), \tag{3.47}$$

The entries on both sides of (3.47) are given by

$$\begin{aligned} \mathcal{D} &= \operatorname{diag}(d_1, d_2, \ldots, d_{i-2}, d_{i-1}y, d_i/y, d_{i+1}, \ldots, n), \\ x' &= d_i x/(d_{i-1}y). \end{aligned} \tag{3.48}$$

For the lower bidiagonal matrices L we have the following transformation:

$$J_{k+1}(x', 1) \cdot \mathcal{L} = L \cdot J_k(x, 1), \tag{3.49}$$

where $x, x' > 0$, with the convention $J_{n+1} = I_n$, and

$$L = \begin{bmatrix} 1 & & & & \\ l_1 & 1 & & & \\ & \ddots & \ddots & & \\ & & l_{n-2} & 1 & \\ & & & l_{n-1} & 1 \end{bmatrix},$$

and

$$\mathcal{L} = \begin{bmatrix} 1 & & & & \\ l'_1 & 1 & & & \\ & \ddots & \ddots & & \\ & & l'_{n-2} & 1 & \\ & & & l'_{n-1} & 1 \end{bmatrix}.$$

The entries on both sides of (3.49) are given by

$$\begin{aligned} l'_j &= l_j, \text{ if } j \notin \{k-1, k\}, \\ l'_{k-1} &= l_{k-1} + x, \\ l'_k &= l_{k-1} l_k / l'_{k-1}, \ if\ k \leq n-1, \\ x' &= x l_k / l'_{k-1}, \ if\ k \leq n-1. \end{aligned}$$

Example 3.5. *For the given Pascal matrix*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix},$$

the bidiagonal factorization of A is given by

$$\begin{aligned} A &= L^{(1)} L^{(2)} L^{(3)} D U^{(3)} U^{(2)} U^{(1)} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &\quad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

When we multiply the matrix A by $J_3(4,2)$, for the upper bidiagonal matrices U we have

$$\begin{aligned} J_3(x',y')\cdot\mathcal{U}^{(1)} &= U^{(1)}\cdot J_3(4,2) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 4 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

by using (3.46) we get

$$\begin{aligned} x' &= x = 4, \\ y' &= y + u_2 x = 2 + (0)4 = 2, \\ u_1' &= u_1 y = (0)2 = 0, \\ u_2' &= u_2/(yy') = 0, \\ u_3' &= u_3 y' = (1)2 = 2, \end{aligned}$$

also, we obtain

$$\begin{aligned} J_3(x',y')\cdot\mathcal{U}^{(2)} &= U^{(2)}\cdot J_3(4,2) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 4 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

the entries of both sides are given by

$$\begin{aligned} x' &= x = 4, \\ y' &= y + u_2 x = 2 + (1)4 = 6, \\ u_1' &= u_1 y = (0)2 = 0, \\ u_2' &= u_2/(yy') = 1/(2.6) = 1/12, \\ u_3' &= u_3 y' = (1)6 = 6, \end{aligned}$$

and for the upper bidiagonal matrix $U^{(3)}$, we have

$$\begin{aligned} J_3(x',y')\cdot\mathcal{U}^{(3)} &= U^{(3)}\cdot J_3(4,6) \\ &= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 4 & 1/6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

have

$$\begin{aligned} J_3(x', y') \cdot \mathcal{U}^{(1)} &= U^{(1)} \cdot J_3(4, 2) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 4 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

by using (3.46) we get

$$\begin{aligned} x' &= x = 4, \\ y' &= y + u_2 x = 2 + (0)4 = 2, \\ u_1' &= u_1 y = (0)2 = 0, \\ u_2' &= u_2/(yy') = 0, \\ u_3' &= u_3 y' = (1)2 = 2, \end{aligned}$$

also, we obtain

$$\begin{aligned} J_3(x', y') \cdot \mathcal{U}^{(2)} &= U^{(2)} \cdot J_3(4, 2) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 4 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

the entries of both sides are given by

$$\begin{aligned} x' &= x = 4, \\ y' &= y + u_2 x = 2 + (1)4 = 6, \\ u_1' &= u_1 y = (0)2 = 0, \\ u_2' &= u_2/(yy') = 1/(2.6) = 1/12, \\ u_3' &= u_3 y' = (1)6 = 6, \end{aligned}$$

and for the upper bidiagonal matrix $U^{(3)}$, we have

$$\begin{aligned} J_3(x', y') \cdot \mathcal{U}^{(3)} &= U^{(3)} \cdot J_3(4, 6) \\ &= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 4 & 1/6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

By using (3.46) we get

$$\begin{aligned}
x' &= x = 4,\\
y' &= y + u_2 x = 6 + (1)4 = 10,\\
u_1' &= u_1 y = (1)6 = 6,\\
u_2' &= u_2/(yy') = 1/(6.10) = 1/60,\\
u_3' &= u_3 y' = (1)10 = 10.
\end{aligned}$$

For the diagonal matrix D we have

$$\begin{aligned}
J_3(x',1)\cdot\mathcal{D} &= D\cdot J_3(4,10)\\
&= \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 10 & 0 & 0\\ 0 & 4 & 1/10 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix},
\end{aligned}$$

by application of (3.48) we get

$$\begin{aligned}
\mathcal{D} &= \operatorname{diag}(d_1,, d_2y, d_3/y, d_4) = \operatorname{diag}(1, 10, 1/10, 1),\\
x' &= d_3x/(d_2y) = 4/10.
\end{aligned}$$

For the lower bidiagonal matrices L we have

$$\begin{aligned}
J_4(x',1)\cdot\mathcal{L}^{(3)} &= L^{(3)}\cdot J_3(4/10,1)\\
&= \begin{bmatrix} 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 4/10 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix},
\end{aligned}$$

the entries on both sides are given by

$$\begin{aligned}
l_1' &= l_1 = 1,\\
l_2' &= l_2 + x = 1 + 4/10 = 14/10,\\
l_3' &= l_2 l_3/l_2' = 10/14,\\
x' &= x l_3/l_2' = 4/14,
\end{aligned}$$

and for the lower bidiagonal matrix $L^{(2)}$, we have

$$\begin{aligned}
J_5(x',1)\cdot\mathcal{L}^{(2)} &= L^{(2)}\cdot J_4(4/14,1)\\
I_5\cdot\mathcal{L}^{(2)} &= \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 4/14 & 1 \end{bmatrix},
\end{aligned}$$

the entries on both sides are given by

$$\begin{aligned} l'_1 &= l_1 = 0, \\ l'_2 &= l_2 = 1, \\ l'_3 &= l_3 + x = 1 + 4/14 = 18/14. \end{aligned}$$

Therefore, the bidiagonal factorization of $AJ_3(4,2)$ is

$$AJ_3(4,2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 18/14 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 14/10 & 1 & 0 \\ 0 & 0 & 10/14 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 1/10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 0 & 0 \\ 0 & 1 & 1/60 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/12 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If we want to find the bidiagonal factorization of $AJ_3(4,2)$ by using Theorem 3.3, we firstly apply the condensed form of the Cauchon algorithm on $G = (A^{\#})^T$ to obtain

$$\tilde{G} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let

$$A' = AJ_3(4,2) = \begin{bmatrix} 1 & 6 & 1/2 & 1 \\ 1 & 16 & 3/2 & 4 \\ 1 & 30 & 3 & 10 \\ 1 & 48 & 5 & 20 \end{bmatrix}.$$

Thus, the matrix

$$B = (A'^{\#})^T = \begin{bmatrix} 20 & 10 & 4 & 1 \\ 5 & 3 & 3/2 & 1/2 \\ 48 & 30 & 16 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

is the matrix obtained from G by multiplying the second row by 4 and add it to third row multiplied by 2, and multiply the second row by 1/2.

By the application of Theorem 3.3 to find the entries of $\tilde{B}$ with $x = 4$ and $y = 4$, the entries in the first and last row do not change. The entries in the third row are changed as follows

$$\tilde{b}_{34} = 6,\ \tilde{b}_{33} = 10,\ \tilde{b}_{32} = 14,\ \tilde{b}_{31} = 18.$$

Also, the entries in the second row are changed as follows

$$\tilde{b}_{24} = 1/2,\ \tilde{b}_{23} = 1/6,\ \tilde{b}_{22} = 1/10,\ \tilde{b}_{21} = 1/14.$$

Therefore, the matrix $\tilde{B}$ is given as follows

$$\tilde{B} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1/14 & 1/10 & 1/6 & 1/2 \\ 18 & 14 & 10 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

By the application of Algorithm 2.4, the bidiagonal factorization of $AJ_3(4,2)$ is

$$A' = AJ_3(4,2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 18/14 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 14/10 & 1 & 0 \\ 0 & 0 & 10/14 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 1/10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 0 & 0 \\ 0 & 1 & 1/60 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/12 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

which coincides with the bidiagonal factorization of A' by the application of Theorem 3.4.

Chapter 4

Spectral Properties of Totally Nonnegative Matrices

In this chapter, we employ the obtained results in Chapter 3 to create a new algorithm that computes the eigenvalues of a nonsingular totally nonnegative matrices.

This chapter consists of two sections. In Section 4.1, we present the Gantmacher–Krein Theorem which gives the main spectral properties of totally positive and oscillatory matrices. In Section 4.2, we introduce an algorithm that computes all the eigenvalues of a nonsingular totally nonnegative matrices to high relative accuracy using the results that obtained in Section 3.1.

4.1 Properties of Eigenvalues and Eigenvectors of Totally Nonnegative Matrices

In this section, we are going to discuss some properties of eigenvalues and eigenvectors of totally nonnegative matrices. Recall that, we say that λ is an *eigenvalue* of a real matrix A if it satisfies $Au = \lambda u$, where λ is a real number and u is a nonzero column vector. We also say that u is an *eigenvector corresponding to the eigenvalue* λ.

To explain some spectral properties of totally positive, totally nonnegative, and oscillatory matrices, we define two counts for the number of sign changes of given vector $u = (u_1, u_2, \dots, u_n)^T \in \mathbb{R}^n$, see [Pin10]. These counts are

$S^-(u)$ is the number of sign changes in the sequence $u_1, u_2, \dots, u_n$ with zero terms are discarded (with the convention $S^-(0) := 0$), and $S^+(u)$ is the maximum number of sign changes in the sequence $u_1, u_2, \dots, u_n$ where zero terms are arbitrarily assigned values $+1$ or -1 (with the convention $S^-(0) := n$).

For example, if $u = (-1, 0, -2, 5, 0, 3, -1)$, then $S^-(u) = 2$ and $S^+(u) = 6$.

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For example, if $u = (-1, 0, -2, 5, 0, 3, -1)$, then $S^-(u) = 2$ and $S^+(u) = 6$.

Definition 4.1. *An $n \times n$ matrix A is called* irreducible *if either $n = 1$ and A is a nonzero matrix or $n \geq 2$ and there is no permutation matrix P such that*

$$PAP^T = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where 0 is an $(n-r) \times r$ zero matrix $(1 \leq r \leq n-1)$. Otherwise is called reducible.

Lemma 4.1. *[FJ11] A TN square matrix A with no zero rows or columns is irreducible if and only if $a_{ij} > 0$, for all i, j such that $|i - j| \leq 1$.*

Definition 4.2. *[GK02] Let $A \in \mathbb{R}^{n \times n}$. Then A is said to be* oscillatory *matrix if A is TN and some power of A is TP.*

Theorem 4.1. *[GK02] Let $A \in \mathbb{R}^{n \times n}$. Then A is oscillatory matrix if and only if A is TN, nonsingular, and irreducible. Furthermore, if A is an oscillatory matrix, then A^{n-1} is TP.*

The following theorem states important and interesting special spectral properties of a TP and oscillatory matrices.

Theorem 4.2. *[Pn98](Gantmacher-Krein) Let A be a square oscillatory matrix. Then the n eigenvalues of A are simple and positive. Let u^k be the eigenvector (unique up to multiplication by a nonzero constant) associated to the eigenvalue λ_k, where $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$. Then*

$$q - 1 \leq S^- \left(\sum_{i=q}^{p} c_i u^i \right) \leq S^+ \left(\sum_{i=q}^{p} c_i u^i \right) \leq p - 1,$$

for each $1 \leq q \leq p \leq n$, and c_i not all zero. In particular, $S^-(u^k) = S^+(u^k) = k - 1$, for $k = 1, 2, \ldots, n$.

Example 4.1. *Let A be the Pascal matrix of order 3*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

The eigenvalues of the matrix A are

$$\lambda_1 = 4 + \sqrt{15},\ \lambda_2 = 1,\ \lambda_3 = 4 - \sqrt{15},$$

which are simple and positive.
and we may take the corresponding eigenvectors to be

$$u^1 = \begin{bmatrix} 5-\sqrt{15} \\ 2\sqrt{15}-5 \\ 5 \end{bmatrix},\ u^2 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix},\ u^3 = \begin{bmatrix} 5+\sqrt{15} \\ -2\sqrt{15}-5 \\ 5 \end{bmatrix}.$$

Observe that,

$$S^-(u^1) = S^+(u^1) = 0,\ S^-(u^2) = S^+(u^2) = 1,\ S^-(u^3) = S^+(u^3) = 2.$$

4.2 Computing Accurate Eigenvalues of Totally Nonnegative Matrices

In Section 2.3, we apply the condensed form of the Cauchon Algorithm on $G = (A^{\#})^T$ to get the bidiagonal factorization of a nonsingular TN matrix A. In this section, we use the results obtained in Section 3.1 to present an algorithm for computing eigenvalues of a nonsingular TN matrix A, given $\tilde{G}$ with high accuracy.

Let $A = [a_{ij}]$ be a nonsingular TN matrix. We start by reducing the matrix A to a tridiagonal matrix [GHW99]. The reduction uses the following elementary operations:

1. subtracting of a positive multiple of a row (column) from the next in order to create a zero;
2. adding of a positive multiple of a row (column) to the previous one.

To reduce the matrix $A = [a_{ij}]$ to a tridiagonal matrix T, we multiply the row $n-1$ by the positive scalar $\frac{a_{n1}}{a_{n-1,1}}$ and subtract it from the row n to a create zero in the position $(n, 1)$. Then we multiply the same scalar to the column n, and add it to the column $n-1$ to complete the similarity. Next, to create a zero in position $(1, n)$, we multiply the column $n-1$ by the positive scalar $\frac{a_{1n}}{a_{1,n-1}}$ and subtract it from the column n, we complete the similarity by multiplying the same scalar to the row n and add it to the row $n-1$. In the same manner, we create zeros in positions $(n-1, 1)$ and $(1, n-1)$. We apply the same process until the matrix is reduced to a tridiagonal matrix [Koe05].

To preserve the accuracy, instead of performing the above elementary operations on A, we perform the elementary operations on $\tilde{G}$ such that subtractions are not required.

Our algorithm takes $\tilde{G}$ as an input. We apply the second elementary operations in Section 3.1 on $\tilde{G}$ to obtain the matrix B which we get by running the condensed form of the Cauchon algorithm on the matrix $G' = (T^{\#})^T$, where T is the tridiagonal matrix that we obtain by applying the so-called *Neville elimination* on A, see, e.g.,

[FJ11], [Adm16]. We obtain from the matrix B the bidiagonal factorization of the tridiagonal matrix T, which has only three nontrivial factors:

$$T = L^{(n-1)} \cdot D \cdot U^{(n-1)}.$$

The matrix T may be not symmetric, so we take the symmetric tridiagonal matrix $\bar{T} = \bar{L}^{(n-1)} \cdot D \cdot \bar{U}^{(n-1)}$, where

$$\bar{l}_i^{(n-1)} = \bar{u}_i^{(n-1)} = \sqrt{l_i^{(n-1)} u_i^{(n-1)}}, i = 1, 2, \ldots, n-1.$$

T and $\bar{T}$ have the same eigenvalues since they have the same characteristic polynomial, see [Koe05], we compute the eigenvalues of $\bar{T}$ as the squares of the singular values of its Cholesky factor $C = D^{1/2}\bar{U}^{(n-1)}$ using the function `bidsvd` [Per]. The eigenvalues of A are the same as the eigenvalues of $\bar{T}$.

Algorithm 4.1. *Let $\tilde{G}$ be given, which is the matrix obtained by running the condensed form of the Cauchon Algorithm on $G = (A^{\#})^T$, the following algorithm computes the eigenvalues of A accurately.*

```
function TNEigenvalues($TildeG$)
$n$ =size($TildeG, 1$)
for $i = n : -1 : 3$ do
    for $i = n : -1 : 3$ do
        $x = TildeG(i, k)/TildeG(i, k+1)$
        $TildeG(i, k) = 0$
        $TildeG$ =AddToNextRow($TildeG, x, 1, k$)
        $x = TildeG(k, i)/TildeG(k+1, i)$
        $TildeG(k, i) = 0$
        $TildeG$ = (AddToNextRow($(TildeG)^T, x, 1, k$))$^T$
    end for
end for
$B = TildeG$
for $i = 1 : n$ do
    $D(i)$ =sqrt($B(n-i+1, n-i+1)$)
end for
for $i = 1 : n-1$ do
    $C(i)$ =sqrt($B(n-i, n-i+1) * B(n-i+1, n-i)/B(n-i+1, n-i+1)$)
end for
Eigenvalues= (bidsvd($D, C$))$^2$
```

If we compare our algorithm with Koev's algorithm [Koe05], the two algorithms compute the eigenvalues of a given nonsingular TN matrices with the same accuracy, but the execution time in Koev's algorithm is less than our execution time since we start in our algorithm with $\tilde{G}$ while Koev starts with $BD(A)$. Figure 4.1 shows the execution time for both algorithms.

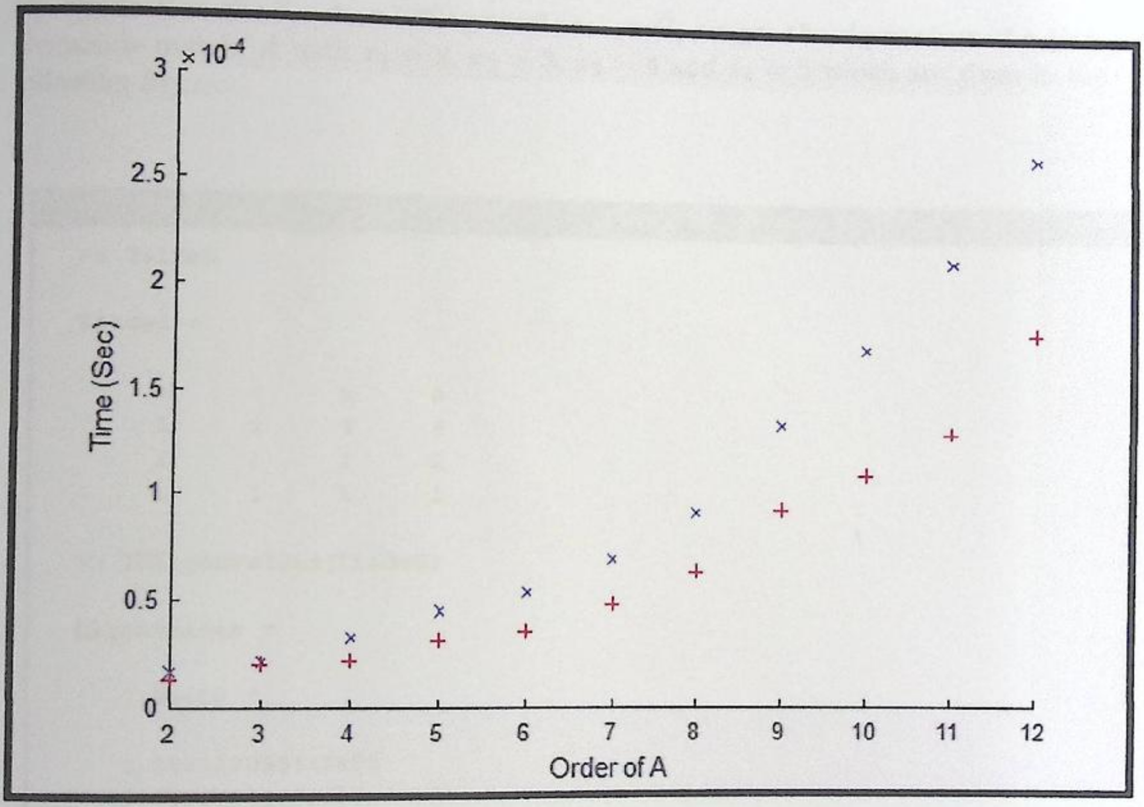
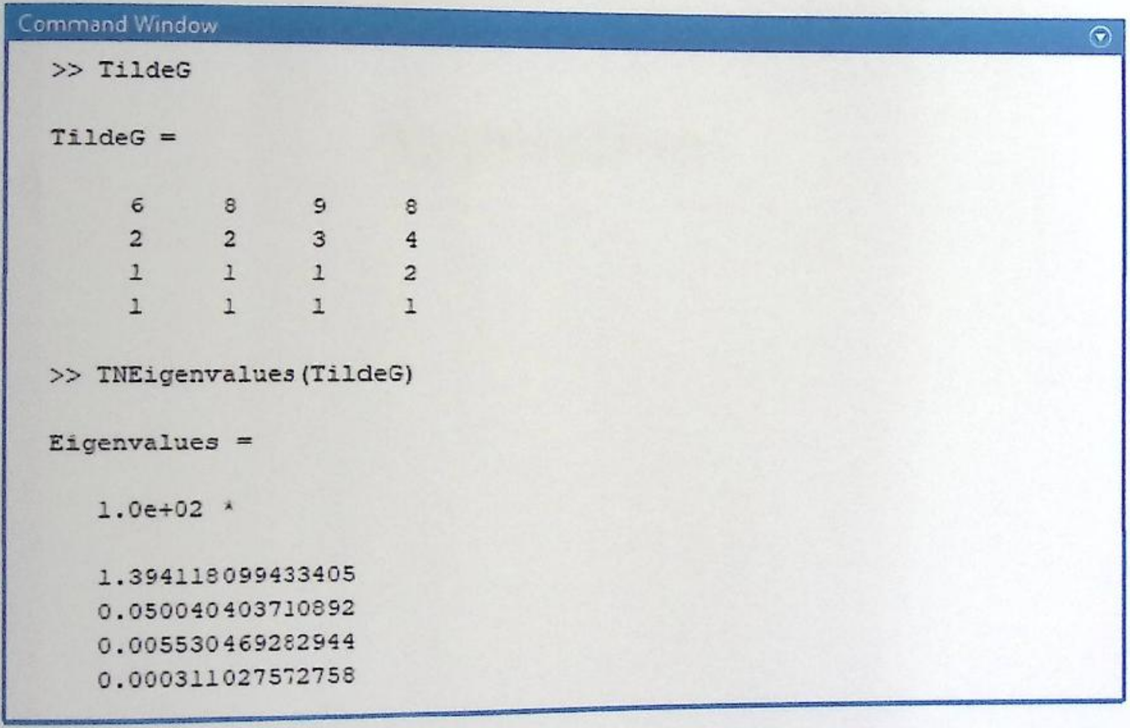


Figure 4.1: Plots of the execution time for computations the eigenvalues of a Vandermonde matrix; "×": Our algorithm, "+": Koev's algorithm.

Example 4.2. *Let $\tilde{G}$ be given as follows*

$$\tilde{G} = \begin{bmatrix} 6 & 8 & 9 & 8 \\ 2 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

By application the function `TNEigenvalues` on $\tilde{G}$, we get the eigenvalues of a Vandermonde matrix A with $x_1 = 2$, $x_2 = 3$, $x_3 = 4$ and $x_4 = 5$ which are given in the following figure.



```
>> TildeG

TildeG =

     6     8     9     8
     2     2     3     4
     1     1     1     2
     1     1     1     1

>> TNEigenvalues(TildeG)

Eigenvalues =

   1.0e+02 *

   1.394118099433405
   0.050040403710892
   0.005530469282944
   0.000311027572758
```

Figure 4.2: An example of the application of Algorithm 4.1 .

Appendices

.1 Condensed Form of the Cauchon Algorithm Using Matlab

```
% function [TildeA]=CFofCA(A)
% Computes the TildeA by application the condenced form of the Cauchon
% Algorithm
function [TildeA]=CFofCA(A)
[n,m]=size(A);
for k=n-1:-1:1
    for i=1:k
        for j=1:m-1
            for x=j+1:m
                  x=j+1;
                while A(k+1,x)==0 && x<m
                     x=x+1;
                end
                if  x==m  && A(k+1,x)==0
                     x=0;
                end
                h=x;
            end
            if h==0
                A1(i,j)=A(i,j);
            else
                A1(i,j)=A(i,j)-A(k+1,j)*A(i,h)/A(k+1,h);
            end
        end
    end
    for i=k+1:n
        for j=1:m
            A1(i,j)=A(i,j);
        end
    for i=1:k
        for j=m
            A1(i,j)=A(i,j);
        end
    end
    end
    A=A1;
end
TildeA=A;
```

.2 Condensed Form of the Restoration Algorithm Using Matlab

```
% function [BarA]=CFofRA(A)
% Computes the BarA by application the condenced form of the Restoration
% Algorithm
function [BarA]=CFofRA(A)
[n,m]=size(A);
for k=2:n
    for i=1:k-1
                if A(k,m)==0
                    A1(i,m-1)=A(i,m-1);
                else
                   A1(i,m-1)= A(i,m-1)+A(k,m-1)*A(i,m)/A(k,m);
                end
           for j=m-2:-1:1
                  h=j+1;
                 while A(k,h)==0 && h<m-1
                       h=h+1;
                 end
                 if  h==m-1  && A(k,h)==0
                       h=0;
                 end
                 if h==0
                      A1(i,j)=A(i,j);
                 else
                       A1(i,j)=A(i,j)+A(k,j)*A1(i,h)/A(k,h);
                 end
          end
    end
 for i=k:n
    for j=1:m
           A1(i,j)=A(i,j);
    end
 end
 for i=1:k-1
    for j= m
           A1(i,j)=A(i,j);
    end
 end
A=A1;
end
BarA=A;
```

.3 Elemantary Operations and Cauchon Algorithm Using Matlab

```
% function TildeB=TNAddToNextRow(TildeA,x,y,i)
% Given TildeA, the following algorithm computes TildeB where B is
   obtained by multiplying the ith row in the matrix A
% by a positive scalar x and add it to the next row multiplied by y,
    and multiply row
% i by 1/y
function TildeB=AddToNextRow(TildeA,x,y,i)
n=size(TildeA,1);
TildeB=TildeA;
      % The entries in the row i and i+1
TildeB(i+1,n)=y*TildeA(i+1,n)+x*TildeA(i,n);
TildeB(i,n)=TildeA(i,n)/y;
     for j=n-1:-1:1
            if TildeA(i+1,j+1)~=0
             TildeB(i+1,j)=x*TildeA(i,j)+TildeA(i+1,j)*TildeB(i+1,j
  +1)/TildeA(i+1,j+1);
             TildeB(i,j)=TildeA(i,j)*TildeA(i+1,j+1)/TildeB(i+1,j+1);
            else
              if TildeA(i+1,j)==0
                   TildeB(i+1,j)=x*TildeA(i,j);
                   TildeB(i,j)=0;
                else
                    h=j+1;
                    while TildeA(i+1,h)==0 && h<n
                        h=h+1;
                    end
                    if h==n && TildeA(i+1,h)==0
                     TildeB(i+1,j)=y*TildeA(i+1,j)+x*TildeA(i,j);
                     TildeB(i,j)=TildeA(i,j)/y;
                    else
               TildeB(i+1,j)=x*TildeA(i,j)+TildeA(i+1,j)*TildeB(i+1,
  h)/TildeA(i+1,h);
               TildeB(i,j)=TildeA(i,j)*TildeA(i+1,h)/TildeB(i+1,h);
                    end
              end
           end
    end
```

.4 Computing Accurate Eigenvalues of Totally Nonnegative Matrices Using Matlab

```
% function TNEigenvalues(B)
% Given B=TildeG, the following algorithm computes the eigenvalues of A
% accurately
function TNEigenvalues(B)
n=size(B,1);
for i=n:-1:3
    for k=1:i-2
        x=B(i,k)/B(i,k+1);
        B(i,k)=0;
        B=AddToNextRow(B,x,1,k);
        x=B(k,i)/B(k+1,i);
        B(k,i)=0;
        B=(AddToNextRow((B).',x,1,k)).';
    end
end
for i=1:n
D(i)=sqrt(B(n-i+1,n-i+1));
end
for i=1:n-1
    C(i)=sqrt(B(n-i,n-i+1)*B(n-i+1,n-i)/B(n-i+1,n-i+1));
end
Eigenvalues=(bidsvd(D.',C)).^2;
```

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