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Generalizations of Reversible and Symmetric Rings

By

Abdelhadi A. Shabaan

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Supervisor : Dr. Iyad Alhribat

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Generalizations of Reversible and Symmetric Rings

By

Student Name : Abdelhadi A. Shabaan

Supervisor : Dr. Iyad Alhribat

Master thesis submitted and accepted,

Date : 20 / 3 / 2016

The name and signatures of the examining committee members are as follows :

1. Dr. Iyad Alhribat, Head of committee Signature ………………
2. Dr. Nureddin Rabie. Internal Examiner Signature ………………
3. Dr. Ali Twaihaa. External Examiner Signature Ali A. Tawaiha

The dean of graduate studies, signature ……………………

Dr. Murad Abusubaih

Palestine Polytechnic University

2016

Declaration

I certify that this thesis, submitted for the degree of master, is the results of my own research except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed :.............................

Abdelhadi A. Shabaan

Date : ..20../..3../.2016...

Dedication

All praise to Allah, today we fold the days' tiredness and the errand summing up between the cover of this humble work .

To the utmost knowledge lighthouse, to our greatest and most honored prophet Mohamed - May peace and grace from Allah be upon him .

To the Spring that never stops giving, to my mother who weaves my happiness with strings from her merciful heart …… to my mother.

To whom he strives to bless comfort and welfare and never stints what he owns to push me in the success way who taught me to promote life stairs wisely and patiently, …… to my dearest father.

To whose love flows in my veins, and my heart always remembers them, …… to my brothers and sisters and friends .

To those who taught us letters of gold and words of jewel of the utmost and sweetest sentences in the whole knowledge. Who reworded to us their knowledge simply and from their thoughts made a lighthouse guides us through the knowledge and success path, …… To my honored teachers and professors.

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Abstract

In this thesis we study central reversible and central symmetric rings, which are generalizations of reversible and symmetric rings respectively. We study their main characterizations and properties and its relationship with other classes of rings.

Moreover, many ring extensions of those rings are considered. We introduce new classes of rings which are central $\alpha-$reversible and central $\alpha-$ symmetric rings as a generalizations of $\alpha-$reversible and $\alpha-$ symmetric rings respectively, some results in this case were introduced.

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Introduction

Throughout this thesis all rings are associative with identity . A ring R is said to be *reversible* if for any $a, b \in R, ab = 0$ implies $ba = 0$ and it is called a *central reversible* if for any $a, b \in R,\ ab = 0$ implies ba is central in R and it is said to be *weakly reversible*, if for any$a, b, r \in R$ such that $ab\ =\ 0$, then $Rbra$ is a nil left ideal of R.

In **chapter one**, we study *central reversible rings* as a generalization of reversible rings. Clearly, reversible rings are central reversible and central reversible rings are weakly reversible. We supply some examples to show that all central reversible rings need not be reversible and all weakly reversible rings need not be central reversible. Therefore, we show that the class of central reversible rings lies strictly between classes of reversible rings and weakly reversible rings.

Among others we prove that central reversible rings are abelian and there exists an abelian ring, but it is not central reversible. It is shown that every central reversible ring is weakly reversible, 2-primal, abelian and so directly finite.

For an Armendariz ring R, we prove that R is central reversible if and only if the polynomial ring $R[x]$ is central reversible if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is central reversible. Moreover, it is also proven that if R is central reversible, then the Dorroh extension of R is central reversible.

A ring R is said to be symmetric if for any $a, b, c \in R,\ abc = 0$ implies $acb = 0$ and it is called central symmetric if for any$a, b, c \in R$,$abc = 0$ implies bac belongs to the center of R.

In **chapter two**, we study *central symmetric rings*, which is a generalization of symmetric rings. We investigate which properties of symmetric rings hold for the central case. Clearly, symmetric rings are central symmetric and central symmetric rings are central reversible. We supply an example to show that all central symmetric rings need not be symmetric. We provide another example to show that all central reversible rings may not be central symmetric. Therefore the class of central symmetric rings lies strictly between classes of symmetric rings and central reversible rings. Sufficient conditions for central symmetric rings to be symmetric are indicated.

It is shown that the class of central symmetric rings is closed under finite direct sums. We have an example which shows that the homomorphic image of a central symmetric ring is not central symmetric. Then we determine under what conditions a homomorphic image of a ring is central symmetric.

Finally in **chapter three**, we consider a skew version of reversible and Symmetric rings with respect to a ring endomorphism α , in particular we study $\alpha-$reversible and $\alpha-$ symmetric rings ,they are defined as follows:

A ring R is called α-reversible if it is both right and left α-reversible that is whenever $ab = 0$ for $a, b \in R$ then $b\alpha(a) = 0$ and $\alpha(b)a = 0$,and it is called α-symmetric if it is both right and left α-symmetric that means whenever $abc = 0$then$ac\alpha(b) = 0$ and $\alpha(b)ac = 0$ for $a, b, c \in R$.Moreover, we introduce new classes of rings, called central $\alpha-$reversible and central $\alpha-$ symmetric rings as a generalization of $\alpha-$reversible and $\alpha-$ symmetric rings . We prove some results in this general setting.

Chapter 1.

Generalization of Reversible Rings

In this chapter we study a class of rings, called central reversible rings, which is a generalization of reversible rings. We prove that some results of reversible rings can be extended to central reversible rings. In this chapter all rings are associative with identity.

Section 1.1. Central Reversible Rings

Definition 1.1.1.

A non zero element a is called nilpotent if there exist $n \in \mathbb{Z}^{+}$ such that $a^{n} = 0$.

Definition 1.1.2.

A ring R is said to be reduced if it has no non-zero nilpotent elements.

Definition 1.1.3. [2]

A ring R is said to be reversible if for any$a, b \in R, ab = 0$ implies $ba = 0$.

We now give the main definition in this chapter.

Definition 1.1.4. [12]

A ring R is called central reversible if for any $a, b \in R,\ ab = 0$ implies ba is central in R.

One may suspect that central reversible rings are reversible. But the following example erases the possibility.

Example 1.1.5. [12]

Let R be a commutative reduced ring and consider $S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c, d \in R \right\}$ be the ring of 3×3 upper triangular matrices with usual addition and multiplication of matrices , we prove that S is central reversible but not reversible. Let

$x = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix}, y = \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \in S$ with $xy = 0$. Then we have the following equations ;

(1) $a_1a_2 = 0,$

(2) $a_1b_2 + b_1a_2 = 0,$

(3) $a_1c_2 + b_1d_2 + c_1a_2 = 0,$

(4) $a_1d_2 + d_1a_2 = 0$.

Since R is commutative, we have $a_2a_1 = 0$. Multiplying equation (2) from the left by b_1a_2 then $(b_1a_2)^2 = 0$ and so $b_1a_2 = 0$ since R is reduced. Using a similar method we get $a_1d_2 = 0 = d_1a_2$. Multiplying equation (3) from the right by a_1c_2, then $(a_1c_2)^2 = 0$ it follows that $a_1c_2 = 0 = c_2\,a_1$ then we get $b_1d_2 + c_1a_2 = 0$. Multiplying this equation from the right by b_1d_2, then $b_1d_2 = d_2b_1 = 0$, consequently

$$yx = \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix}\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b_2d_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0 .$$

Now we claim that yx is central , let $c = \begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & a_3 & d_3 \\ 0 & 0 & a_3 \end{pmatrix} \in S,$

$$yxc = \begin{pmatrix} 0 & 0 & b_2d_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & a_3 & d_3 \\ 0 & 0 & a_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b_2d_1a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$cyx = \begin{pmatrix} a_3 & b_3 & c_3 \\ 0 & a_3 & d_3 \\ 0 & 0 & a_3 \end{pmatrix}\begin{pmatrix} 0 & 0 & b_2d_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_3b_2d_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & b_2d_1a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore S is central reversible.

On the other hand let $x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, note that $xy = 0$ but

$yx = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$. Hence S is not reversible.

Definition 1.1.6.[1]

A ring R is semi-prime if $aRa = 0$ implies $a = 0$for $a \in R$.

Definition 1.1.7.[1]

A ring R is right (left) principally projective if the right (left) annihilator of an element of R is generated by an idempotent.

Definition 1.1.8.[1]

A ring R is right (left) principally Quasi-Baer if the right (left) annihilator of principal right (left) ideal of R is generated by an idempotent.

Lemma 1.1.9.

If R is a reversible ring, then R is central reversible.

Proof.

Let R be a reversible ring with $\mathrm{ab} = 0$. And so, $\mathrm{ba} = 0$, since R is reversible then ba is central in R since 0 is central.

We now investigate under what conditions central reversible rings are reversible.

Proposition 1.1.10. [12]

Let R be a central reversible ring, then R is reversible if R satisfies any of the following conditions.

(1) R is semi-prime ring.
(2) R is a right (left) principally projective ring.
(3) R is a right (left) principally Quasi-Baer ring.

Proof .

Assume that R is a central reversible ring and $\mathrm{a}, \mathrm{b} \in \mathrm{R}$ with $\mathrm{ab} = 0$. Then ba is central. Now consider the following cases.

(1) Let R be a semi-prime ring. Since ba is central, $\mathrm{baRba} = 0$ and so $\mathrm{ba} = 0$. Thus R

is reversible.

(2) Let R be a right principally projective ring. Then there exists an idempotent $e \in R$ such that r_R (a) = eR. Hence b = eband ae = 0. Since ba is central, ba = eba = bae = 0 and so R is reversible. A similar proof may be given for left principally projective rings.

(3) Let R be a right principally quasi-Baer ring. Then there exists an idempotent $e \in R$ such that r_R (aR) = eR. Hence b = eb and ae = 0. Since ba is central, ba = eba = bae = 0 related on 1. And so R is reversible. A similar proof may be given for left principally projective rings.

The following is a consequence of Proposition 1.1.9.

Corollary 1.1.11.

If R is a central reversible ring. Then the following conditions are equivalent .

(1) R is a right principally projective ring.

(2) R is a left principally projective ring.

(3) R is a right principally Quasi-Baer ring.

(4) R is a left principally Quasi-Baer ring.

Proof.

(1)⇒(2) Let R be a right principally projective ring and $a \in R$. Then there exists an idempotent $e \in R$ such that $r_R(a) = eR$. Also by Proposition1.1.9, R is reversible. Let $b \in l_R(a)$. R Being reversible ,$ab = 0$ and so $b = eb$. Since e is central, we have $b = be$. Hence $l_R(a) \subseteq Re$. Obviously, $Re \subseteq l_R(a)$. Thus R is a left principally projective ring.

(2)⇒(3)Let R be a left principally projective ring and $a \in R$. Then there exists an idempotent $e \in R$ such that $l_R(a) = Re$. Since e is central, we have $ea = ae = 0$. Let$b \in r_R(aR)$.Then $axb = 0$ for all $x \in R$. Hence $bax = 0$, and $ba = 0$ due to$1 \in R$. Thus $b = be = eb$, and so $r_R(aR) \subseteq eR$. Hence $eR = r_R(aR)$ since$e R \subseteq r_R(aR)$ holds also. Therefore R is a right principally quasi-Baer ring.

(3)⇒(4) and (4)⇒(1) are similar to the proofs of preceding cases.

Recall that a ring with identity $1 \neq 0$ in which every element is unit is called division ring.

Note that the homomorphic image of a central reversible ring need not be central reversible. Consider the following example.

Example 1.1.12. [12]

Let D be a division ring, $R = D[x, y]$ and $I = (xy)$, where $xy \neq yx$. Since R is a domain, R is central reversible. On the other hand, $(x + I)(y + I)$ is zero but $(y + I)(x + I)$ is not central in R/I. Hence R/I is not central reversible.

Definition 1.1.13.[10]

A ring R is called unit central if all unit elements are central in R.

Our next endeavor is to determine conditions when the homomorphic image of a ring is central reversible.

Recall that an ideal of a ring is said to be **nil ideal** if each of its elements is nilpotent .

Lemma 1.1.14. [12]

Let R be a unit central ring. If I is a nil ideal of R, then R/I is central reversible.

Proof.

Let $a, b \in R$ with $(a + I)(b + I) = 0 + I$. Then $ab \in I$ and so there exists a positive integer n such that $(ab)^n = 0$. Hence $(ba)^{n+1} = 0$. It follows that $1 - ba$ is unit and so central by hypothesis. Thus $rba = bar$ for any $r \in R$. Therefore $(b + I)(a + I)$ is central in R/I.

The next example shows that for a ring R and an ideal I, if R/I is central reversible, then R need not be central reversible.

Example 1.1.15.[12]

Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ where F is any field. Consider the ideal $I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$of R. Then R/I is central reversible because of the commutativity property of R/I . For $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R$, we have $AB = 0$. Consider $C = \begin{pmatrix} c_1 & c_2 \\ 0 & c_3 \end{pmatrix} \in R$ with $c_1 \neq c_3$. It is clear that $CBA \neq BAC$. Therefore R is not central reversible.

Lemma1.1.16. [12]

Let R be a ring. If R/I is a central reversible ring with a reduced ideal I, then R is central reversible.

Proof.

Let R/I be a central reversible ring. Let $a, b \in R$ with $ab = 0$. Since $(a + I)(b + I) = 0 + I$, $(b + I)(a + I)$ is central in R/I. It follows that $bar - rba \in I$ for any $r \in R$. Then $I\,(bar - rab) = 0$. Hence we have $(bar - rba)^2 = 0$. Since I is reduced, $bar = rba$ and so R is central reversible.

Definition 1.1.17. [15]

A ring R is said to be weakly reversible, if for all $a, b, r \in R$ such that $ab = 0$, $Rbra$ is a nil left ideal of R.

We show that the class of central reversible rings lies strictly between classes of reversible and weakly reversible rings.

Theorem1.1.18. [12]

Let R be a ring. Consider the following conditions.

(1) R is reversible.
(2) R is central reversible.
(3) R is weakly reversible.

Then $(1) \rightarrow (2) \rightarrow (3)$.

Proof.

(1) $\Longrightarrow$ (2) By lemma 1.1.16.

(2) $\Longrightarrow$ (3) Let $a, b \in R$ with $ab = 0$. Then for all $x \in R, abx = 0$. Since R is central reversible, clearly bxa is central. Then we have $(rbxa)^2 = (rbxa)(rbxa) = r(bxa)rbxa = rrbx(ab)xa = 0$ for all $r, x \in R$. This implies that R is weakly reversible.

Lemma 1.1.19.[15]

A ring R is a weakly reversible ring if and only if, for any n, the $n \times n$ upper triangular matrix ring $T_n(R)$ is a weakly reversible ring.

The next example shows that weakly reversible need not be central reversible.

Example1.1.20. [12]

Let R be a weakly reversible ring and consider the ring

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \middle| a, b, c, d, f \in R \right\}$$

by last proposition S is weakly reversible. We now prove that S is not central reversible . For $x = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \in S$, we have $xy = 0$ but $yx = x$ which is not central in S, let $z = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$, $xz = \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ but $zx = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ .Therefore S is not central reversible.

Definition 1.1.21.[12]

A ring R is said to be abelian if every idempotent of the ring R is central.

Lemma 1.1.22. [12]

Every reversible ring is abelian.

Proof.

Let $e^2 = e \in R$. For any $r \in R, (1-e)(er - ere) = 0$ implies $(er - ere)(1-e) = er - ere = 0$ then $er = ere$. Similarly for any $r \in R, (re - ere)(1-e) = 0$ implies $re - ere = 0$. Therefore R is abelian.

In addition to lemma 1.1.22, we have the following proposition when we deal with central case.

Lemma1.1.23. [12]

If R is central reversible ring , then it is abelian .

Proof.

Let $e^2 = e \in R$. For any $r \in R, (1-e)(er - ere) = 0$ implies $(er - ere)(1-e) = er - ere$ is central. Commuting $er - ere$ by ewe have $er - ere = 0$. Similarly, for any $r \in R, (re - ere)(1-e) = 0$ implies $re - ere = 0$. Therefore R is abelian.

The next example shows that there exists an abelian ring which is not reversible and so not central reversible.

Example1.1.24. [12]

Consider the ring $R = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} | a \equiv d(mod2), b \equiv c \equiv 0(mod2),\ a, b, c, d \in \mathbb{Z}\right\}$.

Since $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are the only idempotent of R, R is abelian. On the other hand, consider $x = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, y = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \in R$ with $xy = 0$ but yx is not central for $z = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \in R$. $yxz = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 12 \\ 0 & 0 \end{pmatrix}$ and $zyx = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$. Hence R is not central reversible.

Definition 1.1.25.[12]

A ring R is called directly finite whenever $a, b \in R, ab = 1$ implies $ba = 1$.

Lemma. 1.1.26.

Every abelian ring is directly finite.

Proof.

Let R be abelian ring, let $a, b \in R$ with $ab = 1$. $(ba)^2 = baba = b1a = ba$ means that ba is idempotent hence central. Now, $bac = cba = cbaab = cabab = c$. Implies that $ba = 1$ since 1 is unique in R.

Corollary 1.1.27. [12]

Every central reversible ring is directly finite.

Proof.

Clear since every central reversible ring is abelian and every abelian is directly finite.

Definition 1.1.28.[14]

A ring R is said to be semi-commutative if for any $a, b \in R,\ ab = 0$ implies $aRb = 0$.

Definition 1.1.29. [14]

A ring R is said to be *weakly semi-commutative* if for any $a, b\ \in R, ab = 0$ implies arb is a nilpotent element for each $r\ \in R$.

Lemma 1.1.30.

Every reversible ring is semi-commutative.

Proof.

Let R be reversible ring with $ab = 0$ then $ba = 0$ since R is reversible . For any $\in R$, $bar = 0$ hence $arb = 0$ therefore, R is semi-commutative.

Lemma 1.1.31. [12]

Every central reversible ring is weakly semi-commutative.

Proof.

Let $a, b \in R$ with $ab = 0$, then ba is central. Thus we have $(arb)^2 = (arb)(arb) = ar(ba)rb = a(ba)rrb = 0$, and so arb is nilpotent for all $r \in R$. This shows that R is weakly semi-commutative.

Corollary 1.1.32.[12]

Let R be a right principally projective ring. If R is central reversible, then it is semi-commutative.

Proof.

It follows from the fact that every central reversible and right principally projective ring is reversible and so, semi-commutative.

Definition 1.1.33.

A non zero ring R is a prime ring if for any $a, b \in R$, $arb = 0$, implies $a = 0 \; or \; b = 0$.

Definition 1.1.34.

A ring R is a domain in which $ab = 0$ implies $a = 0$ or $b = 0$.

Lemma 1.1.35. [12]

Let R be a ring, then R is a prime and reversible ring if and only if it is a domain.

Proof.

Let R be reversible and prime, let $ab = 0$ then $abr = 0, \forall r \in R$, so $bra = 0$ since R is reversible. Implies that $a = 0$ or $b = 0$.

Conversely, let R be a domain (1) let $arb = 0$ for some $r \in R$ then either $a = 0$ or $rb = 0$ hence either $r = 0$ or $b = 0$ but since r is arbitrary $b = 0$. Hence either $a = 0$ or $b = 0$. Therefore, R is a prime .

(2) Let $ab = 0$ then at least one of a or b is zero, implies that $ba = 0$. Hence R is reversible.

We have the following proposition when we deal with central reversible rings.

Lemma1.1.36. [12]

Let R be a ring, then R is a prime and central reversible ring if and only if it is a domain.

Proof.

Let $a, b \in R$ with $ab = 0$. Then $abr = 0$ for any $r \in R$ and so bra is central. By commuting bra with b, we have $b^2ra = brab = 0$. By hypothesis, $bratb$ is central for any $t \in R$. Since R is prime, $a = 0$ or $b = 0$. The rest is clear.

Definition 1.1.37.

(1) The prime radical of a ring R is defined by $P(R) = \bigcap\{P: P \text{ } is \text{ } a \text{ } prime \text{ } ideal \text{ } of \text{ } R\}$

(2)The nil radical of a ring R is defined by $N(R) = \{a \in R: a \text{ } is \text{ } nilotent \text{ } element \text{ } in \text{ } R\}$

(3) The ring R is called 2-primal if $P(R) = N(R)$.

In (21; Theorem 1.5) it is proved that every reversible ring is 2- primal. In this direction we have the following theorem.

Theorem1.1.38. [12]

If R is a central reversible ring, then it is 2-primal. The converse holds for semi-prime rings.

Proof.

Let R be a central reversible ring. We always have $P(R) \subseteq N(R)$, since $P(R)$ is a nil ideal of R. For the converse inclusion, let $a \in N(R)$ with $a^n = 0$ for some positive integer . Assume that $a \notin Q$ for a prime ideal Q. Since R is central reversible, a is central. For any $r_{n-1}, r_{n-2}, \dots, r_2, r_1 \in R$ we have $a\, r_{n-1} a\, r_{n-2} \dots a r_2 a\, r_1 a = r_{n-1} r_{n-2} \dots r_2 r_1 a^n = 0$. For all prime ideals P we have $aR(a\, r_{n-2} \dots a r_2 a\, r_1 a) \subseteq P$.

Since $a \notin Q, a\, r_{n-2} \ldots a r_2 a\, r_1 a \in P$ for all prime ideals P and $r_{n-2}, \ldots, r_2, r_1 \in R$. Hence $aR(a\, r_{n-3} \ldots a r_2 a\, r_1 a) \subseteq P$ for all prime ideals P and $r_{n-3}, \ldots, r_2, r_1 \in R,$ using a similar reason, since $a \notin Q,$ $aR(a\, r_{n-4} \ldots a r_2 a\, r_1 a) \subseteq P$ for all prime ideals P and for all $r_{n-4}, \ldots, r_2, r_1 \in R$. By going downward induction , we may reach $aRa \subseteq P$ for all prime ideals P. This is the required contradiction. Thus if a is nilpotent, then $a \in P(R)$ and so $N\,(R) \subseteq\ P\,(R)$.

Conversely, let R be a semi-prime and 2-primal ring. Then $P(R) = 0$ and so $N\,(R) = 0$. Hence R is reduced and so central reversible. This completes the proof.

Section 1.2. Some Extensions of Central Reversible Rings

In this section, we study many ring extensions of central reversible rings.

Definition 1.2.1.

A ring R is said to be a direct sum of two sub-rings R_1, R_2 denoted by $R = R_1 \oplus R_2$ if and only if

1) $\forall r \in R,\ r = r_1 + r_2$ where $r_1 \in R_1, r_2 \in R_2$.
2) $R_1 \cap R_2 = \{0\}$.

Now, we prove that central reversible rings are closed under finite direct sum.

Proposition 1.2.2.

Let $\{R_i\}_{i \in I}$ be a class of rings for a finite index set. Then R_i is central reversible for all $i \in I$ if and only if $\oplus_{i \in I} R_i$ is central reversible.

Proof.

Let R_i be central reversible for all $i \in I$, and let $a, b \in \oplus_{i \in I} R_i$ with $ab = 0$ where $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n)$ such that $a_i, b_i \in R_i$ for all $\in I$, with $ab = 0$. Now, $ab = (a_1 b_1, \dots, a_n b_n) = 0$ implies $a_i b_i = 0$ for all $i \in I$, but $a_i b_i \in R_i$ for all $i \in I$, and so, $b_i a_i$ is central in R_i.

Now,$bac = (b_1 a_1, \dots, b_n a_n)(c_1, \dots, c_n) = (b_1 a_1 c_1, \dots, b_n a_n c_n) = (c_1 b_1 a_1, \dots, c_n b_n a_n) = (c_1, \dots, c_n)(b_1 a_1, \dots, b_n a_n) = cba$.Hence$ba$ is central in $\oplus_{i \in I} R_i$.

Conversely, let $\oplus_{i \in I} R_i$ be central reversible . Let $a_i b_i = 0$ for some $a_i, b_i \in R_i$. now, $a_i = (0, \dots, a_i, 0, \dots, 0), b_i = (0, \dots, b_i, 0, \dots, 0)$ then $a_i b_i = (0, \dots, a_i b_i, 0, \dots, 0) = 0$ implies $b_i a_i = (0, \dots, b_i a_i, 0, \dots, 0)$ is central in $\oplus_{i \in I} R_i$.implies $b_i a_i$is central in R_i , in particular, let $c \in \oplus_{i \in I} R_i$ so, . $(0, \dots, b_i a_i c_i, 0 \dots, 0) = (0, \dots, c_i b_i a_i, 0 \dots, 0)$ implies $b_i a_i c_i = c_i b_i a_i$ for some $i \in I$ means $b_i a_i$ is central in R_i. Therefore, R_i is central reversible.

Corollary 1.2.3.

Let R be a ring. Then eR and $(1-e)\ R$ are central reversible for some central idempotent e in R if and only if R is central reversible.

Proof.

Firstly, We going to prove that R is a direct finite of eR and $(1-e)\ R$ .Let e be a central idempotent , so $(1-e)^2 = 1-2e+e^2 = 1-2e+e = 1-e$ and $(1-e)r = r-er = r-re = r(1-e)$ therefore $(1-e)$is central idempotent.

Now, eR and $(1-e)\ R$ are both two sided ideals of R. Let $r \in R$, then $r = er + (1-e)r \in eR + (1-e)\ R$ (1). Let $x \in eR \cap (1-e)\ R$, $\exists r_1, r_2 \in R$ such that $x = er_1 = (1-e)r_2$ implies $e^2 r_1 = e(1-e)r_2$ so, $er_1 = (e-e^2)r_2 = 0$ therefore, $x = 0$. By last proposition eR and $(1-e)\ R$ are central reversible.

Proposition 1.2.4.

Let R be reduced ring, then it is reversible.

Proof.

Let R be reduced ring with $ab = 0$. Now, $(ba)^2 = baba = 0$, hence $ba = 0$. Therefore, R is reversible.

Let R and M be rings. Recall that the trivial extension of R by M is defined to be the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication $(r_1, m_1)(m_2, r_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$ This ring is isomorphic to the ring $\left\{\begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \mid r \in R, m \in M\right\}$ with the usual matrix operations .

Theorem 1.2.5.[11]

If R is a reduced ring, then $T(R, R)$ is reversible.

Proof.

Let $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \in T(R,R)$ with $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = 0$ then $ac = 0 = ad + bc$. Since R is reduced we have $ca = 0$ and $cad + cbc = 0 = cbc$ which implies $bc = 0$ and so $ad = 0$. Hence $cb = 0 = da$. Therefore, $\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = 0$.

For central case we have the following.

Definition 1.2.6. [13]

A ring R is said to be central reduced if every nilpotent element of R is central.

Proposition 1.2.7. [12]

If R is a central reduced ring, then $T(R,R)$ is central reversible.

Proof.

Let $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} \in T(R,R)$ with $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = 0$ then $ac = 0 = ad + bc$. By hypothesis, R is central reversible, then ca is central.

Hence $(ad)^3 = (-bc)(ad)(-bc) = b(ca)dbc = bdbcac = 0$, which implies ad is central. Hence $(da)^4 = 0$ and $(cb)^4 = 0$ which implies da, cb are central. Therefore $\begin{pmatrix} c & d \\ 0 & c \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ is central.

Definition 1.2.8.

A multiplicatively closed set S is a subset of a ring R such that the following two conditions hold,

(1) $1 \in S$,

(2) for x and $y \in S$, the product $xy \in S$.

Remark 1.2.9.

Let $S \subseteq R$ be a multiplicatively closed set, then on $R \times S$,we define the relation $(a, s) \backsim (a', s') \Leftrightarrow$ there is an element $u \in S$ such that $u(as' - a's) = 0$. This relation is an equivalence relation and each equivalence class $[(a, s)]$ is denoted by $\frac{a}{s}$ .The set of all equivalence classes $S^{-1}R = \left\{\frac{a}{s} \mid a \in R, s \in S\right\}$ is a ring together with the following addition and multiplication $\frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'}$ and $\frac{a}{s} . \frac{a'}{s'} = \frac{aa'}{ss'}$.

Proposition 1.2.10.[12]

A ring R is central reversible if and only if $S^{-1}R$ is central reversible.

Proof.

Let R be central reversible ring and $\frac{a}{r}$,$\frac{b}{s}$ $\in S^{-1}R$ where $a, b \in R, r, s \in S$ with $\frac{a}{r}\frac{b}{s} = 0$. Since $\frac{a}{r}\frac{b}{s} = \frac{ab}{rs} = 0$ we have $ab = 0$. By hypothesis ba is central, so $\left(\frac{b}{s}\right)\left(\frac{a}{r}\right)\left(\frac{c}{t}\right) = \left(\frac{c}{t}\right)\left(\frac{b}{s}\right)\left(\frac{a}{r}\right)$ for every $\frac{c}{t} \in S^{-1}R$ where $c \in R, t \in S$.

Therefore $S^{-1}R$ is central reversible.

Conversely, assume that $S^{-1}R$ is central reversible ring. Since R may be embedded in $S^{-1}R$, the rest is clear.

Recall that a nonzero element of a ring that is neither a left nor a right zero divisor is called **Regular element**.

Definition 1.2.11.

Let R be ring, the set of polynomials $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_i \in R$,n is non-negative integer ,with usual addition and multiplication of polynomials is called the ring of polynomials over R in the indeterminate x and is denoted by $R[x]$.

Definition 1.2.12.

The Laurent polynomial ring $R[x, x^{-1}]$ is a ring whose elements are expressions of the form $\sum_k a_k x^k$, $a_k \in R$, where x is the indeterminate and $k \in Z$ and only finitely many coefficients a_k are non zero, with the following addition and multiplication of Laurent polynomials :

$$\sum_i a_i x^i + \sum_i b_i x^i = \sum_i (a_i + b_i) x^i$$

$$\left(\sum_i a_i x^i\right)\left(\sum_j b_j x^j\right) = \sum_k \left(\sum_{\substack{i,j \\ i+j=k}} a_i b_j\right) x^k$$

Corollary 1.2.13.[12]

Let R be a ring, then $R[x]$ is central reversible if and only if $R[x, x^{-1}]$ is central reversible.

Proof.

Consider the subset $S = \{1, x, x^2, x^3, \dots\}$ of $R[x]$ consisting of central regular elements. Then the result follows from Proposition 1.2.10. by taking $S^{-1}R[x] = R[x, x^{-1}]$.

Definition 1.2.14.[20]

A ring R is said to be Armendariz if whenever $f(x) = a_0 + a_1 x + \dots + a_n x^n$ and $g(x) = b_0 + b_1 x + \dots + b_m x^m \in R[x]$ satisfies $f(x)g(x) = 0$ implies $a_i b_j = 0 \ \forall i, j$.

It is well known that reduced rings are Armendariz rings.

Theorem1.2.15.[12]

Let R be an Armendariz ring. Then the following statements are equivalent.

(1) R is central reversible.

(2) $R[x]$ is central reversible.

(3) $R[x, x^{-1}]$ is central reversible.

Proof.

(1)⟹(2) Let $f(x) = \sum_{i=1}^{n} a_i x^i$, $g(x) = \sum_{j=1}^{m} b_j x^j \in R[x]$ with $f(x)g(x) = 0$. Since R is Armendariz $a_i b_j = 0$ for each i and j. But R is central reversible so $b_j a_i$ is central for each i and j. It follows that $g(x)f(x)$ is central in $R[x]$. Therefore, $R[x]$ is central reversible.

(2)⟹(1) Clear .

(2) ⇔ (3) Follows from corollary 1.2.13.

Definition 1.2.16.[12]

Let R be a ring and let $\mathbb{Z}$ be the set of integers, the Dorroh extension is the ring $D(R, \mathbb{Z}) = \{(r, n) | r \in R, n \in \mathbb{Z}\}$ with operations $(r_1, n_1) + (r_2, n_2) = (r_1 + r_2, n_1 + n_2)$ and $(r_1, n_1).(r_2, n_2) = (r_1 r_2 + n_1 r_2 + n_2 r_1, n_1 n_2)$.

Proposition 1.2.17.[12]

A ring R is central reversible if and only if the Dorroh extension $D(R, \mathbb{Z})$of R is central reversible.

Proof.

The sufficiency is clear. For necessity, let $(r_1, n_1), (r_2, n_2) \epsilon D(R.\mathbb{Z})$ with$(r_1, n_1)(r_2, n_2) = 0$. Then $n_1 n_2 = 0$ and assume that $n_1 = 0$. Since R is central reversible, $(r_2 + 1n_2)r_1$ is central in R and so, $(r_2, n_2)(r_1, n_1)$ is central in $D(R.\mathbb{Z})$. Hence $D(R.\mathbb{Z})$is central reversible . A similar proof may be given for $n_2 = 0$.

Chapter 2.

A Generalization of Symmetric Rings

In this chapter we study a class of rings, called central symmetric rings, which is a generalization of symmetric rings. We prove that some results of symmetric rings can be extended to central symmetric rings. All rings in this chapter are associative with identity.

Section 2.1.Central Symmetric Rings.

Definition 2.1.1.[9]

A ring R is said to be symmetric if for any $a, b, c \in R$,$abc = 0$ implies $acb = 0$.

An equivalent condition on a ring that whenever $abc = 0$ implies $bac = 0$.

We now give the main definition in this chapter.

Definition2.1.2.[9]

A ring R is called central symmetric if for any $a, b, c \in R$,$abc = 0$ implies bac belongs to the center of R.

One may suspect that central symmetric rings are symmetric. But the following example erases the possibility.

Example 2.1.3.[9]

Letx, y and z be indeterminates and consider the set $R = \{a_0 + a_1x + a_2y + a_3z | a_0, a_1, a_2, a_3 \in \mathbb{Z}\}$ with component wise addition and defining multiplication by

$(a_0 + a_1x + a_2y + a_3z)(b_0 + b_1x + b_2y + b_3z) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_2b_0)y + (a_0b_3 + a_3b_0 + a_{1b_2})z.$

Then R is a ring with identity. From this multiplication all products are zero except that $xy = z$ and that 1 acts as an identity (see [12 Example 5.1]). Since $x^2 = y^2 = 0$, $xz = xxy = 0 = xyx = zx$ and $zy = xyy = 0 = yxy = yz$, z is central. Hence R is central symmetric. On the other hand, $yx = yx1 = 0$, but $xy1 = z$ and so R is not symmetric.

Our next aim is to find conditions under which a central symmetric ring is symmetric.

Proposition 2.1.4.[9]

If R is a symmetric ring, then R is central symmetric. The converse holds if R satisfies any of the following conditions.

(1) R is a semi-prime ring.

(2) R is a right (left) principally projective ring.

(3) R is a right (left) principally quasi-Baer ring.

Proof.

Clearly every symmetric ring is central symmetric.

Conversely, assume that R is a central symmetric ring and $a, b, c \in R$ with $abc = 0$.It implies that bca is central due to1 $\in R$. Now consider the following cases.

(1) Let R be a semi-prime ring.

Since$(bca)x(bca) = 0$ for all $x \in R$, it follows that $bca = 0$.Therefore R is symmetric.

(2) Let R be a right principally projective ring. Then there exists an idempotent $e \in R$ such that $r_R(a) = eR$. Thus $ae = 0$. Since $bc \in r_R(a) = eR$, we have $bc = ebc$. It follows that $bca = ebca = bcae = 0$.Therefore R is symmetric. A similar proof may be given for left principally projective rings.

(3) Let R be a right principally quasi-Baer ring. Then there exists an idempotent $e \in R$ such that $r_R(aR) = eR$. Hence $ae = 0$. On the other hand, $axbca = abcax = 0$ for all $x \in R$.This implies that $bca \in r_R(aR) = eR$, and so $bca = ebca = bcae = 0$. Thus R is symmetric. In a similar way every left principally quasi-Baer central symmetric ring is symmetric.

The following is an apparent consequence of Proposition 2.1.4.

Corollary 2.1.5.[9]

If R is acentral symmetric ring, then the following conditions are equivalent.

(1) R is a right principally projective ring.

(2) R is a left principally projective ring.

(3) R is a right principally quasi-Baer ring.

(4) R is a left principally quasi-Baer ring.

Proof.

(1)⇒(2) Let R be a right principally projective ring and $a \in R$. Then there exists an idempotent $e \in R$ such that $r_R(a) = eR$.

Also by Proposition2.1.4, R is symmetric. Let $b \in l_R(a)$. R Being symmetric ,$ab = 0$ and so $b = eb$. Since e is central, we have $b = be$. Hence $l_R(a) \subseteq Re$. Obviously, $Re \subseteq l_R(a)$. Thus R is a left principally projective ring.

(2)⇒(3) Let R be a left principally projective ring and $a \in R$. Then there exists an idempotent $e \in R$ such that $l_R(a) = Re$. Since e is central, we have $ea = ae = 0$. Let $b \in r_R(aR)$. Then $axb = 0$ for all $x \in R$. Hence $bax = 0$, and $ba = 0$ dueto1 $\in R$. Thus $b = be = eb$, and so $r_R(aR) \subseteq eR$. Hence $eR = rR(aR)$ since $eR \subseteq r_R(aR)$ holds also. Therefore R is a right principally quasi-Baer ring.

(3)⇒(4) and (4)⇒(1) are similar to the proofs of preceding cases.

Lemma 2.1.6.

All reduced rings are symmetric.

Proof.

Let R be reduced ring with $abc = 0$, $a, b, c \in R$ then $(bca)^2 = bcabca = 0$, implies $bca = 0$. Hence R is symmetric.

In the central case, we have the following result.

Lemma 2.1.7. [9]

If R is a central reduced ring, then it is central symmetric.

Proof.

Let $abc = 0$ for some $a, b, c \in R$.Then$(cab)^2 = cabcab = 0$ so cab is central. On the other hand, $(bca)^2 = bcabca = 0$ so bca is central. For any $r \in R$, $(arbc)^2 = arbcarbc = abcar^2bc = 0$ since bca is central. So $arbc$ is central. For any r∈R, $(abrc)^3 = abrcabrcabrc = abcabrcabrrc = 0$ since cab is central, and $(bcra)^2 = bcrabcra = 0$ implies $bcra$ is central. $(carb)^2 = carbcarb = cabcarrb = 0$.
So $carb$ is central. For any $r, s \in R$, $(arbsc)^2 = arbscarbsc = arbcarbs^2c = abcar^2bs^2c = 0$ since $carb$ and then bca is central. Hence $arbsc$ is central. $(bac)^4 = b(acbac)b(acbac) = 0$ since $acbac$ is central and $(acbac)^2 = 0$. Hence bac iscentral.

In the following we prove when a central symmetric ring is reduced.

Theorem 2.1.8.[9]

Let R be a central symmetric ring. Then we have

(1) If R is a semi-prime ring, then R is reduced.

(2) If R is a right (left) principally projective ring, then R is reduced.

Proof.

(1) Let $a \in R$ with $a^2 = 0$ .For any $r \in R, ra^2 = 0$. By hypothesis ara is central, $(ara)^2 = 0$ and so $aRa = 0$. Since R is semi-prime, $a = 0$.

(2) Let R be a right principally projective ring and $a \in R$ with $a^2 = 0$. There exists $e^2 = e \in R$ such that $r_R(a) = eR$. Then $a \in r_R(a)$ and $ae = 0$ and $a = ea$. By Proposition 2.1.4, e is central and so $a = ea = ae = 0$.

This proof is left-right symmetric since idempotents are central.

It is well known that every symmetric ring is reversible and the converse holds for semi-prime rings. In this direction we have the following.

Proposition 2.1.9.[9]

Let R be a central symmetric ring. Then R is central reversible. The converse statement holds if R is a semi-prime ring.

Proof.

Let $a, b \in R$ with $ab = 0$. Then $1ab = 0$. Hence $1ba = b1a = ba$ is central. Conversely, assume that R is a semi-prime central reversible ring. Let $a, b, c \in R$ with $abc = 0$. We may suppose that a, b and c are non zero. For any $r \in R, abcr = 0$ implies $crab$ is central, and $(crab)^2 = 0$. By assumption $crab = 0$. For any $s \in R$, $crabs = 0$ implies $bscra$is central and $(bscra)^2 = 0$.

By similar reason $bscra = 0$. Hence $(bac)^2 = bacbac = 0$. For any $t \in R, bactbac$ is central and$(bactbac)^2 = 0$. Then $bactbacRbactbac = 0$. Hence $bactbac = 0$ for all $t \in R$. Being R semi-prime we have $bac = 0$.

Corollary 2.1.10.

Every central reduced ring is central reversible.

Definition 2.1.11.

Let G be a group, and let R be a ring with unity. We define the group ring $R[G]$ as the set of all linear combinations of the form $\alpha = \sum_{g \in G} a(g)\, g$,where$a(g) \in R$ and $a(g) = 0$ except a finite number of coefficients ,with addition and multiplication defined as the following:

$$\sum_{g\in G} a(g)\, g + \sum_{g\in G} b(g)\, g = \sum_{g\in G} [a(g) + b(g)]\, g$$

$$\left(\sum_{g\in G} a(g)\, g\right)\left(\sum_{h\in G} b(h)\, h\right) = \sum_{h,g\in G} [a(g)b(h)]\, gh$$

Example 2.1.12.

Let $G = \mathbb{Z}_3$, the cyclic group of three elements with generator a and identity element 1_G. An element r of $\mathbb{C}[G]$ may be written as $r = z_0 1_G + z_1 a + z_2 a^2$ where z_0, z_1 and z_2 are in $\mathbb{C}$, the complex numbers. Writing a different element s as $s = w_0 1_G + w_1 a + w_2 a^2$ their sum is $r + s = (z_0 + w_0)1_G + (z_1 + w_1)a + (z_2 + w_2)a^2$ and their product is $rs = (z_0 w_0 + z_1 w_2 + z_2 w_1)1_G + (z_0 w_1 + z_1 w_0 + z_2 w_2)a + (z_0 w_2 + z_2 w_0 + z_1 w_1)a^2$
Notice that the identity element 1_G of G induces a canonical embedding of the coefficient ring (in this case $\mathbb{C}$) into $\mathbb{C}[G]$; however strictly speaking the multiplicative identity element of $\mathbb{C}[G]$ is $1{\cdot}1_G$ where the first 1 comes from $\mathbb{C}$ and the second from G. The additive identity element is of course zero.

The next example provides that there exists a central reversible ring which is not a central symmetric ring.

Example 2.1.13.[9]

Let $Q_8 = \{1, x_{-1}, x_i, x_{-i}, x_j, x_{-j}, x_k, x_{-k}\}$ be the quaternion group and consider the group ring $R = \mathbb{Z}_2[Q_8]$. The elements of $\mathbb{Z}_2[Q_8]$ as $\mathbb{Z}_2$-linear combinations of $\{x_g : g \in Q_8\}$. By Courter's result in [13,Corollary2.3], R is reversible and so central reversible. But R is not symmetric as in [14,Example7] by taking $a = 1x_j$, $b = 1 + x_i$ and $c = 1 + x_i + x_j + x_k$. Then $abc = 0$ but $bac \neq 0$. In fact $bac = x_i + x_j + x_k + x_{-i} + x_{-k}$ and it is easily checked that $x_i(bac) \neq (bac)x_i$. Hence R is not central symmetric.

Note that the homomorphic image of a central symmetric ring need not be central symmetric. Consider the following example.

Example 2.1.14.[9]

Let $\mathbb{Z}_2$ denote the field of integers modulo 2 and $\mathbb{Z}_2(y)$ denotes the rational functions field of polynomial ring $\mathbb{Z}_2[y]$ and $R = \mathbb{Z}_2(y)[x]$ the ring of polynomials in x over $\mathbb{Z}_2(y)$ subject to the relation $xy + yx = 1$.

It is well known that R is a principal ideal domain and so is a non-commutative domain (see[8,p.30],[5,Note3.9],[21,Example5.3]) . Let $I = x^2R$. Then I is a maximal ideal of R. Consider the ring $S = R/I$. We write $\bar{x}$ and $\bar{y}$ for the images of x and y respectively under the natural epimorphism from R onto S. Let $a, b, c \in R$ with $abc = 0$. Since R is a domain, at least one of a, b and c is zero. Therefore $bac = 0$ and so bac is central and R is central symmetric. For $\bar{x}, \bar{y} \in S$, we have $\bar{x}^2 = 0$ and $\bar{x}\bar{y}+\bar{y}\bar{x} = \bar{1}$. Multiplying the last equality from the right by $\bar{x}$ and using $\bar{x}^2 = 0$, we have $\bar{x}\bar{y}\bar{x} = \bar{x}$. If S were central symmetric, $(\bar{x}\bar{y}-\bar{1})\bar{x} = \bar{0}$ would imply $\bar{x}\,(\bar{x}\bar{y}-\bar{1}) = -\bar{x}$ is central in S and so $\bar{x}$ is central in S. This is a contradiction since $\bar{x}$ is not central.

Our next goal is to find conditions when the homomorphic image of a ring is central symmetric. It is proven that every idempotent of a unit-central ring is central.

Lemma 2.1.15.[9]

Let R be a unit-central ring. If I is a nil ideal of R, then R and R/I are central symmetric.

Proof.

Let $a \in R$ with $a^n = 0$ for some positive integer n. Then $(1 + a)(1 - a + a^2 - a^3 + \ldots + (-1)^{n-1}a^{n-1}) = 1$ Hence $1 + a$ and therefore a is central. Let a, b and $c \in R$ with $\bar{a}\bar{b}\bar{c} = \bar{0}$ in R/I. Then $abc \in I$. Hence abc is nilpotent. So $1 + abc$, and therefore abc is central. Now $(\bar{c}\bar{a}\bar{b})^2 = \bar{0}$ and $(\bar{b}\bar{c}\bar{a})^2 = \bar{0}$ imply $(cab)^2 \in I$ and $(bca)^2 \in I$, and therefore cab and bca are central. $\bar{a}\bar{b}\bar{c}\bar{r} = \bar{0}$ for all $r \in R$ implies $(\bar{c}\bar{r}\bar{a}\bar{b})^2 = \bar{0}$. So $(crab)^2 \in I$ and $(crab)^2$ is nilpotent and $crab$ is central for all $r \in R$. Similarly, $(\bar{b}\bar{s}\,\bar{c}\,\bar{a})^2 = \bar{0}$ implies $bsca$ is central for all $s \in R$.

Let $\bar{s}, \bar{r} \in R/I$. Then $(\bar{b}\bar{s}\bar{c}\bar{r}\bar{a})^2 = \bar{0}$ since $\bar{c}\bar{r}\bar{a}\bar{b}$ is central nilpotent. Hence $bscra$ is central for $s, r \in R$. Now $(\bar{b}\overline{ac})^4 = \bar{0}$ since $\bar{b}\overline{ac}\bar{b}\bar{a}$ is central nilpotent. Hence bac is central. Thus $\bar{b}\overline{ac}$ is central.

The next example shows that for a ring R and an ideal I, if R/I is central symmetric, then R need not be central symmetric.

Example 2.1.16.[9]

Let F be a field and R the ring of all 2×2 upper triangular matrices over F and e_{ij} matrix units with 1 at the entry (i, j) and zeros elsewhere. Let $I = e_{12}R$. Then I is an ideal of R and R/I is a commutative ring, therefore central symmetric.

Consider $A = e_{22}, B = e_{11} + e_{12}$ and $C = A + B$. Then $ABC = 0$ but $BAC = e_{12}$ is not central since $e_{11}e_{12} = e_{12}$ but $e_{12}e_{11} = 0$. Hence R is not central symmetric.

Let R be a ring with an ideal I. Then I is said to be **prime** if $aRb \subseteq I$ implies a or b is in I,while I is called **completely prime** if $ab \in I$ implies that a or b is in I . Completely prime ideals are prime ideals, but the converse is not true. For example, for any positive integer n, the zero ideal in the ring of all $n \times n$ matrices over a field is a prime ideal, but it is not completely prime. For a ring R and an ideal, we show that if R/I is a central symmetric ring with a completely prime reduced ideal I,then R is a symmetric ring.

Lemma 2.1.17.[9]

Let R be a ring. If R/I is a central symmetric ring with a completely prime reduced ideal I, then R is symmetric and so central symmetric.

Proof.

Let $a, b, c \in R$ with $abc = 0$ and $\bar{a}$ will be the image of a under natural epimorphism from R onto R/I.Then $\bar{a}\bar{b}\bar{c} = \bar{0}$ Since R/I is central symmetric and $\bar{a}\bar{b}\bar{c}\bar{r} = \bar{0}$ for any $r \in R$, $\bar{b}\overline{ac}\bar{r}$ is central in R/I (1) also, $\bar{r}\bar{a}\bar{b}\bar{c} = \bar{0}$ for any $r \in R$ implies $\overline{b}\bar{r}\overline{ac}$ is central in R/I (2) .

By using (1) and (2) we prove $(\bar{b}\overline{ac})^4 = \bar{0}$. By(1), we have,

$(\bar{b}\overline{ac})^4 = (\bar{b}\overline{ac}\bar{b})\overline{ac}\bar{b}\overline{ac}\bar{b}\overline{ac} = \overline{ac}\bar{b}\bar{a}(\bar{b}\overline{ac}\bar{b})\bar{c}\bar{b}\overline{ac}$. Hence we have $\overline{ac}\bar{b}\bar{a}(\bar{b}\overline{ac}\bar{b})\bar{c}\bar{b}\overline{ac} = \overline{ac}(\bar{b}\bar{a}\bar{b}\overline{ac})\bar{b}\bar{c}\bar{b}\overline{ac} = \overline{ac}\bar{b}\bar{c}\bar{b}\bar{a}(\bar{b}\bar{a}\bar{b}\overline{ac})\bar{c}$. Thus $\overline{ac}\bar{b}\bar{c}\bar{b}\bar{a}(\bar{b}\bar{a}\bar{b}\overline{ac})\bar{c} = \overline{ac}(\bar{b}\bar{c}\bar{b}\bar{a}\bar{b}\bar{a}\bar{b}\overline{ac})\bar{c} = \bar{a}(\bar{b}\bar{c}\bar{b}\bar{a}\bar{b}\bar{a}\bar{b}\overline{ac})\overline{cc} = \bar{0}$. It follows $(bac)^4 \in I$ and $bac \in I$ and so one of a, b and c belongs to I since I is completely prime. So $cab \in I$ and $(cab)^2 = 0$ implies $cab = 0$ since I is reduced. Similarly,$(bsca)^2 = 0$ implies $bsca = 0$, and $(arbsc)^2 = 0$ implies $arbsc = 0$, and $(cuarbs)^2 = 0$ implies $cuarbs = 0$. Hence $bscuar = 0$ for all $r, s, u \in R$. This implies $(bac)^2 = bacbac = 0$. Thus $bac = 0$. This completes the proof.

Definition 2.1.18.[9]

A ring R is said to be weak symmetric, if for all $a, b, c \in R$, if abc is nilpotent, then acb is nilpotent.

We now give an example to show that there exists a weak symmetric ring which is not a central symmetric ring.

Example 2.1.19.[9]

Consider the ring $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{bmatrix}$ and the elements $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ of R then, $ABC = 0$. But $BAC = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is not central in R. Hence R is not central symmetric. However R is weak symmetric by [18,Proposition 2.3].

Proposition 2.1.20.[9]

Let R be a central symmetric ring. Then R is an abelian ring.

Proof.

Let e be an idempotent of R and $x \in R$. Since $e(xe - exe) = 0$ and $(ex - exe)e = 0$, being R central symmetric, $(xe - exe)e$ and $e(ex - exe)$ are central. Then $(xe - exe)e =$

$e(xe - exe) = 0$ and $e(ex - exe) = (ex - exe)e = 0$. Hence we have $ex = xe$ for all $x \in R$. Therefore R is abelian.

The converse of Proposition 2.1.20. does not hold in general, that is, every abelian ring need not be central symmetric, as the following example shows.

Example 2.1.21.[2]

Consider the ring $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a, b, d, c \in \mathbb{Z}, a \equiv d(mod2), b \equiv c \equiv 0(mod2) \right\}$, with the usual matrix operations. Since 0 and the identity matrices are the only idempotents of R, R is an abelian ring. let $A = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -4 \\ 0 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \in R$. Then $ABC = 0$ but BAC is not central.

The next result shows that the converse statement of Proposition 2.1.20 holds for right principally projective rings.

Proposition 2.1.22.[9]

Let R be a right principally projective ring. If R is abelian, then it is central symmetric.

Proof.

Let $a, b, c \in R$ with $abc = 0$. By hypothesis, there exists an idempotent e of R such that $r_R(a) = eR$. Hence $ae = 0$ and $bc = ebc$. It follows that $bca = ebca = bcae = 0$. Thus R is symmetric and so central symmetric.

Corollary 2.1.23.[9]

Every central symmetric ring is directly finite.

Proof.

It is clear from Proposition 2.1.20, since every abelian ring is directly finite.

Proposition 2.1.24.[9]

Let R be a central symmetric ring. Then the followings hold.

(1) R is central semi-commutative.

(2) R is weakly semi-commutative.

Proof.

(1) Let $a, b \in R$ with $ab = 0$. For any $r \in R, rab = 0$ implies arb is central in R. Hence R is central semi-commutative.

(2) Let $a, b \in R$ with $ab = 0$.Since R is central symmetric, ba is central in R. Hence for each $r \in R, (arb)^2 = arbarb = ar^2bab = 0$. Therefore R is weakly semi-commutative.

It is well known that a ring is a domain if and only if it is prime and Symmetric, we have the following proposition when we deal with central case.

Proposition 2.1.25.[9]

Let R be a ring. Then R is a domain if and only if R is a prime and central symmetric ring.

Proof.

First assume R is a domain. It is clear that R is prime and symmetric and so central symmetric. Conversely, assume R is a prime and central symmetric ring.

Let $a, b \in R$ with $ab = 0$. Then $rab = 0$ and $abr = 0$ for all $r \in R$. Being R central symmetric arb and $bar$are contained in the center of R. Hence we have $(arb)R(arb) = 0$ for any $r \in R$. Since R is prime, $arb = 0$ for any $r \in R$ and so $aRb = 0$. This implies that $a = 0$ or $b = 0$. Therefore R is a domain.

Theorem2.1.26.[9]

If R is a central symmetric ring, then it is 2-primal. The converse holds for semi-prime rings.

Proof.

To complete the proof it is enough to show that $N(R) \leq P(R)$ since $P(R)$ is a nil ideal. Let $a \in N(R)$. We first assume that $a^2 = 0$. By hypothesis ara is central for any $r \in R$. Commuting ara with $sa$for any $s \in R$ we have $arasa = 0$. It follows that $a \in P$ for any prime ideal P and so $a \in P(R)$. Assume now $a^3 = 0$. Then a^2ra and ara^2 are central. Commuting a^2ra by sa for any $s \in R$ we have $a^2rasa = 0$. By hypothesis $arasata$ is central for any $t \in R$. Again commute $arasata$ with az for any $z \in R$ and use the centrality of ara^2 for all $r \in R$ to obtain
$(az)(arasata) = (arasata)(az) = aras(ata^2)z = (ata^2)arasz = 0$. Since z, t, r and s are arbitrary in R, $a \in P(R)$. By induction on the index of nilpotency we may conclude that $P(R)$ consists of all nilpotent elements of R. Hence R is 2-primal. Conversely, let R be a semi-prime and 2-primal ring. Then R is symmetric and so central symmetric.

Corollary 2.1.27.[9]

Let R be a central symmetric ring. Then the ring $R/P(R)$ is central symmetric.

Recall that a ring R is said to be **von Neumann regular** [2] if for every $a \in R$ there exists $b \in R$ with $a = aba$. A ring R is called **strongly regular** [2] if for any $a \in R$ there exists $b \in R$ such that $a = a^2b$. Now we give some relations between symmetric, central symmetric, regular, strongly regular and abelian rings. Also the following theorem provides some conditions for the converses of Proposition 2.1.4 and Proposition 2.1.22.

Theorem 2.1.28.[9]

Let R be a ring. Then the following conditions are equivalent.

(1) R is strongly regular.

(2) R is von Neumann regular and symmetric.

(3) R is von Neumann regular and central symmetric.

(4) R is von Neumann regular and abelian.

Proof.

(1)⇒(2) The first assertion is clear. Let $a, b, c \in R$ with $abc = 0$. Since $(bca)^2 = 0$ and $bac = (bac)^2 r$ for some $r \in R$, we have $bca = 0$. Then R is symmetric.

(2)⇒(3) Obvious.

(3)⇒(4) By Proposition 2.1.20.

(4) ⇒(1) Let $a \in R$. By hypothesis, there exists $b \in R$ such that $a = aba$. Since ab is an idempotent, ab is central. Hence $a = a^2 b$ and therefore R is strongly regular.

Section 2.2.Some Extensions of Central Symmetric Rings.

In this section, we discuss some ring extensions and localization of central symmetric rings .We start with an application of central symmetric rings concerns Modules, so we give a brief introduction about modules.

Definition 2.2.1. [17]

Suppose R is a ring , a left R- module is an additive abelian group M together with a map R×M ⟶M , (r, m) ⟶r m satisfying

1- r (m + n) = r m+ r n .

2- $(r_1 + r_2)$ m = r_1m + r_2 m .

3- $(r_1 r_2)$ m = $r_1(r_2$m) .

4- l.m = m . for all r, $r_1, r_2 \in$ R and all m , n ∈ M .

A right R- module is an additive abelian group M together with a map

M ×R ⟶M , (m , r) ⟶m r , satisfying :

1- (m + n) r = m r + n r .

2 - m $(r_1 + r_2)$ = mr_1 + m r_2 .

3- m $(r_1 r_2)$ = (m r_1) r_2 .

4- m .1 = m . for all r, $r_1, r_2 \in$ R and all m , n ∈ M .

Example 2.2.2.[17]

When R is non- commutative ring ,we can find a right R-module which is not a left R-module .

Let $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c \text{ and } d \in \mathbb{Z} \right\}$ be a ring , $A = \left\{ \begin{bmatrix} r & m \\ 0 & 0 \end{bmatrix} : r, m \in \mathbb{Z} \right\}$ is a right R-module . Note that , $\begin{bmatrix} r & m \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ z & u \end{bmatrix} = \begin{bmatrix} rx+mz & ry+mu \\ 0 & 0 \end{bmatrix} \in A$. But A isnot a left R-module , since if we take $\begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} \in A$, and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in R$. Then $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \notin A$.

Definition 2.2.3. [17]

Let M be a left R- module , then we say that N is a submodule of M if :

1- $N$is a subgroup of M .

2- For $r \in R$, $n \in N$, $rn \in N$.

Example 2.2.4. [17]

If M is a left R-module and $x \in M$, then the set $Rx = \{ rx , r \in R \}$ is an R- sub-module of M .

Definition 2.2.5. [3]

A sub-module N of M is called a direct summand of M if there is a sub-module L of M such $N \oplus L = M$.

We note that L is also a direct summand of M.

Let R be a ring . Then R itself can be regarded as a left R-module by defining rm , $m \in$ R ,$r \in$ R, to be the product of r and m as elements of the ring R. And sub-modules of the left R- module R are just left ideals of R.

A module M has the **summand intersection property** if the intersection of two direct summands is again a direct summand of M. A ring R is said to have the summand intersection property if the right R -module R has the summand intersection property.[3]

A module M has the **summand sum property** if the sum of two direct summands is a direct summand of M and a ring R is said to have the summand sum property if the right R –module R has the summand sum property. [3]

In this case we have the following.

Proposition 2.2.6.[9]

Let R be a central symmetric ring. Then we have the following.

(1) R has the summand intersection property.

(2) R has the summand sum property.

Proof.

(1) Let e and f be idempotents of R. e and f are central, we have $eR \cap fR = efR = feR$ and $(ef)^2 = ef$. This completes the proof.

(2) Let eR and fR be right ideals of R with $e^2 = e, f^2 f \in R$. Then $e + f - ef$ is an idempotent of R. Since R is abelian, it is easy to check that $eR + fR = (e + f - ef)R$. So $eR + fR$ is a direct summand of R.

Now we show that the class of central symmetric rings is closed under finite direct sums.

Proposition 2.2.7.[9]

let I be a finite index set and $\{R_i\}_{i\epsilon I}$ a class of rings. Then R_i is central symmetric for all $i \in I$ if and only if $\oplus_{i\in I} R_i$ is central symmetric.

Proof.

Let R_i be central symmetric for all $i \in I$ and $(a_i)_{i\in I}, (b_i)_{i\in I}, (c_i)_{i\in I} \in \oplus_{i\in I} R_i$ with $(a_i)\ (b_i)(c_i) = 0$. Then $a_i b_i c_i = 0$ and by hypothesis $b_i a_i c_i$ is central in R_i for all $i \in I$.

Hence $(b_i)(a_i)(c_i)$is central in R_i. Therefore R_i is central symmetric. The sufficiency is clear since a sub ring of a central symmetric ring is central symmetric.

The following result is a direct consequence of Proposition 2.2.7.

Corollary 2.2.8.[9]

Let R be a ring. Then eR and $(1-e)R$ are central symmetric for some idempotent element e in R if and only if R is central symmetric.

Proposition 2.2.9.[9]

A ring R is central symmetric if and only if $S^{-1}R$ is central symmetric.

Proof.

Let R be a central symmetric ring and $\frac{a}{s_1}, \frac{b}{s_2}, \frac{c}{s_3} \in S^{-1}R$ where $a, b, c \in R, s_1, s_2, s_3 \in S$ with $\frac{a}{s_1}\frac{b}{s_2}\frac{c}{s_3} = 0$. Since $\frac{abc}{s_1s_2s_3} = 0$, we have $abc = 0$. So bac is central in R, then $\left(\frac{b}{s_2}\right)\left(\frac{a}{s_1}\right)\left(\frac{c}{s_3}\right)$ is also central in $S^{-1}R$. Therefore $S^{-1}R$ is central symmetric .

Conversely, assume that $S^{-1}R$ is central symmetric ring. Since R may be embedded in $S^{-1}R$, the rest is clear.

Corollary 2.2.10.[9]

Let R be a ring. Then $R[x]$ is central symmetric if and only if $R[x, x^{-1}]$ is central symmetric.

Proof.

Consider the sub set $S = \{1, x, x^2, x^3, \dots\}$ of $R[x]$ consisting of central regular elements. Then the result follows from Proposition 2.2.9.

Chapter 3

Skew version of Reversible and Symmetric Rings

In this chapter we consider a skew version of reversible rings and symmetric rings, called α −reversible and α −symmetric rings respectively, with respect to a ring endomorphism $\boldsymbol{\alpha}$. Next we introduce new classes of rings which are generalizations of α −reversible and α −symmetric rings. In this chapter all rings are associative with identity.

Section 3.1. α-Reversible and α −Symmetric Rings

Definition 3.1.1.[19]

(1) A ring R is called right α-reversible if whenever $ab = 0$ for $a, b \in R$ then $b\alpha(a) = 0$.

(2) A ring R is called left α-reversible if whenever $ab = 0$ for $a, b \in R$ then $\alpha(b)a = 0$.

(3) A ring R is called α-reversible if it is both right and left α-reversible.

There exists a reversible ring R with endomorphism α such that R is not α -reversible ring.

Example 3.1.2.[19]

Consider the ring $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ with endomorphism $\alpha: R \rightarrow R$ defined by $\alpha\big((a, b)\big) = (b, a)$ with the usual addition and multiplication. The ring R is commutative reduced hence it is reversible. However, R is not α - reversible. Indeed, $(0,1)(1,0) = 0$ but $\alpha\big((1,0)\big)(0,1) = (0,1)(0,1) = (0,1) \neq 0$.

Definition 3.1.3.[6]

A ring R is α −compatible if for each $a, b \in R$, $a\alpha(b) = 0$ if and only if $ab = 0$.

In the next proposition we give the relationship between α −reversibility and reducibility.

Proposition 3.1.4.[19]

Let R be a reduced α −compatible ring then R is α −reversible.

Proof.

Let $a, b \in R$, and $ab = 0$. By the hypothesis R is α–compatible so $a\alpha(b) = 0$ and then $\alpha(b)a = 0$ (by reducibility). Hence, R is left α–reversible. The right α–reversibility is obtained similarly. Therefore, R is α −reversible.

The next theorem gives the relationship between reversible and α −reversibe rings.

Theorem 3.1.5.[19]

Let R be an α −compatible ring. R is reversible if and only if R isα −reversible ring.

Proof.

Let R be a reversible ring and $ab = 0$ for $a, b \in R$. Then $ba = 0$ and $b\alpha(a) = 0$

(by α–compatibility). Therefore, R is right α– reversible.

On the other hand, $ab = 0$ we have $a\alpha(b) = 0$, so $\alpha(b)a = 0$ (by reversibility) hence R is left α–reversible. Therefore, R is α– reversible.

Conversely, let $ab = 0$ for $a, b \in R$, then $b\alpha(a) = 0$ (by right α– reversibility) and $ba = 0$ (by α–compatibility). Therefore, R is reversible.

Theorem 3.1.6.[19]

A ring R is α −rigid if and only if R is reduced ,α − reversible , and α is monomorphism.

Proof.

Let R be an α −rigid, then R is reduced and α is monomorphism by [6]. Let $ab = 0$ for $a, b \in R$ so, we have $(ba)^2 = baba = 0$ implies that $ba = 0$ "since R is reduced" then $a\alpha(ba)\alpha^2(b) = a\alpha(b)\alpha\big(a\alpha(b)\big) = 0$, hence $a\alpha(b) = 0$ " since R is α −rigid" , and $(\alpha(b)a)^2 = \alpha(b)a\alpha(b)a = 0$ implies $\alpha(b)a = 0$ "since R is reduced".

Therefore, R is left $\alpha -$ reversible.

On the other hand, $b\alpha(ab)\alpha^2(a) = b\alpha(a)\alpha\big(b\alpha(a)\big) = 0$ implies $b\alpha(a) = 0$, then R is right $\alpha -$ reversible. Therefore, R is $\alpha -$ reversible.

Conversely, assume $a\alpha(a) = 0$ for $a \in R$, then $\alpha(a)a = 0$ and $\alpha(a)\alpha(a) = \big(\alpha(a)\big)^2 = 0$ " since R is $\alpha -$ reversible" . Implies that $\alpha(a) = 0$ " since R is reduced" and since R is monomorphism $a = 0$.

Definition 3.1.7.[7]

(1) An endomorphism α of a ring R is called rigid if $a\alpha(a) = 0$ implies $a = 0$, for $a \in R$.
(2) A ring R is called α −rigid if there exists a rigid endomorphism α of R.

The following examples show that neither the conditions α to be monomorphism nor R to be reduced can be dropped.

Example 3.1.8.[19]

(1) Let $R = F[x]$ be the polynomial ring over a field F, and α be an endomorphism of R defined by $\alpha\big(f(x)\big) = f(0)$ where $f(x) \in R$.
R is a commutative domain so it is reduced and $\alpha -$ reversible but $\alpha$is not monomorphism , hence R is not α −rigid.

(2) Let $R=\left\{\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}, a,b\in\mathbb{Z}\right\}$ and α be an endomorphism of R defined by $\alpha\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}=\begin{pmatrix} a & 0 \\ -b & a \end{pmatrix}$. R is $\alpha-$ reversible but is not α $-$rigid.

Definition 3.1.9.[19]

(1) An endomorphism α of a ring R is called right (left) α-symmetric if whenever $abc=0$ implies $ac\alpha(b)=0, (\alpha(b)ac=0)$ for $a,b,c\in R$.

(2) A ring R is called right (left) α-symmetric if there exists a right (left) α-symmetric endomorphism $\alpha$of R. A ring R is called α-symmetric if it is both right and left α-symmetric.

The next theorem gives the relationship between symmetric and α $-$symmetric rings.

Theorem3.1.10.[19]

Let R be α $-$compatible ring. R is symmetric if and only if R is α $-$symmetric.

Proof.

Let R be a symmetric ring and $abc=0$, for $a,b,c\in R$. Then $acb=0$ (by symmetricity) and $ac\alpha(b)=0$ (by α-compatibility), hence R is right α $-$symmetric. Since R is symmetric so it is reversible then we have $\alpha(b)ac=0$, hence R is left α $-$symmetric. Therefore, R is α $-$symmetric ring.

Conversely, let R be an α $-$symmetric ring and $abc=0$, for $a,b,c\in R$. So we have $ac\alpha(b)=0$, and $acb=0$ (by α-compatibility). Therefore R is symmetric ring.

Now, we study the relationship between α-symmetric andα-reversiblerings.

Proposition 3.1.11.[19]

An α-symmetric ring is α-reversible.

Proof.

Let R be an α-symmetric ring. Suppose that $ab = 0$ for $a, b \in R$. Obviously, $1ab = 0$, since R is right α-symmetric, then $b\alpha(a) = 0$. Hence R is right α-reversible. It is easily can be shown that R is left α-reversible as above. Therefore, R is α-reversible.

Definition 3.1.12.[19]

(1) An endomorphism α of a ring R is called semi-commutative if $ab = 0$ implies $aR\alpha(b) = 0$ for $a, b \in R$.
(2) A ring R is called $\alpha -$ semi-commutative if there exists a semi-commutative endomorphism α of R .

Theorem 3.1.13.[19]
Let R be an $\alpha -$compatible ring. Then R is semi-commutative if and only if R is $\alpha -$semi-commutative.

Proof.
Let R be a semi-commutative ring and $ab = 0$ for $a, b \in R$, so $aRb = 0$. Since R is $\alpha -$compatible it implies that $aR\alpha(b) = 0$. Therefore, R is $\alpha -$semi-commutative ring. The "only if " part is obvious.

Proposition 3.1.14.[19]

A reduced $\alpha -$ reversible ring is $\alpha -$ semi-commutative.

Proof.

Let R be a reduced $\alpha -$ reversible ring. Let $ab = 0$ for $a, b \in R$ and c an arbitrary element of R. then$\alpha(b)a = 0$ " by $\alpha -$ reversible" and $\alpha(b)ac = 0$, and $\left(ac\alpha(b)\right)^2 = ac\alpha(b)ac\alpha(b) = 0$, hence $ac\alpha(b) = 0$ " by reduced " . Therefore, R is $\alpha -$ semi-commutative.

Section 3.2. Central α −Reversibleand Central α −Symmetric Rings

In this section, we introduce new classes of rings which are generalizations of α −reversible and α −symmetric rings.

Definition 3.2.1.

(1) A ring R is called central right α −reversible if whenever $ab = 0$ for $a, b \in R$ then $b\alpha(a)$ is central in R.

(2) A ring R is called central left α −reversible if whenever $ab = 0$ for $a, b \in R$ then $\alpha(b)a$ is central in R.

(3) A ring R is called central α −reversible if it is both right and left central α −reversible.

The following example distinguishing the concepts of central right and left α −reversible.

Example 3.2.2.

Consider a ring $R = \left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{Z}\right\}$.

1. Consider the endomorphism α of R defined by $\alpha\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ and $B = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \in R$, such that $AB = 0$, then we get $aa' = 0, cc' = 0$ and $ab' + bc' = 0$. $B\alpha(A) = \begin{pmatrix} aa' & 0 \\ 0 & 0 \end{pmatrix} = 0$. Therefore, R is central right α-reversible since 0 is central in R.

Now, let $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}, B = \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix}$ so, $\alpha(A) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $B\alpha(A) = 0$. But $\alpha(B)A = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 9 \\ 0 & 0 \end{pmatrix}$ which is not central in R. Therefore R is not central right α-reversible.

2. Consider the endomorphism β of R defined by $\beta\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$. By the same way as above we can see that , R is central left β-reversible but is not central right β −reversible.

Theorem 3.2.3.

Let R be α-reversible then R is central α-reversible.

Proof.

Let $a, b \in R$ with $ab = 0$ then $\alpha(b)a = b\alpha(a) = 0$ which is central in R.

The converse is not true in general i.e. there exist central α-reversible ring which is not α-reversible.

Example 3.2.4.

In example 1.1.4 define $\alpha: S \longrightarrow S$ such that $\alpha(a) = a$. Let $A, B \in S$ with $AB = 0$ then $\alpha(B)A = B\alpha(A) = BA$ is central in S since S is central reversible. On the other hand let $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $AB = 0$ but $\alpha(B)A = BA \neq 0$ and so, S is not α-reversible .

In the following theorems, we study the relationships between α −reversible and central α −reversible rings

Theorem 3.2.5.

Let R be $\alpha -$ compatible and semi-prime ring . If R is central α −reversible then R is α-reversible.

Proof.

Let $a, b \in R$ with $ab = 0$ then $\alpha(b)a$ and $b\alpha(a)$ are central in R. And $a\alpha(b) = 0$ " by compatibility " , $\alpha(b)aR\alpha(b)a = \alpha(b)a\alpha(b)aR = 0$ then $\alpha(b)a = 0$ "by semi-prime" . Therefore, R is left α-reversible.Now, $ab = 0 \leftrightarrow \alpha(ab) = \alpha(a)\alpha(b) = 0 \leftrightarrow \alpha(a)b = 0$. So, $b\alpha(a)Rb\alpha(a) = b\alpha(a)b\alpha(a)R = 0$ then $b\alpha(a) = 0$ "by semi-prime". Therefore, R is left α-reversible. Hence, R is α-reversible.

Theorem 3.2.6.

Let R be right (left) principally projective and α $-$compatible ring .If R is central α-reversible then it is α-reversible.

Proof .

Since R is right principally projective ring , then there exist an idempotent $e \in R$such that $r_R(a) = eR$. Let $a, b \in R$ with $ab = 0$ then $a\alpha(b) = 0$ " by compatibility " . So, $ae = 0$ and $e\alpha(b) = \alpha(b)$.

Now, $\alpha(b)a = e\alpha(b)a = \alpha(b)ae = 0$, and so R is left α-reversible. And by the same way we may prove the right α-reversible. Hence R is α-reversible.

Theorem 3.2.7.

Let R be right (left) principally Quasi -Baer and α $-$compatible ring . If R is central α-reversible then it is α-reversible.

Proof .

Let R be right principally Quasi –Baer. And let $ab = 0$ then $a\alpha(b) = 0$. Then there exists an idempotent $e \in R$ such that $r_R(aR) = eR$. By the same proof of right principally projective ring we may prove that R is α-reversible .

Proposition 3.2.8.

Any domain is central α-reversible.

Proof.

Let $ab = 0$ for any $a, b \in R$ then either $a = 0$or $b = 0$.

Case I . $a = 0$then$\alpha(a) = 0$. so, $b\alpha(a) = 0 = \alpha(b)a$.

Case II . $b = 0$then$\alpha(b) = 0$. so, $\alpha(b)a = 0 = b\alpha(a)$.

Therefore R is α-reversible, and so central α-reversible.

Here is an example showing that there exists a central α-reversible ring which is not domain.

Example 3.2.9.

Let $R = \left\{\begin{pmatrix} a & 0 \\ b & a \end{pmatrix} | a, b \in \mathbb{Z}\right\}$ be a ring with endomorphism α defined by $\alpha\left(\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ -b & a \end{pmatrix}$, the ring R is central α-reversible, indeed let $A = \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$ and $B = \begin{pmatrix} c & 0 \\ d & c \end{pmatrix} \in R$ be such that $AB = 0$. So we have $ac = 0$ and $bc + ad = 0$, it implies that either $a = b = 0$ or $a = c = 0$ or $c = d = 0$. However, in all the cases above $\alpha(B)A = B\alpha(A) = 0$. So that R is α-reversible, hence central α-reversible.

In the next proposition we show that central α-reversible is closed under finite direct sum.

Proposition 3.2.10.

Let $\{R_i\}_{i\in I}$ be a class of rings for a finite index set I. And α be an endomorphism defined on R_i into R_i then , R_i is central α–reversible for all $i \in I$ if and only if $\oplus_{i\in I} R_i$ is central α–reversible.

Proof.

Let R_i be central α–reversible for all $i \in I$ and let $(a_1, \dots, a_n), (a_1^{'}, \dots, a_n^{'}) \in \oplus_{i\in I} R_i$

with $(a_1, \dots, a_n)(a_1^{'}, \dots, a_n^{'}) = 0$ =$(a_1 a_1^{'}, \dots, a_n a_n^{'}) = 0$ iff $a_i a_i^{'} = 0 \forall\, i \in I$, then

i) $a_i^{'}\alpha(a_i)$ is central in R_i $\forall\, i \in I$ so, $\left(a_1^{'}\alpha(a_1), \dots, a_n^{'}\alpha(a_n)\right)$ is central in $\oplus_{i\in I} R_i$ and which $\left(a_1^{'}\alpha(a_1), \dots, a_n^{'}\alpha(a_n)\right)$= $(a_1^{'}, \dots, a_n^{'})\left(\alpha(a_1), \dots, \alpha(a_n)\right) = (a_1^{'}, \dots, a_n^{'})\alpha(a_1, \dots, a_n)$ is central in $\oplus_{i\in I} R_i$.

ii) $\alpha(a_i^{'})a_i$ is central in R_i $\forall\, i \in I$ by the same way $\alpha(a_1^{'}, \dots, a_n^{'})(a_1, \dots, a_n)$ is central in $\oplus_{i\in I} R_i$.

By i) & ii) $\oplus_{i\in I} R_i$ is central α–reversible.

Conversely, let $\oplus_{i\in I} R_i$ be central α–reversible and $a_i a_i^{'} = 0$ for some $i \in I$ then $a_i a_i^{'} = (0, \dots a_i, 0, \dots, 0)(0, \dots, a_i{'}, 0, \dots, 0) = 0$ so, $\alpha(0, \dots, a_i{'}, 0, \dots, 0)(0, \dots, a_i, 0, \dots, 0) \& (0, \dots, a_i{'}, 0, \dots, a_n^{'})\alpha(0, \dots a_i, 0, \dots, 0)$ are central in $\oplus_{i\in I} R_i$. And so, $a_i^{'}\alpha(a_i) \& \alpha(a_i^{'})a_i$ are central in R_i. Hence R_i is central α–reversible.

Here, we study central α-reversible on a localization of a ring R.

Proposition 3.2.11.

Let α be an onto endomorphism then a ring R is central α–reversible if and only if $S^{-1}R$ is central α–reversible.

Proof.

Let R be central α–reversible and $\frac{a}{r}, \frac{b}{s} \in S^{-1}R$ where $a, b \in R$,$r, s \in S$ with $\frac{a}{r}\frac{b}{s} = 0$ iff $\frac{ab}{rs} = 0$ iff $ab = 0$, then $b\alpha(a) \&\ \alpha(b)a$ are central in R now, let $\frac{c}{d} \in S^{-1}R$ so

$$\frac{b}{s}\alpha\left(\frac{a}{r}\right)\frac{c}{d} = \frac{b\alpha(a)}{s\alpha(r)}\frac{c}{d} = \frac{c}{d}\frac{b\alpha(a)}{s\alpha(r)} = \frac{c}{d}\frac{b}{s}\alpha\left(\frac{a}{r}\right).$$

By the same way $\frac{c}{d}\alpha\left(\frac{b}{s}\right)\frac{a}{r} = \alpha\left(\frac{b}{s}\right)\frac{a}{r}\frac{c}{d}$. Therefore $S^{-1}R$ is central α–reversible.

Conversely, let $S^{-1}R$ be central α–reversible and $ab = 0$ for $a, b \in R$ then $a = \frac{a}{1}$,$b = \frac{b}{1}$,$1 \in S$ and $\alpha(1) = 1$ then the rest is clear.

Now, we introduce another new class of rings, which is called central α-symmetric rings.

Definition 3.2.12.

(1) A ring R is central right α-symmetric if whenever $abc = 0$ then $ac\alpha(b)$ is central in R for $a, b, c \in R$.

(2) A ring R is central left α-symmetric if whenever $abc = 0$ then $\alpha(b)ac$ is central in R for $a, b, c \in R$.

(3) A ring R is central α-symmetric if it is both central right and left α-symmetric.

It is well known that if R is α-symmetric, then it is α-reversible ring. For central case we have :

Theorem 3.2.13.

Let R be any ring. If R is central α-symmetric then it is central α-reversible ring.

Proof.

Let $ab = 0$ for any $a, b \in R$ then $ab = ab1 = 0$. Implies that $\alpha(b)a1 = \alpha(b)a$ is central in R and$1ab = 0$ then $1b\alpha(b) = b\alpha(b)$ is central in R. Therefore R is central α-reversible ring.

Remark 3.2.14.

If R is α–symmetric then, it is central α −symmetric.

Next proposition shows that central α −symmetric is closed under finite direct sum.

Proposition 3.2.15.

Let $\{R_i\}_{i \in I}$ be a class of rings for a finite index set I. And α be an endomorphism defined onR_i into R_i then, R_i is central α–symmetric for all $i \in I$ if and only if $\oplus_{i \in I} R_i$ is central α–symmetric.

Proof.

Let R_i be central $\alpha-$ symmetric for all $i \in I$ and let$(a_1, \dots, a_n), (b_1, \dots, b_n), (c_1, \dots, c_n) \in \oplus_{i \in I} R_i$ with $(a_1, \dots, a_n)(b_1, \dots, b_n)(c_1, \dots, c_n) = 0$ then $(a_1 b_1 c_1, \dots, a_n b_n c_n) = 0$ if and only if $a_i b_i c_i = 0 \quad \forall i \in I$ implies both $\alpha(b_i) a_i c_i$ and $a_i c_i \alpha(b_i)$ are central in $R_i \forall i \in I$. So, $(\alpha(b_1) a_1 c_1, \dots, \alpha(b_n) a_n c_n)$ is central in $\oplus_{i \in I} R_i$ if and only if $\big(\alpha(b_1), \dots, \alpha(b_n)\big)(a_1, \dots, a_n)(c_1, \dots, c_n)$ is central in $\oplus_{i \in I} R_i$ if and only if $\alpha\big((b_1, \dots, b_n)\big)(a_1, \dots, a_n)(c_1, \dots, c_n)$ is central in $\oplus_{i \in I} R_i$. Hence, $\oplus_{i \in I} R_i$ is central left α-symmetric.

And, $\big(a_1 c_1 \alpha(b_1), \dots, a_n c_n \alpha(b_n)\big)$ is central in $\oplus_{i \in I} R_i$ if and only if $(a_1, \dots, a_n)(c_1, \dots, c_n)\big(\alpha(b_1), \dots, \alpha(b_n)\big)$ is central in $\oplus_{i \in I} R_i$ if and only if $(a_1, \dots, a_n)(c_1, \dots, c_n)\alpha\big((b_1, \dots, b_n)\big)$ is central in $\oplus_{i \in I} R_i$. Hence, $\oplus_{i \in I} R_i$ is central right α-symmetric. Therefore, $\oplus_{i \in I} R_i$ is central α-symmetric.

Conversely, let ,$\oplus_{i \in I} R_i$ be central α-symmetric and let $a_i, b_i, c_i \in R_i$ for some $i \in I$

with $a_i b_i c_i = 0$ but $a_i = (0, \dots, a_i, 0, \dots, 0)$, $b_i = (0, \dots, b_i, 0, \dots, 0)$ and $c_i =$

$(0, \dots, c_i, 0, \dots, 0)$ then $a_i b_i c_i = (0, \dots, a_i, 0, \dots, 0)(0, \dots, b_i, 0, \dots, 0)(0, \dots, c_i, 0, \dots, 0) =$

$(0, \dots, a_i b_i c_i, 0, \dots, 0) = 0$ hence, both

$\alpha\big((0, \dots, b_i, 0, \dots, 0)\big)(0, \dots, a_i, 0, \dots, 0)(0, \dots, c_i, 0, \dots, 0)$ and

$(0, \dots, a_i, 0, \dots, 0)(0, \dots, c_i, 0, \dots, 0)\alpha\big((0, \dots, b_i, 0, \dots, 0)\big)$ are central in $\oplus_{i \in I} R_i$ if and

only if $(0, \dots, \alpha(b_i) a_i c_i, 0, \dots, 0)$ and $(0, \dots, a_i c_i \alpha(b_i), 0, \dots, 0)$ are both central in

$\oplus_{i \in I} R_i$ if and only if $\alpha(b_i) a_i c_i$ and $a_i c_i \alpha(b_i)$ are both central in R_i. Therefore, R_i is

central α-symmetric.

Proposition 3.2.16.

Let α be an onto endomorphism then ,a ring R is central α–symmetric if and only if $S^{-1}R$ is central α– symmetric.

Proof.

Let R be central α– symmetric and $\frac{a}{r},\frac{b}{s},\frac{c}{t} \in S^{-1}R$ where $a,b,c \in R$,$r,s,t \in S$ with $\frac{a}{r}\frac{b}{s}\frac{c}{t} = 0$ iff $\frac{abc}{rst} = 0$ iff $abc = 0$ then $ac\alpha(b)$& $\alpha(b)ac$ are central in R now, let $\frac{w}{d} \in S^{-1}R$ so

$$\alpha\left(\frac{b}{s}\right)\frac{a}{r}\frac{c}{t}\frac{w}{d} = \frac{\alpha(b)ac}{\alpha(s)rt}\frac{w}{d} = \frac{w}{d}\frac{\alpha(b)ac}{\alpha(s)rt} = \frac{w}{d}\alpha\left(\frac{b}{s}\right)\frac{a}{r}\frac{c}{t}.$$

By the same way $\frac{w}{d}\frac{a}{r}\frac{c}{t}\alpha\left(\frac{b}{s}\right) = \frac{a}{r}\frac{c}{t}\alpha\left(\frac{b}{s}\right)\frac{w}{d}$. Therefore $S^{-1}R$ is central α– symmetric.

Conversely, let $S^{-1}R$ be central α– symmetric and $ab = 0$ for $a,b \in R$ then $a = \frac{a}{1}$, $b = \frac{b}{1}$, $1 \in S$ and $\alpha(1) = 1$, then the rest is clear.

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