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Ideals In Almost Distributive Lattice

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Supervisor

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2017

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Dedications

To my precious source of happiness and success,

to my lovely supporter,

my adviser.

To the one who takes care of me endlessly despite everything,

to the one whose tough words wake me up,

to the person who teaches me not to give up,

to her as she does all of these for free,

to my mom,

I dedicate this work

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Abstract

This thesis aims to develop a better understanding of Almost distributive lattice and its ideals. We present the definition of Almost distributive lattice, ideals and filters. Also we study some basic properties of Almost distributive lattice and give some examples on this class which includes almost all the existing ring theoretic generalisations of a Boolean ring like regular rings.

The concepts of α-ideals, annihilator ideals, O-ideals and minimal prime ideals are defined, and we furnish the relation between these concepts. In addition It is proved that the set of all annihilator ideals of an Almost distributive lattice forms a complete Boolean algebra and the set of all α-ideals forms a complete distributive lattice. Also characterization theorems for each of these concepts are proved.

Furthermore we present the definition of regular and π-regular ring, which are type of rings included by ADLs, The properties and characterizations theorems connecting between these concepts are studied, and several examples are given.

Contents

Introduction

In 1854, George Boole introduced an important class of algebraic structures in his research work on mathematical logic. In his honour these structures have been named as Boolean algebras. Boole approached logic in a new way reducing it to a simple algebra, incorporating logic into mathematics. By the end of that century, Charles S. Pierce and Ernst Schroeder found it useful to introduce the concept of lattice as a generalization of a Boolean algebra.

It was Garrett Birkhoff's work in the mid thirties of the 20-century that started the general development of the lattice theory. In a brilliant series of papers, he demonstrated the importance of the lattice theory and showed that it provides a unified framework for unrelated developments in many mathematical disciplines. V. Glivenko, Karl Menger, John von Neumann, Oystein Ore, George Gratzer, P.R. Halmos, E.T. Schmidt, G. Szasz, M.H. Stone, R.P. Dilworth and many authors have developed enough for this field.

In 1981, U.M. Swamy and G.C. Rao, introduced another type of generalization of a Boolean algebra called Almost Distributive Lattice(ADL) which is neither complemented nor distributive, and not even a lattice. The class of distributive lattices has occupied a major part of the present lattice theory since lattices were abstracted from Boolean algebras through the class of distributive lattices and the class of distributive lattices has many interesting properties which lattices, in general, do not have. On the other hand, as observed by M.H.

Stone, a Boolean algebra also has a ring structure (called a Boolean ring). Several mathematicians have worked on the ring theoretic generalization of a Boolean algebra. Prominent among them are p-rings (Mc Coy and Montgomery), regular rings (Von Neumann), π-regular rings (Arens and Kaplansky), associate rings (Sussman), p_1-rings (Subrahmanyam), triple systems(Subrahmanyam), Baer rings(Meuborn) and m-domain rings(Subrahmanyam). In order to obtain common abstraction to almost all the existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other, in 1981, Swamy and Rao introduced the concept of an Almost Distributive Lattice as an algebra $(A, \vee, \wedge, 0)$ of type (2, 2, 0) satisfying some identities.

Many important fundamental concepts like ideals, prime ideals, maximal ideals, maximal filters etc. were extended to the class of ADLs. It was observed that the set of all principal ideals of an ADL forms a distributive lattice. This provided a path to extend many existing concepts of lattice theory to the class of ADLs.
In 2008, G.C. Rao and S. Ravi Kumar introduced the concept of Normal ADLs, Relatively normal ADLs and derived a number of characterizations for an ADL to became a Normal. Later, in 2010, G.C. Rao and M. Sambasiva Rao introduced the concepts of annihilator ideals in an ADLs, annulets in an ADLs, α-ideals in an ADLs and prime ideals in an ADLs and studied their properties and proved many characterization theorems.

The material of this thesis lies in three chapters, each contains basic definitions, examples and important theorems.

Chapter one: In this chapter we begin with basic definitions needed in this work. It consists of four sections. Section-1 contains definitions and results related to partially ordered sets, totally orderd sets and duallity. In section-2, we give the definition of lattice, sublattice and distributive lattice, also we study some basic properties of these concepts, on other hand two concepts are mentioned, ideals and filters are defined and studied. In section-3 we give the definition of an ADL and state important properties of an ADL that will be used in developing the further theory, these are taken from [21, 20, 7]. In section-4, we collect definitions and some preliminary results related to ideals, filters, principal ideals and maximal ideals of an ADL.

Chapter two : In this chapter which is also divided into four sections we talk about certain types of ideals in ADL obtained from restrictions on an ideal and thereby a special kind with properties not obtained in a general ideal. In section-1 we present minimal, maximal, and prime ideals, also we derive the properties and the relations between these concepts. In section-2 the concept of Annihilator ideals in an ADL R is given with suitable examples, and we prove some basic properties of the annihilator ideals, analogous to that in a distributive lattice. Furthermore we confirm that the set of all annihilator ideals of an ADL forms a complemented lattice and that if every proper ideal is an annihilator ideal, then R is relatively complemented. In section-3 we discuss various theorems about α-ideal, and derive the properties of these types, also we obtain necessary and sufficient conditions for primes and annihilator ideals to become an α-ideals. Section-4 includes the definition of O-ideals. Examples are presented to illustrate this concept. Also their properties are studied, and a set of equivalent conditions are established for every O-ideal of an ADL to become a principal ideal.

Chapter three: In this chapter regular and π-regular rings are defined as a classes of ADLs. We mentioned some properties and characterizations of each concept. In addition theorems connecting between these concepts are studied, and several examples are given.

Chapter 1

preliminaries

This chapter is devoted to listing some definitions and results that will be used in succeeding chapters. It is not intended to be an exhaustive study of any topic nor it is a complete list of all of the facts which will be used in later chapters. Instead, it is intended to be a collection of those results which will play important roles in what follows.

1.1 Partially Ordered Sets

This section describes the basic theory of partially ordered sets, we give the definition of a poset and totally ordered set. Also we discuss some examples. Duality which is a very important concept will be defined.

Definition 1.1.1 *Let A be a nonempty set. Then a binary relation $\leq$ on A satisfying the following properties for all x, y, $z \in A$ is called a partial order on A :*

- *Reflexivity: $x \leq x$*
- *Antisymmetric: $x \leq y$ and $y \leq x$ imply that $x = y$*
- *Transitivity: $x \leq y$ and $y \leq z$ imply that $x \leq z$.*

In this case, $(A, \leq)$ is called a partial ordered set or simply a poset. In a poset $(A, \leq)$, if $x \leq y$ and $x \neq y$, then we write $x < y$.

Definition 1.1.2 *Let A be a nonempty set and $\leq, \leq'$ be two partial orders on A. Then we say that $\leq$ and $\leq'$ are dual to each other if, for any x, $y \in A$, $x \leq y$ if and only if $y \leq' x$. Note that the dual of any partial order $\leq$ on a nonempty set A is again a partial order on A and is unique. The poset $(A, \leq')$ is called the dual of the poset $(A, \leq)$.*

we must note that to every theorem that concerns an ordered set A there is a corresponding theorem that concerns the dual ordered set A^d. This is obtained by replacing each statement that involves $\leq$, explicitly or implicitly, by its dual and this is the principle of duality

Definition 1.1.3 *Let $(A, \leq)$ be a poset and x, $y \in A$. Then we say that x and y are comparable if either $x \leq y$ or $x \geq y$. Otherwise we say that x and y are incomparable. A poset $(A, \leq)$ in which there are no incomparable elements is called a chain or a totally ordered set.*

The notion of an ordered sets plays an important role not only throughout mathematics but also in adjacent disciplines such as computer science, so now we introduce some examples of partially orderd sets

Example 1.1.1 *On the set $p(E)$ of all subsets of a non-empty set E the relation $\subseteq$ of set inclusion is a partial order, and $(p(E), \subseteq)$ is a poset*

Example 1.1.2 *On the set N of natural numbers the relation $\mid$ of divisibilit, defined by $m \mid n$ if and only if m divides n, is a partial order, and so $(N, \mid)$ is a poset.*

Example 1.1.3 *If $(E_1, \leq_1)$, . . . , $(E_n, \leq_n)$ are ordered sets then the Cartesian product set $\prod_{i=1}^{n} E_i$ can be given the Cartesian order $\leq$ defined by:*

$(x_1, \ . \ . \ . \ , x_n) \leq (y_1, \ . \ . \ . \ , y_n) \leftrightarrow x_i \leq_i y_i.$

Definition 1.1.4 *Let $(A, \leq)$ be a poset, $H \subseteq A$ and $x \in A$. Then*

- *x is called a lower bound of H if $x \leq h$ for all h in H, and dually x is called an upper bound of H if $h \leq x$ for all $h \in H$.*
- *x is called the greatest lower bound or infimum of H if x is a lower bound of H and for any lower bound y of H, we have $y \leq x$, and dually x is called the least upper bound or supremum of H if x is an upper bound of H and for any upper bound y of H, we have $x \leq y$.*

Definition 1.1.5 *Let $(A, \leq)$ be a poset,and $x \in A$. Then:*

- *x is called a minimal element if $a \in A$ and $a \leq x$ implies that $a = x$.*
- *x is called a maximal element if $a \in A$ and $x \leq a$ implies that $x = a$.*

Definition 1.1.6 *Let $(A, \leq)$ be a poset,and $x \in A$. Then:*

- *x is called least element of A if $x \leq a$ for all $a \in A$. If A has least element, then it is unique and is denoted by 0.*
- *x is called greatest element of A if $x \leq a$ for all $a \in A$. If A has greatest element,then it is unique and is denoted by 1.*

A poset $(A, \leq)$ with 0 and 1 is called a bounded poset.

Theorem 1.1.1 *(Zorn's Lemma): If $(A, \leq)$ is a poset in which every chain has an upper bound, then there exists a maximal element in A.*

1.2 Lattice and Distributive Lattice

In this section we present the concept of lattice and sublattice in two ways. Examples are inserted to illustrate this concept, also we discuss the concept of lattice homomorphism, distributive lattice, semilattice. Furthermore we talk about ideals and filters in lattices.

1.2.1 Definition Of Lattices

Many important properties of a poset P are expressed in terms of the existence of certain upper bounds or lower bounds of subsets of P. The most important class of posets defined in this way is lattices.

Lattices can be defined in two ways: one based on the existence of an order relation satisfying certain properties, and one based on the existence of binary operations satisfying certain algebraic properties.

In partially ordered sets,the least upper bound x$\bigvee$ of {x, y } may fail to exist for different reasons, one of them that x and y may have no common upper bound.

A special structure arises when every pair of elements in a posets has a least upper bound and a greatest lower bound.

Definition 1.2.1 *A lattice is partially ordered set (L, $\leq$) in which every pair of element x, $y \in L$ has an infimum and a supremum (in L).*

Theorem 1.2.1 *Let L be a lattice, then the following identities hold for all a, b and $c \in L$*

(L1) Idempotency: $x \bigwedge x = x$ *and* $x \bigvee x = x$.

(L2) Commutativity: $x \bigwedge y = y \bigwedge x$ *and* $x \bigvee y = y \bigvee x$.

(L3) Associativity: $(x \bigwedge y) \bigwedge z = x \bigwedge (y \bigwedge z)$ *and* $(x \bigvee y) \bigvee z = x \bigvee (y \bigvee z)$.

(L4) Absorption laws: $x \bigwedge (x \bigvee y) = x$ *and* $x \bigvee (x \bigwedge y) = x$.

We introduced lattices as partially ordered sets, however we may adopt an alternative viewpoint. In mathematics the term algebraic structure generally refers to a set with one or more operations defined on it. Lattices can also be characterized as algebraic structures satisfying certain axioms.

Theorem 1.2.2 *If (L, $\bigvee$, $\bigwedge$) is a nonempty set with two binary operations $\wedge$ and $\vee$ satisfying (L1)-(L4), then L is a lattice where meet is $\wedge$, join is $\vee$ and the order relation is given by:*

$$a \leq b \quad if \quad a \vee b = b$$

or equivalently,

$$a \leq b \quad if \quad a \wedge b = a.$$

Moreover, since the set of axioms (L1)-(L4) is self-dual, it follows that if a statement holds in every lattice, then any dual statement holds in every lattice.

Now we can give the following definition of a lattice.

Definition 1.2.2 *An algebraic structure (L, $\bigvee$, $\bigwedge$) consisting of a non empty set L and two binary operations $\bigvee$ and $\bigwedge$ on L is called a lattice if following identities hold for all*

a, b, c $\in$ L

(L1) Idempotency: $x \bigwedge x = x$ and $x \bigvee x = x$.

(L2) Commutativity: $x \bigwedge y = y \bigwedge x$ and $x \bigvee y = y \bigvee x$.

(L3) Associativity: $(x \bigwedge y) \bigwedge z = x \bigwedge (y \bigwedge z)$ and $(x \bigvee y) \bigvee z = x \bigvee (y \bigvee z)$.

(L4) Absorption laws: $x \bigwedge (x \bigvee y) = x$ and $x \bigvee (x \bigwedge y) = x$.

After giving the two definitions of lattices we insert some examples to clarify this concept

Example 1.2.1 *Every chain is a lattice; here we have inf $\{x, y\}$ = min $\{x, y\}$ and sup $\{x, y\}$ = max $\{x, y\}$.*

Example 1.2.2 *For every set E, $(p(E), \cap, \cup)$ is a lattice.*

Example 1.2.3 *$(N_0, |)$the natural numbers with zero under devision is a bounded lattice, the bottom element is 1 and the top element is 0. Here we have inf $\{m, n\}$ = gcd $\{m, n\}$ and sup $\{m, n\}$ = lcm $\{m, n\}$.*

Definition 1.2.3 *A sublattice of a lattice L is a nonempty subset M of L that is a lattice with the same operations on L. That is, if L is a lattice and $M \neq \phi$ is a subset of L such that for every pair of elements a, b in M both $a \bigwedge b$ and $a \bigvee b$ are in M, then M is a sublattice of L.*

Example 1.2.4 *Let $L = \{1, 2, 3, 6, 12\}$ under division,the subset $S = \{1, 2, 3, 12\}$ is a lattice under division but not a sublattice of L, the subset $T = \{1, 2, 3, 6\}$ is a sublattice of L*

Proposition 1.2.1 *In any lattice $(L, \bigvee, \bigwedge)$, if $y \leq z$ then $x \bigwedge y \leq x \bigwedge z$, $\forall x, y, z \in L$*

Remark 1.2.1 *In any lattice $(L, \bigvee, \bigwedge)$, if $x \leq z$ then:*
$x \bigvee (y \bigwedge z) \leq (x \bigvee y) \bigwedge z, \quad \forall x, y, z \in L$

Remark 1.2.2 *In any lattice $(L, \bigvee, \bigwedge)$,we have the distributive inequalities:*
$x \bigwedge (y \bigvee z) \geq (x \bigwedge y) \bigvee (x \bigwedge z)$.
$x \bigvee (y \bigwedge z) \leq (x \bigvee y) \bigwedge (x \bigvee z)$.

Remark 1.2.3 *In any lattice $(L, \bigvee, \bigwedge)$, the following are equivalent:*

- $c \leq a \Rightarrow a \bigwedge (b \bigvee c) = (a \bigwedge b) \bigvee c$.
- $c \leq a \Rightarrow a \bigwedge (b \bigvee c) = (a \bigwedge b) \bigvee (a \bigwedge c)$.
- $a \bigwedge (b \bigvee (a \bigwedge c)) = (a \bigwedge b) \bigvee (a \bigwedge c)$.

$\forall a, b, c \in L$

Remark 1.2.4 *In any lattice* $(A, \bigvee, \bigwedge)$, *the following identities* ***"distributive laws"*** *are equivalent:*

- $x \bigwedge (y \bigvee z) = (x \bigwedge y) \bigvee (x \bigwedge z)$.
- $(x \bigvee y) \bigwedge z = (x \bigwedge z) \bigvee (y \bigwedge z)$.
- $x \bigvee (y \bigwedge z) = (x \bigvee y) \bigwedge (x \bigvee z)$.
- $(x \bigwedge y) \bigvee z = (x \bigvee z) \bigwedge (y \bigvee z)$.

Definition 1.2.4 *A lattice* $(A, \bigvee, \bigwedge)$ *satisfying the* ***distributive laws*** *is called a distributive lattice.*

Note 1.2.1 *In a distributive lattice, if* $c \bigwedge x = c \bigwedge y$ *and* $c \bigvee x = c \bigvee y$ *then* $x = y$

Note 1.2.2 *L is a distributive lattice, if and only if* $\forall\ x, y, z \in L$
$(x \bigwedge y) \bigvee (y \bigwedge z) \bigvee (z \bigwedge x) = (x \bigvee y) \bigwedge (y \bigvee z) \bigwedge (z \bigvee x)$.

Definition 1.2.5 *A partially ordered set P is complete if for every subset A of P both sup A and inf A exist (in P).*
The elements sup A and inf A will be denoted $\bigvee A$ *and* $\bigwedge A$ *respectively.*

Definition 1.2.6 *A bounded lattice is an algebraic structure of the form* $(L, \bigvee, \bigwedge, 0, 1)$ *such that* $(L, \bigvee, \bigwedge)$ *is a lattice, 0 is(the lattice's bottom) the identity element of the join operation* $\bigvee$ *and 1 (the lattice's top) is the identity element of meet operation* $\bigwedge$.

Example 1.2.5 *Let X=* $\{a, b, c\}$ *and let* $L = (X, \bigvee, \bigwedge, 0, 1)$ *be the power set of X, then L is a bouned lattice with* $0 = \phi$ *and* $1 = X$*. Where the join and meet here are the union and intersection respectively.*

Definition 1.2.7 *A bounded lattice (L, $\bigvee$, $\bigwedge$) with 0 and 1 is said to be complemented if to each $x \in L$ there exists $y \in L$ such that $x \bigvee y = 1$ and $x \bigwedge y = 0$.*

Example 1.2.6 *The lattice of subsets of a set A is complemented lattice, for we identify the whole set A as 1 and the empty set as 0, then define the complement of any subset of A as the collection of all elements of A which are not in the subset*

Definition 1.2.8 *Let A be a lattice. For any $x, y \in A$ with $x \preceq y$,*
the set $[x, y] = \{ a \in A : x \leq a \leq y \}$ is called a closed interval in A.

Definition 1.2.9 *A lattice A is called relatively complemented if and only if every closed interval in A is complemented, that is, for any $x, y \in A$ with $x \leq y$ and for any $a \in [x, y]$, there exists $b \in [x, y]$ such that $a \bigwedge b = x$ and $a \bigvee b = y$.*
A is called sectionally complemented if every interval $[0, x]$, $(x \in A)$ is complemented.

Example 1.2.7 *The lattice of subsets of a set A is complemented lattice and relativly complemented lattice, for we identify the whole set A is 1 and the empty set is 0, and then define the complement of any subset of A as the collection of all elements of A which are not in the subset.*

1.2.2 Ideals and Filters

Ideals are of fundamental importance in algebra. Filters the order duals of lattice ideals have a variety of applications, so in this brief subsection we submit definitions and some theorems

Definition 1.2.10 *A non-empty subset I of a lattice A is called an ideal of A if it satisfies the following :*

- x, $y \in I$ *implies* $x \bigvee y \in I$.
- $x \in I$, $a \in A$ *implies* $x \bigwedge a \in I$.

Every ideal I of a lattice A is a sublattice, every lattice is an ideal of it self, and every intersection of ideals of A is an ideal.

Definition 1.2.11 *A dual ideal is called a filter, specifically a filter F in A is a nonempty set in A that inherits finite meets, that is*

- x, $y \in F$ *implies* $x \bigwedge y \in F$.
- $x \in F$, a *in* A *implies* $x \bigvee a \in F$.

The set of all ideals of A is denoted by Id(A) and the set of all filters of A is denoted by F(A).

Example 1.2.8 *The following are ideals in* $(P(X), \cup, \cap)$

- *All subsets not containing an arbitrary element of* X.
- *All finite subsets.*

And if we let (X, τ) *be a topological space and let* $x \in X$, *then the set* $\{V \subseteq X : (\exists\ U \in \tau): x \in U \subseteq V\}$ *is a filter in* $P(X)$.

Theorem 1.2.3 *If A is a lattice, ordered by set inclusion, the set $Id(A)$ of all ideals of A forms a lattice in which the lattice operations are given by* $inf\{J, K\} = J \cap K$,
$sup\{J, K\} = \{x \in A\colon (\exists\, j \in J)(\exists\, k \in K),\ x \leq j \bigvee k\}$.

There are many important concepts in ideals, like minimal, maximal, principle ideals, which we illustrate them briefly in the following definition

Definition 1.2.12 *Let A be a lattice.*

1) A proper ideal J of A is maximal if for any ideal I, $J \subseteq I \subseteq A \Rightarrow I = J$ or $I = A$

2) A proper ideal J is prime if $a \bigwedge b \in J \Rightarrow a \in J$ or $b \in J$, the set of prime ideals of A is called the spectrum of A and is denoted by $spec(A)$

3) For any prime ideal P of A, define
$O(P) = \{\ x \in A : x \bigwedge y = 0, \text{ for some } y \notin P\ \}$. *Then clearly $O(P)$ is an ideal in A such that $O(P) \subseteq P$.*

4) Two ideals I, J of A are called co-maximal if $I \bigvee J =$
$\{\ x \in A\colon (\exists\, j \in J)(\exists\, k \in K),\ x \leq j \bigvee k\ \} = A$.

5) $(a] = \{\ a \bigwedge x : x \in A\ \}$ is the principal ideal generated by a. The set $PI(A)$ of all principal ideals of A is a sublattice of $Id(A)$.

1.3 Almost Distributive Lattice

In this section we introduced the concept of an **almost distributive lattice** as a common abstraction of all existing ring theoretic generalizations of a Boolean algebra on one hand and distributive lattices on the other. In this section we give the definition of an Almost Distributive Lattice (**ADL**) and certain elementary properties of an ADL's.

1.3.1 Definition of ADL

Definition 1.3.1 *[10] An algebraic structure $(R, \bigvee, \bigwedge, 0)$ is called an Almost Distributive Lattice, abbreviated as* ***(ADL)*** *with 0 if it satisfies the following axioms:*

- (L_1) $x \bigvee 0 = x$
- (L_2) $0 \bigwedge x = 0$
- (L_3) $(x \bigvee y) \bigwedge z = (x \bigwedge z) \bigvee (y \bigwedge z)$
- (L_4) $x \bigwedge (y \bigvee z) = (x \bigwedge y) \bigvee (x \bigwedge z)$
- (L_5) $x \bigvee (y \bigwedge z) = (x \bigvee y) \bigwedge (x \bigvee z)$
- (L_6) $(x \bigvee y) \bigwedge y = y$.

  For any x, y, $z \in A$.

An Almost Distributive Lattice(ADL) satisfies almost all the properties of a distributive lattice except possibly commutativity of $\bigwedge$, commutativity of $\bigvee$ and right distributivity of $\bigvee$ over $\bigwedge$, so It can be seen directly that every distributive lattice is an ADL.

Example 1.3.1 *Let X be a non-empty set. Fix $x_0 \in X$. For any x, $y \in X$, define:*

$$x \bigwedge y = \begin{cases} x_0 & \text{if x} = x_0 \\ y & \text{if x} \neq x_0 \end{cases} \qquad x \bigvee y = \begin{cases} y & \text{if } x = x_0, \\ x & \text{if x} \neq x_0 \end{cases}$$

Then (X, $\bigvee$, $\bigwedge$, x_0) is an ADL with x_0 as its 0, this ADL is called a discrete ADL, one can directly note that this example is not a distributive lattice since the commutativity does not hold, also this example shows that every non-empty set can be made into an ADL with any arbitrarily preassigned element as its zero.

Example 1.3.2 *Let R = { 0, a, b, c } and define $\bigvee$ and $\bigwedge$ as follows:*

$\bigvee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

$\bigwedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

clearly (R, $\bigvee$, $\bigwedge$, 0) is an ADL with 0

From now onwards by R we mean an ADL (R, $\bigvee$, $\bigwedge$, 0) unless otherwise mentioned. Now we give some basic results.

Lemma 1.3.1 *[7] For any $a \in R$, we have:*

1. $a \bigwedge 0 = 0$
2. $a \bigwedge a = a$
3. $a \bigvee a = a$

4. $0 \bigvee a = a$.

Proof

(1) We know $0 \bigwedge 0 = 0$ now $(a \bigvee b) \bigwedge b = b$, so $(a \bigvee 0) \bigwedge 0 = 0 \rightarrow (a \bigwedge 0) \bigvee (0 \bigwedge 0)$

$= 0 \rightarrow (a \bigwedge 0) \bigvee 0 = 0 \rightarrow a \bigwedge 0 = 0$

similarly we can prove the fourth one.

(2) $(a \bigvee b) \bigwedge b = b$, so $(0 \bigvee a) \bigwedge a = a \rightarrow (0 \bigwedge a) \bigvee (a \bigwedge a) = a \rightarrow 0 \bigvee (a \bigwedge a) = a$

$\rightarrow a \bigwedge a = a$, in the same way we can prove the third one. ■

Theorem 1.3.1 *[11] For any* a , $b \in R$, *we have:*

1. $(a \bigwedge b) \bigvee b = b$

2. $a \bigvee (a \bigwedge b) = a = a \bigwedge (a \bigvee b)$

3. $a \bigvee (b \bigwedge a) = a = (a \bigvee b) \bigwedge a$.

Proof

(1) Since $b = (a \bigvee b) \bigwedge b$ "by L(6)"

$= (a \bigwedge b) \bigvee (b \bigwedge b) = (a \bigwedge b) \bigvee b$

(2) $a \bigvee (a \bigwedge b) = (a \bigvee a) \bigwedge (a \bigvee b) = a \bigwedge (a \bigvee b)$

also Since $a = a \bigvee 0 = a \bigvee (0 \bigwedge b) = (a \bigvee 0) \bigwedge (a \bigvee b) = a \bigwedge (a \bigvee b)$

(3) $a \bigvee (b \bigwedge a) = (a \bigvee b) \bigwedge (a \bigvee a) = (a \bigvee b) \bigwedge a$

also Since $a = a \bigvee 0 = a \bigvee (b \bigwedge 0) = (a \bigvee b) \bigwedge (a \bigvee 0) = (a \bigvee b) \bigwedge a$ ■

Corollary 1.3.1 *[11] For any* $a, b \in R$,

(1) $a \bigvee b = a$ *if and only if* $a \bigwedge b = b$;

(2) $a \bigvee b = b$ *if and only if* $a \bigwedge b = a$.

In view of the above corollary, we give the following definition.

Definition 1.3.2 *[11] For any a, b ∈ R, we say that a is less than or equal to b and write* $a \leq b$ *if* $a \bigwedge b = a$ *or equivalently,* $a \bigvee b = b$.

From the above definition we remark that "$\leq$" is a partial ordering on R.

In theorem 1.3.1, we mentioned the absorption laws that are valid in ADLs in general. Regarding the remaining absorption laws we have the following:

Theorem 1.3.2 *[10] For any a, b ∈ R, the following are equivalent:*

(1) $(a \bigwedge b) \bigvee a = a$.

(2) $a \bigwedge (b \bigvee a) = a$.

(3) $(b \bigwedge a) \bigvee b = b$.

(4) $b \bigwedge (a \bigvee b) = b$.

(5)$a \bigwedge b = b \bigwedge a$.

(6) $a \bigvee b = b \bigvee a$.

(7) The supremum of a and b exists in R and equals $a \bigvee b$.

(8) There exists x in R such that $a \leq x$ *and* $b \leq x$.

(9) The infimum of a and b exists in R and equals $a \bigwedge b$.

Proof

The equivalence of (1) and (2) as well as that of (3) and (4) follow from (L4).

(5) → (1) and (6) → (2) are clear by using theorem 1.3.1.

We prove (1) → (5):

Assume (1) now $b \bigwedge a = b \bigwedge ((a \bigwedge b) \bigvee a) = (b \bigwedge (a \bigwedge b)) \bigvee (b \bigwedge a)$

$= (a \bigwedge b) \bigvee (a \bigwedge (b \bigwedge a)$ [since $b \bigwedge (a \bigwedge b) = a \bigwedge b$]

$= a \bigwedge (b \bigvee (b \bigwedge a)) = a \bigwedge b$.

Now assume (2). Then

$a \bigvee b = (a \bigwedge (b \bigvee a) \bigvee (b \bigwedge (b \bigvee a) = (a \bigvee b) \bigwedge (b \bigvee a)$

$= ((a \bigvee b) \bigwedge b) \bigvee ((a \bigvee b) \bigwedge a) = b \bigvee a$. Thus (2)$\Rightarrow$ (6).

By interchanging the roles of a and b, we get the equivalence of (1) through (6)

Now assume (6) since, for any a, b $\in$ R, $a \leq a \bigvee b$, by (6), we have $a \bigvee b$ is an upper bound of a and b. Let c be an upper bound of a and b. Then

$(a \bigvee b) \bigwedge c = (a \bigwedge c) \bigvee (b \bigwedge c) = a \bigvee b$ and hence, $a \bigvee b$ is the supremum of a and b.

(7) $\Rightarrow$ (8) is clear.

Now we prove (8) $\Rightarrow$ (1):

Assume (8), then $(a \bigwedge b) \bigvee a = (a \bigwedge b) \bigvee (a \bigwedge x) = a \bigwedge (b \bigvee x) = a \bigwedge x = a$.

The equivalence of (5) and (9) follows, dually, from that of (6) and (7). ■

Remark 1.3.1 *[11] For any a, b $\in$ R, $(a \bigvee b) \bigvee a = a \bigvee b = a \bigvee (b \bigvee a)$*

Proof

$a \bigvee b = (a \bigvee (b \bigwedge (b \bigvee a)) = (a \bigvee b) \bigwedge (a \bigvee (b \bigvee a))$

$= ((a \bigvee b) \bigwedge a) \bigvee (((a \bigvee b) \bigwedge b) \bigvee ((a \bigvee b) \bigwedge a)) = a \bigvee (b \bigvee a)$ ■

Now, since for any a, b, c $\in$ R, we have $a \bigwedge c \leq c$ and $b \bigwedge c \leq c$, hence the following is a consequence of the equivalence of (6) and (8)in Theorem 1.3.2

Corollary 1.3.2 *[10] For any a, b, c $\in$ R*

$(a \bigvee b) \bigwedge c = (b \bigvee a) \bigwedge c$.

Theorem 1.3.3 *[11] The meet operation ($\bigwedge$) is associative*

Proof

$(a \bigwedge b) \bigwedge c = (a \bigwedge b) \bigwedge (c \bigvee (a \bigwedge (b \bigwedge c))$

$= ((a \bigwedge b) \bigwedge c) \bigvee ((a \bigwedge b) \bigwedge (a \bigwedge (b \bigwedge c))) = ((a \bigwedge b) \bigwedge c) \bigvee (a \bigwedge (b \bigwedge c))$

"by Corollary 1.3.1, since $(a \wedge b) \vee (a \wedge (b \wedge c)) = a \wedge (b \vee (b \wedge c)) = a \wedge b$"

$= (((a \wedge b) \wedge c) \vee a) \wedge (((a \wedge b) \wedge c) \vee (b \wedge c)) = a \wedge (b \wedge c)$ ■

As a consequence of theorem 1.3.3 and the equivalence of (5) and (8)in Theorem 1.3.2 we get this corollary.

Corollary 1.3.3 *[11] For any a, b, c $\in R$, $a \wedge b \wedge c = b \wedge a \wedge c$ More generally, we have:*

$a_1, \ldots, a_n$, $b \in R$ and ($i_1, i_2, \ldots, i_n$) is any permutation of (1, 2,....,n), then

$a_{i1} \wedge a_{i2} \wedge \cdots \wedge a_{in} \wedge b = a_1 \wedge a_2 \wedge \cdots \wedge a_n \wedge b$.

If $\vee$ is right distributive over $\wedge$ in R, then for any a, b $\in$ R we have,

$a = (0 \wedge b) \vee a = a \wedge (b \vee a)$, so that $\vee$ is commutative and hence we have the following:

Theorem 1.3.4 *[7] Let (R, $\vee$, $\wedge$, 0) be an ADL. Then the following are equivalent.*

(1) (R, $\vee$, $\wedge$, 0) is a distributive lattice with smallest element 0.

(2) $\wedge$ is commutative in R.

(3) $\vee$ is commutative in R.

(4) $\vee$ is right distributive over $\wedge$ in R.

Remark 1.3.2 *If $a \leq b$ and $x \in R$, then $a \wedge x \leq b \wedge x$, $x \wedge a \leq x \wedge b$, and $x \vee a \leq x \vee b$. But it is not true in general, that is $a \leq b$ need to imply that $a \vee x \leq b \vee x$, for this consider the example 1.3.2 where $0 \leq b$, but $0 \vee a \geq b \vee a$; in fact, R is a distributive lattice with 0 if and only if for any a, b $\in R$, $a \leq b$ implies that $a \vee x \leq b \vee x$ for all $x \in R$.*

Definition 1.3.3 *[11] For any a, b $\in R$, a is said to be compatible with b (written $a \sim b$) if $a \wedge b = b \wedge a$ or equivalently, $a \vee b = b \vee a$.*

A subset S of R is said to be compatible if $a \sim b$ for all a, b $\in S$.

Example 1.3.3 *In example 1.3.2 a, c are compatible since* $a \bigwedge c = c \bigwedge a = c$, *but a, b are incompatible since* $a \bigwedge b \neq b \bigwedge a$.

Definition 1.3.4 *[6]An element* $m \in R$ *is called maximal if it is maximal as in the partially ordered set* $(R, \leq)$. *That is, for any* $x \in R$, $m \leq x$ *implies* $m = x$.

Theorem 1.3.5 *[6] Let R be an ADL and* $m \in R$ *Then the following are equivalent:*

(i) m is a maximal element with respect to $\leq$.

(ii) $m \bigvee x = m$, *for all* $x \in R$.

(iii) $m \bigwedge x = x$, *for all* $x \in R$.

Theorem 1.3.6 *[11] If R has a maximal element, then for any element* $x \in R$, *there exists a maximal element* $y \in R$ *such that* $x \leq y$ *and hence, R is a distributive lattice with 1 if and only if R has a unique maximal element.*

proof:

Let y be a maximal element in R. Then, for any x ∈ R, x ⋁ y is a maximal element and $x \leq x \bigvee y$. ■

Definition 1.3.5 *Let* $(R, \bigvee, \bigwedge, 0)$ *be an ADL with 0. Let* $x, y \in R$ *such that* $x \leq y$. *Then the set*

$[x, y] = \{ a \in R : x \leq a \leq y \}$ *is called an interval in R.*

Remark 1.3.3 *Since for any a,* $b \in [x, y]$, $a \leq y$, *and* $b \leq y$, *then from theorem 1.3.2 by the equivalence of (8) with the other statements, every interval in an ADL R is a bounded distributive lattice under the induced operations* $\bigvee$ *and* $\bigwedge$.

In ADLs homomorphisms are defined in the same way as homomorphisms in lattices.

Definition 1.3.6 *[6] Let R and R' be two ADLs with zeros 0 and 0′ respectively. Then a mapping $f : R \to R'$ is called a homomorphism if it satisfies the following :*

(i) $f(x \bigvee y) = f(x) \bigvee f(y)$.

(ii) $f(x \bigwedge y) = f(x) \bigwedge f(y)$.

Example 1.3.4 *Let R be an ADL as in example 1.3.2 and $R' = A \times B$, where $A = \{ 0, a' \}$, $B = \{0', b_1, b_2 \}$ be two discrete ADLs. Then $(R', \bigwedge, \bigvee, \bar{0})$ is an ADL with respect to the point wise operations and $\bar{0} = (0, 0')$.*

Now define a mapping f: $R \to R'$,as follows: $f(0) = \bar{0}$, $f(a) = (a', b_1)$, $f(b) = (a', b_2)$, and $f(c) = (a', 0')$, one can easily verify that f is an ADL homomorphism

Example 1.3.5 *Let $R = \{ 0, a, b, c \}$ be a discrete ADL. Define a mapping $f : R \to R$ by $f(x) = 0$ for all $x \in R$. Then clearly f is a homomorphism on R.*

Remark 1.3.4 *As in group theory a one-to-one homomorphism is called a monomorphism, an onto homomorphism is called an epimorphism, and a bijective homomorphism is called an isomorphism.*

Also the kernel of the homomorphism f is defined by:

$Ker\, f = \{ x \in R : f(x) = 0' \}$.

Example 1.3.6 *In example 1.3.4 $Ker\, f = \{ x \in R : f(x) = \bar{0} \} = \{ 0 \}$. Which mean that f is a monomorphism.*

In example 1.3.5 $ker\, f = R$, so f here not one-to-one also we can notice that it is not onto.

We end this section by the definition of subADL which also define in analogue way as in lattice

Definition 1.3.7 *A subADL of an ADL R is a nonempty subset of R that is an ADL with the same meet and join operations as R.*

1.4 Ideals and Filters in ADL

In this section we recall certain definitions and results about ideals coincides with the usual concept of an ideal in a lattice, also we present the concept of principle ideals. Finally we prove that the set of all principal ideals of R is a distributive lattice with smallest element 0.

Definition 1.4.1 *[7] A non-empty subset I of an ADL R is called an ideal if and only if it satisfies the following :*

(i) $x,\ y \in I \Rightarrow x \bigvee y \in I$.

(ii) $x \in I,\ a \in R \Rightarrow x \bigwedge a \in I$.

Note the similarity between this definition and the definition of an ideal in a ring (except in a ring with 1, an ideal is almost never a subring)

If I is an ideal of R, then $x \bigwedge a \in I$ for any $a \in I$, and $x \in R$, since $x \bigwedge a = x \bigwedge (a \bigwedge a) = (a \bigwedge x) \bigwedge a \in I$. Therefore in this case, any right ideal in the usual sense is a left ideal two, and hence a two sided ideal in the usual sense. However a left ideal may not be a right ideal, and to illustrate this consider the following example.

Example 1.4.1 *Let D be a discrete ADL. For any* $0 \neq x \in D$, *the set* $\{0, x\}$ *is a left ideal but not a right ideal of D.*

Remark 1.4.1 *Every ideal of ADL R is a subADL .*

Since $x \bigwedge a \in I,\ \forall\ x,\ a \in R$ *and* $x \in I$.

Definition 1.4.2 *[7] A non-empty subset F of an ADL R is called a filter if and only if it satisfies the following :*

(i) $x,\ y \in F \Rightarrow x \bigwedge y \in F$.

(ii) $x \in F,\ a \in R \Rightarrow a \bigvee x \in F$.

If F is a filter of R, then a $\bigvee$ x $\in$ R for any a $\in$ R and x $\in$ R, and since a $\bigvee$ x = (a $\bigvee$ x) $\bigwedge$ (a $\bigvee$ x) = (x $\bigvee$ a) $\bigwedge$ (a $\bigvee$ x) = (x $\bigwedge$ (a $\bigvee$ x)) $\bigvee$ (a $\bigwedge$ (a $\bigvee$ x)) = (x $\bigwedge$ (a $\bigvee$ x)) $\bigvee$ a $\in$ R. Therefore in any ADL, every left filter in the usual sense is a right filter and hence a two sided filter. However a right filter may not be a left filter. For, consider the following example.

Example 1.4.2 *Let D be a discrete ADL. For any $0 \neq x \in D$, the set $\{ x \}$ is a right filter but not a left filter of D.*

Theorem 1.4.1 *[6] For any nonempty subset S of R,*
$(S] := \{ (\bigvee_1^n s_i) \bigwedge x : s_i \in S, x \in R$, n is a positive integer$\}$ is the smallest ideal of R containing S and is called the ideal generated by S.

In particular, for any x $\in$ R, (x] := (x) = $\{x \bigwedge t : t \in R\}$.
The set I(R) of all ideals of R is closed under arbitrary intersections and contains R. Thus I(R) is a complete lattice in which the I $\bigvee$ J of any two ideals I and J of R is:
$\{x \bigvee y$: x $\in$ I and y $\in$ J$\}$.

Theorem 1.4.2 *[7] Let R be an ADL and I an ideal of R.*
Then for any x, $y \in R$, we have the following:
(i) $(x] = \{ x \bigwedge a : a \in R\}$.
(ii) $x \in (y]$ if and only if $y \bigwedge x = x$.
(iii) $x \bigwedge y \in I$ if and only if $y \bigwedge x \in I$.
(iv) $(x \bigwedge y] = (y \bigwedge x]$
(v) If $x \in I$ and $a \in R$, such that $a \leq x$ implies $a \in I$.

proof

(i) If we take S = { x } and applied theorem 1.4.1
(ii) ($\rightarrow$) x $\in$ (y] then x = y $\bigwedge$ a, a $\in$ R so y $\bigwedge$ x = y $\bigwedge$ (y $\bigwedge$ a) = (y $\bigwedge$ y) $\bigwedge$ a = y $\bigwedge$ a = x

(←) Since x = y $\bigwedge$ x and y ∈ (y], from definition of ideal x ∈ (y]

(iii)(→) x $\bigwedge$ y ∈ I then (x $\bigwedge$ y) $\bigwedge$ x ∈ I but (x $\bigwedge$ y) $\bigwedge$ x = y $\bigwedge$ (x $\bigwedge$ x) = y $\bigwedge$ x

(←) y $\bigwedge$ x ∈ I, then (y $\bigwedge$ x) $\bigwedge$ y ∈ I but (y $\bigwedge$ x) $\bigwedge$ y = x $\bigwedge$ (y $\bigwedge$ y) = x $\bigwedge$ y

(iv) a ∈ (x $\bigwedge$ y] ↔ a = (x $\bigwedge$ y) $\bigwedge$ t , t ∈ R ⇔ a = y $\bigwedge$ (x $\bigwedge$ t), t ∈ R ⇔ a = (y $\bigwedge$ x) $\bigwedge$ t, t ∈ R ⇔ a ∈ (y $\bigwedge$ x] ■

Theorem 1.4.3 *[7] For any a, b ∈ R, we have the following:*

1. *(a] $\bigvee$ (b] = (a $\bigvee$ b] = (b $\bigvee$ a]*
2. *(a] $\bigwedge$ (b] = (a $\bigwedge$ b] = (b $\bigwedge$ a]*

proof

1. x ∈ (a $\bigvee$ b] ↔ x = (a $\bigvee$ b) $\bigwedge$ t, t ∈ R.

↔ x = (a $\bigwedge$ t) $\bigvee$ (b $\bigwedge$ t), t ∈ R.

↔ x = x_1 $\bigvee$ y_1, x_1 ∈ (x] and y_1 ∈ (y].

↔ x ∈ (x] $\bigvee$ (y].

similarly we can prove 2. ■

Filters which are the duall of ideals have a duall properties, so we recite the following without proof.

Definition 1.4.3 *[6] Let F be any non-empty subset of R. Then the set*

[F) = $\{ x \bigvee (\bigwedge_{i=1}^{n} a_i) : a_i \in F,\ x \in R,\ n \in N \}$ is the smallest filter of R containing F and is called the filter generated by F.

If F = $\{x\}$, then we write [x) for [F). Therefore for any $x \in R$ [x) is the filter generated by x which is called the principal filter generated by x.

Theorem 1.4.4 *[6] Let R be an ADL and F a filter of R. Then for any x, y ∈ R, we have the following:*

(i) [x) = $\{ a \bigvee x : a \in R \}$.

(ii) $x \in$ [y) if and only if $x = x \bigvee y$.

(iii) $x \bigvee y \in F$ *if and only if* $y \bigvee x \in F$.

(iv) $[x \bigvee y) = [y \bigvee x)$.

(v) If $x \in F$ *and* $a \in R$ *such that* $x \leq a$ *implies* $a \in F$.

we must note that the set F(R) of all filters of R forms a distributive lattice under set inclusion in which the g.l.b and l.u.b of any two filters F,G of R are given respectively by:

$F \bigwedge G = F \cap G$ and

$F \bigvee G = \{ f \bigwedge g : f \in F \text{ and } g \in G \}$.

Corollary 1.4.1 *[6] Let* $x, y \in R$. *Then the following are equivalent:*

(i) $[x) \bigvee [y) = [x \bigwedge y) = [y \bigwedge x)$.

(ii) $[x) \bigwedge [y) = [x \bigvee y) = [y \bigvee x)$.

After proving theorem 1.4.3, one can immediately prove the following theorem:

Theorem 1.4.5 *[6] The class PI(R) (PF(R)) of all principal ideals (filters) of an ADL R is a sublattice of the distributive lattice I(R) (F(R)) of ideals (filters) of R.*

Now by recalling definition 1.3.6 of a mapping f : $R_1 \rightarrow R_2$ to be a homomorphism, and defining j^c and j^e to any ideal J to be: $f^{-1}(J) = \{x \in R_1 : f(x) \in J \subseteq R_2\}$ and $\{ (f(x)] : x \in J \}$ respectively we end this section by proving the following theorem

Theorem 1.4.6 *[7] Let* R_1 *and* R_2 *be two ADLs and* $f : R_1 \rightarrow R_2$ *a homomorphism.*

1. If J is an ideal of R_2*, then* J^c *is an ideal of* R_1.

2. If $I, I_1 \subseteq R_1$*, and* $I \subseteq I_1 \Rightarrow I^e \subseteq I^e$ *in* R_2

3. If $J, J_1 \subseteq R_2$*, and* $J \subseteq J_1 \Rightarrow J^c \subseteq J^c$ *in* R_1

4. For every $I \in I(R_1)$, $I \subseteq (I^e)^c$ *and* $I = (I^e)^c$ *if f is a bijection.*

5. For every $J \in I(R_2)$, $(J^c)^e \subseteq J$ *and* $(J^c)^e = J$ *if f is onto.*

proof:

1. Let J be an ideal of R_2 and let x, y $\in f^{-1}(J)$, then f(x), f(y) $\in$ J. Since J is an ideal f(x) $\bigvee$ f(y) = f(x $\bigvee$ y) $\in$ J, thus x $\bigvee$ y $\in f^{-1}(J)$.

now if x $\in f^{-1}(J)$ and r $\in R_1$, then f(x) $\in$ J, f(r) $\in R_2$ and hence f(x $\bigwedge$ r) = f(x) $\bigwedge$ f(r) $\in$ J(since J is an ideal). thus x $\bigwedge$ r $\in f^{-1}(J)$, therefore, J^c is an ideal of R_1.

2. Let I $\subseteq I_1$, then f(I) $\subseteq$ f(I_1) $\Rightarrow$ (f(I)] $\subseteq$ (f(I_1)] $\Rightarrow I^e \subseteq I^e$.

3. Let J $\subseteq J_1$, then f^{-1}(J) $\subseteq f^{-1}(J_1)$, therefore $J^c \subseteq J^c$.

4. By definition f(I) $\subseteq I^e$, hence I $\subseteq f^{-1}(I^e) = (I^e)^c$.

Now suppose that f is a bijection and x $\in (I^e)^c$, then f(x) $\in$ (f(I)] this gives f(x) = f(y) $\bigwedge$ r, for some y $\in$ I and r $\in R_2$, since f is onto, there exists s $\in R_2$ such that r = f(s), therefore f(x) = f(y) $\bigwedge$ f(s) = f(y $\bigwedge$ s), since f is one-one, x = y $\bigwedge$ s, and since y $\in$ I, we get x $\in$ I. therefore $(I^e)^c \subseteq$ I. Hence, $(I^e)^c$ = I.

5. Now, $(J^c)^e$ = (f(J^c)] = (f($f^{-1}(J)$)] = { f(x) $\bigwedge$ r : x $\in f^{-1}(J)$, r $\in R_2$ } = { f(x) $\bigwedge$ r : f(x) $\in$ J, r $\in R_2$ } $\subseteq$ J.

Now, suppose f is onto and y $\in$ J, choose, x in R_1 such that f(x) = y, so x $\in f^{-1}(J) = J^c$, now y = f(x) $\in$ f(J^c) $\subseteq$ (f(J^c)] = $(J^c)^e \Rightarrow$ y $\in (J^c)^e$, therefore J = $(J^c)^e$.

Chapter 2

Types Of Ideals In ADL

It is usually possible to put a set of postulates or restrictions on an ideal and thereby obtain a special kind of ideals with properties not in a general ideals. In this chapter we will consider some special types of ideals in ADL, such as : prime ideals, α - ideals , o - ideals and other types,also we derive some relations between them and some conditions for one to become another one .

2.1 Minimal, Maximal, and Prime Ideals

Definition 2.1.1 *[8] A proper ideal P of R is said to be prime if for any x, $y \in R$, $x \bigwedge y \in P$ then either $x \in P$ or $y \in P$.*
Analogously, we can define the concept of prime filter, i.e A proper filter F of R is said to be prime if for any x, $y \in R$, $x \bigvee y \in F$ then either $x \in F$ or $y \in F$.

Example 2.1.1 *Let $R = \{ 0, a, b, c \}$ and define $\bigvee$ and $\bigwedge$ as follows:*

$\bigvee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

$\bigwedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then clearly (R, $\bigvee$, $\bigwedge$, 0) is an ADL with 0, and $P=\{0, c\}$ is a prime ideal of R and $F=\{ a, b \}$ is a prime filter.

Definition 2.1.2 *[8] A proper ideal M is said to be maximal if it is not properly contained in any proper ideal of R, and P is said to be minimal ideal if there exists no ideal of R contained in it.*

Example 2.1.2 *In example 2.1.1 the ideal $P = (0] = \{ 0 \}$ is minimal prime ideal, and $P=\{0, c\}$ is maximal ideal in R.*

Now we include some properties of prime ideals that can be routinely proved by this lemmas:

Lemma 2.1.1 *[17] A subset P of R is a prime ideal if and only if R/P is a prime filter*

Proof :

At first we must prove that R-P is a filter, so let x, y $\in$ R-P then x, y $\in$ R, x, y $\notin$ P, so x, y $\in$ R and x $\bigwedge$ y $\notin$ P. "since if x $\bigwedge$ y $\in$ P and P is prime then either x or y $\in$ P which is a contradiction." now we get that x$\bigwedge$y $\in$ R-P.

Now let x $\in$ R and a $\in$ R-P, then x $\bigvee$ a $\in$ R but $\notin$ P " since if x $\bigvee$ a $\in$ P then (x $\bigvee$ a) $\bigwedge$ a = a $\in$ P which is a contradiction " so x $\bigvee$ a $\in$ R-P , i.e R-P is a filter.

($\Rightarrow$)Assume P is prime and R-P not a prime so there exist x, y $\in$ R such that x $\bigvee$ y $\in$ R-P but x, y $\notin$ R-P, so x, y $\in$ P, but since P is prime ideal then x $\bigvee$ y $\in$ P which is a contradiction.

($\Leftarrow$) Let R-P is a prime filter but P is not prime, so there exist x, y $\in$ R such that x $\bigvee$ y $\in$ R but x, y $\notin$ P, so x, y $\in$ R-P, but since R-P is prime filter then x $\bigwedge$ y $\in$ R-P, which is a contradiction.

Also we can directly verified that if P is minimal then R-P is maximal. ■

Lemma 2.1.2 *[6] Every maximal ideal of R is a prime ideal.*

Proof :

Suppose I $\subseteq$ R is maximal and a $\bigwedge$ b $\in$ I with a $\notin$ I, then the ideal generated by I and a must be R, so that b = p $\bigvee$ a for some p $\in$ I, then

b = b $\bigwedge$ b = (p $\bigvee$ a) $\bigwedge$ b = (p $\bigwedge$ b) $\bigvee$ (a $\bigwedge$ b)$\in$ I, which means b $\in$ I. So I is prime. ■

Of course not every prime is maximal and example 2.1.1 insure that (0] is prime but not maximal since it is included in the prime ideal P = { 0 , c }, also we must note that the above lemma is true in ring theory, but not true in general for lattices, to ensure this consider the following example:

Example 2.1.3 *In the lattice M_3 in the following figure, the ideal I = { 0, a } is maximal. However, b $\bigwedge$ c $\in$ I, but b $\notin$ I and z $\notin$ I, so I is not prime.*

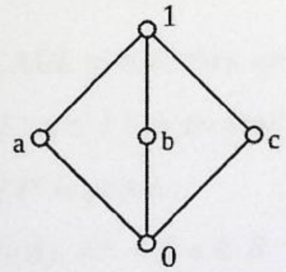

Figure 2.1:

The following lemma is routinely proved by Zorn's lemma

Lemma 2.1.3 *[10] If R is an ADL with maximal element then every proper ideal of R is contained in a maximal ideal of R*

proof:

We can see that Zorn's Lemma may be useful, because the theorem calls for finding a maximal element, so we take P = { proper ideals of R that contain I }, a maximal element of P will finish the proof, so we need only show that P satisfies the condition of Zorn's Lemma. Let L be a chain of P, and let M $=\bigvee_{j\in L} j$, now it just still to prove that M is an ideal containing I.

So at first suppose that x ∈ R, and a ∈ M, then a must be contained in some J ∈ L, Thus, a $\bigwedge$ x ∈ J, and so a $\bigwedge$ x ∈ M.

Now suppose that a, b ∈ M, then there is some J_a, J_b ∈ L with a ∈ J_a and b ∈ J_b, but since L is a chain, so either $J_a \subseteq J_b$ or $J_b \subseteq J_a$, without loss of generality, suppose that $J_b \subseteq J_a$, then both a, b ∈ J_a, hence a $\bigvee$ b ∈ J_a, and so a $\bigvee$ b ∈ M. ■

Since the mapping x → (x] is a homomorphism of the ADL R into the lattice PI(R) of all principal ideals of R, the following theorem is an immediate consequence of the theorem 1.4.6.

Theorem 2.1.1 *[10] Let R be an ADL with 0, then we have the following:*

1. For any ideal I of R, $I^e = \{\ (a] : a \in I\ \}$ is an ideal of PI(R).

Moreover, I is prime if and only if I^e is prime.

2. For any ideal K of the lattice PI(R), $K^c = \{\ a \in R : (a] \in K\ \}$ is an ideal of R, further, K is prime if and only if K^c is prime.

3. For any ideals I_1, I_2 of R, $I_1 \subseteq I_2$ if and only if $(I_1)^e \subseteq I_2^e$.

4. For any ideals K_1, K_2 of PI(R), $K_1 \subseteq K_2$ if and only if $K_1^c \subseteq K_2^c$.

5. $(I^e)^c = I$, for all ideals I of R.

6. $(K^c)^e = K$, for all ideals K of PI(R).

We know that the concept of ideal was introduced in an ADL analogous to that in a distributive lattice, this enables us to extend many existing concepts and results in distributive lattice to the class of ADLs, so now we study many important properties of minimal prime ideals of R which was studied in distributive lattice, and we begin by this definition.

Definition 2.1.3 *[6] Let I be an ideal of R. A prime ideal P in R is said to be a minimal prime ideal belonging to an ideal I if:*

i) $I \subseteq P$

ii) there is no prime ideal Q such that $I \subseteq Q \subseteq P$.

That is P is minimal among the prime ideals of R containing I.

Note that a minimal prime ideal belonging to the zero ideal of R is a minimal prime ideal of R.

Lemma 2.1.4 *[6] Every prime ideal of R contains a minimal prime ideal.*

proof:

Let P be a prime ideal of R and let F = R-P, then F is a prime filter, then by Zorn's lemma, there is a maximal prime filter G in R. Now, $F \subseteq G \Rightarrow$ R-G $\subseteq$ R-F = P. Therefore by lemma 2.1.1 R-G is a minimal prime ideal contained in P. ■

Remark 2.1.1 *we say a subset S is multiplicatively closed subset of R is if $S \neq \phi$ and for any a, $b \in S$, $a \bigwedge b \in S$.*

Theorem 2.1.2 *[10] Let I be an ideal and S be a multiplicatively closed subset of R such that $I \cap S = \phi$. Then there is a prime ideal M of R such that $I \subseteq M$ and $M \cap S = \phi$.*

proof:

Let A = { J : J is an ideal of R, I $\subseteq$ J and S $\cap$ J = ϕ }, since I $\in$ A, A $\neq \phi$, clearly, A satisfies the hypothesis of Zorn's lemma, therefore A has a maximal element M.

Now, we prove M is prime. Let a, b $\in$ R and a $\notin$ M, b $\notin$ M, then (M $\bigvee$ (a]) $\cap$ S $\neq \phi$ and (M $\bigvee$ (b]) $\cap$ S $\neq \phi$.

Let x $\in$ (M $\bigvee$ (a]) $\cap$ S and y $\in$ (M $\bigvee$ (b]) $\cap$ S, since S is a multiplicatively closed subset of R, x $\bigwedge$ y $\in$ S and x $\bigwedge$ y $\in$ (M $\bigvee$ (a]) $\cap$ (M $\bigvee$ (b]) = M $\bigvee$ (a $\bigwedge$ b].

If a $\bigwedge$ b $\in$ M then x $\bigwedge$ y $\in$ M $\cap$ S which is not true as M $\in$ A.

Therefore a $\bigwedge$ b $\notin$ M. Thus M is prime. ■

Since any filter and any sub ADL are multiplicatively closed subsets of R, then the following corollaries are an immediate consequence of theorem 2.1.2

Corollary 2.1.1 *[10] Let S be a sub ADL of R which does not meet the ideal I, then I is contained in an ideal M which is maximal with respect to the property of not meeting S. Moreover, M is prime.*

Corollary 2.1.2 *[17] Let I be an ideal and F be a filter of R such that $I \cap F = \phi$. Then there is a prime ideal M of R such that $I \subseteq M$ and $M \cap F = \phi$.*

By corollary 2.1.2 and since P is prime ideal if and only if R-P is prime filter, so if we take G = R-M we get the following :

Corollary 2.1.3 *[10] Let I be an ideal and F be a filter of R such that $I \cap F = \phi$. Then there is a prime filter G of R such that $F \subseteq G$ and $G \cap I = \phi$.*

Now we insert some theorems to give some important proprieties of the prime ideals in ADLs .

Theorem 2.1.3 *[10] Let I be an ideal and S be a multiplicatively closed subset of an ADL R such that*

$I \cap S = \phi$ then there is a minimal prime ideal T of R such that $I \subseteq T \subseteq$ R-S.

proof:

Let I be an ideal and S be a multiplicatively closed subset of R such that I ∩ S = ϕ, then from theorem 2.1.2, there exists a prime ideal P of R such that I ⊆ P and S ∩ P = ϕ, since P is a prime ideal of R, R-P is a filter of R and S ⊆ R-P, also I ∩ (R-P) = ϕ.

Now let ξ = { F : F is a filter of R, S ⊆ F and I ∩ F = ϕ }, clearly, R-P ∈ ξ and hence $\xi \neq \phi$.

Therefore by Zorn's lemma there exists a maximal element G ∈ ξ. It is easy to verify that G is a prime filter of R and hence R-G is a prime ideal of R. Also I ⊆ (R-G) and S ∩ (R-G) = ϕ. Clearly S ⊆ G.

Now let Q be any other prime ideal of R such that I ⊆ Q and Q ⊆ (R-G), this gives I ∩ (R-Q) = ϕ and also S ⊆ G ⊆ R-Q, but G is a maximal element of ξ. Therefore we get R-Q = G. This gives Q = R-G. ■

Theorem 2.1.4 *[10] Let I, S be any two ideals of R. Then S is a minimal prime ideal belonging to I if and only if R-S is a maximal multiplicatively closed sub set of R with respect to the property of not meeting I.*

proof:

Assume that R-S is a maximal multiplicatively closed subset of R with respect to the property of not meeting I.

Now (R-S) ∩ I = ϕ ⇒ I ⊆ S.

Let x, y ∈ R such that x ∉ S and y ∉ S. Then x, y ∈ R-S and hence x $\wedge$ y ∈ R-S. This gives x $\wedge$ y ∉ S. Therefore S is a prime ideal of R and I ⊆ S, let Q be any other prime ideal of R such that I ⊆ Q and Q ⊆ S. Then R-Q is a multiplicatively closed subset of R and I ∩ (R-Q) = ϕ, but from our assumption R-S is maximal such that (R-S) ∩ I = ϕ.

Therefore we get R-Q ⊆ R-S and hence S ⊆ Q, this gives Q = S, therefore S is a minimal prime ideal of R.

Conversely, let S be a minimal prime ideal belonging to I. Then R-S is a multiplicatively closed subset of R and (R-S) $\cap$ I = ϕ. let T be any other multiplicatively closed subset of R such that T $\cap$ I = ϕ and R-S $\subseteq$ T. Then from theorem 2.1.3, there exists a minimal prime ideal P of R such that I $\subseteq$ P $\subseteq$ R-T.

Clearly, R-T $\subseteq$ S, therefore we get I $\subseteq$ P $\subseteq$ R-T $\subseteq$ S, since S is a minimal prime ideal belonging to I, we get P = S and hence R-T = S.

This gives R-S = T, therefore R-S is maximal multiplicatively closed subset of R with respect to the property of not meeting I. ■

Theorem 2.1.5 *[10] Let I be an ideal of R, and let P be a prime ideal containing I, then P is a minimal prime ideal belonging to I if and only if for each $x \in P$ there is a $y \notin P$ such that $x \bigwedge y \in I$.*

proof:

Let P be a minimal prime ideal belonging to I, then from theorem 2.1.4, R-P is a prime filter which is maximal with respect to the property of not meeting I.

Let x $\in$ P, then x $\notin$ R-P, let F = (R-P) $\bigvee$ [x), suppose F $\cap$ I = ϕ, then there is a prime filter G such that F $\subseteq$ G and G $\cap$ I = ϕ, therefore R-G is a prime ideal and I $\subseteq$ R-G, since R-P $\subseteq$ F $\subseteq$ G, we get R-G $\subseteq$ P and hence R-G = P, that is G = R-P, so that x $\in$ R-P, and this is a contradiction, therefore F $\cap$ I $\neq$ ϕ.

Choose t $\in$ [(R-P) $\bigvee$ [x)] $\cap$ I, then t $\in$ I and t $\in$ (R-P) $\bigvee$ [x), therefore, t = y $\bigwedge$ s, where y $\in$ R-P, s $\in$ [x).

Now t = y $\bigwedge$ (s $\bigvee$ x) = (y $\bigwedge$ s) $\bigvee$ (y $\bigwedge$ x) = t $\bigvee$ (y $\bigwedge$ x). Thus y $\bigwedge$ x $\in$ I, that is for every x $\in$ P, there is y $\notin$ P such that y $\bigwedge$ x $\in$ I.

Conversely, assume the condition, let Q be any prime ideal belonging to I and Q $\subseteq$ p, and let x $\in$ P, then from our assumption, there is y $\notin$ P such that x $\bigwedge$ y $\in$ I, now, Q $\subseteq$ P $\Rightarrow$ x $\bigwedge$ y $\in$ Q and hence x $\in$ Q since y $\notin$ Q.

Therefore P is a minimal prime ideal belonging to I. ■

We end this section by two corollaries that are direct consequence of theorems 2.1.4 and 2.1.5 respectively

Corollary 2.1.4 *[10] Let I be an ideal of R, and let P be a minimal prime ideal belonging to I, then R/P is a prime filter which is maximal with respect to the property of not meeting I.*

Corollary 2.1.5 *[17] A prime ideal P of R is a minimal prime ideal if and only if for each $x \in P$, there is $y \notin P$ such that $x \bigwedge y = 0$.*

2.2 Annihilator Ideals

In this section the concept of Annihilator ideals in an ADL is given with suitable examples, and we prove some basic properties of the annihilator ideals, analogous to that in a distributive lattice.

Definition 2.2.1 *[6] For any non-empty subset A of an ADL R with 0, define* $A^* = \{ x \in R : a \bigwedge x = 0, \text{ for all } a \in A \}$*,* A^* *is called the annihilator of A.*
For any $a \in R$*, we have* $(\{a\})^* = (a]^*$*, where (a] is the principal ideal generated by a.*

Remark 2.2.1 *For any* $\phi \neq A \subseteq R$*, we have clearly* $A \cap A^* = (0]$*, since if there exist* $x \in A \cap A^*$ *then* $x \in A$ *and* $x \in A^*$*, so* $a \bigwedge x = 0, \forall a \in A$*, thus* $x = x \bigwedge x = 0$

Example 2.2.1 *Clearly* $(0]^* = R$ *and* $R^* = (0]$*.*

The following two results can be directly verified by using definition 2.2.1, so we give them without proof.

Remark 2.2.2 *[16] For any non-empty subset A of R,* A^* *is an ideal of R.*

Lemma 2.2.1 *[17] For any non-empty subsets I and J of R, we have the following:*

1. *If* $I \subseteq J$*, then* $J^* \subseteq I^*$*.*
2. $I \subseteq (I^*)^*$
3. $((I^*)^*)^* = I^*$
4. $I^* \cap (I^*)^* = (0]$
5. $I^* \subseteq J^* \Leftrightarrow (J^*)^* \subseteq (I^*)^*$
6. $I \cap J = (0] \Leftrightarrow I \subseteq J^*$

Theorem 2.2.1 *[16] For any two ideals I and J of R, we have the following:*

1. $(I \bigvee J)^* = I^* \cap J^*$
2. $((I \cap J)^*)^* = (I^*)^* \cap (J^*)^*$

proof:

1. Suppose x $\in I^* \cap J^*$ and t = a $\bigvee$ b $\in$ I $\bigvee$ J, where a $\in$ I and b $\in$ J.

Then t $\bigwedge$ x = (a $\bigvee$ b) $\bigwedge$ x = (a $\bigwedge$ x) $\bigvee$ (b $\bigwedge$ x) = 0 $\bigvee$ 0 = 0. Therefore $I^* \cap J^* \subseteq (I \bigvee J)^*$.

The converse follows from Lemma 2.2.2(1).

2. Let x $\in (I^*)^* \cap (J^*)^*$, y $\in (I \cap J)^*$, i $\in$ I and j $\in$ J. Since i $\bigwedge$ j $\in$ I $\cap$ J and y $\in (I \cap J)^*$, we get that (y $\bigwedge$ i) $\bigwedge$ j = 0. Which implies that y $\bigwedge$ i $\in (j]^*$ for all j $\in$ J. Hence y $\bigwedge$ i $\in J^*$. Since x $\in (J^*)^*$, we get (x $\bigwedge$ y) $\bigwedge$ i = 0 for all i $\in$ I. Hence x $\bigwedge$ y $\in I^*$. Since x $\in (I^*)^*$, we get x $\bigwedge$ y $\in (I^*)^*$, thus x $\bigwedge$ y $\in I^* \cap (I^*)^* = (0]$. Hence x $\bigwedge$ y = 0 for all y $\in (I \cap J)^*$. Therefore x $\in ((I \cap J)^*)^*$.

Thus $(I^*)^* \cap (J^*)^* \subseteq ((I \cap J)^*)^*$.

Converse follows from Lemma 2.2.2(1). ■

We now define the concept of annihilator ideal in an ADL R

Definition 2.2.2 *[16] Let R be an ADL, an ideal I of R is called an annihilator ideal if* $I = S^* = \{ y \in R : y \bigwedge s = 0$, *for all* $s \in S \}$ *for some non-empty subset S of R, or equivalently,* $I = (I^*)^*$.

We denote the set of all annihilator ideals of R by A(R).

Example 2.2.2 *Let X be a discrete ADL with 0 and with at least two elements, other than 0. Then* $(X^n, \bigvee, \bigwedge, 0')$ *is an ADL with zero* $0' = (0, 0, ..., 0)$, *where* $\bigvee$, $\bigwedge$ *are defined coordinate-wise.*

Now, let $I = \{(0, a_1, a_2, ..., a_{n-1}) : a_i \in X\}$. *Then it can be observed that I is an ideal of R, now it is clear that* $I^* = \{(x, 0, 0, ..., 0) : x \in X\}$ *and* $(I^*)^* = \{(0, a_1, a_2, ..., a_{n-1}) : a_i \in X\} = I$.

Hence I is an annihilator ideal of R.

Example 2.2.3 *In example 2.1.1 consider the ideal* $I = \{ 0, c \}$. *Then* $(I^*)^* = (0]^* = R$. *Therefore I is not an annihilator ideal in R.*

Example 2.2.4 *Let $R = \{ 0, a, b, c \}$ and define $\bigvee$ and $\bigwedge$ as follows:*

$\bigvee$	0	a	b	c
0	0	a	b	c
a	a	a	b	b
b	b	b	b	b
c	c	b	b	c

$\bigwedge$	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	a	b	c
c	0	0	c	c

Then clearly $(R, \bigvee, \bigwedge, 0)$ is an ADL with 0. Consider the set $I = \{ 0, a \} \subseteq R$, then clearly I is an ideal in R.

Now $I^* = \{ 0, c \}$ and also $(I^*)^* = \{ 0, a \} = I$, thus I is an annihilator ideal in R.

Similarly, the ideal $J = \{ 0, c \}$ of R, is another annihilator ideal in R.

Lemma 2.2.2 *[16] For $I, J \in A(R)$, we have $I \cap J = (I^* \bigvee J^*)^*$.*

proof:

Since $I^* \subseteq I^* \bigvee J^*$, we get $(I^* \bigvee J^*)^* \subseteq I(^*)^* =$ I.

Similarly $(I^* \bigvee J^*)^* \subseteq$ J. Hence $(I^* \bigvee J^*)^* \subseteq$ I $\cap$ J, let x $\in$ I $\cap$ J and y $\in I^* \bigvee J^*$, then y = t $\bigvee$ s for some t $\in I^*$ and s $\in J^*$, now x $\bigwedge$ y = x $\bigwedge$ (t $\bigvee$ s) = (x $\bigwedge$ t) $\bigvee$ (x $\bigwedge$ s) = 0 $\bigvee$ 0 = 0.

Hence x $\in (I^* \bigvee J^*)^*$. Therefore I $\cap$ J = $((I^* \bigvee J^*)^*$. ■

By theorem 2.2.1 page 36, one can immediately prove that the intersection "the meet $\bigwedge$" of two annihilator ideals is again an annihilator ideal, but the join "$\bigvee$" of two annihilator ideals is not necessarily an annihilator ideal, for this consider the following example.

Example 2.2.5 *Consider the distributive lattice $R = \{ 0, a, b, c, 1 \}$ whose Hasse diagram is given in the following figure :*

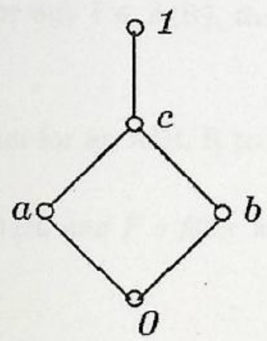

Figure 2.2:

Consider the ideals $I = \{ 0, a \}$ and $J = \{ 0, b \}$. Now $I^* = \{ 0, b \} = J$, and $J^* = \{ 0, a \} = I$. Hence $I^{**} = \{ 0, a \} = I$ and $J^{**} = \{ 0, b \} = J$.

Thus I and J are both annihilator ideals in R, now $I \bigvee J = \{ 0, a, b, c \}$.

So $(I \bigvee J)^* = \{ 0 \}$, hence $(I \bigvee J)^{**} = R$, therefore $I \bigvee J$ is not an annihilator ideal in R.

So we know that if R is an ADL, then we know that (I(R), $\bigvee$, $\bigwedge$) is a distributive lattice, whereas the set A(R) is not a sublattice of I(R) of all ideals of R.

However if we define I $\bigvee$ J = $(I^* \cap J^*)^*$ we get the following theorem :

Theorem 2.2.2 *[15] Let R be an ADL with 0. Then the set A(R) of all annihilator ideals of R forms a complemented lattice.*

proof:

Let I, J $\in$ A(R), then $I^{**} = I$ and J^{**} = J, hence $(I \cap J)^{**} = I^{**} \cap J^{**}$ = I $\cap$ J, by theorem 2.2.1, thus I $\cap$ J $\in$ A(R), we have also I $\bigvee$ J $\in$ A(R). It can be easily observed that (A(R), $\bigwedge$, $\bigvee$) is a lattice.

Since $(0]^*$ = R and R^* = (0], we get that (0], R $\in$ A(R) are the least and the greatest elements of A(R), therefore, (A(R), $\bigvee$, $\bigwedge$) is a bounded lattice.

Now let I $\in$ A(R), then clearly I^* $\in$ A(R) and I $\bigwedge$ I^* = I $\cap$ I^* = (0], I $\bigvee$ I^* = $(I^* \cap I^{**})^*$ = $(I^* \cap I)^*$ = $(0]^*$ = R.

Thus I^* is the complement of I for any I $\in$ A(R), therefore (A(R), $\bigvee$, $\bigwedge$, *, (0], R) is a complemented lattice. ■

Now we derive a sufficient condition for an ADL R to become relatively complemented.

Lemma 2.2.3 *[15] Let R be an ADL and F a filter in R. If $a \leq c$ and $a \in F \bigvee (c]$, then: $a = f \bigwedge c$ for some suitable $f \in F$.*

proof:

Since a $\in$ F $\bigvee$ (c], we can write a = f $\bigwedge$ x for some f $\in$ F and x $\in$ (c], since x $\in$ (c], we get x = x $\bigvee$ c, which implies x $\bigwedge$ c = c. Now a $\leq$ c $\Rightarrow$ a = a $\bigwedge$ c = f $\bigwedge$ x $\bigwedge$ c = f $\bigwedge$ c. ■

Lemma 2.2.4 *[15] Let R be an ADL. If R is not relatively complemented then there exists two distinct prime ideals in R, one of them contains the other.*

proof:

Suppose R is not relatively complemented, then there exists three elements a, b, c $\in$ R such that b $\leq$ c $\leq$ a and c has no complement in the interval [b, a].

Write F = { x $\in$ R : (c $\bigvee$ x) $\bigwedge$ a = a }, we first prove that F is a filter.

Clearly a $\in$ F, let x, y $\in$ F, then [c $\bigvee$ (x $\bigwedge$ y)] $\bigwedge$ a = [(c $\bigvee$ x) $\bigwedge$ (c $\bigvee$ y)] $\bigwedge$ a = (c $\bigvee$ x) $\bigwedge$ [(c $\bigvee$ y) $\bigwedge$ a] = (c $\bigvee$ x) $\bigwedge$ a = a, hence x $\bigwedge$ y $\in$ F, now let x $\in$ F and r $\in$ R, then [c $\bigvee$ r $\bigvee$ x] $\bigwedge$ a = [r $\bigvee$ c $\bigvee$ x] $\bigwedge$ a = (r $\bigwedge$ a) $\bigvee$ [(c $\bigvee$ x) $\bigwedge$ a] = (r $\bigwedge$ a) $\bigvee$ a = a, hence r $\bigvee$ x $\in$ F. Therefore F is a filter.

Now consider the Filter E = F $\bigvee$ [c), suppose b $\in$ E, since b $\leq$ c and b $\in$ F $\bigvee$ [c), by lemma 2.2.3, we have b = f $\bigwedge$ c for some f $\in$ F.

Now b = b $\bigwedge$ a = (f $\bigwedge$ c) $\bigwedge$ a = (c $\bigwedge$ f) $\bigwedge$ a = c $\bigwedge$ (f $\bigwedge$ a) .... (1)

again c $\bigvee$ (f $\bigwedge$ a) = (c $\bigwedge$ a) $\bigvee$ (f $\bigwedge$ a) = (c $\bigvee$ f) $\bigwedge$ a = a, because f $\in$ F ... (2)

From (1) and (2) we can obtain that f $\bigwedge$ a is a relative complement of c in [b, a]. which is a contradiction.

Hence b $\notin$ E, thus (b] $\cap$ E = ϕ, there exists a prime ideal P of R such that (b] $\subseteq$ P and P $\cap$ E = ϕ. Now P $\cap$ E = $\phi \Rightarrow$ P $\cap$ { F $\bigvee$ [c) } = $\phi \Rightarrow$ (P $\cap$ F) $\bigvee$ (P $\cap$ [c)) = $\phi \Rightarrow$ P $\cap$ F = ϕ and P $\cap$ [c) = ϕ.

Now consider the ideal I = (c] $\bigvee$ P, suppose a $\in$ I, since c $\leq$ a and a $\in$ I = (c] $\bigvee$ P, we can write a = c $\bigvee$ p for some p $\in$ P.

Thus (c $\bigvee$ p) $\bigwedge$ a = a, which implies p $\in$ F, hence p $\in$ P $\cap$ F, which is a contradiction to that P $\cap$ F = ϕ, hence I $\cap$ [a) = ϕ. Thus there exists a prime ideal Q such that I $\subseteq$ Q and [a) $\cap$ Q = ϕ. Hence P $\subseteq$ I $\subseteq$ Q. ■

Theorem 2.2.3 *[15] Let R be an ADL with 0, in which every proper ideal is an annihilator ideal. Then R is relatively complemented.*

proof:

Assume that I^{**} = I for all I $\in$ I(R), suppose R is not relatively complemented, then by lemma 2.2.4, there exists two distinct prime ideals say P, Q in R such that P $\subseteq$ Q, choose q $\in$ Q/P, let x $\in$ Q^*, then x $\bigwedge$ q = 0 $\in$ P, so x $\in$ P, because P is a prime ideal and q $\notin$ P.

Hence Q^* $\subseteq$ P, thus Q^* $\subseteq$ P $\subseteq$ Q. Therefore Q^* = Q $\cap$ Q^* = (0], hence Q^{**} = $(0]^*$ = R. Thus Q $\subsetneq$ Q^{**}, because Q is a proper ideal. Which is a contradiction.

Hence R must be relatively complemented. ■

Recall definition 1.2.5 of a homomorphisms in ADLs, now we will study some properties of annihilator preserving homomorphisms and derive a sufficient condition for a homomorphism to be annihilator preserving, so we begin with some lemmas and definition.

Lemma 2.2.5 *[17] Let $f : R \rightarrow R'$ be a homomorphism, then for any nonempty subset A of R, we have: $f(A^*) \subseteq (f(A))^*$.*

proof:

Let a $\in$ f(A^*) and y $\in$ f(A), then there exists b $\in$ A^* and x $\in$ A such that a = f(b) and y = f(x). Now a $\bigwedge$ y = f(b) $\bigwedge$ f(x) = f(b $\bigwedge$ x) = f(0) = $0'$.

That is a $\bigwedge$ y = $0'$ for all y $\in$ f(A), hence a $\in$ $(f(A))^*$. Thus f(A^*) $\subseteq$ $(f(A))^*$. ■

Note that $(f(A))^* \subseteq$ f(A^*) is not true in general, for this consider the following example:

Example 2.2.6 *Let $R = \{\ 0,\ a,\ b,\ c\ \}$ be a discrete ADL, define a mapping $f:\ R \to R$ by $f(x) = 0$ for all $x \in R$, then clearly f is a homomorphism on R.*

Take $A = \{\ a,\ b\ \}$ then $A^* = \{\ 0\ \}$ and $f(A) = \{\ 0\ \}$, so $f(A^*) = \{\ 0\ \}$ and $(f(A))^* = R$. Thus $(f(A))^* \nsubseteq f(A^*)$.

Definition 2.2.3 *[15] Let $f:\ R \to R'$ be a homomorphism, then f is called annihilator preserving if $f(A^*) = (f(A))^*$ for any $(0] \subseteq A \subseteq R$.*

Example 2.2.7 *Let $A = \{\ 0,\ a\ \}$ and $B = \{\ 0,\ b_1,\ b_2\ \}$ be two discrete ADLs. Write $R = A \times B$, then $(R, \bigvee, \bigwedge, 0)$ is an ADL under point-wise operations. Also the zero element in R is (0, 0). Let*

$R' = \{0', a', b', c'\}$ be another ADL in which the operations $\bigvee'$, $\bigwedge'$ are defined as followes:

$\bigvee'$	$0'$	a'	$'b$	c'
$0'$	$0'$	a'	b'	c'
a'	a'	a'	c'	c'
b'	b'	c'	b'	c'
c'	c'	c'	c'	c'

$\bigwedge'$	$0'$	a'	$'b$	c'
$0'$	$0'$	$0'$	$0'$	$0'$
a'	$0'$	a'	$0'$	a'
b'	$0'$	$0'$	b'	b'
c'	$0'$	a'	$'b$	c'

Now define the mapping $f:\ R \to R'$ as follows: $f((0,\ 0)) = 0'$; $f((a,\ 0) = a'$; $f((0,\ b_1)) = f((0,\ b_2)) = b'$; $f((a,\ b_1)) = f((a,\ b_2)) = c'$.

Then clearly f is a homomorphism from R onto R', also it can be verified that f is annihilator preserving.

Example 2.2.8 *Let R and R' be two dense ADLs(i.e. an ADL R in which* $(a]^* = (0]$, *for all* $0 \neq a \in R$).

Then every homomorphism from R into R' is annihilator preserving, since for any $A \subseteq R$, *we have* $f(A^*) = f((0]) = (0'] = (f(A))^*$.

We know that in ring theory if f is a homomorphism which is onto and ker$f = \{ 0 \}$ then f is an isomorphism, but this is not true in general in ADLs, and this is clear in the following example:

Example 2.2.9 *Let* $R = \{0, a, b \}$ *and* $R' = \{ 0', c \}$ *be two disctrete ADLs. Define a mappping* $f : R \to R'$ *by* $f(0) = 0'$ *and* $f(a) = f(b) = c$. *Then clearly f is a homomorphism from R into* R', *also f is onto, and* $Kerf = \{ 0 \}$. *But f is not one to one.*

However, we have the following:

Theorem 2.2.4 *[15] Let R and R' be two ADLs with zero elements 0 and $0'$ respectively and* $f : R \to R'$ *a homomorphism. If* $Kerf = \{ 0 \}$ *and f is onto, then f is annihilator preserving.*

Proof:

Let A be a subset of R such that $(0] \subseteq A \subseteq R$. We have always $f(A^*) \subseteq \{ f(A)\}^*$. Now, let $x \in \{ f(A)\}^* \subseteq R'$. Since f is onto, there exists $y \in R$ such that $f(y) = x \in \{ f(A)\}^* \Rightarrow$ $f(y) \bigwedge f(a) = 0'$ for all $a \in A$.

$\Rightarrow f(y \bigwedge a) = 0' \Rightarrow y \bigwedge a \in Kerf = \{ 0 \} \Rightarrow y \bigwedge a = 0$ for all $a \in A \Rightarrow y \in A^*$

$\Rightarrow x = f(y) \in f(A^*)$ Hence $\{f(A)\}^* \subseteq f(A^*)$. Therefore $\{f(A)\}^* = f(A^*)$. ■

Theorem 2.2.5 *[15] Let R and R' be two ADLs with zero elements 0 and $0'$ respectively and* $f : R \to R'$ *a homomorphism. Then we have the following:*

(1) If f is annihilator preserving and onto, then f(I) is an annihilator ideal of R' for every annihilator ideal I of R.

(2) If f^{-1} preserves annihilators, then $f^{-1}(I)$ is an annihilator ideal of R for every annihilator ideal I of R.

Proof:

(1) Let I be an annihilator ideal of R. Then f(I) is an ideal of R', since f is annihilator preserving, $(f(I))^{**} = f(I^{**}) = f$(I). Therefore f(I) is an annihilator ideal in R'.

(2) Let J be an annihilator ideal of R'. Then f^{-1}(J) is an ideal of R, Since f^{-1} preserves annihilators, we get $(f^{-1}(J))^{**} = f^{-1}(J^{**}) = f^{-1}$(J). ■

Since Ker$f = f^{-1}(0')$ and { $0'$ } is an annihilator ideal in R', we have the following corollary.

Corollary 2.2.1 *[15] Let R and R' be two ADLs with zero elements 0 and $0'$ respectively and $f : R \to R'$ an annihilator preserving homomorphism. Then Kerf is an annihilator ideal of R.*

Recall that two ideals I and J of R are said to be co-maximal if I $\bigvee$ J = R, now we study and prove necessary and sufficient conditions for every prime ideal to be minimal in R through the following theorems.

Theorem 2.2.6 *[8] For any $a \in R$ if the ideals (a] and $(a]^*$ are co- maximal, then $(a] = (a]^{**}$.*

Proof:

For a ∈ R let R = (a] $\bigvee$ $(a]^*$. We have $(a]^{**} = (a]^{**} \cap$ R = $(a]^{**} \cap$ ((a] $\bigvee$ $(a]^*$) =
($(a]^{**} \cap$ (a]) $\bigvee$ ($(a]^{**} \cap (a]^*$) = (a] $\bigvee \phi$ = (a]. since (a] $\subseteq (a]^{**}$). ■

Lemma 2.2.6 *[8] A prime ideal of an ADL R is minimal if and only if $a \in P \Rightarrow (a]^* \nsubseteq P$.*

Proof:

By corollary 2.1.5 P is minimal prime ideal if and only if for each a ∈ P, there is y ∉ P such that x $\bigwedge$ y = 0, so there exist y ∈ $(a]^*$ but y ∉ P, thus $(a]^* \nsubseteq$ P. ■

Theorem 2.2.7 *[8] Every prime ideal in R is minimal prime if and only if the ideals (a] and $(a]^*$ are co-maximal for each $a \in R$.*

Proof:

Let every prime ideal in R be minimal prime, and let there exists a ∈ R such that $(a] \bigvee (a]^* \subseteq R$, select x ∈ R such that $x \notin (a] \bigvee (a]^*$. Hence there exists a prime ideal, say P, in R such that $(a] \bigvee (a]^* \subseteq P$ and $P \cap [x) = \phi$. P being minimal by assumption, (a] and $(a]^*$ can not be contained in P simultaneously (by lemma 2.2.8) . This in turn shows that $R = (a] \bigvee (a]^*$ for each a ∈ R.

Conversely, let $R = (a] \bigvee (a]^*$ for each a ∈ R, and let there exists a prime ideal P in R which is not minimal, then there exists a minimal prime ideal, say M, in R such that $M \subseteq P$. Select x *in* P/M, as $x \notin M$, $(x]^* \subseteq M$ (by lemma 2.2.8). But then $(x] \subseteq P$ and $(x]^* \subseteq P$ will give $R = (x] \bigvee (x]^* \subseteq P$; a contradiction.

Hence every prime ideal in R must be minimal prime. ■

2.3 α - Ideals

In this section we introduce the concept of an α - ideal in an ADL R, analogous to that in a distributive lattice. We study some basic properties of the class of α - ideals, also we derive some conditions for prime ideals to become α - ideals, and to annihilator ideals also. Finally we characterize α-ideals in terms of annulets.

Definition 2.3.1 *[17] Let R be an ADL with 0. An ideal I of R is called an α - ideal if* $(x]^{**} \subseteq I$ *for all* $x \in I$.

Example 2.3.1 *Let* $A = \{0, a\}$ *and* $B = \{0, b_1, b_2\}$ *be two discrete ADLs. Write* $R = A \times B = \{(0, 0), (0, b_1), (0, b_2), (a, 0), (a, b_1), (a, b_2)\}$. *Then* $(R, \bigvee, \bigwedge, 0')$ *is an ADL where* $0' = (0, 0)$, *under point-wise operations. Take* $I = \{(0, 0), (0, b_1), (0, b_2)\}$. *Clearly I is an ideal of R, and* $((0,0)]^{**} = \{(0, 0)\} \subseteq I$.
Also $((0,b_1)]^{*} = ((0,b_2)]^{*} = \{(0, 0), (a, 0)\}$. *So* $((0,b_1)]^{**} = ((0,b_2)]^{**} = ((0,0)]^{*} \cap ((a,0)]^{*}$ $= R \cap \{(0, 0), (0,b_1), (0, b_2)\} = I$. *Thus I is an* α *- ideal of R.*

Example 2.3.2 *Let R be an ADL with 0 and S a multiplicatively closed subset of R. Then* $I = \{ x \in R : x \bigwedge y = 0 \text{ for some } y \in S \}$ *is an* α *- ideal of R.*
Observe that $I = \bigcup_{x \in S}(x]^{*}$.
Now, let $a \in I$, *then* $a \in (x]^{*}$ *for some* $x \in S$. *So* $(a]^{**} \subseteq (x]^{*}$ *for some* $x \in S$. *Hence* $(a]^{**} \subseteq \bigcup_{x \in S}(x]^{*} = I$. *Therefore I is an* α *- ideal of R.*

Now we insert some useful characterization of α - ideals, but first we begin by this definition

Definition 2.3.2 *[16] Let R be an ADL with 0. For any ideal I of R, define*
$I^{e} = \{ x \in R : (a]^{*} \subseteq (x]^{*} \text{ for some } a \in I \}$

Lemma 2.3.1 *[17] Let R be an ADL with 0. Then for any ideals I, J of R, we have the following :*

(a). $I \subseteq I^e$

(b). $I \subseteq J \Rightarrow I^e \subseteq J^e$

(c). $(I \cap J)^e = I^e \cap J^e$

(d). $I^e \bigvee J^e \subseteq (I \bigvee J)^e$

(e). $(I^e)^e = I^e$

(f). $(I^*)^e = I^*$.

Proof:

(a). Since $\forall$ i $\in$ I, $(i]^* \subseteq (i]^*$ so by the definition I $\subseteq I^e$.

(b). Let I $\subseteq$ J and x $\in I^e$, then

$\exists$ i $\in$ I such that $(i]^* \subseteq (x]^*$, but since I $\subseteq$ J, then $\exists$ i $\in$ J such that $(i]^* \subseteq (x]^*$, so x $\in J^e$, therefore $I^e \subseteq J^e$.

(c). Let x $\in (I \cap J)^e \Leftrightarrow \exists$ a $\in$ (I $\cap$ J) such that

$(a]^* \subseteq (x]^* \Leftrightarrow \exists$ a $\in$ I such that $(a]^* \subseteq (x]^*$, and $\exists$ a $\in$ J such that $(a]^* \subseteq (x]^* \Leftrightarrow$ x $\in I^e$, and x $\in J^e \Leftrightarrow$ x $\in I^e \cap J^e$.

(d). Let x $\in I^e \bigvee J^e$, then x $= i' \bigvee j'$ for some $i' \in I^e$, $j' \in J^e$, so $\exists$ i $\in$ I such that $(i]^* \subseteq (i']^*$, and $\exists$ j $\in$ J such that $(j]^* \subseteq (j']^*$, thus $(i]^* \bigvee (j]^* \subseteq (i']^* \bigvee (j']^*$.

So $(i \bigvee j]^* \subseteq (i' \bigvee j']^*$, therefore $(i \bigvee j]^* \subseteq (x]^*$ for some i $\bigvee$ j $\in$ I $\bigvee$ J, so x $\in (I \bigvee J)^e$.

(e). From (a) we know that $I^e \subseteq (I^e)^e$. Now let x $\in (I^e)^e$, then $\exists$ a $\in I^e$ such that $(a]^* \subseteq (x]^*$, and since a $\in I^e$, $\exists$ b $\in$ I such that $(b]^* \subseteq (a]^*$.

So $\exists$ b $\in$ I such that $(b]^* \subseteq (a]^* \subseteq (x]^*$, therefore x $\in I^e$, and $(I^e)^e \subseteq I^e$, so $(I^e)^e = I^e$.

(f). Let x $\in (I^*)^e$, then $\exists$ a $\in I^*$ such that $(a]^* \subseteq (x]^*$, but I $\subseteq (a]^* \subseteq (x]^*$, so $(x] \subseteq (x]^{**} \subseteq I^*$, thus x $\in I^*$, hence $(I^*)^e \subseteq I^*$

Now, since $(x]^* \subseteq (x]^*$ $\forall$ x $\in I^*$, so we have $I^* \subseteq (I^*)^e$.

Therefor $(I^*)^e = I^*$. ■

Lemma 2.3.2 *[16] Let R be an ADL and I an ideal of R. Then I^e is the smallest α - ideal containing I.*

Proof:

Clearly I^e is an ideal containing I. Suppose x $\in I^e$, then $(a]^* \subseteq (x]^*$ for some a $\in$ I. Now, let t $\in (x]^{**}$, then $(x]^* \subseteq (t]^*$, thus $(a]^* \subseteq (x]^* \subseteq (t]^*$, where a $\in$ I. Hence t $\in I^e$. Thus $(x]^{**} \subseteq I^e$. Therefore I^e is an α - ideal containing I.

Now let K be any α - ideal in R such that I $\subseteq$ K, let x $\in I^e$. Then $(a]^* \subseteq (x]^*$ for some a $\in$ I $\subseteq$ K. Since K is an α - ideal and a $\in$ K, we get x $\in (x]^{**} \subseteq (a]^{**} \subseteq$ K. Hence $I^e \subseteq$ K.

Therefore I^e is the smallest α - ideal containing I. ■

We know that (I(R), $\bigvee$, $\bigwedge$) is a distributive lattice, but the set I_α(R)"the set of all α - ideals of R" is not a sublattice of I(R), however it forms a distributive lattice, this is illustrated by the following example and theorem.

Example 2.3.3 *Consider the distributive lattice R= {0, a, b, c, 1} whose Hasse diagram is given as in figure 2.2*

Consider the ideals I = {0, a} and J = {0,b }. Now $(a]^*$ = {0, b} and $(b]^*$ = {0, a}. Hence $(a]^{**}$ = {0, a} = I and $(b]^{**}$ = {0, b} = J.

Thus I and J are both α-ideals in R, now I $\bigvee$ J = {0, a, b, c}.

So $((c])^*$ = { 0 }, hence $((c])^{**}$ = R, thus I $\bigvee$ J is not an α- ideal in R, therefore $I_\alpha(R)$ is not a sublattice of I(R).

Theorem 2.3.1 *[16] Let R be an ADL with 0. Then $I_\alpha(R)$ forms a distributive lattice.*

Proof:

For I, J $\in I_\alpha$(R), define I $\bigwedge$ J = I $\cap$ J and I $\breve{\bigvee}$ J = $(I \bigvee J)^e$. Then clearly I $\cap$ J is an α-ideal and the infimum of both I and J in I_α(R). Hence I $\cap$ J $\in I_\alpha$(R).

Also I $\breve{\bigvee}$ J is an α-ideal. Clearly I, J $\subseteq$ I $\bigvee$ J $\subseteq$ I $\breve{\bigvee}$ J.

Now let K be any upper bound for I, J in I_α(R). Hence I $\bigvee$ J $\subseteq$ K, which implies that $(I \bigvee J)^e \subseteq K^e =$ K. Therefore I $\breve{\bigvee}$ J is the supremum of both I and J in I_α(R). Hence (I_α(R), $\bigwedge$, $\breve{\bigvee}$) is a lattice.

We now prove the distributivity. Let I, J, K $\in I_\alpha$(R). Now (I $\breve{\bigvee}$ J) $\cap$ (I $\breve{\bigvee}$ K) =

$(I \bigvee J)^e \cap (I \bigvee K)^e = [(\text{I} \bigvee \text{J}) \cap (I \bigvee K)]^e = [I \bigvee (J \cap K)]^e =$ I $\breve{\bigvee}$ (J $\cap$ K).

Thus (I_α(R), $\bigwedge$, $\breve{\bigvee}$) is a distributive lattice. ■

Corollary 2.3.1 *[17] The set of all α-ideals of R is a complete lattice, ordered by set inclusion.*

Proof:

For I, J $\in I_\alpha$(R), define I $\leq$ J $\Leftrightarrow$ I $\subseteq$ J. Clearly (I_α(R), $\leq$) is a partially ordered set.

(0] and R are the α-ideals in R and they are the bounds for I_α(R). Let { I_i : i $\in \Delta$ } be a family of α-ideals in I_α(R). By the extention of lemma 2.3.1(c), we get $(\bigcap_{i\in\Delta} I_i)^e =$ $(\bigcap_{i\in\Delta} I_i^e)$.

Clearly $(\bigcap_{i\in\Delta} I_i)$ is the infimum of { I_i : i $\in \Delta$ } in I_α(R).

Therefore I_α(R) is a complete lattice. ■

Lemma 2.3.3 *[17] Let R be an ADL with 0. Then for any $x \in R$,*

$(x]^* = \bigcap$ { *P : P is a minimal prime ideal and* $x \notin P$ }

Proof:

Clearly $(x]^* \subseteq \bigcap$ { P : P is a minimal prime ideal and x $\notin$ P }, since $\forall$ y $\in (x]^*$, x $\bigwedge$ y = 0 $\in$ P, which is any minimal prime ideal, and since P is prime and x $\notin$ P therefore y $\in$ P. Conversely, suppose that a $\notin (x]^*$, then a $\bigwedge$ x $\neq 0$, hence there exists a minimal prime ideal P such that a $\bigwedge$ x $\notin$ P. Therefore a $\notin$ P and x $\notin$ P. Thus a $\notin \bigcap$ { P : P is a minimal prime ideal and x $\notin$ P }. Hence $\bigcap$ { P : P is a minimal prime ideal and x $\notin$ P } $\subseteq (x]^*$. Therefore $(x]^* = \bigcap$ { P : P is a minimal prime ideal and x $\notin$ P }. ■

The following theorem characterize important conditions for α-ideals.

Theorem 2.3.2 *[17] Let R be an ADL, then for any ideal I of R, the following are equivalent:*

(1). I is an α-ideal

(2). $I = I^e$

(3). for $x, y \in R$, $(x]^* = (y]^*$ and $x \in I$ imply $y \in I$

(4). $I = \bigcup_{x \in I}(x]^{**}$.

Proof:

(1) $\Longrightarrow$ (2): Assume that I is an α-ideal, clearly I $\subseteq I^e$. Now let x $\in I^e$, then $(a]^* \subseteq (x]^*$ for some a $\in$ I. Hence x $\in (x]^{**} \subseteq (a]^{**} \subseteq$ I. Thus $I^e \subseteq$ I.

(2) $\Longrightarrow$ (3): Assume that I = I^e. Let x, y $\in$ R such that $(x]^* = (y]^*$. Suppose x $\in$ I, then by (2), $(a]^* \subseteq (x]^*$ for some a $\in$ I. Hence $(a]^* \subseteq (x]^* = (y]^*$. Thus $(a]^* \subseteq (y]^*$ for some a $\in$ I. Therefore y $\in I^e =$ I.

(3) $\Longrightarrow$ (4): Assume the condition (3). Clearly I $\subseteq (\bigcup_{x\in I}(x]^{**})$.

Let x $\in$ I and y $\in (x]^{**}$, then $(x]^* \subseteq (y]^*$. Now we prove that $(y]^* = (x \bigwedge y]^*$.

We know that $(y]^* \subseteq (x \bigwedge y]^*$. Let t $\in (x \bigwedge y]^*$. Then x $\bigwedge$ y $\bigwedge$ t = 0. So y $\bigwedge$ t $\in (x]^* \subseteq (y]^*$. Thus y $\bigwedge$ t = 0. Hence t $\in (y]^*$. Therefore $(y]^* = (x \bigwedge y]^*$. By the condition (3) we can get that y ∈ I. Hence $(x]^{**} \subseteq$ I for all x ∈ I.Thus $(\bigcup_{x\in I}(x]^{**}) \subseteq$ I. Therefore I = $(\bigcup_{x\in I}(x]^{**})$

(4) $\Longrightarrow$ (1):Suppose x ∈ I. Then $(x]^{**} \subseteq (\bigcup_{x\in I} I_i)(x]^{**}$ = I. Thus I is an α-ideal. ■

We now prove the one-to-one correspondence between the set I_α(R) of all α-ideals of R and the set of all α-ideals of PI(R).

Definition 2.3.3 *[13] Let R be an ADL. Then for any ideal I in R, define* $\bar{I} = \{(a]:\ a \in I\}$

Definition 2.3.4 *[13] Let R be an ADL with 0. Then for any principal ideal (a] in PI(R), define* $\{(a]\}^* = \{(x]:\ (a] \wedge (x] = (0]\ \}$.

In the following lemma, we state certain properties in ADLs which can be proved directly.

Lemma 2.3.4 *[17] For any a, b ∈ R, we have the following:*

(i). $x \in (a]^* \Leftrightarrow (x] \in \{(a]\}^*$

(ii). $(a]^* = (b]^* \Leftrightarrow \{(a]\}^* = \{(b]\}^*$.

Theorem 2.3.3 *[17] Let R be an ADL and I an ideal of R. Then I is an α-ideal in R if and only if $\bar{I}$ is an α-ideal in PI(R).*

Proof:

Assume that I is an α-ideal in R. Let (a], (b] ∈ PI(R) such that $\{(a]\}^* = \{(b]\}^*$ and (a] ∈ $\bar{I}$. Then $(a]^* = (b]^*$ and (a] = (t] for some t ∈ I. Since a ∈ (a] = (t], then a = t $\wedge$ x, for some x ∈ R, and because I is an ideal, then t $\wedge$ x = a ∈ I. Also since I is an α-ideal, by theorem 2.3.2 we get b ∈ I. Hence (b]∈ $\bar{I}$, therefore $\bar{I}$ is an α ideal in PI(R).

Conversely, assume that $\bar{I}$ is an α-ideal in PI(R), and let a, b ∈ R such that $(a]^* = (b]^*$ and a ∈ I. Then $\{(a]\}^* = \{(b]\}^*$ and (a] ∈ $\bar{I}$. Since $\bar{I}$ is an α-ideal in PI(R), we get (b] ∈ $\bar{I}$. Hence b ∈ I. Therefore I is an α-ideal in R. ■.

Now we derive some conditions for prime ideals to become an α-ideals, but before, we insert some definitions which are used in the text.

Definition 2.3.5 *[12] An element $a \in R$ is called dense if $(a]^* = (0]$.*

The set of all dense elements of R is denoted by D, also an ideal I of R is called dense if $I^* = (0]$.

Definition 2.3.6 *[12] An ADL R with 0 is called a $\star$-ADL, if for each $x \in R$, there exists an element $a \in R$ such that $(x]^{**} = a^*$.*

Example 2.3.4 *Let $R = \{ 0, a, b, c \}$ and define $\bigvee$ and $\bigwedge$ on R as in example 2.1.1, then $(R, \bigvee, \bigwedge, 0)$ is an ADL, consider the ideal $P = \{ 0, c \}$.*
Now, we make the following observations, first, P is a prime ideal of R and $P^* = (0]$. So P is a dense prime ideal, second, P is not a minimal prime ideal, because 0 is also a prime ideal, and third, R is a $\star$-ADL and c is a dense element in P.

Lemma 2.3.5 *[16] An ADL R with 0 is a $\star$-ADL if and only if to each $x \in R$, there exists $y \in R$ such that $x \bigwedge y = 0$ and $x \bigvee y \in D$.*

Definition 2.3.7 *[12] An ADL R with 0, is called disjunctive if and only if for all a, $b \in R$, $(a]^* = (b]^*$ implies $a = b$.*

Example 2.3.5 *Let $R = \{0, a, b, c\}$ be a set. Define $\bigvee$ and $\bigwedge$ on R as follows:*

$\bigvee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	a	b	a
c	c	a	a	c

$\bigwedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	b	b	0
c	0	c	0	c

Now, $(a]^* = (0]$, $(b]^* = \{0, c\}$ and $(c]^* = \{0, b\}$. Thus $x \neq y$ implies that $(x]^* \neq (y]^*$ for all $x, y \in R$. Hence R is disjunctive.

Definition 2.3.8 *[16] An ADL R with 0 is called a weakly disjunctive ADL if and only if for all $x, y \in R$, $(x]^* = (y]^*$ implies that $(x] = (y]$.*

Example 2.3.6 *Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ be two discrete ADLs. Write $R = A \times B$, we have $((0, b_1)]^* = ((0, b_2)]^* = \{(0, 0), (a, 0)\}$ and $((0, b_1)] = ((0, b_2)]$ $= \{(0, 0), (0, b_1), (0, b_2)\}$. Also $((a, b_1)]^* = ((a, b_2)]^* = \{(0, 0)\}$ and, $((a, b_1)] = ((a, b_2)] = R$. Therefore R is a weakly disjunctive ADL.*

It can be easily observed that every disjunctive ADL is weakly disjunctive. But the converse is not true. For, the ADL R in the above example is weakly disjunctive but not a disjunctive ADL.

Theorem 2.3.4 *[16] Let R be an ADL with 0 and D be the set of all dense elements of R. Then we have:*

(a). Every non-dense prime ideal of R is an α-ideal.

(b). Every minimal prime ideal of R is an α-ideal.

(c). If R is a $\star$-ADL, then every prime ideal P of R with $P \cap D = \phi$ is an α-ideal.

Proof:

(a). Let P be a non-dense prime ideal of R. So, there exists an element $0 \neq x \in R$ such that $x \in P^*$. Hence $P \subseteq P^{**} \subseteq (x]^*$. Now, let $a \in (x]^*$. Then $a \bigwedge x = 0 \in P$. Since $x \in P^*$ and $x \neq 0$, we get that $a \in P$. Hence $(x]^* \subseteq P$. Thus $P = (x]^*$. Now, If $a \in P = (x]^*$, then clearly $(a]^{**} \subseteq (x]^{***} = (x]^* = P$. Therefore P is an α-ideal of R.

(b). Let P be a minimal prime ideal of R. Let $x, y \in R$ such that $(x]^* = (y]^*$ and $x \in P$. Suppose $y \notin P$. Then $(y]^* \subseteq P$. Hence $(x]^* \subseteq P$ which implies that $x \notin P$. Which is a contradiction. Hence $y \in P$. Therefore P is an α-ideal.

(c). Let R be a $\star$-ADL and P a prime ideal of R such that $P \cap D = \phi$. Now, let $x \in P$. Since R is a $\star$-ADL, there exists $y \in R$ such that $x \bigwedge y = 0$ and $x \bigvee y \in D$. Since $P \cap D = \phi$, we get $x \bigvee y \notin P$. So $x \notin P$ and $y \notin P$. That is, to each $x \in P$, there exists $y \notin P$ such that $x \bigwedge y = 0$. Hence P is a minimal prime ideal.

Therefore by (b), P is an α-ideal of R. ■

Theorem 2.3.5 *[16] Let R be an ADL with 0. Then the following conditions are equivalent:*

(a). R is weakly disjunctive.

(b). Each ideal is an α-ideal.

(c). Each prime ideal is an α-ideal.

Proof:

$(a) \Longrightarrow (b)$: Assume that R is weakly disjunctive. Let I be an ideal of R, and suppose $x, y \in R$ such that $(x]^* = (y]^*$. Hence we get that $(x] = (y]$. Suppose $x \in I$. Then $y \in (y] = (x] \subseteq I$. Therefore I is an α-ideal.

$(b) \Longrightarrow (c)$: Since every ideal is an α-ideal, so every prime ideal is an α-ideal.

$(c) \Longrightarrow (a)$: Assume that every prime ideal of R is an α-ideal, and let $x, y \in R$ such that $(x] \neq (y]$.

Without loss of generality assume that $(x] \nsubseteq (y]$. Let $\gamma = \{I \in I(R): x \bigwedge y \in I \text{ and } x \notin I\}$. Clearly $(x \bigwedge y] \in \gamma$ "since $(x \bigwedge y]$ is an ideal and contained $x \bigwedge y$" . Let P be a maximal element of γ, and $a, b \in R$ with $a \notin P$ and $b \notin P$.

Therefore by the maximality of P, $P \bigvee (a]$ and $P \bigvee (b]$ are not in γ, hence $x \in P \bigvee (a]$ and $x \in P \bigvee (b]$. So $x \in \{P \bigvee (a]\} \cap \{P \bigvee (b]\} = P \bigvee (a \bigwedge b]$.

If $a \bigwedge b \in P$, then $x \in P \bigvee (a \bigwedge b] = P$. Which is a contradiction. So $a \bigwedge b \notin P$, hence P is a prime ideal, so by the hypothesis, P is an α-ideal. Now $x \bigwedge y \in P$ implies that $y \in P$ because of $x \notin P$. Now suppose $(x]^* = (y]^*$. Since $y \in P$ and P is an α-ideal, we get that $x \in P$. Which is a contradiction. Hence $(x]^* \neq (y]^*$. Thus R is weakly disjunctive. ■

Theorem 2.3.6 *If no proper α-ideal of R is dense, then R is $\star$-ADL*

Proof:

Let $x \in R$ and put $I = (x]^* \bigvee (x]^{**}$. Then I^e is an α-ideal in R, further, $(x]^* \subseteq I \subseteq I^e$, we get $(I^e)^* \subseteq (x]^{**}$ and $(x]^{**} \subseteq I \subseteq I^e$ imply $(I^e)^* \subseteq (x]^*$, hence $(I^e)^* \subseteq (x]^* \cap (x]^{**} = (0]$. Therefore I^e is a dense α-ideal in R, by the assumption $I^e = R$. As $D \neq \phi$, there exists $d \in D$ such that $d \in I^e$.

There exists $t \in I$ *such that* $(t]^* \subseteq (d]^* = (0]$ *and hence* $(t]^* = (0]$, *as* $t \in I = (x]^* \bigvee (x]^{**}$, *we have* $t \leq a \bigvee b$ *for some* $a \in (x]^*$ *and* $b \in (x]^{**}$. *Hence* $a \bigwedge b = 0$, *further* $(a \bigvee b]^* \subseteq (t]^* = (0]$ *gives* $(a]^* \cap (b]^* = (0]$ *and hence* $(a]^* \subseteq (b]^{**}$. *As* $b \in (x]^{**}$, *we get* $(b]^{**} \subseteq ((x]^{**})^{**} = (x]^{**}$. *Thus we have* $(a]^* \subseteq (x]^{**}$, *at the same time* $a \in (x]^*$ *gives* $(a]^* \supseteq (x]^{**}$, *combining both the inclusions we get* $(a]^* = (x]^{**}$. *Thus for any* $x \in R$, *there exists* $a \in R$ *such that* $(a]^* = (x]^{**}$. *Hence* R *is* $\star$*- ADL* ■

Theorem 2.3.7 *[17] Let* R *be an ADL. If every* α*-ideal is a principal ideal in* R*, then* R *is a* $\star$*-ADL.*

Proof:

Assume that every α*-ideal is a principal ideal in* R*. Let* $x \in R$*. Then for any* $a \in (x]^*$*, we get* $(a]^{**} \subseteq (x]^{***} = (x]^*$*. Hence* $(x]^*$ *is an* α*-ideal of* R*. So by hypothesis,* $(x]^* = (a]$ *for some* $a \in R$*. Thus* $(x]^{**} = (a]^*$*. Therefore* R *is a* $\star$*-ADL.* ■

We now define relative annihilators in an ADL R*, which lead to another important characterization of* α*-ideals.*

Definition 2.3.9 *[16] Let* A *be a non-empty subset of* R *and* J *an ideal of* R*. Then define the annihilator of* A *relative to* J *as follows:*

$< A, J > = \{x \in R : x \bigwedge a \in J, \text{ for all } a \in A\}$.

Remark 2.3.1 *If* $J = \{0\}$*, then we get that* $< A, 0 > = A^*$*.*

Theorem 2.3.8 *[16] Let* R *be an ADL with 0. If* J *is an* α*-ideal of* R*, so is* $< A, J >$ *for any subset* A *of* R*.*

Proof:

Let J *be an* α*-ideal of* R*. We have clearly* $< A, J >$ *is an ideal of* R*. Let* $x, y \in R$ *such that*

$(x]^* = (y]^*$ *and* $x \in \langle A, J \rangle$.

Now $x \in \langle A, J \rangle \Rightarrow x \bigwedge a \in J$ *for all* $a \in A \Rightarrow (x]^{**} \cap (a]^{**} = (x \bigwedge a]^{**} \subseteq J$ *for all* $a \in A$

$\Rightarrow (y]^{**} \cap (a]^{**} \subseteq J$ *for all* $a \in A$

$\Rightarrow y \bigwedge a \in (y \bigwedge a]^{**} \subseteq J$ *for all* $a \in A$

$\Rightarrow y \in \langle A, J \rangle$ *for all* $a \in A$ *Hence* $\langle A, J \rangle$ *is an* α*-ideal of* R. ■

Corollary 2.3.2 *[12] Every annihilator ideal is an* α*-ideal.*

The converse of the above corollary is not true. That is every α*-ideal of* R *need not be an annihilator ideal. For example, a proper dense* α*-ideal is not an annihilator ideal.*

Theorem 2.3.9 *If every dense ideal in* R *contains a dense element, then every* α*-ideal in* R *is an annihilator ideal.*

Proof:

Let I *be an* α*-ideal of* R*, clearly* $I \subseteq I \bigvee I^*$ *gives* $(I \bigvee I^*)^* \subseteq I^*$ *and* $I^* \subseteq I \bigvee I^*$ *yields* $(I \bigvee I^*)^* \subseteq I^{**}$*, hence* $(I \bigvee I^*)^* \subseteq I^* \cap I^{**} = (0]$ *showing that* $(I \bigvee I^*)$ *is a dense ideal of* R*. By hypothesis,* $(I \bigvee I^*) \cap D \neq \phi$*. Let* $d \in (I \bigvee I^*) \cap D$*. As* $d \in I \bigvee I^*$ *we have* $d \leq a \bigvee b$ *for some* $a \in I$ *and* $b \in I^*$*. Hence,* $(a]^* \cap (b]^* \subseteq (d]^* = (0]$ *gives* $(b]^* \subseteq (a]^{**}$*. Let* $x \in I^{**}$*. Then* $b \bigwedge x = 0$ *as* $b \in I^*$*. Thus* $x \in (b]^* \subseteq (a]^{**}$*. As* $a \in I$ *and* I *is an* α*-ideal, we get* $(a]^{**} \subseteq I$*. But then* $x \in I$ *shows that* $I^{**} \subseteq I$*. But as* $I \subseteq I^{**}$ *always holds, we get* $I = I^{**}$ *and the result follows.* ■

Definition 2.3.10 *[7] An ideal* $J \neq (0]$ *of* R *is a semi-complement of an ideal* I *of* R *if* $I \cap J = (0]$*.*

A family κ *of ideals of* R *is said to be semi-complemented if every element of* κ *has a semi-complement in it.*

Theorem 2.3.10 [16] *Let R be an ADL with 0 in which $D \neq \phi$. Then the following conditions are equivalent:*

(1). R is $\star$-ADL and $I^* \neq (0]$ for each proper α-ideal I of R.

(2). $I \cap D \neq \phi$ for each $I \in DI(R)$"dense ideals".

(3). Every α-ideal is an annihilator ideal.

(4). $I_\alpha(R)$ is semi-complemented.

(5). $I_\alpha(R)$ has a unique dense element.

Proof:

(1) $\Rightarrow$ (2): Assume the condition (1), and let $I \in DI\ (R)$.

Suppose $I \cap D = \phi$, then there exists a prime ideal P such that $I \subseteq P$ and $P \cap D = \phi$, now suppose x, $y \in R$ such that $(x]^* = (y]^*$, and assume $x \in P$.

Since R is $\star$-ADL, there exists $x' \in R$ such that $x \bigwedge x' = 0$ and $x \bigvee x' \in D$. Hence $x' \in (x]^* = (y]^*$ and $x \bigvee x' \notin P$. Since $x \in P$, we get $x' \notin P$. Hence $x' \bigwedge y = 0 \in P$ and $x' \notin P$. So y must be in P. Thus P is a proper α-ideal. Therefore by (1), $P^* \neq (0]$. But $I \subseteq P$ implies that $P^* \subseteq I^* = (0]$. Which is a contradiction. Therefore $I \cap D \neq \phi$, for each $I \in DI(R)$.

(2) $\Rightarrow$ (3): Assume the condition (2), let I be an α-ideal of R. Always we have $I \subseteq I^{**}$, so let $x \in I^{**}$. Since $I \bigvee I^*$ is a dense ideal, by (2) it has a dense element, say d.

Hence $d = r \bigvee s$ where $r \in I$ and $s \in I^*$, thus $(r]^* \cap (s]^* = (r \bigvee s]^* = (d]^* = (0]$. Therefore $(s]^* \subseteq (r]^{**}$, now $x \in I^{**}$ and $s \in I^*$ imply that $x \bigwedge s = 0$. Hence $x \in (s]^* \subseteq (r]^{**} \subseteq I$, because of $r \in I$ and I is an α-ideal. Hence $I^{**} \subseteq I$. Therefore $I = I^{**}$.

(3) $\Rightarrow$ (4): Assume the condition (3), and let $I \in I_\alpha(R)$. Then by (3), I is an annihilator ideal of R, since the set of all annihilator ideals forms a complemented lattice, it follows that $I_\alpha(R)$ is semi-complemented.

(4) $\Rightarrow$(5): Assume that $I_\alpha(R)$ is semi-complemented. Clearly $R \in I_\alpha(R)$ and R is dense in$I_\alpha(R)$. Suppose $I(\neq R)$ is a dense element in $I_\alpha(R)$. By (4), there exists an α-ideal $J \neq (0]$

such that $I \cap J = (0]$. *Hence* $J \subseteq I^* = (0]$ *implies that* $J = (0]$. *Which is a contradiction. Thus* $I_\alpha(R)$ *has a unique dense element, precisely* R.

(5) $\Rightarrow$ (1): *Assume that* $I_\alpha(R)$ *has a unique dense element, precisely* R. *It is clear that every proper* α*-ideal is non-dense. So it is enough to prove that* R *is a* $\star$*-ADL.*

Let $x \in R$. *Clearly* $((x]^* \bigvee (x]^{**})^e \in I_\alpha(R)$. *Now* $(x]^*, (x]^{**} \subseteq ((x]^* \bigvee (x]^{**})^e$

$\Rightarrow ((x]^* \bigvee (x]^*)^e)^* \subseteq (x]^{**}, (x]^* \Rightarrow (((x]^* \bigvee (x]^{**})^e)^* \subseteq (x]^* \cap (x]^{**} = (0]$ *That is* $((x]^* \bigvee (x]^{**})^e$ *is a dense element in* $I_\alpha(R)$. *Hence by (5), we get that* $((x]^* \bigvee (x]^{**})^e = R$. *Therefore* $((x]^* \bigvee (x]^{**})^e$ *has a dense element, say* d.

Then $(r \bigvee s]^* \subseteq (d]^* = (0]$ *for some* $r \in (x]^*$ *and* $s \in (x]^{**}$, *thus* $(r]^* \cap (s]^* = (r \bigvee s]^* \subseteq (0]$, *hence* $(r]^* \subseteq (s]^{**} \subseteq (x]^{**}$, *since* $s \in (x]^{**}$. *Again, since* $r \in (x]^*$, *we get that* $(x]^{**} \subseteq (r]^*$. *Hence* $(x]^{**} = (r]^*$. *Therefore* R *is a* $\star$*-ADL.* ■

We now prove that the image and the inverse image of an α*-ideal under an annihilator preserving homomorphism of ADLs, are again* α*-ideals, but first we begin by this lemma:*

Lemma 2.3.6 *[13] Let* R *and* R' *be two ADLs with zeroes* 0 *and* $0'$ *respectively and* $f : R \to R'$ *a homomorphism. Then* $f((a]) \subseteq (f(a)]$ *for any* $a \in R$. *Moreover, if* f *is onto, then* $f((a]) = (f(a)]$.

Proof:

Let $a \in R$ *and* $x \in (a]$. *Then* $x = a \bigwedge x$.

Now $f(x) = f(a \bigwedge x) = f(a) \bigwedge f(x)$, *hence* $f(x) \in (f(a)]$, *therefore* $f((a]) \subseteq (f(a)]$.

Now, suppose f *is onto, let* $t \in (f(a)]$, *choose* $x \in R$ *so that* $f(x) = t$.

Then $f(x) = f(a) \bigwedge f(x) = f(a \bigwedge x) \in f((a])$, *since* $a \bigwedge x \in (a]$.

Therefore $f((a]) = (f(a)]$. ■

Theorem 2.3.11 *[17] Let* $f : R \to R'$ *be an annihilator preserving epimorphism. If* J *is an* α*-ideal in* R, *then* $f(J)$ *is an* α*-ideal in* R'.

Proof:

Let J be an α*-ideal of R. Let* $a \in J$*, then* $(a]^{**} \subseteq J$*, hence* $f((a]^{**}) \subseteq f(J)$*, since f is annihilator preserving, we get* $f((a]^{**}) = (f((a])^{**} = (f(a)]^{**}$ *(by above lemma).*
Hence $(f(a)]^{**} \subseteq f(J)$*, therefore* $f(J)$ *is an* α*-ideal in* R'*.* ■

Theorem 2.3.12 *[17] If* $f : R \to R'$ *is an annihilator preserving epimorphism, and J is an* α*-ideal of* R' *then* $f^{-1}(J)$ *is an* α*-ideal in R.*

Proof:

Suppose J is an α*-ideal of* R'*. Let* $a, b \in R$ *such that* $(a]^* = (b]^*$*, since f is annihilator preserving then by lemma 2.3.6, we get* $(f(a)]^* = (f(b)]^*$*.*
Suppose $a \in f^{-1}(J)$*, then* $f(a) \in J$*, since J is an* α*-ideal in* R'*, we get* $f(b) \in J$*.*
Hence $b \in f^{-1}(J)$*, thus* $f^{-1}(J)$ *is an* α*-ideal in R.* ■

Since $\{ 0' \}$ *is an* α*-ideal in* R' *and* $Kerf = f^{-1}(\{ 0' \})$*, we have:*

Corollary 2.3.3 *[17] If f is an annihilator preserving homomorphism of R into* R'*, then Kerf is an* α*-ideal of R.*

2.4 O - Ideals

In this section, the concept of O-ideals is given in an ADL and some properties of these O-ideals are studied. A set of equivalent conditions are established for every O-ideal of an ADL to become a principal ideal.

Definition 2.4.1 *[18] For any filter F of an ADL R, define* $O(F) = \{ x \in R : x \bigwedge f = 0 \text{ for some } f \in F \}$

Example 2.4.1 *Let* $A = \{ 0, a \}$ *and* $B = \{ 0, b_1, b_2 \}$ *be two discrete ADLs. Write* $R = A \times B$*, under point-wise operations. Consider the filter* $F = \{(a, 0), (a, b_1), (a, b_2) \}$*, then* $O(F) = \{(0, 0), (0, b_1), (0, b_2) \}$

Remark 2.4.1 *[9] For any filter F of an ADL R, O(F) is an ideal in R.*

Proof:

Clearly $0 \in O(F)$*, now let* $a, b \in O(F)$*, then* $a \bigwedge f = b \bigwedge g = 0$ *for some* $f, g \in F$ *. Now* $(a \bigvee b) \bigwedge (f \bigwedge g) = (a \bigwedge f \bigwedge g) \bigvee (b \bigwedge f \bigwedge g) = (0 \bigwedge g) \bigvee (f \bigwedge b \bigwedge g) = 0 \bigvee 0 = 0$*. Hence* $a \bigvee b \in O(F)$ *. Again, let* $a \in O(F)$ *and* $x \in R$*, then* $a \bigwedge f = 0$ *for some* $f \in F$*, now* $(a \bigwedge x) \bigwedge f = x \bigwedge a \bigwedge f = x \bigwedge 0 = 0$*. So* $a \bigwedge x \in O(F)$ *. Thus O(F) is an ideal in R.* ■

Theorem 2.4.1 *[9] For any two filters F,G of an ADL R, we have the following:*

(1) $O(F) = \bigcup_{x \in F} (x]^*$.

(2) $F \subseteq G$ *implies* $O(F) \subseteq O(G)$.

(3) $O(F \cap G) = O(F) \cap O(G)$.

(4) $O(F) \bigvee O(G) \subseteq O(F \bigvee G)$.

Theorem 2.4.2 [18] *Let R be an ADL with dense elements. Then for any filter F of R, O(F) = R if and only if F = R.*

Proof:

Let d be a dense element of R. Assume that F = R. Then $O(F) = O(R) = \bigcup_{x \in F} (x]^* = R$ *(since* $0 \in R$*).*

Conversely, assume that O(F) = R, then $d \in O(F)$*, hence* $d \bigwedge f = 0$ *for some* $f \in F$*, so f = 0. Therefore F = R.* ■

Definition 2.4.2 [18] *An ideal I of an ADL R is called an O-ideal if and only if I = O(F) for some filter F of R.*

The set of all O-ideals of R is denoted by $I_o(R)$

Example 2.4.2 In example 2.4.1 consider the ideal $I = \{(0,\ 0),\ (0,\ b_1),\ (0,\ b_2)\}$ *and filter* $F = \{(a,\ 0),\ (a,\ b_1),\ (a,\ b_2)\}$. *Now* $O(F) = \bigcup_{x \in F} (x]^* = ((a, 0)]^* \cup ((a, b_1)]^* \cup ((a, b_2)]^* = \{(0,\ 0),\ (0,\ b_1),\ (0,\ b_2)\} \cup \{(0,\ 0)\} \cup \{0,\ 0)\} = \{(0,\ 0),\ (0,\ b_1),\ (0,\ b_2)\}$ *Therefore I = O(F) and hence I is an O-ideal of R.*

Example 2.4.3 Let $R = \{0,\ a,\ b,\ c,\ 1\}$ *be a distributive lattice whose Hasse diagram is given in figure 2.1.*

Consider $I = \{0,\ a\}$ *and* $J = \{0,\ a,\ b,\ c\}$*, clearly I and J are ideals of R, also* $F = \{b,\ c,\ 1\}$ *is a filter of R.* $O(F) = \bigcup_{x \in F} (x]^* = \{0,\ a\} = I$*, therefore I is an O-ideal in R, but J is not an O-ideal of R.*

Theorem 2.4.3 [13] *Every O-ideal is an* α*-ideal.*

Proof:

Let I is an O-ideal, $x \in I$*, and* $(x]^* = (y]^*$ *for some* $y \in R$*, then* $x \bigwedge f = 0$ *for some* $f \in F$*, so* $f \in (x]^*$*, thus* $f \in (y]^*$*, so* $y \bigwedge f = 0$ *for some* $f \in F$*, hence* $y \in I$*.*

Therefore I is an α*-ideal.* ■

Theorem 2.4.4 *[18] A proper O-ideal of an ADL R contains no dense elements.*

Proof:

Let I be an O-ideal and d a dense element of R, since I is an O-ideal, we get that $I = O(F)$ *for some filter F of R, now suppose* $d \in I = O(F)$*, then* $d \bigwedge f = 0$ *for some* $0 \neq f \in F$*, thus* $f \in (d]^*$*, which is a contradiction to that d is a dense element of R.* ■

It was already observed that the zero ideal $\{0\}$ *is an annihilator ideal as well as an* α*-ideal in an ADL R , but, in general, it is not an O-ideal in R as demonstrated by the following example:*

Example 2.4.4 *Let* $R = \{0, a, b\}$ *be a discrete ADL and X is any infinite set. Take* $R' = \{ f \in R^X : \text{support } f \text{ is finite} \}$ *where support* $f = \{x \in X : f(x) \neq 0\}$*. Define* $\bigvee$ *and* $\bigwedge$ *on* R' *point-wise. Then* $(R', \bigvee, \bigwedge, f_0)$ *is an ADL with* f_0 *as zero where* $f_0(x) = 0$ *for all* $x \in X$*. We first observe that* R' *has no dense elements. Suppose* $f \in R'$ *. Choose*
$x \in X -$ *Support f and define* $g : X \to R$ *by:*

$$g(t) = \begin{cases} a & \text{if } t = x, \\ 0 & \text{otherwise} \end{cases}$$

Then clearly support $g = \{ x \}$ *and hence* $f_0 \neq g \in R'$*. Now,* $(f \bigwedge g)(t) = f(t) \bigwedge g(t) = 0$ *for all* $t \in X$*. Hence* $g \in (f]^*$*. Thus* $(f]^* \neq (f_0]$ *. Therefore* R' *has no dense elements. Suppose the zero ideal* $(f_0]$ *of* R' *is an O -ideal. Then* $(f_0] = O(F)$ *for some filter F of* $R' =$
$\bigcup_{x \in F} (f]^*$ *Hence we get* $(f]^* = (f_0]$ *for all* $f \in F$ *. Thus we have obtained that each*
Which is a contradiction to the fact that R' *has no dense elements.*
$f \in F$ *is a dense element.*
Therefore the zero ideal $(f_0]$ *is not an O-ideal of* R' *.*

The following theorem derives a necessary and sufficient condition for the zero ideal of an ADL R to become an O-ideal.

Theorem 2.4.5 *[18] The zero ideal is an O-ideal if and only if R has a dense element*

Proof:

Assume that $\{0\}$ *is an O-ideal of R, then* $\{0\} = O(F) = \bigcup_{x\in F}(x]^*$ *for some filter F of R. Hence* $(x]^* = (0]$ *for each* $x \in F$. *Thus R has a dense element, since* $F \neq \phi$. *Conversely, assume that R has a dense element. Then the set D of all dense elements of R is a filter of R. Also* $O(D) = \bigcup_{x\in F}(x]^* = \{0\}$. *Therefore* $\{0\}$ *is an O-ideal of R.* ■

We know that the set of all annihilators ideals form a complete lattice, and the set of all α*-ideals form a distributive lattice, so what about the set of all O-ideal?*
In lemma 2.4.1, it is observed that $O(F) \bigvee O(G) \subseteq O(F \bigvee G)$ *for any two filters F,G of R, in general, equality does not hold in an ADL, and the following example illustrate that.*

Example 2.4.5 *In example 2.4.2 Consider the two filters* $F = \{b, c, 1\}$ *and* $G = \{a, c, 1\}$, *then* $O(F) = \{0, a\}$, $O(G) = \{0, b\}$ *and* $O(F) \bigvee O(G) = \{0, a, b, c\}$, *but* $O(F \bigvee G) = O(R) = R$. *Hence* $O(F) \bigvee O(G) \neq O(F \bigvee G)$.

Thus $I_o(R)$ *is not a sublattice of the distributive lattice* $I(R)$ *of all ideals of R. However, in the following theorem, it is proved that* $I_o(R)$ *forms a distributive lattice.*

Theorem 2.4.6 *[18] For any ADL R,* $I_o(R)$ *is a distributive lattice.*

Proof:

For any two filters F,G of R, define binary operations $\bigwedge$ *and* $\check{\bigvee}$ *as follows:*
$O(F) \bigwedge O(G) = O(F \cap G)$ *and* $O(F) \check{\bigvee} O(G) = O(F \bigvee G)$.
Now by theorem 2.4.1, $O(F \cap G)$ *is the infimum of* $O(F)$ *and* $O(G)$ *in* $I_o(R)$, *also* $O(F) \check{\bigvee} (G)$ *is an O-ideal of R, clearly* $O(F), O(G) \subseteq O(F \bigvee G) = O(F) \check{\bigvee} (G)$ *now let* $O(H)$ *be an O-ideal of R such that* $O(F) \subseteq O(H)$ *and* $O(G) \subseteq O(H)$, *where H is a filter of R.Now:* $x \in O(F \bigvee G)$, *then* $x \bigwedge a = 0$ *for some* $a \in F \bigvee G$, *so* $x \bigwedge f \bigwedge g = 0$ *for some* $f \in F$ *and* $g \in G$, *thus* $x \bigwedge f \in O(G) \subseteq O(H)$, *then* $x \bigwedge f \bigwedge h_1 = 0$ *for some* $h_1 \in H$,

so $x \bigwedge h_1 \in O(F) \subseteq O(H)$, *hence* $x \bigwedge h_1 \bigwedge h_2 = 0$ *for some* $h_2 \in H$, *therefore* $x \in O(H)$ *(since* $h_1 \bigwedge h_2 \in H$*).*

Therefore $O(F \bigvee G)$ *is the supremum of* $O(F)$, $O(G)$ *in* $I_o(R)$, *so* $(I_o(R), \bigwedge, \check{\bigvee})$ *is a lattice, and one can easily verified that it is a distributive lattice.* ■

Theorem 2.4.7 *[18] Let R be an ADL with dense elements. Then the lattice* $I_o(R)$ *is bounded and complete.*

Proof:

Let F,G be two filters of R. Then $O(F)$, $O(G) \in I_o(L)$.

Define $O(F) \leq O(G) \Leftrightarrow O(F) \subseteq O(G)$ *. Clearly* $(I_o(R), \leq)$ *is a partially ordered set, now* $\{0\}$, *and R are the O-ideals in R and they are the bounds for* $I_o(R)$ *. Therefore by the extension of theorem 2.4.1(3), we get that* $I_o(R)$ *is a complete lattice.* ■

Now we give a set of equivalent conditions for every O-ideal of R to become a principal ideal.

Lemma 2.4.1 *[16] If each* $(x]^*$ *of an ADL R is a principal ideal, then every prime* α*-ideal of R is a minimal prime ideal.*

Proof:

Let P be a prime α*-ideal of R, then* $P \cap D = \phi$, *now let* $x \in P$, *then by hypothesis,* $(x]^* = (y]$ *for some* $y \in R$, *hence* $x \bigwedge y = 0$.

Now $(x \bigvee y]^* = (x]^* \cap (y]^* = (x]^* \cap (x]^{**} = (0]$ *. Hence* $x \bigvee y \notin P$ *. Thus* $y \notin P$ *. Therefore P is a minimal prime ideal.* ■.

Theorem 2.4.8 *[18] The following conditions are equivalent in an ADL R:*

(1) Every α*-ideal is a principal ideal.*

(2) Every O-ideal is a principal ideal.

(3) Each$(x]^*$ is a principal ideal and every minimal prime ideal is non-dense.

(4) Every prime α-ideal is a principal ideal.

Proof:

(1) $\to$ (2) : Since every O-ideal is an α-ideal, it is clear.

(2) $\to$ (3) : Assume that every O-ideal is a principal ideal. Since each $(x]^* = O(R)$ is an O-ideal, it remains to prove that every minimal prime ideal is non-dense.

Let P be a minimal prime ideal of R, then R-P is a filter, since P is minimal, we get $P = O(R\text{-}P)$, hence P is an O-ideal, thus $P = (a]$ for some $a \in R$.

Suppose $P^* = (a]^* = (0]$, then $a \in P$ is a dense element, which is a contradiction to that a proper O-ideal does not contain a dense element. Therefore P is non-dense.

(3) $\to$ (4) : Assume the condition (3), and let P be a prime α-ideal of R, then by lemma 2.4.3, P is a minimal prime ideal.

By hypothesis, P is non-dense, hence $P = (x]^*$ for some $0 \neq x \in R$.

Again by hypothesis $P = (x]^*$ is a principal ideal.

(4) $\to$ (1) : Assume condition (4), and let I be an α-ideal. Suppose that I is not principal, consider $\zeta = \{\ J : J$ is an α-ideal which is not a principal ideal $\}$, then clearly $I \in \zeta$.

Let $\{J_i\}$ $i \in \Delta$ be a chain in ζ, clearly $\bigcup J_i$ is an α-ideal in ζ.

Suppose $\bigcup J_i = (a]$ for some $a \in R$, then $a \in \bigcup J_i$ implies that $a \in J_i$ for some $i \in \Delta$. Hence $(a] \subseteq J_i$ for some $i \in \Delta$. On the other hand, $J_i \subseteq \bigcup J_i = (a]$.

Hence $J_i = (a]$ for some $i \in \Delta$, which is a contradiction, thus $\bigcup J_i$ is an upper bound for $\{J_i\}$ $i \in \Delta$ in ζ.

Let M be a maximal element of ζ, and choose $x,\ y \in R$ such that $x \notin M$ and $y \notin M$.

Then $M \subseteq M \bigvee (x] \subseteq (M \bigvee (x])^e$ and $M \subseteq M \bigvee (y] \subseteq (M \bigvee (y])^e$, by the maximality of M, we get $(M \bigvee (x])^e = (b]$ and $(M \bigvee (y])^e = (c]$ for some $b,\ c \in R$. Hence $(M \bigvee (x \bigwedge y])^e = (M \bigvee (x])^e \cap (M \bigvee (y])^e = (b] \cap (c] = (b \bigwedge c]$.

If $x \bigwedge y \in M$, then $M = M^e = (b \bigwedge c]$, which is a contradiction, hence M is a prime α-ideal which is not a principal ideal, which is a contradiction to the hypothesis. Therefore I is a principal ideal. ■

In the following theorem, some equivalent conditions for every α-ideal of an ADL R to become an O-ideal are derived, which leads to a characterization of $\star$-ADLs.

Theorem 2.4.9 *[18] Let R be an ADL, then the following are equivalent:*

(1). R is a $\star$-ADL.

(2). Every α-ideal is an O-ideal.

(3). Every annihilator ideal is an O-ideal.

(4). For $x \in R$, $(x]^{**}$ is an O-ideal.

Proof:

(1) $\Longrightarrow$ (2) : Assume that R is a $\star$-ADL. Let I be an α-ideal of R, consider the set $I^0 = \{ x \in R : (a]^* \subseteq (x]^{**}$ for some $a \in I \}$, we first prove that I^0 is a filter of R, clearly $\phi \neq D \subseteq I^0$. Let $x, y \in I^0$, then we get $(a]^* \subseteq (x]^{**}$ and $(b]^* \subseteq (y]^{**}$ for some $a, b \in I$. Now $(a \bigvee b]^* = (a]^* \cap (b]^* \subseteq (x]^{**} \cap (y]^{**} = (x \bigwedge y]^{**}$ and $a \bigvee b \in I$.

Hence $x \bigwedge y \in I^0$.

Again, let $x \in I^0$ and $r \in R$, then we get that $(a]^* \subseteq (x]^{**}$ for some $a \in I$.

Now $(r \bigwedge x]^* \subseteq (x]^* \subseteq (a]^{**}$, hence $(a]^* \subseteq (r \bigvee x]^{**}$ and $a \in I$, hence $r \bigvee x \in I^0$. Therefore I^0 is a filter of R.

We now show that $I = O(I^0)$. Let $x \in O(I^0)$, then $x \bigwedge f = 0$ for some $f \in I^0$.

Hence $x \in (f]^*$, now $f \in I^0$, then $(a]^* \subseteq (f]^{**}$ for some $a \in I$, so $(f]^* \subseteq (a]^{**} \subseteq I$ (since I is an α-ideal and $a \in I$), thus $x \in I$ Therefore $O(I^0) \subseteq I$.

Conversely, let $x \in I$. Since R is a $\star$-ADL, there exists $y \in R$ such that $(x]^* = (y]^{**}$, since $x \in I$, we get that $y \in I^0$, also $x \in (x]^{**} = (y]^*$ and $y \in I^0$, hence $x \in O(I^o)$. Thus $I \subseteq O(I^0)$.

Therefore I is an O-ideal.

(2) $\Longrightarrow$ *(3) : Since every annihilator ideal is an* α*-ideal, it is clear.*

(3) $\Longrightarrow$ *(4) : Since* $(x]^{**}$ *is an annihilator ideal, it is obvious.*

(4) $\Longrightarrow$ *(1) : Let* $x \in R$*, hence by (4),* $(x]^{**} = O(F)$ *for some filter F of R, now let* $t \in (x]^{**} = O(F)$*, then* $t \in (y]^*$ *for some* $y \in F$*. Hence* $(x]^{**} \subseteq (y]^*$*. On the other hand, we have* $(y]^* \subseteq \bigcup_{y \in F} (y]^* = O(F) = (x]^{**}$*. Hence we can get* $(x]^{**} = (y]^*$*. Therefore R is a* $\star$*-ADL.* ■

Corollary 2.4.1 *[9] Let L be a* $\star$*-ADL. Then the following are equivalent:*

(a). Every ideal is an α*-ideal.*

(b). Every ideal is an O-ideal.

Now the properties of prime O-ideals are observed, and some characterization theorems are proved with the help of prime O-ideals of an ADL.

Lemma 2.4.2 *[11] Let L be an ADL. Then we have the following:*

(1) For any filter F of R, $F \cap O(F) \neq \phi$ *implies that* $F = O(F) = R$*.*

(2) For any prime ideal P of R, $O(P) = O(R\text{-}P)$*.*

Proof:

(1). Suppose $F \cap O(F) \neq \phi$*. Choose* $x \in F \cap O(F)$*. Then* $x \in F$ *and* $x \bigwedge f = 0$ *for some* $f \in F$*. Hence* $0 = x \bigwedge f \in F$*. Therefore* $F = O(F) = R$*.*

(2). Let P be a prime ideal of R. Then we have $x \in O(P) \Leftrightarrow x \bigwedge y = 0$ *for some* $y \notin P$ $\Leftrightarrow x \bigwedge y = 0$ *for some* $y \in R\text{-}P \Leftrightarrow x \in O(R\text{-}P)$*. Therefore* $O(P) = O(R\text{-}P)$ *for any prime ideal P of R.* ■

Theorem 2.4.10 *[14] Let R be an ADL. Then we have the following conditions:*

(1) Every minimal prime ideal is an O-ideal.

(2) Every non-dense prime ideal is an O-ideal.

(3) If R is a $\star$*-ADL, then every prime ideal P with* $P \cap D = \phi$ *is an O-ideal.*

Proof:

(1). Let P be a minimal prime ideal of R, then R/P is a maximal filter of R. Since P is minimal, we get P = O(P) = O(R-P). Hence P is an O-ideal.

(2). Let P be a non-dense prime ideal of R. Then there exists $0 \neq x \in R$, such that $x \in P^*$. Thus $P \subseteq P^{**} \subseteq (x]^*$. Let $a \in (x]^*$. Then $a \bigwedge x = 0 \in P$ and $x \notin P$, because of $x \in P^*$. Hence $a \in P$. Thus $(x]^* \subseteq P$. Hence we get $P = (x]^* = O([x))$. Therefore P is an O-ideal of R.

(3). Let R be a $\star$-ADL and P a prime ideal of R such that $P \cap D = \phi$. We prove that P = O(R-P). Clearly $O(R\text{-}P) = O(P) \subseteq P$. Conversely, let $x \in P$. Since R is a $\star$-ADL, there exists $y \notin P$ such that $x \bigwedge y = 0$ and $x \bigvee y \in D$. Hence $x \in (y]^*$ and $x \bigvee y \notin P$. Thus $x \in (y]^*$ and $y \notin P$. Therefore $x \in O(R\text{-}P)$. Thus P is an O-ideal. ■

In general, the converse of Theorem 2.4.8 (1) is not true. That is, an O-ideal of an ADL need not be a minimal prime ideal. In fact, it need not even be a prime ideal which is evident from the following example in distributive lattice.

Example 2.4.6 *Let $X = \{1, 2, 3, 4\}$ be a set and R the sublattice of the power set of X, which is generated by the sets $\{1\}$, $\{2\}$ and $\{3\}$. That is, $R = \{\{\phi\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$. Take $I = \{\{\phi\}, \{1\}\}$ and $F = \{\{2, 3\}, \{1, 2, 3\}\}$. Clearly I is an ideal and F a filter of R. Now $O(F) = (\{2,3\}]^* \cup (\{1,2,3\}]^* = \{1\} \cup \{\phi\} = \{\{\phi\}, \{1\}\}$ = I. Hence I is an O-ideal of R. But I is not prime, because of $\{2\} \notin I$ and $\{3\} \notin I$ but $\{2\} \cap \{3\} = \phi \in I$.*

Theorem 2.4.11 *[11] Let I be a proper O-ideal of an ADL R. Then I is prime if and only if I contains a prime ideal.*

Proof:

Assume that I contains a prime ideal, say P. Since I is an O-ideal of R, we get that $I = O(F)$ for some filter F of R. Choose $a, b \in R$ such that $a \notin I$ and $b \notin I$. Then $a \notin P$ and $b \notin P$. Hence $a \bigwedge b \notin P$. Thus $(a \bigwedge b]^* \subseteq P \subseteq I = O(F)$. Suppose $a \bigwedge b \in I = O(F)$. Then $a \bigwedge b \bigwedge f = 0$ for some $f \in F$. Hence $f \in (a \bigwedge b]^* \subseteq O(F)$. Thus $f \in F \cap O(F)$. Therefore $F \cap O(F) \neq \phi$. Thus $I = O(F) = F = R$, which is a contradiction.

Therefore I is prime. ■

Theorem 2.4.12 *[11] Every prime O-ideal of an ADL R is a minimal prime ideal.*

Proof:

Let P be a prime O-ideal of R. Then $P = O(F)$ for some filter F of R. Let $x \in P = O(F)$. Then $x \bigwedge y = 0$ for some $y \in F$. Suppose $y \in P$. Then $y \in F \cap O(F)$. Hence $F \cap O(F) \neq \phi$. By Lemma 2.4.4 (1), $P = O(F) = F = R$, which is a contradiction. Hence $y \notin P$. Therefore P is a minimal prime ideal ■

Let us recall that two ideals I, J of an ADL R are comaximal if $I \bigvee J = R$. We now introduce $\sqcup$-comaximality of O-ideals of an ADL.

Definition 2.4.3 *[14] Two O-ideals I, J of an ADL R are called $\sqcup$-comaximal if $I \bigvee J = R$.*

Any two comaximal O-ideals are $\bigvee$-comaximal. But the converse is not true. This can be illustrated in the following example.

Example 2.4.7 *Consider the distributive lattice $R = \{0, a, b, c, 1\}$ whose Hasse diagram is given in Figure 2.1*

Consider the ideals $I = \{0, a\}$ and $J = \{0, b\}$. Clearly $F = \{b, c, 1\}$ and $G = \{a, c, 1\}$ are filters in R. It can be easily verified that $I = O(F)$ and $J = O(G)$. Thus I and J are two distinct O-ideals of R. Now $I \bigvee J = O(F) \bigvee O(G) = O(F \bigvee G) = O(R) = R$. Hence I and *are* $\bigvee$*-comaximal. But* $I \bigvee J = \{0, a\} \bigvee \{0, b\} = \{0, a, b, c\} \neq R$. *Therefore* I *and* J *re not comaximal in* R.

emma 2.4.3 [14] Let R *be an ADL. Then, for any* $x, y \in R$ *with* $x \bigwedge y = 0$, $(x]^*$ *and* $y]^*$ *are* $\bigvee$*-comaximal in* R.

Proof:

hoose $x, y \in$ *such that* $x \bigwedge y = 0$. *Then we have* $(x]^* \bigvee (y]^* = (O[x)) \bigvee O([y)) =$ $O([x) \bigvee [y)) = O([x \bigwedge y)) = (x \bigwedge y]^* = (0]^* = R$. *Therefore* $(x]^*$ *and* $(y]^*$ *are* $\bigvee$*-comaximal* *n* R. ■

Theorem 2.4.13 [11] Let R *be an ADL. Then any two distinct prime O-ideals of* R *are* $\bigvee$*-comaximal.*

Proof:

Let P *and* Q *be two distinct prime O-ideals of* R. *Then by Theorem 2.4.12,* P *and* Q *are minimal prime ideals. Choose* $a \in P\text{-}Q$ *and* $b \in Q\text{-}P$. *Since* P, Q *are minimal prime ideals, there exist* $x \notin P$ *and* $y \notin Q$ *such that* $a \bigwedge x = b \bigwedge y = 0$. *Since* $b \bigwedge x \notin P$ *and* P *is a prime ideal, we can get* $(b \bigwedge x]^* \subseteq P$. *Similarly, we get* $(a \bigwedge y]^* \subseteq Q$.

Now $(b \bigwedge x) \bigwedge (a \bigwedge y) = b \bigwedge a \bigwedge x \bigwedge y = a \bigwedge x \bigwedge b \bigwedge y = 0 \bigwedge 0 = 0$. *By Lemma 2.4.3, we get* $(b \bigwedge x]^* \bigvee (a \bigwedge y]^* = R$. *Hence* $R = (b \bigwedge x]^* \bigvee (a \bigwedge y]^* \subseteq P \bigvee Q$.

Therefore P *and* Q *are* $\bigvee$*-comaximal in* R. ■

J are $\bigvee$-comaximal. But $I \bigvee J = \{0, a\} \bigvee \{0, b\} = \{0, a, b, c\} \neq R$. Therefore I and J are not comaximal in R.

Lemma 2.4.3 [14] Let R be an ADL. Then, for any $x, y \in R$ with $x \bigwedge y = 0$, $(x]^*$ and $(y]^*$ are $\bigvee$-comaximal in R.

Proof:

Choose $x, y \in$ such that $x \bigwedge y = 0$. Then we have $(x]^* \bigvee (y]^* = (O[x)) \bigvee O([y)) = O([x) \bigvee [y)) = O([x \bigwedge y)) = (x \bigwedge y]^* = (0]^* = R$. Therefore $(x]^*$ and $(y]^*$ are $\bigvee$-comaximal in R. ■

Theorem 2.4.13 [11] Let R be an ADL. Then any two distinct prime O-ideals of R are $\bigvee$-comaximal.

Proof:

Let P and Q be two distinct prime O-ideals of R. Then by Theorem 2.4.12, P and Q are minimal prime ideals. Choose $a \in P\text{-}Q$ and $b \in Q\text{-}P$. Since P, Q are minimal prime ideals, there exist $x \notin P$ and $y \notin Q$ such that $a \bigwedge x = b \bigwedge y = 0$. Since $b \bigwedge x \notin P$ and P is a prime ideal, we can get $(b \bigwedge x]^* \subseteq P$. Similarly, we get $(a \bigwedge y]^* \subseteq Q$.

Now $(b \bigwedge x) \bigwedge (a \bigwedge y) = b \bigwedge a \bigwedge x \bigwedge y = a \bigwedge x \bigwedge b \bigwedge y = 0 \bigwedge 0 = 0$. By Lemma 2.4.3, we get $(b \bigwedge x]^* \bigvee (a \bigwedge y]^* = R$. Hence $R = (b \bigwedge x]^* \bigvee (a \bigwedge y]^* \subseteq P \bigvee Q$.

Therefore P and Q are $\bigvee$-comaximal in R. ■

Chapter 3

Regular and π-Regular Rings

As mentioned before the concept of almost distributive lattice was introduced to include almost all existing rings such as, regular and π- regular rings.

In this chapter, regular, π-regular, semiabealian π-regular, and strong π-regular rings are defined.

The properties, characterizations and theorems connected between these concepts are studied, and several examples are given.

3.1 Regular Rings

Throughout this chapter, given a ring R, we use the symbol Id(R) to denote the set of idempotents in R, U(R) its unit group. The Jacobson radical, and the set of nilpotent elements of a ring R are denoted by J(R), and N(R), respectively.

Definition 3.1.1 *[9] A ring R is called regular if for any element $a \in R$, there exists an element x in R such that $a = axa$. Such an element a is called regular.*

Example 3.1.1 *Every division ring is obviously regular because if $a = 0$, then $a = axa$ for all x and if $a \neq 0$, then $a = axa$ for $x = a^{-1}$.*

Example 3.1.2 *Every direct product of regular rings is clearly a regular ring.*

Example 3.1.3 *A ring R is a Boolean ring if every element $r \in R$ is idempotent. Boolean rings are regular, since $\forall\ r \in R$, $r = rrr$.*

Theorem 3.1.1 *[2] Let R be a ring. Then R is regular if and only if R has the condition $\forall\ a \in R$, $\exists\ e^2 = e \in R$ such that $Ra = Re$.*

Proof :

Suppose that R is regular, then for any $a \in R$, there exists $x \in R$ such that $a = axa$.

Since xa and ax are idempotents in R, taking $xa = e$, $Ra = Raxa = Rae \subseteq Re$ and $Re = Rxa \subseteq Ra$. Hence $Ra = Re$.

Conversely, assume that R has the given condition $\forall\ a \in R$, $\exists\ e^2 = e \in R$ such that $Ra = Re$, and R has an identity. Then $a \in Ra = Re$, so that there exists $y \in R$ such that $a = ye$. From the condition, we see that $e = ee \in Re = Ra$, so that there exists $x \in R$ such that $e = xa$. Thus we obtain that $a = ye = yee = yexa = axa$.

Consequently, R is regular. ■

Theorem 3.1.2 *[26] A ring R is regular if and only if every finitely generated left ideal of R is generated by an idempotent.*

Proof :

Suppose first that every finitely generated left ideal of R can be generated by an idempotent. Let $x \in R$, then $I = Rx = Re$ for some idempotent e. That is $x = re$ and $e = sx$ for some $r, s \in R$. But then $xsx = xe = re^2 = re = x$.

Conversely, suppose that R is regular. We first show that every principle left ideal $I = Rx$ can be generated by an idempotent. So let $y \in R$ be such that $xyx = x$ and let $yx = e$. Clearly e is an idempotent and $xe = x$. Thus $x \in Re$ and so $I \subseteq Re$. Also $e = yx \in I$ and hence $Re \subseteq I$, so $I = Re$.

To complete the proof of the theorem we only need to show that if $J = Rx_1 + Rx_2$, then there exists some idempotent e ∈ R such that J = Re.

To see this, choose an idempotent e_1 such that $Rx_1 = Re_1$. Thus J = $Re_1 + Rx_2(1 - e_1)$.

Now choose an idempotent e_2 such that $Rx_2(1 - e_1) = Re_2$ and put $e_3 = (1 - e_1)e_2$.

See that e_3 is an idempotent, $e_1e_3 = e_3e_1 = 0$ and $Re_2 = Re_3$. Thus J = $Re_1 + Re_3$. Let e = $e_1 + e_3$. Then e is an idempotent and J = Re. ■

Theorem 3.1.3 *[2] Every regular ring R with identity has no non-zero nil left R-subgroup.*

Proof :

Let R be a regular ring and K be a nil left R-subset of R, it suffices to show that K = { 0 }.

Let a ∈ K, since R is regular, R has the condition ∃ e^2 = e ∈ R such that Ra = Re.

Since K is a left R-subset, we have that a ∈ Ra ⊆ K. On the other hand, since K is nil, there exists positive integer m, such that $a^m = 0$.

Next, from the condition e = ee ∈ Re = Ra ⊆ K, also there exists positive integer n, such that e = e^n = 0. From the above two conditions, we have a ∈ R0, so that a = r0 for some r ∈ R. Consequently, a = r0 = $(r0)^m = a^m$ = 0: That is, K = { 0 }. ■

Theorem 3.1.4 *[8] The center of regular ring is regular.*

Proof :

The center of the ring (ξ) is defined as: the set of all a ∈ R which commute with every x ∈ R.

Consider an $a \in \xi$, as R is regular, an x ∈ R with $axa = a$ exists.

As a commutes with every element in R, so $aa^2x^3a = axaxaxa = a$. Again a^2x = xa^2 = axa = a.

Since a commutes with every thing, hence for every z ∈ R, xa^2z = xa^2 z = z xa^2 = a^2zx, so a^2z commutes with x. Therefore x^3 commutes with it too, and so a^2x^3z = x^3a^2z = a^2z x = za^2x^3. Thus $x' = a^2x^3 \in \xi$, and since $a\,x'\,a$ = a, this establishes the regularity of ξ ■

Theorem 3.1.5 *[2] Let R be a ring with identity, such that Id(R) $\subseteq \xi$, and $x \in R$ is regular, then x is a unit regular.(there exist u $\in$ u(R), such that xux = x)*

Proof :

Let x $\in$ R is a regular, then there exist y $\in$ R such that xyx=x, so xy, and yx $\in$ Id(R) $\subseteq \xi$.

Hence xy = x(yx)y = (xy)(yx) = y(xy)x = yx.

Let u = x + xy − 1 and v = xy + xy^2 − 1. since xy = yx, and xyx = x, we have:

uv = vu = x^2y + x^2y^2 − x+$(xy)^2$ + xyxy^2 − xy − xy − xy^2 + 1 =

x + xy − x + xy + xy^2 − xy − xy − xy^2 + 1 = 1.

Moreover, xvx= x^2yx + x^2y^2x - x^2 = x^2 + x − x^2 = x. ■

Proposition 3.1.1 *[8] Let R be a unit regular ring "for every $x \in R$, then there exist u $\in$ u(R), such that xux = x", and $x \in R$, and x = ev, for some e $\in$ Id (R), and v $\in$ u(R).*

Proof:

Let x $\in$ R, then for some u $\in$ u(R), xux = x, thus xy $\in$ Id(R).

let v be the multiplicative inverse of u in R, then x = xuv a product of an idempotent and a unit of R. ■

We know by lemma 2.1.2 that in ADLs every maximal ideal is prime ideal, the following theorem ensure that in a regular ring the converse is also true.

Theorem 3.1.6 *[26] If R is regular ring, then P is prime ideal if and only if Pis maximal ideal.*

Proof:

($\Leftarrow$) Since any regular ring is an ADL then this side follow by lemma 2.1.2.

($\Rightarrow$)Assume P is prime ideal then R/P is integral domain and since R is regular then R/P is regular integral domain, so R/P is a field.

Thus P is a maximal ideal in R ■

We end this section by the relation between ADLs and regular rings which demonstrated by the following example

Example 3.1.4 *Let (R, +, ·, 0, 1) be a commutative regular ring with identity, If we define* $x \bigwedge y = x_0 y$ *and* $x \bigvee y = x + y + x_0 y$, $\forall x, y \in R$. *Then (R,* $\bigvee$, $\bigwedge$, *0) is an ADL.*

Proof:

To show that (R, +, ·, 0, 1) is an ADL we must verify the axioms (L_1-L_6) mentioned in chapter one in definition 1.3.1, remember that to any a in R, $aa_0a = a$

(L_1) $x \bigvee 0 = x + 0 + 0 = x$.

(L_2) $0 \bigwedge x = x.0 = 0$.

(L_4) $x \bigwedge (y \bigvee z) = x \bigwedge (y + z + y_0 z) = x_0(y + z + y_0 z) = x_0 y + x_0 z + x_0 y_0 z$.

Also $(x \bigwedge y) \bigvee (x \bigwedge z) = x_0 y \bigvee x_0 z = x_0 y + x_0 z + (x_0 y)_0\, x_0 z = x_0 y + x_0 z + x_0 y_0 z$, since R is commutative, x_0 is idempotent element and $(x_0 y)_0\, x_0 z = x_0 y_0 z$ $[(x_0 y)_0 = x_0 y_0$, since $(x_0 y) x_0 y_0 (x_0 y) = x_0(y y_0 y) = x_0 y]$

So we get that $x \bigwedge (y \bigvee z) = (x \bigwedge y) \bigvee (x \bigwedge z.)$

(L_3) $(x \bigvee y) \bigwedge z = (x + y + x_0 y) \bigwedge z = (x + y + x_0 y)_0 z = (x_0 + y_0 + x_0 y_0) z =$

$(x_0 z + y_0 z + x_0 y_0 z)$

Also $(x \bigwedge z) \bigvee (y \bigwedge z) = x_0 z \bigvee y_0 z = x_0 z + y_0 z + (x_0 z)_0 y_0 z = x_0 z + y_0 z + x_0 y_0 z =$

$x_0 z + y_0 z + x_0 y_0 z_0 z$. [since $z_0 z = z_0 z z_0 z = z z_0 z = z$]

So we get that $(x \bigvee y) \bigwedge z = (x \bigwedge z) \bigvee (y \bigwedge z)$.

(L_5) $x \bigvee (y \bigwedge z) = x + y_0z + x_0y_0z$.

Also $(x \bigvee y) \bigwedge (x \bigvee z) = (x+y+x_0y)_0(x + z + x_0z) = (x_0+y_0+x_0y_0)(x + z + x_0z)$
$= x + y_0z + x_0y_0z + 2\ x_0z + 2\ y_0x + 2\ x_0y_0z = x + y_0z + x_0y_0z$.[since $2x_0 = 0$]

So we get that $x \bigvee (y \bigwedge z) = (x \bigvee y) \bigwedge (x \bigvee z)$.

(L_6) $(x \bigvee y) \bigwedge y = (x+y+x_0y)_0y = (x_0+y_0+x_0y_0)y = x_0y + y_0y + 2x_0y$
$= 2x_0y + y = y$. ■

3.2 π - Regular Rings

The very first mention of this notion dates back to 1939, when it was introduced by McCoy, as a generalization of von Neumann regular rings, examples of π-regular rings, besides von Neumann regular ones, are Artinian rings and perfect rings.

In this section we define the π-regular rings and study some of their properties.

Definition 3.2.1 *[7] A ring R is said to be π regular ring, if $\forall\ x \in R$, $\exists\ n \geq 1$ and $y \in R$ such that $x^n\ y\ x^n\ = x^n$, in such a case x is called a π-regular element.*

And R is said to be unit π - regular ring, if $\forall\ x \in R$, $\exists\ m \geq 1$ and u a unit in R such that $x^m\ u\ x^m\ = x^m$.

Through out this section R denotes a commutative ring with identity.

The following theorem gives a characterization of all π- regular elements in a ring R such that Id(R)$\subseteq \xi$(R)

Theorem 3.2.1 *[3] Suppose Id(R)$\subseteq \xi(R)$, and $x \in R$, then x is π-regular if and only if there exists $e \in Id(R)$ such that ex is regular and (1-e)$x \in N(R)$*

Proof:

Since x is π-regular, for some n$\geq$1, x^n is regular, hence by theorem 3.1.4 and proposition 3.1.1, we have x^n = eu for some e $\in$ Id(R) and u $\in$ U(R), then ex $(x^{n-1}u^{-1})$ex = (ex^nu^{-1})ex = (euu^{-1})ex = ex, hence ex is regular.

Now $((1-e)x)^n$ = (1$-$ e)x^n = (1$-$e)eu = 0, since (1$-$e) $\in \xi$(R), so (1$-$e)x $\in$ N(R).

For the converse, suppose for some e $\in$ Id(R) such that ex is regular and (1$-$e)x $\in$ N(R), then $((1-e)x)^n$ = (1$-$e)x^n = 0, hence $ex^n = x^n$, since ex is regular by proposition 3.1.1 ex = cu for some c $\in$ Id(R) and u $\in$ U(R).

Hence $(ex)^n = (cu)^n = cu^n$, but $(ex)^n = ex^n = x^n$, thus $x^n = cu^n$, let y = cu^{-n}, then x^ny$x^n = x^n$, therefore x is π-regular. ■

Corollary 3.2.1 *[3] Suppose Id(R)⊆ ξ(R), and $x \in R$, such that x is π-regular, then for some e ∈ Id(R) and u ∈ U(R), we have $ex = eu$.*

Proof:

By the proof of theorem 3.2.1,for some e ∈ Id(R), v ∈ U(R), and m ≥ 1, we have x^m = ev and ex is regular.

Hence by theorem 3.1.4 and proposition 3.1.1 ex = cw for some c ∈ Id(R) and w ∈ U(R), in fact e = c, for e(ex) = e(cw), but e(ex) = ex = cw, thus ecw = cw and therefore ec = c, since c, e ∈ ξ(R), we have $(ex)^m = ex^m = cw^m$, since x^m = ev, ex^m = ev = cw^m.

hence e = $cw^m v^{-1}$, thus ec = $cw^m v^{-1}$c = $cw^m v^{-1}$, since c∈ ξ(R), hence ec = e, since ec = c and ec = c, e = c, therefore ex = ew ■

Theorem 3.2.2 *[2] The set N(R) of an abelian π-regular ring R is two sided ideal of R*

Proof:

Let w ∈ N(R) and r ∈ R, now suppose rw not in N(R), then by corollary 3.2.1 there exists e ∈ Id(R), u ∈ U(R) such that erw = rew = eu, observe that e ≠ 0, for if e = 0 then (1-e)rw = rw ∈ N(R), by theorem 3.2.1 and this contradicts the assumption that rw is not in N(R). Since ew ∈ N(R), let n be the smallest integer such that $(ew)^n = 0$, then n ≥ 2, since e ≠ 0, thus 0 = rew$(ew)^{n-1}$ = eu$(ew)^{n-1}$ = u$(ew)^{n-1}$, hence $(ew)^{n-1} = 0$, which is a contradition. Thus, for any w ∈ N(R) we have rw ∈ N(R).

Now let z, w ∈ N(R) and suppose w+z not in N(R), then there exists c ∈ Id(R), and v ∈ U(R) such that c(w+z) = cv, hence cw = cv - cz = cv(1-v^{-1}z), since $-v^{-1}$z ∈ N(R), 1-v^{-1}z = u ∈ U(R), thus cw = cvu, but cw ∈ N(R) and cvu is not in N(R), hence w+z ∈ N(R).

Hence N(R) is a two-sided ideal of R. ■

Lemma 3.2.1 *[2] Let R be a ring with 1, and I be two sided ideal in R, if [c] ∈ Id(R/I), then there exist e ∈ Id(R), such that [c]=[e] in R/I*

Lemma 3.2.2 *[3] Let K = R/N(R), and u ∈ R, then [u] ∈ U(K) if and only if u ∈ U(R)*

Theorem 3.2.3 *[3] suppose Id(R) $\subseteq \xi(R)$, the ring R is π - regular if and only if R/N(R) is regular.*

Proof :

Suppose R is π - regular, and let [x] $\in$ R/N(R), then for some y $\in$ R and n $\geq$ 1, x^n y $x^n = x^n$, thus e = x^n y $\in$ Id(R), and therefore 1 - e $\in$ Id(R).

Now $((1-e)x)^n$ =(1-e)x^n = (1-(x^ny)) x^n =0, thus (1 - e)x = (1-x^ny)x $\in$ N(R), so [x][x^{n-1}y][x] = [x^ny][x] = [x], hence R/N(R) is regular.

Let K = R/N(R) is regular, and x $\in$ R, then there exist [y] $\in$ K, such that [x][y][x] = [x] in K, also [x] is unit regular in K and [x] = [c][u] for some [c] $\in$ Id(R)and u $\in$ U(R), then by lemma 3.2.1 there exist e $\in$ Id(R) such that [c] = [e] and by lemma 3.2.2 u $\in$ U(R), thus x = eu+w for some w $\in$ N(R).

Now ex = e(u+w), since N(R) $\subseteq$ J(R), w $\in$ J(R), then u+w $\in$ U(R), thus ex is regular, further (1-e)x = x-ex = (eu+w)-(eu+ew) = w-ew $\in$N(R).

Hence (1-e)x $\in$ N(R), therefore x is π-regular. ■

Theorem 3.2.4 *[2] Let R be a ring with 1, and R is π - regular then R is a unit π - regular*

Proof :

By theorem 3.1.4 , since x^m is regular, then the claim is evident. ■

Theorem 3.2.5 *[2] The ring R is π - regular ring, if and only if for all $x \in$ R, there exist e $\in$ Id(R), u $\in$ U(R), w $\in$ N(R), such that $x = eu + w$*

Proof:

Let R is π - regular ring, then K = R/N(R) is unit regular, then by lemma 3.2.2 [x] = [c][u] for some [c] $\in$ Id(K) and [u] $\in$ U(K), now by lemma 3.2.1 there is e $\in$Id(R) such that [e] = [c] in K, thus by lemma 3.2.2 we have [u] $\in$ U(R), so [x] = [e][u] implies that x - eu = w, for some w $\in$ N(R), so x = w + eu .

Conversely suppose x = eu + w, for some e $\in$ Id(R), u $\in$ U(R), w $\in$ N(R), then for some n ≥ 1, $w^n = 0$. Now consider the expansion of $x^n = (w + eu)^n$, observe that $(eu)^n = eu^n$, since e $\in \xi$(R) and the sum of other terms are in N(R), hence $x^n = (w + eu)^n = eu^n + ed$, for some d $\in$ N(R), but u^n + d = v $\in$ U(R), hence x^n = ev, so if r is the multiplicative inverse of v in R then $x^n r x^n = x^n$. ■

Corollary 3.2.2 *A ring R is abelian π-regular ring if and only if Id(R) $\subseteq \xi$(R), N(R) is two-sided ideal of R, and for every $x \in R$ there exist e $\in$ Id(R), $u \in U(R)$, and $w \in N(R)$ such that $x = eu+w$*

Theorem 3.2.6 *[3] Suppose Id(R) $\subseteq \xi$(R), then R is π-regular if and only if for some two-sided nil ideal I of R, R/I is π-regular*

Proof:

Suppose R is π-regular, by theorem3.2.2 I = N(R) is two-sided ideal, then by theorem 3.2.3 R/I is regular and hence is π-regular.

Now assume that R/I is π-regular for some two-sided nil ideal I, then N(R/I) = N(R)/I is two-sided ideal of R/I by theorem 3.2.3, so N(R) is two-sided ideal of R, since R/I is π-regular, so is R/N(R), therefore R is π-regular. ■

Now we insert some examples of π - regular rings.

Example 3.2.1 *Let K be a principle ideal domain with 1, and m be a non-zero, non-unit element in K, then $m = p_1^{a_1}\ p_2^{a_2} \cdots\ p_n^{a_n}$, where a_i's ≥ 1 and p_i's are distinct primes in K. Let S = K/(m), and I = ($p_1\ p_2 \cdots p_n$) then N(S) = I/(m), thus S/N(S) isomorphic to K/I, hence S/N(S) is regular, therefor S is π - regular.*

Note *that an element p of a ring R is said to be* ***primes*** *if it is not zero or a unit and whenever p divides ab for some a and b in R, then p divides a or p divides b.*

Example 3.2.2 *All finite commutative rings are π -regular rings*

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