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# Arithmetic Progression of more Squares

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**Abstract**. The aim of this paper is to present new arithmetic progressions among squares, i.e., more than three squares. Moreover we will introduce arithmetic progression of higher power.

## 1 Introduction

Referring to my research [3] it is possible to produce triplets of the type $(a, b, c)$ of positive integers so that $a^2, b^2, c^2$ are in Arithmetic Progression (AP) such that : $b^2 - a^2 = c^2 - b^2$.

In 2008 [2] Peth'o and Ziegler have found an arithmetic progression of length 4 that lies on some curve $X^2 - dY^2 = m$ and they have found an arithmetic progression such that there does not exist such a curve.

In 2010 [1] Enrique Gonz and J. ÃĎorn Steuding gave a partial answer to this question: Let $d$ be a squarefree integer. Does there exist four squares in arithmetic progression over $\mathbb{Q}(\sqrt{d})$ depending on the value of $d$? In the affirmative case, they construct explicit arithmetic progressions consisting of four squares over $\mathbb{Q}(\sqrt{d})$.

It is well known that the right triangles whose sides are integers $X, Y, Z$(a Pythagorean triple) formed Arithmetical Progressions (AP) among three squares such that the following equation holds:

$$(X + Y)^2 + (X - Y)^2 = 2Z^2$$

For example the Pythagorean triple $(3, 4, 5)$ represents the AP $(1, 5, 7)$, i.e. $7^2 + 1^2 = 2(5)^2$, or more familiars $5^2 - 1^2 = 7^2 - 5^2$, in fact equation $(X + Y)^2 + (X - Y)^2 = 2Z^2$ has infinitely many solutions, and chains with common difference 24.

## 2 The Extension of $(X + Y)^2 + (X - Y)^2 = 2Z^2$

We begin with the following important extension of $(X + Y)^2 + (X - Y)^2 = 2Z^2$.

**Theorem 2.1.** *With slight modification of* $(X + Y)^2 + (X - Y)^2 = 2Z^2$*, we got the following new equation:*

$$(X + Y)^2 + (X - Y)^2 + (-X + Y)^2 + (-X - Y)^2 = 4Z^2$$

**Proof.** Assume that $X = 2mn$, $Y = m^2 - n^2$, and $Z = m^2 + n^2$, then by direct substitutions we got $(2mn + m^2 - n^2)^2 + (2mn - (m^2 - n^2))^2 + (-2mn + m^2 - n^2)^2 + (-2mn - (m^2 - n^2))^2 = 4(m^2 + n^2)^2 = 4Z^2$.

We show the first sixteen consecutive AP of this equation in the following table , for examples: i) $7^2 + 1^2 + (-7)^2 + (-1)^2 = 100 = 4(25) = 4(5)^2$, i.e.$7^2 + (-7)^2 - (5)^2 - (5)^2 = (5)^2 + (5)^2 - (-1)^2 - 1^2$ ,so we have the AP $(-7, -1, 1, 5, 7)$. ii) $137^2 + 7^2 + (-7)^2 + (-137)^2 = 37636 = 4(9409) = 4(97)^2$ , i.e. $137^2 + (-137)^2 - (97)^2 - (97)^2 = (97)^2 + (97)^2 - (-7)^2 - 7^2$ , so we have the AP $(-137, -7, 7, 97, 137)$.

| X | Y | Z | (X+Y) | (X-Y) | (-X+Y) | (-X-Y) | 4(Z) |
|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 7 | 1 | -1 | -7 | 4(5) |
| 5 | 12 | 13 | 17 | 7 | -7 | -17 | 4(13) |
| 7 | 24 | 25 | 31 | 17 | -17 | -31 | 4(25) |
| 8 | 15 | 17 | 23 | 7 | -7 | -23 | 4(17) |
| 9 | 40 | 41 | 49 | 31 | -31 | -49 | 4(41) |
| 11 | 60 | 61 | 71 | 49 | -49 | -71 | 4(61) |
| 12 | 35 | 37 | 47 | 23 | -23 | -47 | 4(37) |
| 13 | 84 | 85 | 97 | 71 | -71 | -97 | 4(85) |
| 16 | 63 | 65 | 79 | 47 | -47 | -79 | 4(65) |
| 20 | 21 | 29 | 41 | 1 | -1 | -41 | 4(29) |
| 28 | 45 | 53 | 73 | 17 | -17 | -73 | 4(53) |
| 33 | 56 | 65 | 89 | 23 | -23 | -89 | 4(65) |
| 36 | 77 | 85 | 113 | 41 | -41 | -113 | 4(85) |
| 39 | 80 | 89 | 119 | 41 | -41 | -119 | 4(89) |
| 48 | 55 | 73 | 103 | 7 | -7 | -103 | 4(73) |
| 65 | 72 | 97 | 137 | 7 | -7 | -137 | 4(97) |

**Theorem 2.2.** *If $X, Y, Z$ are the sides of right triangle then;*

$$(X+Y+Z)^2 + (X-Y+Z)^2 + (X+Y-Z)^2 + (-X+Y+Z)^2 = 8Z^2$$

**Proof.** Assume that $X = 2mn, Y = m^2 - n^2$, and $Z = m^2 + n^2$. Then by direct substitutions we got: $(2mn + m^2 - n^2 + m^2 + n^2)^2 + (2mn - m^2 + n^2 + m^2 + n^2)^2 + (2mn + m^2 - n^2 - m^2 - n^2)^2 + (-2mn + m^2 - n^2 + m^2 + n^2)^2 = 8(m^2 + n^2)^2 = 8Z^2$.

Table below shows the first sixteen AP of $(X+Y+Z)^2 + (X-Y+Z)^2 + (X+Y-Z)^2 + (-X+Y+Z)^2 = 8Z^2$.

| X | Y | Z | (X+Y+Z) | (X-Y+Z) | (X+Y-Z) | (-X+Y+Z) | 8(Z) |
|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 12 | 4 | 2 | 6 | 8(5) |
| 5 | 12 | 13 | 30 | 6 | 4 | 20 | 8(13) |
| 7 | 24 | 25 | 56 | 8 | 6 | 42 | 8(25) |
| 8 | 15 | 17 | 40 | 10 | 6 | 24 | 8(17) |
| 9 | 40 | 41 | 90 | 10 | 8 | 72 | 8(41) |
| 11 | 60 | 61 | 132 | 12 | 10 | 110 | 8(61) |
| 12 | 35 | 37 | 84 | 14 | 10 | 60 | 8(37) |
| 13 | 84 | 85 | 182 | 14 | 12 | 156 | 8(85) |
| 16 | 63 | 65 | 144 | 18 | 14 | 112 | 8(65) |
| 20 | 21 | 29 | 70 | 28 | 12 | 30 | 8(29) |
| 28 | 45 | 53 | 126 | 36 | 20 | 70 | 8(53) |
| 33 | 56 | 65 | 154 | 42 | 24 | 88 | 8(65) |
| 36 | 77 | 85 | 198 | 44 | 28 | 126 | 8(85) |
| 39 | 80 | 89 | 208 | 48 | 30 | 130 | 8(89) |
| 48 | 55 | 73 | 176 | 66 | 30 | 80 | 8(73) |
| 65 | 72 | 97 | 234 | 90 | 40 | 104 | 8(97) |

For example the Pythagorean triple $(3, 4, 5)$ represents the AP $(2, 4, 6, 12)$, i.e.$2^2 + 4^2 + 6^2 + 12^2 = 8(5)^2$, hence: $12^2 + 6^2 - 5^2 - 5^2 - 5^2 - 5^2 = 5^2 + 5^2 + 5^2 + 5^2 - 2^2 - 4^2$,if we cancel 2 from the AP $(2,4,6,12)$ , then we get $(1,2,3,6)$ , which is of course AP $(1^2+2^2+3^2+6^2=2(5)^2)$ , this example has new ideas that maybe provoke for more new results which is open for Mathematicians . Table above shows the first sixteen AP of equation:

$$(X+Y+Z)^2+(X-Y+Z)^2+(X+Y-Z)^2+(-X+Y+Z)^2=8Z^2$$

**Theorem 2.3.** *If $X,Y,Z$ are the sides of right triangle then;* $(X+Y+Z)^2+(X-Y+Z)^2+(X+Y-Z)^2+(-X+Y+Z)^2+(-X-Y-Z)^2+(-X+Y-Z)^2+(X-Y-Z)^2+(-X-Y+Z)^2=16Z^2$

**Proof.** Assume that$X=2mn, Y=m^2-n^2$, and $Z=m^2+n^2$, then by direct substitutions we got: $(2mn+m^2-n^2+m^2+n^2)^2+(2mn-m^2+n^2+m^2+n^2)^2+(2mn+m^2-n^2-m^2-n^2)^2+(-2mn+m^2-n^2+m^2+n^2)^2+(-2mn-m^2+n^2-m^2-n^2)^2+(-2mn+m^2-n^2-m^2-n^2)^2+(2mn-m^2+n^2-m^2-n^2)^2+(-2mn-m^2+n^2+m^2+n^2)^2=16(m^2+n^2)^2=16Z^2$.

Table below shows selected solutions of equation : $(X+Y+Z)^2+(X-Y+Z)^2+(X+Y-Z)^2+(-X+Y+Z)^2+(-X-Y-Z)^2+(-X+Y-Z)^2+(X-Y-Z)^2+(-X-Y+Z)^2=16Z^2$
For the following tabular: $t_1=(X+Y+Z), t_2=(X-Y+Z), t_3=(X+Y-z), t_4=(-X+Y+Z), t_5=(-X-Y-Z), t_6=(-X+Y-Z), t_7=(X-Y-Z), t_8=(-X-Y+Z), t_9=16Z$

| X | Y | Z | $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $t_6$ | $t_7$ | $t_8$ | $t_9$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 12 | 4 | 2 | 6 | -12 | -4 | -6 | -2 | 16(5) |
| 5 | 12 | 13 | 30 | 6 | 4 | 20 | -30 | -6 | -20 | -4 | 16(13) |
| 7 | 24 | 25 | 56 | 8 | 6 | 42 | -56 | -8 | -42 | -6 | 16(25) |
| 8 | 15 | 17 | 40 | 10 | 6 | 24 | -40 | -10 | -24 | -6 | 16(17) |
| 9 | 40 | 41 | 90 | 10 | 8 | 72 | -90 | -10 | -72 | -8 | 16(41) |
| 11 | 60 | 61 | 132 | 12 | 10 | 110 | -132 | -12 | -110 | -10 | 16(61) |
| 12 | 35 | 37 | 84 | 14 | 10 | 60 | -84 | -14 | -60 | -10 | 16(37) |
| 13 | 84 | 85 | 182 | 14 | 12 | 156 | -182 | -14 | -156 | -12 | 16(85) |
| 16 | 63 | 65 | 144 | 18 | 14 | 112 | -144 | -18 | -112 | -14 | 16(65) |
| 20 | 21 | 29 | 70 | 28 | 12 | 30 | -70 | -28 | -30 | -12 | 16(29) |
| 28 | 45 | 53 | 126 | 36 | 20 | 70 | -126 | -36 | -70 | -20 | 16(53) |
| 33 | 56 | 65 | 154 | 42 | 24 | 88 | -154 | -42 | -88 | -24 | 16(65) |
| 36 | 77 | 85 | 198 | 44 | 28 | 126 | -198 | -44 | -126 | -28 | 16(85) |
| 39 | 80 | 89 | 208 | 48 | 30 | 130 | -208 | -48 | -130 | -30 | 16(89) |
| 48 | 55 | 73 | 176 | 66 | 30 | 80 | -176 | -66 | -80 | -30 | 16(73) |
| 65 | 72 | 97 | 234 | 90 | 40 | 104 | -234 | -90 | -104 | -40 | 16(97) |

It seems that equation 4 is the general form of the previous equations, so we may plug some variables by zero to get them.

**Theorem 2.4.** *If $X,Y,Z$ are the sides of right triangle then;*

$$X^8+Y^8+Z^8=2W^2$$

**Proof.** Let $m,n$ integers, with $(m,n)=1$, then the solution of equation $X^8+Y^8+Z^8=2W^2$: $X=2mn, Y=m^2-n^2, Z=m^2+n^2$ , and $W=(m^4+2m^3n+2m^2n^2-2mn^3+n^4)(m^4-2m^3n+2m^2n^2+2mn^3+n^4)$ It can be proved by direct substitution .

Table below shows selected solutions of equation $X^8+Y^8+Z^8=2W^2$:

| X | Y | Z | (X)^8 | (Y)^8 | (Z)^8 | 2W^2 |
|---|---|---|---|---|---|---|
| 3 | 4 | 5 | 6,561 | 65,536 | 390,625 | 2(481)^2 |
| 5 | 12 | 13 | 390,625 | 429,981,696 | 815,730,721 | 2(24961)^2 |
| 7 | 24 | 25 | 5,764,801 | 110,075,314,176 | 152,587,890,625 | 2(362401)^2 |
| 8 | 15 | 17 | 16,777,216 | 2,562,890,625 | 6,975,757,441 | 2(69121)^2 |
| 9 | 40 | 41 | 43,046,721 | 6,553,600,000,000 | 7,984,925,229,121 | 2(2696161)^2 |
| 11 | 60 | 61 | 214,358,881 | 167,961,600,000,000 | 191,707,312,997,281 | 2(13410241)^2 |
| 12 | 35 | 37 | 429,981,696 | 2,251,875,390,625 | 3,512,479,453,921 | 2(1697761)^2 |
| 13 | 84 | 85 | 815,730,721 | 2,478,758,911,082,500 | 2,724,905,250,390,620 | 2(51008161)^2 |
| 16 | 63 | 65 | 4,294,967,296 | 248,155,780,267,521 | 318,644,812,890,625 | 2(16834561)^2 |

Conclusion: From the equations above we can conclude such a sequence of patterns that began from equation $(X+Y)^2+(X-Y)^2=2Z^2$ with $2Z^2$, continuo in equation $(X+Y)^2+(X-Y)^2+(-X+Y)^2+(-X-Y)^2=4Z^2$ with $4Z^2$, same of equation $(X+Y+Z)^2+(X-Y+Z)^2+(X+Y-Z)^2+(-X+Y+Z)^2=8Z^2$ with $8Z^2$, and $16Z^2$ with equation $(X+Y+Z)^2+(X-Y+Z)^2+(X+Y-Z)^2+(-X+Y+Z)^2+(-X-Y-Z)^2+(-X+Y-Z)^2+(X-Y-Z)^2+(-X-Y+Z)^2=16Z^2$, so what about $32Z^2$, $64Z^2$, and so on. Are there common pattern connected AP with $2nZ^2$?

## References


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