Palestine Polytechnic University
Deanship of Graduate Studies and Scientific Research
Master of Applied Mathematics

Distribution of Functions of Normal Random Variables

Submitted by:

Doaa Zuhair Ibraheem Salim

Thesis submitted in partial fulfillment of requirements of the degree
Master of Applied Mathematics
Feb, 2021

The undersigned hereby certify that they have read, examined and recommended to the Deanship of Graduate Studies and Scientific Research at Palestine Polytechnic University the approval of a thesis entitled: **Distribution of Functions of Normal Random Variables**, submitted by **Doaa Zuhair Ibraheem Salim** in partial fulfilment of the requirements for the degree of Master in Applied Mathematics.

Graduate Advisory Committee:

Dr. Monjed H. Samuh (Supervisor), Palestine Polytechnic University.

Signature:____________________ Date:______________

Dr. Ahmad Khamayseh (Co-supervisor), Palestine Polytechnic University.

Signature:____________________ Date:______________

Dr. Mohammad Adam (Internal committee member), Palestine Polytechnic University.

Signature:____________________ Date:______________

Dr. Inad Nawajah (External committee member), Hebron University.

Signature:____________________ Date:______________

Thesis Approved

Dr. Murad Abusubaih
Dean of Graduate Studies and Scientific Research
Palestine Polytechnic University

Signature:____________________ Date:______________

DECLARATION

I declare that the Master Thesis entitled **"Distribution of Functions of Normal Random Variables"** is my original work, and hereby certify that unless stated, all work contained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgement is made in the text.

Doaa Zuhair Ibraheem Salim

Signature:________________________ Date:________________

STATEMENT OF PERMISSION TO USE

In presenting this thesis in partial fulfilment of the requirements for the master degree in Applied Mathematics at Palestine Polytechnic University, I agree that the library shall make it available to borrowers under rules of the library.

Brief quotations from this thesis are allowable without special permission, provided that accurate acknowledgement of the source is made.

Permission for extensive quotation from, reproduction, or publication of this thesis may be granted by my main supervisor, or in his absence, by the Dean of Graduate Studies and Scientific Research when, in the opinion of either, the proposed use of the material is for scholarly purposes.

Any coping or use of the material in this thesis for financial gain shall not be allowed without my written permission.

Doaa Zuhair Ibraheem Salim

Signature:________________________ Date:________________

ACKNOWLEDGEMENT

First and foremost, all praise and thanks are due to Allah for helping me to complete this work. I could never have finished this thesis without his help and guidance.

I would like to express my sincere gratitude to my supervisor, Dr. Monjed H. Samuh, for the continuous support of my Master study and related research, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I also would like to express my gratitude and thanks to Dr. Ahmad Khamayseh for his valuable guidance and all the support he provided me.

I am also thankful to Dr. Inad Nawajah and Dr. Mohammad Adam for serving on my thesis committee and for providing constructive comments which have improved my thesis.

Last but not the least, I would like to express my gratitude and appreciation to my family, my sisters and brother for supporting me spiritually throughout writing this thesis and my life in general.

Abstract

The need for the distribution of combination of random variables arises in many areas of the sciences and engineering. In this thesis, the distributions of two different combination of Gaussian random variables will be investigated. It is established that such expressions can be represented in their most general form as the sum of chi-square and the product of two normal random variables. The distribution of the product of two normal random variables studied by some authors in the literature in different ways. The first expression assumed two independent and identical random variables with zero mean and the same variance. The second one assumed two dependent random variables with zero mean, same variance, and with a specific correlation coefficient. Closed forms of the probability density function and cumulative distribution function will be derived, as well as some statistical properties such as mean, variance, moments, moment generating function, and order statistics will be studied. All derivation ascertained by using Monte Carlo simulation. Additionally, method of moment estimation and the maximum likelihood estimation will be used to estimate the parameter of the derived distribution. Simulation study carried out using R software. Finally, a practical application well be presented.

Table of Contents

List of Figures

List of Tables

List of Abbreviations

IoT Internet of Things 2, 80

I/Q In-phase/Quadrature-phase 2, 68, 78-83

RV Random Variable 6-8, 11-12, 15, 20, 23-26, 29, 31, 34, 69, 72

PDF Probability Density Function 2, 7-8, 11, 15-16, 21, 24, 26-28, 30-34, 39, 60, 68-69, 71-72, 81-82, 85

CDF Cumulative Density Function 7, 11, 30, 65, 68, 72, 76, 85

JPDF Joint Probability Density Function 7-8, 12, 31-33, 35, 69-71

MGF Moment Generating Function 9, 15-16, 24-25, 57, 75, 82

o.s order statistics 10-11, 60

MME Method of Moment Estimation 13, 60-61, 64-66, 75

MLE Maximum Likelihood Estimation 14, 29, 60, 63-65, 67, 75-79

SM Slash Maxwell 28

CI Confidence Interval 15

ODE Ordinary Differential Equation 18

GG Generalized Gamma 29

APEP Average Pairwise Error Probability 82

mmWave millimeter Wave 80

QAM Quadrature Amplitude Modulation 80-81

IF Intermediate Frequency 78-79

Chapter 1

Introduction

1.1 Motivation and General Purpose

Engineers use statistical data in variety of applied settings. For instance, an industrial and system engineers may be interested in determining the overall throughout for a certain manufacturing process, e.g., a chemical engineer may be interested in determining the expected yield in a certain chemical reaction, while a transportation engineer may be interested in determining the average volume of traffic flow through an intersection in order to increase driver safety by improving traffic control devices. In each of these settings, the engineer require, to physically collect and analyze data which consists of observations of a system. It can be argued that the behaviour of most real-world systems is stochastic rather than deterministic. Therefore, a responsible analysis of any phenomenon should include the inherent probabilistic behaviour of a system. Measures such as the mean and variance are certainly useful in describing the main features of a distribution. However, in decision making processes, the most valuable pieces of information available are the probability density function or the cumulative distribution function. In fact, when the probability distribution of a given phenomenon will be known, more intelligent decisions can be made regarding the performance of a given system. For these reasons, probability density derivation or estimation become an important tool to the engineering analyst. In real-world systems, multiple random variables will be usually involved which are interacting in some way to produce a new random response. Hence, it is expected that this random response follows some probability distribution measure.

Many responses may be modelled as the sum, difference, product or quotient of random variables or combinations of some or all of these metrics. Thus, it is a major issue to investigate the probability distribution of new resulted random variables. In some cases, a closed-form (analytical solution) probability distribution can be obtained. In some other cases, it is difficult, if not impossible, to solve for a closed-form for such a distribution. For the latter, it is desirable to develop a practical technique by which an engineer may closely approximate the probability distribution for the sum, difference, product or quotient of random variables.

To this end, consider an example from the field of communication engineering. Future wireless communication systems, including fifth-generation (5G) networks and the Internet of Things (IoT), require a massive number of inexpensive transceivers. These transceivers come with various hardware impairments, such as phase noise and in-phase/quadrature-phase (I/Q) imbalance. This thesis studies the PDF of an optimal maximum likelihood receiver designed by Alsmadi (2020).

1.2 Statement of the Problem

This thesis investigates the probability density, cumulative distribution functions, and some other statistical properties for the following two formulations of random variables:

(i) Suppose X_1 and X_2 are two independent normal random variables with zero mean and identical variance σ^2. Define

$$Z = X_1^2 + X_2^2 - \theta X_1 X_2, \tag{1.1}$$

where θ is an unknown scale parameter.

(ii) Suppose X_1 and X_2 are two dependent normal random variables with zero mean, identical variance σ^2, and correlation coefficient ρ. Define

$$U = X_1^2 + X_2^2 - 2\rho X_1 X_2. \tag{1.2}$$

1.3 Research Questions

The research focus of this thesis is summarized in the following four main research questions:

(i) How to derive the probability density function of the new defined random variables?

(ii) How to estimate the parameters of the new derived probability distribution functions?

(iii) What are the statistical properties of the new probability distribution functions?

(iv) How do we apply the new probability distribution functions in real life?

1.4 Thesis Objective

The main contribution of this thesis is to develop a new probability distribution based on a combination of some normal random variables. Two cases will be adopted. The first one is for independent normal random variables represented by (1.1), and the second one is for the dependent normal random variables in (1.2). The specific objectives as related to the previous research questions are as follows:

Research Question 1:

(i) List some literature review on the distribution of function of random variables, and in particular of normal random variables.

(ii) Derive the probability distribution of the new defined (dependent and independent) normal random variables.

Research Question 2:

(i) Estimate the parameters of the new probability distribution function using?

  (i) Method of moment estimation.

  (ii) Maximum likelihood estimation.

(ii) Evaluate the derived estimators in terms of bias and mean squared errors.

Research Question 3:
Investigate the properties of the probability distribution of the new random variable. Particularly:

(i) Descriptive measures, such as mean and variance.

(ii) Moments of the distribution.

(iii) Moment generating function.

Research Question 4:
Motivate the new probability distribution by real life applications.

1.5 Thesis Organization

The organization of this thesis is as follows: Chapter 2 introduces the preliminaries and a review of the literature for some similar distributions and their results. In Chapter 3, the distribution of function of independent normal random variables will be derived, as well as its properties and the parameter estimation of the distribution. Chapter 4 provides the distribution of function of dependent normal random variables and its properties. Moreover, the parameter estimation of the distribution will be introduced. Finally, an application will be presented. In Chapter 5, a summary of our investigation and some important conclusions will be presented. We also suggest some potential topics for future research.

Appendix contains the detailed proof of chapter 3.

Chapter 2

Background

This chapter gives some preliminaries that are fundamental for the thesis. Also, it gives some literature review and theoretical background on the product of two normal random variables, product of independent Rayleigh random variables, product of m Gamma and n Pareto random variables, Slash Maxwell distribution, product of Bessel distribution of second kind and Gamma distribution, and distributions involving correlated generalized Gamma variables.

2.1 Preliminaries

In this section, some preliminaries that are fundamental for the thesis will be introduced (The definitions and theorems presented here can be found at Wackerly et al. (2008), Pishro-Nik (2014), Hogg et al. (2015), and Georgiev (2019)).

2.1.1 Random Variable and Some Properties

Definition 2.1.1. *Random Variable*

A random variable (RV) *is a variable whose value is unknown or a function that assigns real values to each of an experiment's outcomes. Random variables often designated by* *letters and can be classified as discrete, if it assumes only a finite or countably infinite number of distinct values, or continuous, if the data can take infinitely many values. For example, a RV measuring the time taken for something to be done is continuous since there are an infinite number of possible times that can be taken.*

Definition 2.1.2. *Probability Density Function*

A probability density function (PDF)*, or* density of a continuous RV*, is a function whose specify the probability of the RV falling within a particular range of values, unlike taking on any one value. This probability is given by the area under the density function, but above the horizontal axis and between the lowest and greatest values of the range. For any continuous RV with PDF $f(x)$, we have that:*

(i) $f(x) \geq 0 \qquad \forall x.$

(ii) $\int_{\forall x} f(x)dx = 1.$

Definition 2.1.3. *Cumulative Distribution Function*

The cumulative distribution function (CDF) *of a random variable is another method to describe the distribution of random variables. The advantage of the CDF is that it can be defined for any kind of random variable (discrete, continuous, and mixed). The CDF of the RV X is defined as*

$$F_X(x) = P(X \leq x), \qquad \forall x \in \mathbb{R}.$$

Note that the subscript X indicates that this is the CDF of the random variable X. Also, note that the CDF is defined for all $x \in \mathbb{R}$.

Definition 2.1.4. *Joint Probability Density Function*

Two RV's X_1 and X_2 are jointly continuous *if there exists a non-negative function $f_{X_1X_2} : \mathbb{R}^2 \to \mathbb{R}$, such that, for any set $A \in \mathbb{R}^2$, we have*

$$P((X_1, X_2) \in A) = \int\int_A f_{X_1X_2}(x_1, x_2)dx_1dx_2.$$

The function $f_{X_1X_2}(x_1, x_2)$ is called the joint probability density function (JPDF) *of X_1 and X_2. It achieves the following properties:*

(i) $f(x_1, x_2) \geq 0, \qquad \forall x_1 \text{ and } \forall x_2.$

(ii) $\int_{\forall x_1} \int_{\forall x_2} f(x_1, x_2) dx_2 dx_1 = 1.$

Definition 2.1.5. *Independence*

Two RV's X_1 and X_2 are called independent *if the JPDF is the product of the individual PDFs, i.e., if $f_{X_1X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$ for all x_1, x_2.*

Note: While the individual (marginal) densities f_{X_1} and f_{X_2} can always be computed from the joint density $f_{X_1X_2}(x_1, x_2)$, only for independent RVs can one go backwards, e.g., obtain the joint density from the marginal densities.

Definition 2.1.6. *Moments*

The n^{th} moment *of a real-valued continuous function $f(x)$ of a RV X is given by the measure,*

$$\mu_n = E(X^n) = \int_{-\infty}^{\infty} x^n f_X(x) dx.$$

Definition 2.1.7. *Expectation*

Let X be a continuous RV with PDF $f(x)$ and

$$\int_{-\infty}^{\infty} |x| f(x) dx < \infty \qquad (finite),$$

then the expectation *of X (sometimes called as the* expected value *of X, or the* mean *of X) is*

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx < \infty \qquad (finite).$$

Definition 2.1.8. *Variance*

The most common measure of variability used in statistics is the variance*, which is a function of the deviations (or distances) of the sample measurements from their mean. If X is a RV with mean $E(X) = \mu$, the variance of a RV X is defined to be the expected value of $(X - \mu)^2$. That is,*

$$\sigma^2 = E[(X - \mu)^2]. \tag{2.1}$$

The standard deviation *of X is the positive square root of σ^2.*

It is worthwhile to observe that σ^2 can be calculated as

$$\sigma^2 = E[X^2] - \mu^2.$$

This frequently affords an easier way of computing the variance of X.

Definition 2.1.9. *Skewness*

The measure of skewness defined here is called the Pearson moment coefficient of skewness*. This measure provides information about the amount and direction of the departure from symmetry. It's value can be positive or negative, or even undefined. The higher the absolute value of the skewness measure, the more asymmetric the distribution. The skewness measure of symmetric distributions is, or near zero. The Pearson's moment coefficient of skewness (or the coefficient of skewness) is the ratio of the third central moment to the cube of the standard deviation, and is denoted by γ_1, i.e.,*

$$\gamma_1 = \frac{E[(Z-\mu)^3]}{\sigma^3} = \frac{\mu_3}{\sigma^3}. \tag{2.2}$$

Definition 2.1.10. *Kurtosis*

Kurtosis *is a measure of the tailedness of the probability distribution of a real-valued random variable. Like skewness, kurtosis describes the shape of a probability distribution. The ratio of the fourth central moment to the fourth power of the standard deviation, is called the* kurtosis*, i.e.,*

$$\gamma_2 = \frac{E[(X-\mu)^4]}{\sigma^4}. \tag{2.3}$$

Definition 2.1.11. *Moment Generating Function*

Let X be a random variable such that, for some $h > 0$, the expectation of $\exp\{tX\}$ *exists for $-h < t < h$. The* moment generating function (MGF) of X *is defined to be the function $M(t) = E(\exp\{tX\})$, for $-h < t < h$.*

Definition 2.1.12. *The Coefficient of Variation*

The coefficient of variation (γ) *is a statistical measure of the dispersion of data points* *in a data series around the mean. The coefficient of variation represents the ratio of the standard deviation to the mean, and it is a useful statistic for comparing the degree of variation from one data series to another, even if the means are drastically different from one another.*

Definition 2.1.13. *Order Statistics*

Order statistics (o.s) *plays a prominent role in L-moment theory. The study of order statistics is the study of the statistics of ordered (sorted) random variables and samples. The random variable* X *for a sample of size* n*, when sorted, forms the order statistics of* $X : X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$*. The sample order statistics from a random sample are created by sorting the sample into ascending order:* $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$*. As we will see, the concept and use of order statistics take into account both the value(magnitude) and the relative relation (order) to other observations. In general, order statistics are already a part of the basic summary statistic repertoire that most individuals including non scientists or statisticians are familiar with. The minimum and maximum are examples of extreme order statistics and are defined by the following notation:*

$$minX_n = X_{1:n} \tag{2.4}$$

$$maxX_n = X_{n:n} \tag{2.5}$$

Remark 1. *(i) The order statistic* $X_{1:1}$*(a single observation) contains information about the location of the distribution on the real-number line.*

(ii) For a sample of $n = 2$*, the order statistics are* $X_{1:2}$*(smallest) and* $X_{2:2}$*(largest). For a highly dispersed distribution, the expected difference between* $X_{2:2} - X_{1:2}$ *would be large, whereas for a tightly dispersed distribution, the difference would be small. The expected difference between order statistics of an* $n = 2$ *can be used to express the variability or scale of a distribution.*

(iii) For a sample of $n = 3$*, the order statistics are* $X_{1:3}$*(smallest),* $X_{2:3}$*(median), and* $X_{3:3}$ *(largest). For a negatively skewed distribution, the difference* $X_{2:3} - X_{1:3}$ *would*

be larger (more data to the left) than $X_{3:3} - X_{2:3}$. The opposite (more data to the right) would occur if a distribution where positively skewed.

Let $F(z)$ be the CDF of a random sample $Z_1, Z_2, \ldots, Z_n$ and $f(z)$ be its PDF. Reorder $Z_1, Z_2, \ldots, Z_n$ such that $Z_{(n)} > Z_{(n-1)} > \cdots > Z_{(1)}$, then $Z_{(j)}$ is called the j^{th} order statistic, and have the following marginal PDF's:

$$f_{Z_{(j)}}(z) = n \begin{pmatrix} n-1 \\ j-1 \end{pmatrix} f(z)(F(z))^{j-1}(1-F(z))^{n-j}. \tag{2.6}$$

The 1^{st} o.s, $Z_{(1)}$, has the following marginal PDF

$$f_{Z_{(1)}}(z) = nf(z)(1-F(z))^{n-1}. \tag{2.7}$$

The n^{th} o.s, $Z_{(n)}$, has the following marginal PDF

$$f_{Z_{(n)}}(z) = nf(z)(F(z))^{n-1}. \tag{2.8}$$

In the following, some theorems related to the distribution of function of normal RV's will be presented.

Theorem 2.1.1. *Let X_1, X_2 are two independent normal RV's with means μ_1, μ_2 and variances σ_1^2, σ_2^2, respectively. Then, the RV $Y = X_1 \pm X_2$ has a normal distribution with mean $E[Y] = \mu_1 \pm \mu_2$, and variance $Var[Y] = \sigma_1^2 + \sigma_2^2$. In case of dependent RV's, the distribution remains normal with mean $E[Y] = \mu_1 \pm \mu_2$, and variance $Var[Y] = \sigma_1^2 + \sigma_2^2 \pm 2Cov(X_1, X_2)$.*

Theorem 2.1.2. *If $X_1, X_2, \ldots, X_n$ are n mutually independent normal variables with means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, respectively, then the linear function $Y = \sum_{i=1}^n c_i X_i$ has the normal distribution with mean $\sum_{i=1}^n c_i \mu_i$ and variance $\sum_{i=1}^n c_i^2 \sigma_i^2$.*

Theorem 2.1.3. *The random variables $X_1 + X_2$ and $X_1 - X_2$ are independent if and only if X_1,X_2 are distributed normal.*

Theorem 2.1.4. *If $X_i \sim N(0, 1)$, $i = 1, 2, \ldots, n$ are mutually independent RV's, then $Y = \sum_{i=1}^n X_i^2$ has the chi-square distribution with n degrees of freedom.*

2.1.2 Methods for Deriving the Probability Distribution of a Function of Random Variables

Two methods for deriving the probability distribution for a given function of random variables will be presented. Any one of these may be employed to find the distribution of a given function of the variables, however one of the methods usually leads to a simpler derivation than the other. The method that works best varies from one application to another.

Definition 2.1.14. *The Distribution Function Method*
If Y has probability density function $f(y)$ and if U is some function of Y, then we can find $F_U(u) = P(U \leq u)$ directly by integrating $f(y)$ over the region for which $U \leq u$. The probability density function for U is found by differentiating $F_U(u)$.

Definition 2.1.15. *The Bivariate Transformation Method*
Suppose that Y_1 and Y_2 are continuous RV's with JPDF $f_{Y_1,Y_2}(y_1, y_2)$ and that for all (y_1, y_2), such that $f_{Y_1,Y_2}(y_1, y_2) > 0$,

$$u_1 = h_1(y_1, y_2) \qquad \textit{and} \qquad u_2 = h_2(y_1, y_2),$$

is a one-to-one transformation from (y_1, y_2) to (u_1, u_2) with inverse

$$y_1 = h_1^{-1}(u_1, u_2) \qquad \textit{and} \qquad y_2 = h_2^{-1}(u_1, u_2).$$

If $h_1^{-1}(u_1, u_2)$ and $h_2^{-1}(u_1, u_2)$ have continuous partial derivatives with respect to u_1 and u_2, and Jacobian

$$J = \begin{vmatrix} \frac{dh_1^{-1}}{du_1} & \frac{dh_1^{-1}}{du_2} \\ \frac{dh_2^{-1}}{du_1} & \frac{dh_2^{-1}}{du_2} \end{vmatrix} = \frac{dh_1^{-1}}{du_1}\frac{dh_2^{-1}}{du_2} - \frac{dh_2^{-1}}{du_1}\frac{dh_1^{-1}}{du_2} \neq 0,$$

then the JPDF of U_1 and U_2 is

$$f_{U_1,U_2}(u_1, u_2) = f_{Y_1,Y_2}(h_1^{-1}(u_1, u_2), h_2^{-1}(u_1, u_2)|J|, \tag{2.9}$$

where $|J|$ is the absolute value of the Jacobian.

2.1.3 Method of Estimation

In this subsection, a more formal and detailed examination of some of the mathematical properties of point estimators will be undertaken. Then two other useful methods for deriving estimators will be introduced. Some properties of estimators derived by these methods will be discussed.

Definition 2.1.16. *Point Estimation*

Point estimation *is the process of finding an approximate value of some parameters such as the mean of a population from random samples of the population. The accuracy of any particular approximation is not known precisely, though probabilistic statements concerning the accuracy of such numbers as found over many experiments can be constructed.*

It is desirable for a point estimate to be:

(i) Consistent. The larger the sample size, the more accurate the estimate.

(ii) Unbiased. The expectation of the observed values of many samples (average observation value) equals the corresponding population parameter. For example, the sample mean is an unbiased estimator for the population mean.

(iii) Most efficient or best unbiased of all consistent, unbiased estimates, the one possessing the smallest variance (a measure of the amount of dispersion away from the estimate).

Several methods are used to derive the estimators. The most often one is the *maximum likelihood method*. It uses differential calculus to determine the maximum of the probability function of a number of sample parameters (see Definition 2.1.19 below). Another method of estimation based on equates values of sample moments (functions describing the parameter) to population moments called *method of moment* (MME)(see Definition 2.1.17).

Definition 2.1.17. *Method of Moments Estimation*

The method of moments *is a method of estimation of population parameters. It starts by expressing the population moments (i.e., the expected values of powers of the random variable under consideration) as functions of the parameters of interest. Those expressions are then set equal to the sample moments. The number of such equations is the same as the number of parameters to be estimated. These equations are then solved for the parameters of interest. The solutions are estimates of such parameters.*

The method of moments was introduced by Pafnuty Chebyshev in 1887 in the proof of the central limit theorem (CLT).

Definition 2.1.18. *Likelihood Function*

In frequentist inference the likelihood *is a quantity proportional to the probability that, from a population having a particular value of θ, a sample having the observed value x_0, should be obtained.* Likelihood, *being the outcome of a* likelihood function *thus defined, describes the plausibility, under a certain statistical model (the null hypothesis in hypothesis testing), of a certain parameter value after observing a particular outcome. Formally: $L(\Theta; x_0) \propto f(x_0; \theta)$, $\forall \theta \in \Theta$.* Likelihood *is central to parametric statistical inference. It is a basis for the maximum likelihood estimate.*

Definition 2.1.19. *Maximum Likelihood Estimation*

Maximum likelihood estimation (MLE) *is a method of estimating the parameters of a probability distribution by maximizing a likelihood function, so that, under the assumed statistical model the observed data is most probable. The point in the parameter space that maximizes the likelihood function is called the maximum likelihood estimate. The logic of maximum likelihood is both intuitive and flexible, and as such the method has become a dominant means of statistical inference.*

If the likelihood function is differentiable, the derivative test for determining maxima can be applied. In some cases, the first-order conditions of the likelihood function can be solved explicitly; for instance, the estimates of the normal RV parameters μ and σ^2 obtained as the sample mean ($\bar{x}$) and the sample variance (s^2). Under most circumstances, however, numerical methods will be necessary to find the maximum of the likelihood function.

Let $Z_1, Z_2, \ldots, Z_n$ be observations selected from Z distribution with parameters θ and σ. Let $\Theta = (\theta, \sigma)$ be a 2×1 parameter vector. The likelihood function of Z for the parameter vector Θ can be written as

$$L(\Theta; z_1, z_2, \ldots, z_n) = \prod_{i=1}^{n} f(z_i; \theta, \sigma). \tag{2.10}$$

Definition 2.1.20. *Confidence Interval*

Let $X_1, X_2, \ldots, X_n$ be a sample on a RV X, where X has PDF $f(x;\theta)$ and $0 < \alpha < 1$ be a pre-assigned level of significance. Assume $L = L(X_1, X_2, \ldots, X_n)$ and $U = U(X_1, X_2, \ldots, X_n)$ be two statistics. Then we say that the interval (L, U) is a $(1-\alpha)100\%$ confidence interval (CI) for θ *if*

$$P_\theta\left[\theta \in (L, U)\right] = 1 - \alpha. \tag{2.11}$$

That is, the probability that the interval includes θ is $1-\alpha$, which is called the confidence coefficient or the confidence level of the interval.

2.1.4 Some Common Discrete and Continuous Random Variables

In this subsection, some common distributions will be listed, focusing on some important statistical properties such as mean, variance, and MGF. Table 2.1 shows two discrete distributions, while Table 2.2 presents three continuous distributions that will be used later.

Table 2.1: Some discrete distributions (Wackerly et al., 2008)

Distribution	PMF	Mean	Variance	MGF
Binomial(n,p)	$\binom{n}{x} x(1-p)^{n-x}, x = 1, \ldots, n$	np	$np(1-p)$	$[(1-p)+pe^t]^n$
Poisson(λ)	$\frac{e^{-\lambda}\lambda^x}{x!}$; $\lambda \geq 0$	λ	λ	$e^{\lambda(e^t-1)}$

Table 2.2: Some continuous distributions (Wackerly et al., 2008)

Distribution	PDF	Mean	Variance	MGF
Normal(μ, σ^2)	$\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, -\infty < x, \mu < \infty, \sigma > 0$	μ	σ^2	$e^{\mu t + \frac{\sigma^2 t^2}{2}}$
Exponential(β)	$\frac{1}{\beta} e^{-\frac{x}{\beta}}$; $0 \leq x < \infty$, $\beta > 0$	β	β^2	$\frac{1}{1-\beta t}$, $t < \frac{1}{\beta}$
Gamma(α, β)	$\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}$; $0 \leq x < \infty$, $\alpha, \beta > 0$	$\alpha\beta$	$\alpha\beta^2$	$\left(\frac{1}{1-\beta t}\right)^\alpha$, $t < \frac{1}{\beta}$

2.1.5 Some Special Functions

In the following, some special functions, that will be referred to them during the thesis, will be introduced (These functions can be found at Abramowitz and Stegun (1965), Gradshteyn et al. (2007), and Arfken et al. (2013)).

Definition 2.1.21. *Gamma Function*

The (complete) Gamma function $\Gamma(n)$ *is defined to be a generalization of the factorial function to nonintegral values, introduced by the Swiss mathematician Leonhard Euler in the 18th century. The Gamma function can be defined as a definite integral for* $\Gamma(z) > 0$ *(Euler's integral form) as*

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp\{-t\}\, dt. \tag{2.12}$$

For integer n, it is easy to show that

$$\Gamma(n) = (n-1)!. \tag{2.13}$$

Definition 2.1.22. *Polygamma Function*

The polygamma functions *are the logarithmic derivatives of the Gamma function. The polygamma function of order* $m+1$ *is defined as*

$$\psi^m(z) = \frac{d^{m+1}}{dz^{m+1}} log\Gamma(z), \qquad m = 1, 2, \ldots \tag{2.14}$$

Thus

$$\psi^{(0)}(z) = \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{2.15}$$

holds where $\psi(z)$ is the digamma function and $\Gamma(z)$ is the gamma function.

Definition 2.1.23. *Bessel Function*

The linear second order ordinary differential equation of type

$$x^2 y'' + xy' + (x^2 - v^2)y = 0, \tag{2.16}$$

for an arbitrary complex number v, is called the Bessel equation*. The number v is called the* order *of the Bessel equation.*

The Bessel equation is named after the German mathematician and astronomer Friedrich Wilhelm Bessel who studied this equation in detail and showed (in 1824) that its solutions are expressed in terms of a special class of functions called *cylinder functions* or *Bessel functions.*

Bessel functions can be first, second or third kind.

Definition 2.1.24. *Modified Bessel Functions: I_α, K_α*

The Bessel functions are valid even for complex arguments x, and an important special case is that of a purely imaginary argument. In this case, the solutions to the Bessel equation are called the modified Bessel functions (or occasionally the hyperbolic Bessel functions) of the first and second kind and are defined as

$$I_0(z) = \sum_{r=0}^{\infty} \frac{z^{2r}}{2^{2r}(r!)^2} = \frac{1}{\pi}\int_0^{\pi} \exp\{zcos(\theta)\}\, d\theta, \tag{2.17}$$

$$K_0(z) = -ln\left(\frac{z}{2}\right) I_0(z) + \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}(k!)^2}\psi(k+1), \tag{2.18}$$

$$K_v(xu) = \sqrt{\frac{\pi}{2u}} \frac{x^v \exp\{-xu\}}{\Gamma(v+\frac{1}{2})} \int_0^{\infty} \exp\{-xt\}\, t^{v-\frac{1}{2}} \left(1 + \frac{t}{2u}\right)^{v-\frac{1}{2}} dt. \tag{2.19}$$

$$\begin{aligned} K_0(xu) &= \sqrt{\frac{\pi}{2u}} \frac{\exp\{-xu\}}{\Gamma(\frac{1}{2})} \int_0^{\infty} \exp\{-xt\}\, t^{-\frac{1}{2}} \left(1 + \frac{t}{2u}\right)^{-\frac{1}{2}} dt \\ &= \exp\{-xu\} \int_0^{\infty} \frac{\exp\{-xt\}}{\sqrt{t^2 + 2tu}} dt, \end{aligned} \tag{2.20}$$

where ψ is the digamma function, see (8.447.1), (8.447.3), (8.431.5) and (8.432.8) in

Gradshteyn et al. (2007).

Definition 2.1.25. *Gaussian Hypergeometric Function*

The Gaussian *or* ordinary hypergeometric function ${}_2F_1(a,b;c;z)$ *is a special function represented by the hypergeometric series, that includes many other special functions as specific or limiting cases. It is a solution of a second-order linear ordinary differential equation (ODE). Every second-order linear ODE with three regular singular points can be transformed into this equation.*

The hypergeometric function is defined for $|z| < 1$ *by the power series*

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{2.21} $$

(see Eq. 9.14.1 in Gradshteyn et al. (2007)). It is undefined (or infinite) if c *equals a non-positive integer.*

Definition 2.1.26. *Generalized Exponential Integral Function*

The exponential integral (E_i) *is a special function on the complex plane. It is defined as one particular definite integral of the ratio between an exponential function and its argument. The exponential integral may also be generalized to*

$$ E_n(z) = -\int_1^{\infty} \frac{\exp\{-zt\}}{t^n} dt, \tag{2.22} $$

where $\mathbb{R}(z) > 0$, $n = 0, 1, 2, \ldots$

Definition 2.1.27. *Meijer G-Function*

The G-function *was introduced by Cornelis Simon Meijer (1936) as a very general function intended to include most of the known special functions as particular cases. This was not the only attempt of its kind: the generalized hypergeometric function and the MacRobert E-function had the same aim, but Meijer's G-function was able to include those as particular cases as well. The first definition was made by Meijer using a series; nowadays the accepted and more general definition is via a path integral in the complex plane, introduced in its full generality by Arthur Erdélyi in 1953.*

A general definition of the Meijer G-function is given by the following line integral in the complex plane:

$$G^{m,n}_{p,q}\left(\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array}\middle| z\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} z^s ds, \tag{2.23}$$

where Γ denotes the gamma function. L is a loop beginning and ending at ∞, encircling all poles of $\Gamma(b_j - s)$, $j = 1, 2, \ldots, m$, exactly once in the negative direction, but not encircling any pole of $\Gamma(1 - a_j + s)$, $j = 1, 2, \ldots, n$. Then the integral converges for all z if $q > p \geq 0$; it also converges for $q = p > 0$ as long as $|z| < 1$. In the latter case, the integral additionally converges for $|z| = 1$ if $\mathbf{R}(v) < 1$, where v is defined as for the first path. Also, L is a loop beginning and ending at $-\infty$ and encircling all poles of $\Gamma(1 - a_j + s)$, $j = 1, 2, \ldots, n$, exactly once in the positive direction, but not encircling any pole of $\Gamma(b_j - s)$, $j = 1, 2, \ldots, m$. Now the integral converges for all z if $p > q \geq 0$; it also converges for $p = q > 0$ as long as $|z| > 1$. As noted for the second path too, in the case of $p = q$ the integral also converges for $|z| = 1$ when $\mathbf{R}(v) < 1$. This integral is of the so-called *Mellin - Barnes type*, and may be viewed as an *inverse Mellin transform.* The definition holds under the following assumptions:

(i) $0 \leq m \leq q$ and $0 \leq n \leq p$, where m, n, p and q are integer numbers.

(ii) $a_k - b_j \neq 1, 2, 3, \ldots$ for $k = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, which implies that no pole of any $\Gamma(b_j - s)$, $j = 1, 2, \ldots, m$, coincides with any pole of any $\Gamma(1 - a_k + s)$, $k = 1, 2, \ldots, n$.

(iii) $z \neq 0$.

Definition 2.1.28. *Beta Function*

Products of Gamma functions can be identified as describing an important class of definite integrals involving powers of sine and cosine functions, and these integrals, in turn, can be further manipulated to evaluate a large number of algebraic definite integrals. These *properties make it useful to define the* beta function*, defined as*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.24}$$

2.2 Review of the Literature

2.2.1 Distribution of the Product of Two Normal Random Variables

The normal distribution is one of the most common in probability theory which arises from the numerous application that it has in many fields in the real world, such as biology, psychology, engineering, physics, and many other fields.

For the past decades, the normal distribution was discovered by Lane (1809). He determined its probability density function, cumulative distribution function, moments, besides many of its properties. Several distributions are derived from the normal distribution such as *chi-square* and *t-distribution*, also its relation to the *exponential*, the *uniform distribution*.

One of the most important theorem about the normal distribution is that the sum of two normal RV's remains normal, as mentioned earlier in Theorem 2.1.1. The product of them was not distinguished like the sum but there are some researches in the literature. Let X_1, X_2 be two normal RV's with means μ_1, μ_2, and standard deviations σ_1, σ_2, respectively, ρ the coefficient of correlation, and the inverse of the variance coefficient $\rho_1 = \mu_1/\sigma_1$, $\rho_2 = \mu_2/\sigma_2$. Graig (1936) found the moment generating function first when the random variables are independent ($\rho = 0$). Then he derived the distribution function of $Z = X_1X_2/\sigma_1\sigma_2$ in a closed form as a difference between two integrals. Let $W := X_1X_2$, the function will be in terms of X, W. Graig (1936) considered two cases for $X(X > 0, X < 0)$, so the cumulative distribution function becomes

$$F_W(w) = \frac{\exp\left\{-\frac{\mu_1^2}{2\sigma_1^2} - \frac{\mu_2^2}{2\sigma_2^2}\right\}}{2\pi\sigma_1\sigma_2}\left(\int_0^{\infty} \phi(w,x_1)\frac{dx_1}{x_1} - \int_{-\infty}^{0} \phi(w,x_1)\frac{dx_1}{x_1}\right), \tag{2.25}$$

where

$$\phi(w, x_1) = \exp\left\{\frac{-\sigma_2^2 x_1^4 - 2\mu_1\sigma_2^2 x_1^3 - 2\mu_2\sigma_1^2 w x_1 + \sigma_1^2 w^2}{2\sigma_1^2\sigma_2^2 x_1^2}\right\}. \tag{2.26}$$

Substituting $Z = \frac{X_1 X_2}{\sigma_1 \sigma_2}$, ρ_1 and ρ_2, reduces the distribution function

$$F_Z(z) = \frac{\exp\left\{-\frac{\rho_1^2+\rho_2^2}{2}\right\}}{2\pi}[\varphi_1(z) - \varphi_2(z)], \tag{2.27}$$

where

$$\varphi_1(z) = \int_0^\infty \exp\left\{\frac{x_1^2}{2} - \rho_1 x_1 - \rho_2 \frac{z}{x_1} + \frac{z^2}{2x_1^2}\right\}\frac{dx_1}{x_1}, \tag{2.28}$$

and $\varphi_2(z)$ is the integral of the same function over the interval $(-\infty, 0)$.

To evaluate this integral, Graig (1936) expanded it in a Laurent series in powers for all values of x except zero. Also, he found the result in term of the Bessel function of a certain kind. Note that for $\rho_1 = 0$ or $\rho_2 = 0$, the distribution is symmetric about the mean, then decreases at the origin.

Graig (1936) derived the distribution function of Z when ($\rho \neq 0$) using the special case ($\rho = 0$). The expansion that found for $F(z)$ is very slowly convergent for large values of ρ_1 and ρ_2. Graig (1936) introduced some special cases.

(i) For $\rho_1 = \rho_2 = \rho = 0$, we have $F(z) = \frac{1}{\pi}K_0(z)$.

(ii) For $\rho_1 = 1$, $\rho_2 = \rho = 0$, the curve is symmetrical about the origin and the mean.

(iii) For $\rho_1 = \rho_2 = \frac{1}{2}$, $\rho = 0$, he constructed tables for some value of X within a given interval.

Aroian (1947) started from the results of Graig (1936). He showed that when ρ_1 and ρ_2 close to infinity, the probability function of $Z = \frac{X_1 X_2}{\sigma_1 \sigma_2}$ approaches to normal. In the case ($\rho = 0$), the Gram-Charlier Type A series are excellent approximations to the Z distribution, where the exact value of the PDF evaluated by numerical method using some infinite series.

The moment generating function of Z, $M_Z(t)$ is given by

$$M_Z(t) = \frac{\exp\left\{\frac{(\rho_1^2+\rho_2^2-2\rho\rho_1\rho_2)t^2+2\rho_1\rho_2)t}{2(1-(1+\rho)t)(1-(1-\rho)t)}\right\}}{\sqrt{[1-(1+\rho)t][1-(1-\rho)t]}}. \tag{2.29}$$

Let μ_Z and σ_Z be the mean and standard deviation of Z. Then

$$\mu_Z = \rho_1\rho_2 + \rho \ \ and \ \ \sigma_Z = \sqrt{\rho_1^2 + \rho_2^2 + 2\rho\rho_1\rho_2 + 1 + \rho^2}. \tag{2.30}$$

Setting $T_z = \frac{Z-\mu_Z}{\sigma_Z}$ and by using (2.29), the moment generating function of T_z is

$$M_{T_z}(t) = \frac{\exp\left\{\frac{-2\rho w+\rho_1^2+\rho_2^2-2\rho\rho_1\rho_2 w^2+4\rho^2 w^2-2w^3(\rho^2-1)(\rho_1\rho_2+\rho))}{2[1-(1+\rho)w][1-(1-\rho)w]}\right\}}{\sqrt{[1-(1+\rho)w][1-(1-\rho)w]}}, \tag{2.31}$$

where $w = \frac{t}{\sigma_Z}$, assuming $\rho \geq 0$. Furthermore,

$$\lim_{\rho_1,\rho_2\to\infty} M_{T_z}(t) = \exp\left\{\frac{t^2}{2}\right\}. \tag{2.32}$$

According to the theorem of Curtiss by Curtiss (1933) on the moment generating function, the probability function of Z closes to be normal with mean μ_Z and variance σ_Z^2 when ($\rho \geq 0$). The theorem investigates some special cases:

(i) If $-1+\epsilon < \rho < 1$, $\epsilon > 0$, the probability function of Z approaches a normal curve, with mean μ_Z and variance σ_Z^2 when ρ_1 and ρ_2 close to ∞. The same result when ρ_1 and ρ_2 approach -∞.

(ii) If $-1 < \rho < 1-\epsilon$, $\epsilon > 0$, $\rho_1 \to \infty$, $\rho_2 \to -\infty$, the distribution of Z approaches normality with mean μ_Z and variance σ_Z^2.

Remark 2. *In any of the cases above ρ_1 and ρ_2 may be interchanged, also the approach to normality is more rapid when ρ_1 and ρ_2 have the same sign as ρ.*

Aroian et al. (1978) developed a general formula for the product of bivariate RV (X_1, X_2) with parameters $\mu_1, \mu_2, \sigma_1, \sigma_2$, and correlation coefficient ρ.

Assuming $\delta_1 = \frac{\mu_1}{\sigma_1}$, $\delta_2 = \frac{\mu_2}{\sigma_2}$, and $Z = \frac{X_1 X_2}{\sigma_1 \sigma_2}$. Then, the distribution function of Z is

$$F_Z(z) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \xi(z, \delta_1, \delta_2, \rho, t) dt, \tag{2.33}$$

where

$$\begin{aligned}
\phi(z, \delta_x, \delta_y, \rho, t) = t^{-1} G^{-1} &\exp -(H + 4\rho\delta_x\delta_y)t^2 + (1-\rho^2)Ht^4)/2G^2 \\
&\times [(0.5(G+I))^{0.5} sint(y - (\delta_x\delta_y I - \rho H t^2)/G^2) \\
&- (o.5(G-I))^{0.5} cost(y - (\delta_x\delta_y I - \rho H t^2)/G^2], \qquad (2.34) \\
H &= \delta_x^2 + \delta_Y^2 - 2\rho\delta_x\delta_y, \qquad (2.35) \\
I &= 1 + (1-\rho^2)t^2, \qquad (2.36) \\
G^2 &= (1 + (1-p^2)t^2)^2 + 4\rho^2 t^2. \qquad (2.37)
\end{aligned}$$

To evaluate this integral, Romberg integration method is used, which evaluates the integral by combining the Composite Trapezoidal Rule with Richardson Extrapolation. Then performed numerical integral from t_i to t_{i-1}, $i = 1, 2, \ldots$, and $t_0 = 0$. The $t_i's$ selected near the zeros of the function ϕ.

Remark 3. *Note that when using Romberg integration method there is no need to the exact zero's by taking advantage of the Euler's transformation for summing the area below and above the t-axis.*

Now,

$$\int_0^\infty \phi(z, \delta_1, \delta_2, \rho, t) = \sum_{i=1}^\infty R_{i,1} dt, \tag{2.38}$$

where $\sum_{i=1}^\infty R_{i,1}$ is a series equivalent to the Euler's transformation, it's useful when dealing with a series that converges slowly.

Aroian et al. (1978) faced a problem in evaluating (2.38). When t is equal to zero, there is an indeterminate form of the function $\phi(z, \delta_1, \delta_2, \rho, t)$.

Rohatgi (1976) derived the PDF of $Z = X_1 X_2$ by assuming X_1 is a continuous RV with PDF $f(x_1)$, definite and positive in (a, b), with $0 < a < b < \infty$. Let X_2 be a RV with PDF $g(x_2)$, definite and positive in (c, d), with $0 < c < d < \infty$. Then the author considered three cases: when $ad < bc$, $ad = bc$, $ad > bc$. He noted that the product is not defined at zero, the range of normal distribution must be bounded. Also, provided a very good approach for the product of two independent $N(0, 1)$ distribution was given by

$$h_Z(z) = \frac{1}{\pi} K_0(|z|), \qquad z \in \mathbb{R}, \tag{2.39}$$

where $K_0(\cdot)$ denotes the modified Bessel function of the second kind of order zero.

Ware and Lad (2003) derived the MGF of $Y = X_1 X_2$ by assuming that (X_1, X_2) are bivariate independent normal distributions with means μ_1, μ_2, and variances σ_1^2, σ_2^2, respectively.

Define $X_1 = X_0 + Z_1$ and $X_2 = X_0 + Z_2$, where

$$\begin{bmatrix} X_0 \\ Z_1 \\ Z_2 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \rho\sigma_1\sigma_2 & 0 & 0 \\ 0 & \sigma_1^2 - \rho\sigma_1\sigma_2 & 0 \\ 0 & 0 & \sigma_2^2 - \rho\sigma_1\sigma_2 \end{bmatrix} \right).$$

To find the MGF of $Y = X_1 X_2 = (X_0 + Z_1)(X_0 + Z_2)$, we know that

$$\begin{aligned} M_Y(t) &= \int_{-\infty}^{\infty} \exp\{ty\} f_Y(y) dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\{t(x_0 + z_1)(x_0 + z_2)\} f_Y(x_0, z_1, z_2) dx_0 dz_1 dz_2 \\ &= \frac{\exp\left\{ \frac{\frac{1}{2}(\mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 - 2\rho\mu_1\mu_2\sigma_1\sigma_2)t^2 + 2\rho_1\rho_2)t}{2(1-\rho\sigma_1\sigma_2 t)^2 - \sigma_1^2\sigma_2^2 t^2} \right\}}{\sqrt{(1 - \rho\sigma_1\sigma_2 t)^2 - \sigma_1^2\sigma_2^2 t^2}}. \end{aligned} \tag{2.40}$$

This MGF found by Aroian et al. (1978) who calculated $M_Z(t)$, where $Z = \frac{X_1 X_2}{\sigma_1 \sigma_2}$. Note that the result is written only in terms of the ratios ρ_1 and ρ_2, which are proportional to the reciprocals of the coefficient of variation, and ρ (see (2.29)).

To confirm the equivalence between (2.40) and (2.29), observe that

$$M_Z(t) = M_{\frac{X_1 X_2}{\sigma_1 \sigma_2}}(t) = M_Y\left(\frac{t}{\sigma_1 \sigma_2}\right). \tag{2.41}$$

The moment generating function can be used to find moments about the origin of Y. By differentiating $M_Y(t)$ and evaluating at $t = 0$, one can find as many moments as required. These moments can be used to calculate the mean, variance, and skewness of the product of two correlated normal variables. Ware and Lad (2003) found that

$$\mu_Y = \mu_1\mu_2 + \rho\sigma_1\sigma_2, \tag{2.42}$$

$$\sigma_Y^2 = \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + \sigma_1^2\sigma_2^2 + 2\rho\mu_1\mu_2\sigma_1\sigma_2 + \rho^2\sigma_1^2\sigma_2^2. \tag{2.43}$$

For the case of two independent normally distributed variables, the limit distribution of the product is normal. These approaches follow the evolution of the ratio (mean/standard deviation), but some important questions remain under consideration?

(i) When the ratio mean/standard deviation is enough to guarantee the normal approach of the product?

(ii) Approximation to normality is more sensitive for individual ratios or combined ratio.

(iii) How is the development of the skewness of the product, is there a specific level of it?

Macías and Oliveira (2012) assumed that X_1 and X_2 are two normal RV's with parameters μ_1, σ_1^2, $r_1 = \frac{\mu_1}{\sigma_1}$, μ_2, σ_2^2, and $r_2 = \frac{\mu_2}{\sigma_2}$. Then, it was concluded that about the product:

(i) When two variables have unit variance ($\sigma^2 = 1$) with different means, the normal approach is a good option for means greater than 1. But, when the mean is lower, normal approach is not correct.

(ii) When two variables have unit mean ($\mu = 1$) with different variances, the normal approach requires that, at least, one variable has a variance lower than 1.

(iii) When, at least, one of the inverse of the variation coefficient δ_1 or δ_2 is high, then normal approach is correct.

(iv) When two normal distributions have same variance $\sigma_1^2 = \sigma_2^2 = \sigma^2$, we define the combined ratio as $\frac{\mu_1\mu_2}{\sigma}$, then a high value for combined ratio produce a good normal approach to product, but when the combined ratio is lower than 1, the normal approach fails.

Nadarajah and Pogťany (2016) found the PDF of $Z = X_1X_2$, when (X_1, X_2) is a bivariate normal distribution random vector with mean zero, variance one, and correlation coefficient ρ. Then, for $Z \in \mathbb{R}$,

$$f_Z(z) = \frac{1}{\pi\sqrt{1-\rho^2}} \exp\left\{\frac{\rho z}{1-\rho^2}\right\} K_0\left(\frac{|z|}{1-\rho^2}\right). \tag{2.44}$$

Cui et al. (2016) derived the exact PDF of the product $Z = X_1X_2$, assuming X_1 and X_2 are two real Gaussian RV's $X_1 \sim N(\mu_1, \sigma_1)$ and $X_2 \sim N(\mu_2, \sigma_2)$ with correlation coefficient ρ, and the mean for the two variables are not equal zero.

$$\begin{aligned} f_Z(z) = \exp\left\{\frac{-1}{2(1-\rho^2)}\left(\frac{\mu_1^2}{\sigma_1^2} + \frac{\mu_2^2}{\sigma_2^2} - \frac{2\rho(x_1 + \mu_1\mu_2)}{\sigma_1\sigma_2}\right)\right\} \times \sum_{n=0}^{\infty}\sum_{m=0}^{2n} \frac{x_1^{2n-m}|x_1|^{m-n}\sigma_1^{m-n-1}}{\pi(2n)!(1-\rho^2)^{\frac{2n+1}{2}}\sigma_1^{m-n-1}} \\ \left(\frac{\mu_1}{\sigma_1^2} - \frac{\rho\mu_2}{\sigma_1\sigma_2}\right)^m \binom{2n}{m} \times \left(\frac{\mu_1}{\sigma_1^2} - \frac{\rho\mu_2}{\sigma_1\sigma_2}\right)^{2n-m} K_{m-n}\left(\frac{|x_1|}{(1-\rho^2)\sigma_1\sigma_1}\right). \end{aligned} \tag{2.45}$$

Nadarajah (2011) identified the product of two zero-mean correlated normal random variables as a variance-gamma random variable, from which an explicit formula for the probability density function is derived.

The variance-gamma distribution with parameters $r > 0$, $\theta \in \mathbb{R}$, $\sigma > 0$, $\mu \in \mathbb{R}$ has PDF

$$f(x_1) = \frac{1}{\sigma\sqrt{\pi}\Gamma\left(\frac{r}{2}\right)} \exp\left\{\frac{\theta}{\sigma^2}(x_1-\mu)\right\}\left(\frac{|x_1-\mu|}{2\sqrt{\theta^2+\sigma^2}}\right)^{\frac{r-1}{2}} K_{\frac{r-1}{2}}\left(\frac{\sqrt{\theta^2+\sigma^2}}{\sigma^2}|x-\mu|\right), \qquad x_1 \in \mathbb{R}.$$

Here, it was provided an alternative proof of the main results of the work of Nadarajah and Pogány by noting that Z has a variance-gamma distribution.
Let $Z = X_1X_2$. Then, $Z \sim VG(1, \rho\sigma_1\sigma_2, \sigma_1\sigma_2\sqrt{1-\rho^2}, 0)$. Consequently, the PDF of Z is given by

$$f_Z(x_1) = \frac{1}{\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{\rho x_1}{\sigma_1\sigma_2(1-\rho^2)}\right\} K_0\left(\frac{|x_1|}{\sigma_1\sigma_2(1-\rho^2)}\right), \qquad x_1 \in \mathbb{R}. \tag{2.46}$$

2.2.2 Distribution of the Product of Independent Rayleigh Random Variables

Salo et al. (2006) derived the exact probability density function and the distribution function of a product of n independent Rayleigh distribution using the *Meijer G-Function*. Let

$$Y = \prod_{i=1}^{n} X_i,$$

where X_i is a Rayleigh distributed random variable with probability density function

$$f_{X_i}(x) = \frac{x}{\sigma_i^2} \exp\left\{-\frac{x^2}{2\sigma_i^2}\right\}, \qquad x \geq 0, \ \ i = 1, 2, \ldots, n. \tag{2.47}$$

The exact probability density function of Y is given by

$$f_Y(y) = 2(2^n\sigma^2)^{-\frac{1}{2}} G_{0,n}^{n,0}\left((2^n\sigma^2)^{-1}y^2 \,\middle|\, \begin{matrix} - \\ \frac{1}{2}, \ldots, \frac{1}{2} \end{matrix}\right). \tag{2.48}$$

Notice that the density depends only on the parameter $\sigma^2 = \prod_{i=1}^{n} \sigma_i^2$. As the *Meijer G-Function* is implemented only in few mathematical software packages, they provided series forms of the distribution functions for $n = 3, 4, 5$. The series form was derived by evaluating the contour integral in the definition of the *Meijer G-Function* using calculus of residues.

For $n = 3$, the exact PDF of Y is given by

$$f_Y(y) = \sqrt{\frac{x}{8\sigma^2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^3} [ln^2(x) - 2ln(x)A_3(k) + A_3^{(1)}(k) + [A_3(k)]^2] x^k. \tag{2.49}$$

2.2.3 Exact Distribution of the Product of m Gamma and n Pareto Random Variables

Salo et al. (2006) and Nadarajah (2007) derived exact expressions for the product of two independent random variables: one assumed to be gamma and the other Pareto. Nadarajah (2008) provided the extension for the product, $Z = X_1 X_2 \dots X_m Y_1 Y_2 \dots Y_n$, of $m + n$ independent probability density functions specified by

$$f_{X_i}(x) = \frac{\lambda_i^{\beta_i} \exp\{-\lambda_i x\}}{\Gamma(\beta_i)}, \tag{2.50}$$

(for $x > 0, \beta_i > 0$, and $\lambda_i > 0$) and Y_i, $i = 1, \dots, n$ have their PDFs specified by

$$f_{Y_i}(y) = \frac{\mu_i \sigma_i^{\mu_i}}{y^{\mu_i+1}}. \tag{2.51}$$

The probability density function of Z is given by

$$f_Z(z) = \frac{\mu_1 \dots \mu_i}{\Gamma(\beta_1) \dots \Gamma(\beta_m) z} \sum_{k=1}^{n} C_{kn} G^{m,1}_{1,m+1} \left(w \middle| \begin{matrix} 1 - \mu_k \\ \beta_1, \dots, \beta_m, -\mu_k \end{matrix} \right). \tag{2.52}$$

2.2.4 Slash Maxwell Distribution

Gaunt (2018) defined Slash Maxwell (SM) distribution as a variate to a power of an independent uniform random variate.

Slash distribution is defined as $Z = \frac{Y}{U^{1/q}}$, where Y and U are independent and have the standard $N(0, 1)$ and $U(0, 1)$ distribution, respectively. Here, $q > 0$ is the shape parameter which controls the kurtosis of distribution. It is clear from this representation that the Slash distribution is an extension of normal distribution. Indeed, the Slash distribution has heavier tails than the normal distribution. During that study, stochastic representation and some distributional properties such as moments, skewness, and kurtosis measures were provided. The MLE used for estimating the unknown parameters.

2.2.5 Distribution of the Product of Bessel Distribution of Second Kind and Gamma Distribution

Acitas et al. (2020) introduced a new distribution by taking the product of the probability density functions of Bessel distribution of second kind and gamma distribution. Some distributional properties of the proposed distribution outlined. The percentile points for some selected values of parameters are provided. It observed that the new distribution is skewed to the right and carries most of the properties of skewed distributions. They established a characterization result of the proposed distribution by using the truncated moment by considering a product of reverse hazard rate and another.

2.2.6 Distributions Involving Correlated Generalized Gamma Variables

Ahsanullah et al. (2015) derived some of the most important statistical properties of the product and the ratio of two correlated generalized Gamma (GG) RV's. The probability density functions of both the product and the ratio of two correlated GG RV's were obtained in closed form, while the cumulative distribution function of the product was derived in terms of an infinite series. Also capitalizing on the distribution of the product and an union upper bound for the distribution of the sum of two correlated GG RV's are also derived.

Chapter 3

Distribution of Function of Independent Normal Random Variables

3.1 Introduction

In this chapter, a specific distribution in an engineering problem specifically in wireless communications will be studied. Let $Z = X_1^2 + X_2^2 - \theta X_1 X_2$, where θ is unknown scale parameter, so it can be any number greater that zero, X_1 and X_2 are independent and identically distributed (iid) normal random variables with zero mean and variance σ^2. In the sequel, the probability density function of Z will be derived in terms of the Bessel functions. This chapter is organized as follows: Section 3.2 describes the derivation of the PDF of Z. Section 3.3 gives the CDF of Z. Also, shows some special cases of the distribution of Z. In Section 3.4, some statistical properties of the distribution will be derived such as the limit behaviour, mean, variance, moments, moment generating function and order statistics. Section 3.5 provides the estimation of the parameters Z distribution, besides a simulation study.

3.2 The Probability Density Function of Z

Our purpose in this section is to derive the PDF of Z. To do so, let us rewrite the RV Z as $Z = Y_1 + Y_2$ where $Y_1 = X_1^2 + X_2^2$ and $Y_2 = -\theta X_1 X_2$. First, we need to find the JPDF of Y_1 and Y_2, then the PDF of Z is to be obtained.

Let us start with the following transformation:

$$Y_1 + \frac{2Y_2}{\theta} = X_1^2 + X_2^2 - \frac{2\theta X_1 X_2}{\theta} = (X_1 - X_2)^2,$$
$$Y_1 - \frac{2Y_2}{\theta} = X_1^2 + X_2^2 + \frac{2\theta X_1 X_2}{\theta} = (X_1 + X_2)^2.$$

Recall that (see Theorem 2.1.2), $(X_1 \pm X_2) \sim N(0, 2\sigma^2)$. Accordingly, let

$$W_1 = \frac{1}{2\sigma^2}\left(Y_1 + \frac{2Y_2}{\theta}\right),$$
$$W_2 = \frac{1}{2\sigma^2}\left(Y_1 - \frac{2Y_2}{\theta}\right).$$

From Theorem 2.1.4, W_1 and W_2 are iid χ^2 RV's each with 1 degree of freedom. Thus, the JPDF of W_1 and W_2 is given by

$$\begin{aligned} g(w_1, w_2) &= g_{W_1}(w_1) g_{W_2}(w_2) \\ &= \frac{w_1^{\frac{-1}{2}} \exp\left\{\frac{-w_1}{2}\right\}}{\sqrt{2}\Gamma\left(\frac{1}{2}\right)} \frac{w_2^{\frac{-1}{2}} \exp\left\{\frac{-w_2}{2}\right\}}{\sqrt{2}\Gamma\left(\frac{1}{2}\right)}. \\ &= \frac{1}{2\pi} w_1^{\frac{-1}{2}} w_2^{\frac{-1}{2}} \exp\left\{\frac{-1}{2}(w_1 + w_2)\right\}. \end{aligned} \tag{3.1}$$

To find the JPDF of Y_1 and Y_2, transformation technique (see Definition 2.1.15) is used. The Jacobian of Y_1 and Y_2 is given as

$$J = \begin{vmatrix} \frac{dw_1}{dy_1} & \frac{dw_1}{dy_2} \\ \frac{dw_2}{dy_1} & \frac{dw_2}{dy_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sigma^2} & \frac{1}{\theta\sigma^2} \\ \frac{1}{2\sigma^2} & \frac{-1}{\theta\sigma^2} \end{vmatrix} = \frac{-2}{2\theta\sigma^4} = \frac{-1}{\theta\sigma^4}.$$

The JPDF of Y_1 and Y_2 is given as

$$\begin{aligned} f(y_1,y_2) &= \frac{1}{2\pi}\left(\frac{1}{2\sigma^2}\left(y_1+\frac{2y_2}{\theta}\right)\right)^{-\frac{1}{2}}\left(\frac{1}{2\sigma^2}\left(y_1-\frac{2y_2}{\theta}\right)\right)^{-\frac{1}{2}}\exp\left\{-\frac{y_1}{2\sigma^2}\right\}\left|\frac{-1}{\theta\sigma^4}\right| \\ &= \frac{\exp\left\{\frac{-y_1}{2\sigma^2}\right\}}{\sigma^4\pi\theta\sqrt{\frac{-2y_2+\theta y_1}{\sigma^2\theta}}\sqrt{\frac{2y_2+\theta y_1}{\sigma^2\theta}}} \\ &= \frac{\exp\left\{\frac{-y_1}{2\sigma^2}\right\}}{\sigma^2\pi\sqrt{-2y_2+\theta y_1}\sqrt{2y_2+\theta y_1}} \\ &= \frac{\exp\left\{\frac{-y_1}{2\sigma^2}\right\}}{\sigma^2\pi\sqrt{\theta^2y_1^2-4y_2^2}}. \end{aligned} \tag{3.2}$$

Recall that W_1 and W_2 follow a chi-square distribution, so both of them are positive.

$$\begin{aligned} W_1 > 0 &\Rightarrow Y_1+\frac{2Y_2}{\theta} > 0 \Rightarrow Y_1 > \frac{-2Y_2}{\theta} \Rightarrow Y_2 > \frac{-\theta Y_1}{2}, \qquad \textit{and} \\ W_2 > 0 &\Rightarrow Y_1-\frac{2Y_2}{\theta} > 0 \Rightarrow Y_1 > \frac{2Y_2}{\theta} \Rightarrow Y_2 < \frac{\theta Y_1}{2}. \\ & \textit{Hence } \frac{-\theta Y_1}{2} < Y_2 < \frac{\theta Y_1}{2}. \end{aligned}$$

Moreover, $Y_1 = X_1^2 + X_2^2 > 0$.

Finally, the JPDF of Y_1 and Y_2 is given by:

$$f(y_1,y_2) = \frac{\exp\left\{\frac{-y_1}{2\sigma^2}\right\}}{\sigma^2\pi\sqrt{\theta^2y_1^2-4y_2^2}} \qquad \text{for } y_1>0, \frac{-\theta y_1}{2} < y_2 < \frac{\theta y_1}{2}.$$

Since $f(y_1,y_2)$ is a JPDF, it has to be non-negative and the integral over its support is one. In the following, we check these two properties:

(i) It is clear that $f(y_1,y_2)$ is non-negative over its support.

(ii) $\int_{\forall y_1}\int_{\forall y_2} f(y_1,y_2)dy_2dy_1 = 1.$

$$\textit{Hence} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y_1,y_2)dy_2dy_1 = \int_0^{\infty}\int_{\frac{-\theta y_1}{2}}^{\frac{\theta y_1}{2}} \frac{\exp\left\{\frac{-y_1}{2\sigma^2}\right\}}{\sigma^2\pi\sqrt{\theta^2y_1^2-4y_2^2}}dy_2dy_1.$$

First, we evaluate the inner integral, which is the PDF of Y_1

$$\begin{aligned} f_{Y_1}(Y_1) &= \int_{\frac{-\theta y_1}{2}}^{\frac{\theta y_1}{2}} \frac{\exp\left\{\frac{-y_1}{2\sigma^2}\right\}}{\sigma^2\pi\sqrt{\theta^2y_1^2-4y_2^2}}dy_2 \\ &= \frac{\exp\left\{\frac{-y_1}{2\sigma^2}\right\}}{2\sigma^2\pi}\sin^{-1}\left(\frac{2y_2}{\theta y_1}\right)\Bigg|_{\frac{-\theta y_1}{2}}^{\frac{\theta y_1}{2}} \\ &= \frac{\exp\left\{\frac{-y_1}{2\sigma^2}\right\}}{2\sigma^2}, \qquad y_1 > 0. \end{aligned}$$

It's worth to note that the PDF of Y_1 is an exponential distribution with rate $\frac{1}{2\sigma^2}$. Integrating $f_{Y_1}(Y_1)$ overall values of Y_1 will results in 1. Thus, $f(y_1, y_2)$ represents a JPDF.

Before we derive the PDF of Z, it is of interest to investigate the PDF of Y_2. To find the marginal distribution PDF of Y_2 alone, we need to integrate the JPDF $f(y_1, y_2)$ with respect to Y_1. As it is shown in Figure 3.1, the integration must be divided into two cases based on the sign of Y_2. Case 1: For $Y_2 > 0$, let

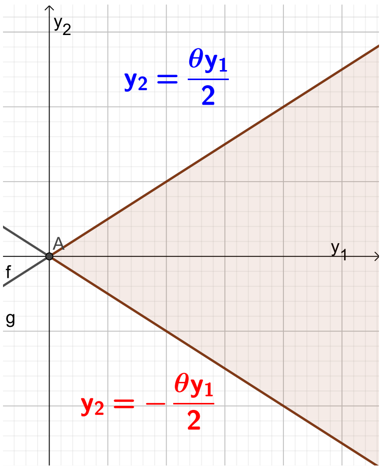


Figure 3.1: Plot of Y_2 vs Y_1

$$
\begin{aligned}
I_1 &:= \int_{\frac{2y_2}{\theta}}^{\infty} f(y_1, y_2) dy_1 \\
&= \int_{\frac{2y_2}{\theta}}^{\infty} \frac{\exp\left\{\frac{-y_1}{2\sigma^2}\right\}}{\sigma^2 \pi \sqrt{\theta^2 y_1^2 - 4y_2^2}} dy_1
\end{aligned}
$$

To evaluate I_1, let $u = \frac{y_1}{y_2}$, then $du = \frac{1}{y_2} dy_1$. When $y_1 = \frac{2y_2}{\theta}, u = \frac{2}{\theta}$, and when $y_1 \to$

$\infty, u \to \infty$. Thus,

$$\begin{aligned}
I_1 &= \frac{1}{\sigma^2\pi}\int_{\frac{2}{\theta}}^{\infty} \frac{\exp\left\{\frac{-uy_2}{2\sigma^2}\right\}}{\sqrt{\theta^2u^2y_2^2 - 4y_2^2}} y_2 du \\
&= \frac{1}{\sigma^2\pi}\int_{\frac{2}{\theta}}^{\infty} \frac{\exp\left\{\frac{-uy_2}{2\sigma^2}\right\}}{|y_2|\sqrt{\theta^2u^2 - 4}} y_2 du \\
&= \frac{1}{\sigma^2\pi}\int_{\frac{2}{\theta}}^{\infty} \frac{\exp\left\{\frac{-uy_2}{2\sigma^2}\right\}}{\sqrt{\theta^2u^2 - 4}} du, \qquad \textit{since}\ \ y_2 > 0.
\end{aligned}$$

For this integral, we use change of variable again. Let $t = u - \frac{2}{\theta}, dt = du$. When $u = \frac{2}{\theta}$, $t = 0$, and when $u \to \infty$, $t \to \infty$. Thus,

$$\begin{aligned}
I_1 &= \frac{1}{\sigma^2\pi}\int_0^{\infty} \frac{\exp\left\{\frac{-(t+\frac{2}{\theta})y_2}{2\sigma^2}\right\}}{\sqrt{\theta^2t^2 + 4\theta t}} dt, \\
&= \frac{1}{\sigma^2\pi\theta} K_0\left(\frac{y_2}{\sigma^2\theta}\right), \qquad (3.3)
\end{aligned}$$

where $K_0(\cdot)$ is the modified Bessel function of the second kind of order zero (see Definition 2.1.24).

Case 2: For $Y_2 < 0$, let $I_2 := \int_{\frac{-2y_2}{\theta}}^{\infty} f(y_1, y_2) dy_1$.

Similarly as in Case 1, we get

$$\begin{aligned}
I_2 &:= \int_{\frac{-2y_2}{\theta}}^{\infty} \frac{\exp\left\{\frac{-y_1}{2\sigma^2}\right\}}{\sigma^2\pi\sqrt{\theta^2y_1^2 - 4y_2^2}} dy_1 \\
&= \frac{1}{\sigma^2\pi\theta} K_0\left(\frac{-y_2}{\sigma^2\theta}\right). \qquad (3.4)
\end{aligned}$$

Thus, from (3.3) and (3.4), the marginal PDF of Y_2 alone is given by

$$f_{Y_2}(y_2) = \frac{1}{\sigma^2\pi\theta} K_0\left(\frac{|y_2|}{\sigma^2\theta}\right), \qquad y_2 \in \mathbb{R}.$$

Recall that $Y_2 = -\theta X_1 X_2$ is the product of two iid RV's multiplied by the scale parameter θ. As discussed in Subsection 2.2.1, the PDF of the product has been studied several times in the literature under some assumptions. It is obvious that our PDF of Y_2 is a general case of Rohatgi's Theorem by Rohatgi (1976) who provides the PDF only when the two independent RV's are $N(0, 1)$.

To find the PDF of $Z = Y_1 + Y_2$, we will solve it in two stages: First, we find the joint density function of Z and Y_1. Second we find the marginal PDF of Z.

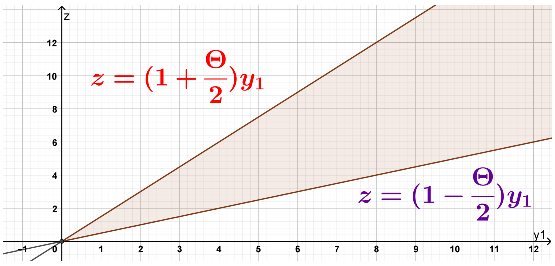
Let Y_1 be fixed at a value $y_1 > 0$. Then $Z = y_1 + Y_2$, and we can consider the onedimensional transformation problem in which $Z = h(Y_2) = y_1 + Y_2$. Letting $g(y_1, z)$ denotes the JPDF of Y_1 and Z, we have, with $Y_2 = Z - y_1 = h^{-1}(z)$, and $|\frac{dh^{-1}}{dz}| = 1$,

$$\begin{aligned} g(y_1, z) &= f(y_1, h^{-1}(z))|\frac{dh^{-1}}{dz}| \\ &= \frac{\exp\left\{\frac{-y_1}{2\sigma^2}\right\}}{\sigma^2\pi\sqrt{\theta^2 y_1^2 - 4(z-y_1)^2}} \qquad \text{for } y_1 > 0 \text{ and } \frac{-\theta y_1}{2} < z - y_1 < \frac{\theta y_1}{2}. \end{aligned}$$

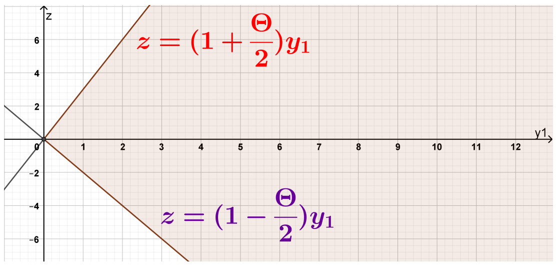
Simplifying, we obtain

$$g(y_1, z) = \frac{\exp\left\{\frac{-y_1}{2\sigma^2}\right\}}{\sigma^2\pi\sqrt{-2(z-y_1)+\theta y_1}\sqrt{2(z-y_1)+\theta y_1}},$$

for $y_1 > 0$ and $(1 - \frac{\theta}{2})y_1 < z < (1 + \frac{\theta}{2})y_1$. Notice that the values of Z depends on θ. Z must produce positive values when $0 < \theta < 2$ as shown in Figure 3.2a, and when $\theta > 2$, Z must produce two values which are positive and negative (see Figure 3.2b). The



(a) When $0 < \theta < 2$



(b) When $\theta > 2$

Figure 3.2: Plot of Z vs Y_1

marginal density of Z is then given by:

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} g(y_1, z) dy_1 \\ &= \begin{cases} \int_{\frac{2z}{2+\theta}}^{\frac{2z}{2-\theta}} g(y_1, z) dy_1 & \text{for } 0 < \theta < 2, \\ \int_{\frac{2z}{2+\theta}}^{\infty} g(y_1, z) dy_1 & \text{for } z > 0 \text{ and } \theta > 2, \\ \int_{\frac{2z}{2-\theta}}^{\infty} g(y_1, z) dy_1 & \text{for } z < 0 \text{ and } \theta > 2. \end{cases} \end{aligned}$$

Case 1: For $0 < \theta < 2$, let

$$
\begin{aligned}
I_1 &= \int_{\frac{2z}{2+\theta}}^{\frac{2z}{2-\theta}} g(y_1, z) dy_1 \\
&= \int_{\frac{2z}{2+\theta}}^{\frac{2z}{2-\theta}} \frac{\exp\left\{\frac{-y_1}{2\sigma^2}\right\}}{\sigma^2\pi\sqrt{-2(z-y_1)+\theta y_1}\sqrt{2(z-y_1)+\theta y_1}} dy_1 \\
&= \int_{\frac{2z}{2+\theta}}^{\frac{2z}{2-\theta}} \frac{\exp\left\{\frac{-y_1}{2\sigma^2}\right\}}{\sigma^2\pi\sqrt{-2z+(\theta+2)y_1}\sqrt{2z+(\theta-2)y_1}} dy_1.
\end{aligned}
$$

Sometimes rather than making a substitution of the form $u =$ function of y_1, we may try a change of variable of the form y_1 as a function of some other variable such as t, and write $dy_1 = y_1'(t)dt$, where $y_1'(t) =$ derivative of y_1 with respect to t.
Here we consider $y_1 = \frac{2z}{\theta+2} + \frac{4\theta z sin^2(t)}{4-\theta^2}$, thus $dy_1 = \frac{8\theta z \sin(t)\cos(t)}{4-\theta^2}dt$. Based on this substitution, we have

$$
-2z + (\theta+2)y_1 = \frac{4\theta z sin^2(t)}{2-\theta},
$$

and

$$
\begin{aligned}
2z + (\theta-2)y_1 &= 2z + (\theta-2)\left(\frac{2z}{(\theta+2)} + \frac{4\theta z \sin^2(t)}{4-\theta^2}\right) \\
&= 2z + (\theta-2)\frac{2z}{\theta+2} - \frac{4\theta z \sin^2(t)}{\theta+2} \\
&= \frac{4\theta z - 4\theta z \sin^2(t)}{\theta+2} \\
&= \frac{4\theta z(1-\sin^2(t))}{\theta+2} \\
&= \frac{4\theta z cos^2(t)}{\theta+2}.
\end{aligned}
$$

To find the limits of the integrand with respect to t, we can write $sin^2(t) = \frac{(2-\theta)(y_1(\theta+2)-2z)}{4\theta z}$. Now, when $y_1 = \frac{2z}{2+\theta}$, $t = 0$, and when $y_1 = \frac{2z}{2-\theta}$, $t = \frac{\pi}{2}$. Hence the original integral becomes

$$\begin{aligned}
I_1 &= \frac{1}{\sigma^2\pi}\int_0^{\frac{\pi}{2}} \frac{\exp\left\{\left(\frac{-1}{2\sigma^2}\right)\left(\frac{2z}{\theta+2}+\frac{4\theta z\sin^2(t)}{4-\theta^2}\right)\right\}}{\sqrt{\frac{4\theta z\sin^2(t)}{2-\theta}}\sqrt{\frac{4z\theta\cos^2(t)}{\theta+2}}}\frac{8\theta z\sin(t)\cos(t)}{4-\theta^2}dt \\
&= \frac{1}{\sigma^2\pi}\exp\left\{\frac{-z}{\sigma^2(\theta+2)}\right\}\int_0^{\frac{\pi}{2}} \frac{\exp\left\{\frac{-2\theta z\sin^2(t)}{\sigma^2(4-\theta^2)}\right\}}{\left(\frac{4\theta z}{\sqrt{4-\theta^2}}\sin(t)\cos(t)\right)}\left(\frac{8\theta z\sin(t)\cos(t)}{4-\theta^2}\right)dt \\
&= \frac{2}{\sigma^2\pi\sqrt{4-\theta^2}}\exp\left\{\frac{-z}{\sigma^2(\theta+2)}\right\}\int_0^{\frac{\pi}{2}}\exp\left\{\frac{-2\theta z\sin^2(t)}{\sigma^2(4-\theta^2)}\right\}dt \\
&= \frac{2}{\sigma^2\pi\sqrt{4-\theta^2}}\exp\left\{\frac{-z}{\sigma^2(\theta+2)}\right\}\int_0^{\frac{\pi}{2}}\exp\left\{\frac{-2\theta z}{\sigma^2(4-\theta^2)}\left(\frac{1-\cos(2t)}{2}\right)\right\}dt \\
&= \frac{2}{\sigma^2\pi\sqrt{4-\theta^2}}\exp\left\{\frac{-z}{\sigma^2(\theta+2)}\right\}\exp\left\{\frac{-\theta z}{\sigma^2(4-\theta^2)}\right\}\int_0^{\frac{\pi}{2}}\exp\left\{\frac{\theta z\cos(2t)}{\sigma^2(4-\theta^2)}\right\}dt.
\end{aligned}$$

Using (8.431.5) in Gradshteyn et al. (2007) we get

$$\begin{aligned}
I_1 &= \frac{2}{\sigma^2\pi\sqrt{4-\theta^2}}\exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\}\frac{\pi}{2}I_0\left(\frac{\theta z}{\sigma^2(4-\theta^2)}\right) \\
&= \frac{1}{\sigma^2\sqrt{4-\theta^2}}\exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\}I_0\left(\frac{\theta z}{\sigma^2(4-\theta^2)}\right),
\end{aligned}$$

where $I_0(\cdot)$ denotes the modified Bessel function of the first kind of order zero (see Definition 2.1.24). It is worth to observe that $I_0(\cdot)$ is an even function.

Case 2: For $\theta > 2$, we have two integrals to evaluate; $I_2 = \int_{\frac{2z}{2+\theta}}^{\infty} g(y_1, z)dy_1$ for $Z > 0$ and $I_3 = \int_{\frac{2z}{2-\theta}}^{\infty} g(y_1, z)dy_1$ for $Z < 0$. We start with I_2:

$$I_2 = \frac{1}{\sigma^2\pi}\int_{\frac{2z}{2+\theta}}^{\infty}\frac{\exp\left\{\frac{-y_1}{2\sigma^2}\right\}}{\sqrt{(-2z+(\theta+2)y_1)}\sqrt{(2z+(\theta-2)y_1)}}dy_1.$$

Here we consider the substitution: $y_1 = zu$, so $dy_1 = zdu$. After adjusting the limits of the integral, we get

$$
\begin{aligned}
I_2 &= \frac{1}{\sigma^2\pi}\int_{\frac{2}{2+\theta}}^{\infty} \frac{\exp\left\{\frac{-zu}{2\sigma^2}\right\}}{\sqrt{-2z+(\theta+2)zu}\sqrt{2z+(\theta-2)zu}} zdu \\
&= \frac{1}{\sigma^2\pi}\int_{\frac{2}{2+\theta}}^{\infty} \frac{\exp\left\{\frac{-zu}{2\sigma^2}\right\}}{\sqrt{(-2z+z\theta u+2zu)(2z-2zu+\theta zu)}} zdu \\
&= \frac{1}{\sigma^2\pi}\int_{\frac{2}{2+\theta}}^{\infty} \frac{\exp\left\{\frac{-zu}{2\sigma^2}\right\}}{\sqrt{z^2(-4+8u+(\theta^2-4)u^2)}} zdu \\
&= \frac{1}{\sigma^2\pi}\int_{\frac{2}{2+\theta}}^{\infty} \frac{\exp\left\{\frac{-zu}{2\sigma^2}\right\}}{|z|\sqrt{-4+8u+(\theta^2-4)u^2}} zdu \\
&= \frac{1}{\sigma^2\pi}\int_{\frac{2}{2+\theta}}^{\infty} \frac{\exp\left\{\frac{-zu}{2\sigma^2}\right\}}{\sqrt{-4+8u+(\theta^2-4)u^2}} du, \qquad \textit{since} \;\; Z>0.
\end{aligned}
$$

The last integral can be computed with the substitution: $u = t + \frac{2}{\theta+2}$, so $du = dt$. After adjusting the limits of the integral, we get:

$$
\begin{aligned}
I_2 &= \frac{1}{\sigma^2\pi}\int_0^{\infty} \frac{\exp\left\{\frac{-z(t+\frac{2}{\theta+2})}{2\sigma^2}\right\}}{\sqrt{(\theta^2-4)t^2+4t\theta}} dt \\
&= \frac{1}{\sigma^2\pi}\int_0^{\infty} \frac{\exp\left\{\frac{-zt}{2\sigma^2}\right\}\exp\left\{\frac{-z}{\sigma^2(\theta+2)}\right\}}{\sqrt{(\theta^2-4)t^2+4t\theta}} dt \\
&= \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}}\exp\left\{\frac{-z(\theta-2)}{\sigma^2(\theta^2-4)}\right\}\int_0^{\infty} \frac{\exp\left\{\frac{-zt}{2\sigma^2}\right\}}{\sqrt{t^2+\frac{4t\theta}{\theta^2-4}}} dt \\
&= \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}}\exp\left\{\frac{2z}{\sigma^2(\theta^2-4)}\right\}\exp\left\{\frac{-\theta z}{\sigma^2(\theta^2-4)}\right\}\int_0^{\infty} \frac{\exp\left\{\frac{-zt}{2\sigma^2}\right\}}{\sqrt{t^2+\frac{4t\theta}{\theta^2-4}}} dt.
\end{aligned}
$$

Employing (8.432.8) in Gradshteyn et al. (2007) we get

$$
I_2 = \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}}\exp\left\{\frac{2z}{\sigma^2(\theta^2-4)}\right\} K_0\left(\frac{\theta z}{\sigma^2(\theta^2-4)}\right).
$$

To find I_3, the same procedure with appropriate substitution must be considered, and the result is

$$
I_3 = \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}}\exp\left\{\frac{2z}{\sigma^2(\theta^2-4)}\right\} K_0\left(\frac{-\theta z}{\sigma^2(\theta^2-4)}\right).
$$

Therefore, the probability density function of Z is given by

$$f_Z(z) = \begin{cases} \frac{1}{\sigma^2\sqrt{4-\theta^2}} \exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\} I_0\left(\frac{\theta z}{\sigma^2(4-\theta^2)}\right) & \text{for } 0 < \theta < 2, \\ \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}} \exp\left\{\frac{2z}{\sigma^2(\theta^2-4)}\right\} K_0\left(\frac{\theta|z|}{\sigma^2(\theta^2-4)}\right) & \text{for } z \in \mathbb{R} \text{ and } \theta > 2. \end{cases} \tag{3.5}$$

Figures 3.3 and 3.4 show the PDF curves of Z for distinct values of θ and σ. It is obvious from Figure 3.3 that for a fixed θ, the PDF curve is more close to the positive x-axis when we increase the value of σ. While the PDF curve at a specific Z is decreased when we increase the value of θ assuming σ constant. From Figure 3.4 notes that the curve is more symmetric about $z = 0$ as θ increases. Moreover, it is obvious that we have a vertical cusp at $z = 0$.

3.3 The Cumulative Distribution Function of Z

The cumulative distribution function is the integration of the probability density function over the interval $(-\infty, z]$; that is,

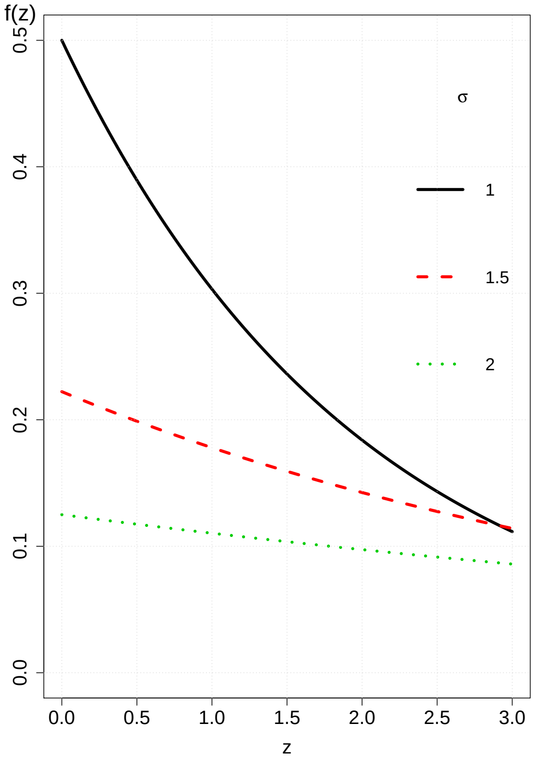
$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{z} f_u(u)du \\ &= \begin{cases} \frac{1}{\sigma^2\sqrt{4-\theta^2}} \int_0^z \exp\left\{\frac{-2u}{\sigma^2(4-\theta^2)}\right\} I_0\left(\frac{\theta u}{\sigma^2(4-\theta^2)}\right) du & \text{for } 0 < \theta < 2, \\ \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}} \int_0^z \exp\left\{\frac{2u}{\sigma^2(\theta^2-4)}\right\} K_0\left(\frac{\theta u}{\sigma^2(\theta^2-4)}\right) du & \text{for } z > 0 \text{ and } \theta > 2, \\ \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}} \int_{-\infty}^0 \exp\left\{\frac{2u}{\sigma^2(\theta^2-4)}\right\} K_0\left(\frac{-\theta u}{\sigma^2(\theta^2-4)}\right) du & \\ +\frac{1}{\sigma^2\pi\sqrt{\theta^2-4}} \int_0^z \exp\left\{\frac{2u}{\sigma^2(\theta^2-4)}\right\} K_0\left(\frac{-\theta u}{\sigma^2(\theta^2-4)}\right) du & \text{for } z < 0 \text{ and } \theta > 2. \end{cases} \end{aligned}$$

Case 1: For $0 < \theta < 2$,

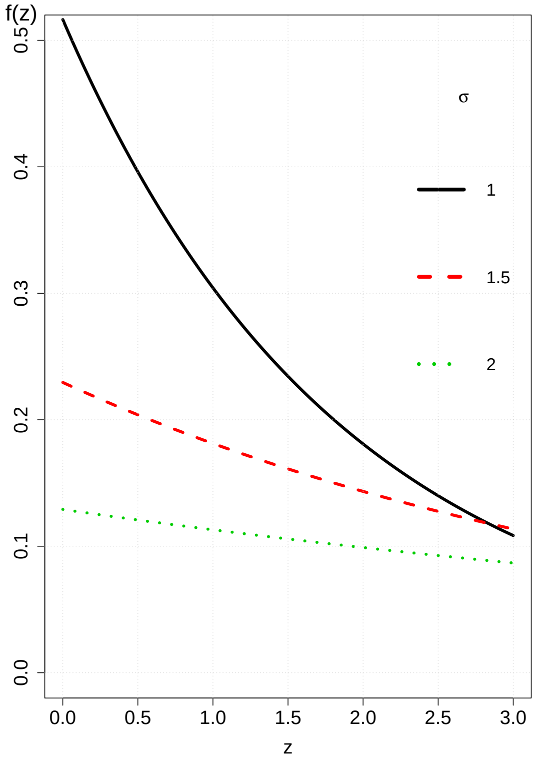
$$F_Z(z) = \frac{1}{\sigma^2\sqrt{4-\theta^2}} \int_0^z \exp\left\{\frac{-2u}{\sigma^2(4-\theta^2)}\right\} I_0\left(\frac{\theta u}{\sigma^2(4-\theta^2)}\right) du. \tag{3.6}$$

From (8.447.1) in Gradshteyn et al. (2007) we have

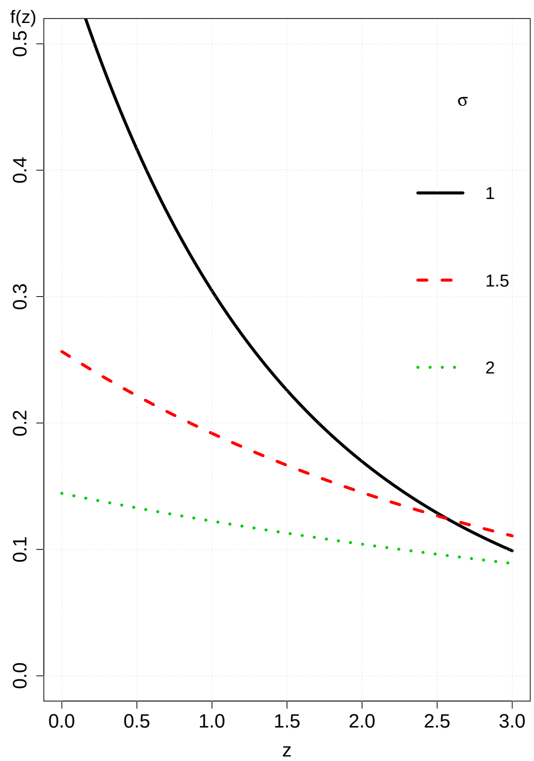
$$I_0\left(\frac{\theta u}{\sigma^2(4-\theta^2)}\right) = \sum_{n=0}^{\infty} \frac{\theta^{2n}u^{2n}}{(\sigma^2)^{2n}(4-\theta^2)^{2n}4^n(n!)^2}. \tag{3.7}$$



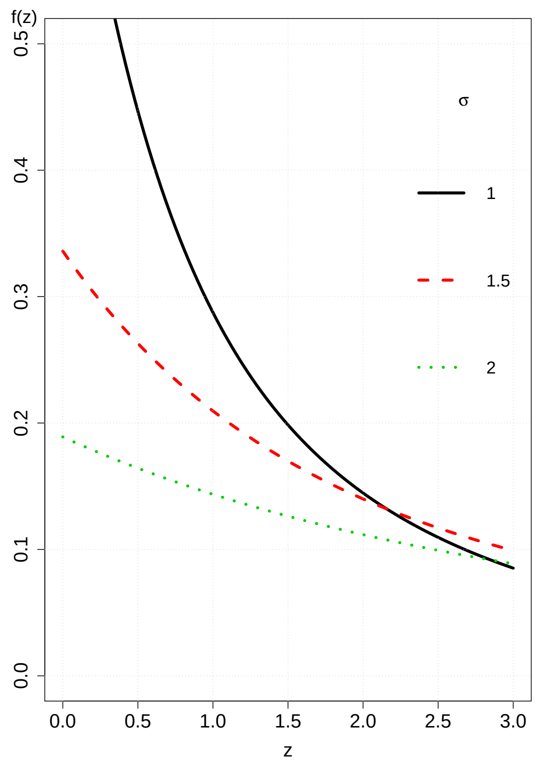
(a) When $\theta = 0$



(b) When $\theta = 0.5$

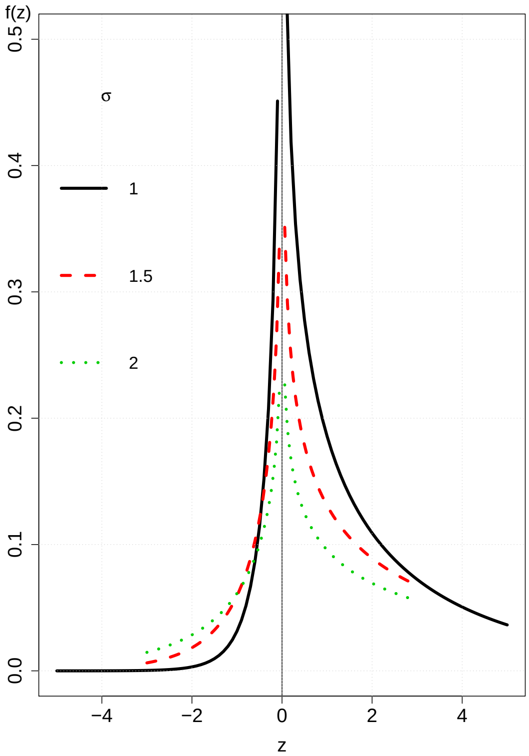


(c) When $\theta = 1$

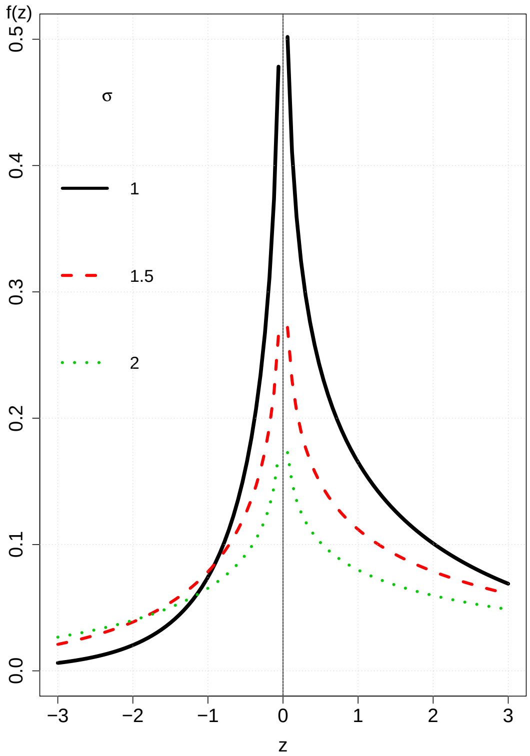


(d) When $\theta = 1.5$

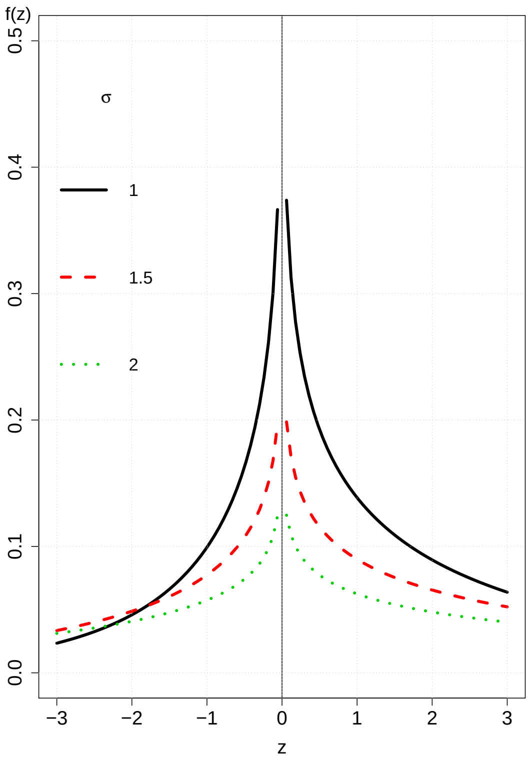
Figure 3.3: Plot of $f_Z(z)$ when $0 \leq \theta < 2$



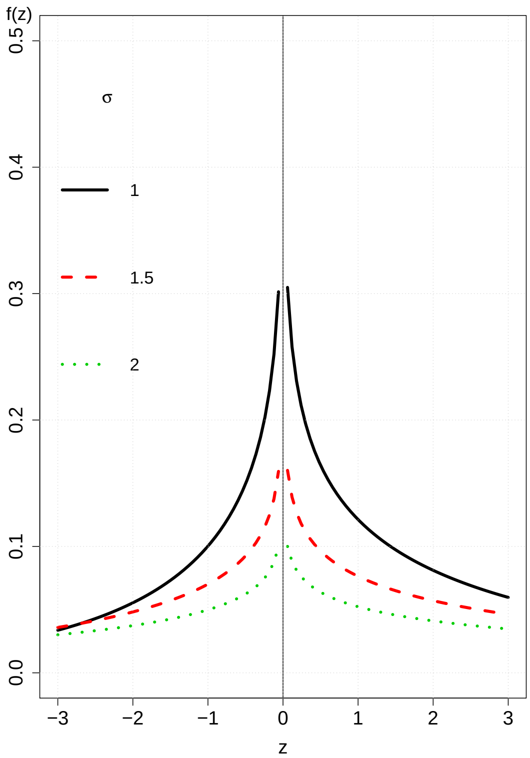
(a) When $\theta = 2.5$



(b) When $\theta = 3$



(c) When $\theta = 4$



(d) When $\theta = 5$

Figure 3.4: Plot of $f_Z(z)$ when $\theta > 2$

Substituting (3.7) in (3.6), we get

$$F_Z(z) = \frac{1}{\sigma^2\sqrt{4-\theta^2}} \int_0^z \exp\left\{\frac{-2u}{\sigma^2(4-\theta^2)}\right\} \sum_{n=0}^{\infty} \frac{\theta^{2n}u^{2n}}{\left(\sigma^2\right)^{2n}(4-\theta^2)^{2n}4^n(n!)^2} du$$
$$= \frac{1}{\sigma^2\sqrt{4-\theta^2}} \sum_{n=0}^{\infty} \frac{\theta^{2n}}{\left(\sigma^2\right)^{2n}(4-\theta^2)^{2n}4^n(n!)^2} \int_0^z \exp\left\{\frac{-2u}{\sigma^2(4-\theta^2)}\right\} u^{2n} du.$$

Consider the substitution $h = \frac{2u}{\sigma^2(4-\theta^2)}$, so $dh = \frac{2}{\sigma^2(4-\theta^2)}du$.

After adjusting the limits of the integral, and employing (3.351.1) in Gradshteyn et al. (2007) we get

$$F_Z(z) = \frac{1}{\sigma^2\sqrt{4-\theta^2}} \sum_{n=0}^{\infty} \frac{\theta^{2n}}{\left(\sigma^2\right)^{2n}(4-\theta^2)^{2n}4^n(n!)^2}$$
$$\times \int_0^{\frac{2z}{\sigma^2(4-\theta^2)}} \exp\{-h\} \left(\frac{h\sigma^2(4-\theta^2)}{2}\right)^{2n} \frac{\sigma^2(4-\theta^2)}{2} dh$$
$$= \sum_{n=0}^{\infty} \frac{\theta^{2n}\sqrt{4-\theta^2}}{2^{4n+1}(n!)^2} \int_0^{\frac{2z}{\sigma^2(4-\theta^2)}} \exp\{-h\} h^{2n} dh$$
$$= \sum_{n=0}^{\infty} \frac{\theta^{2n}\sqrt{4-\theta^2}}{2^{4n+1}(n!)^2} \gamma\left(2n+1, \frac{2z}{\sigma^2(4-\theta^2)}\right),$$

where $\gamma(\cdot,\cdot)$ is the lower incomplete gamma function.

Case 2: For $\theta > 2$,

(i) For $Z < 0$, we have

$$F_Z(z) = \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}} \int_{-\infty}^z \exp\left\{\frac{2u}{\sigma^2(\theta^2-4)}\right\} K_0\left(\frac{-\theta u}{\sigma^2(\theta^2-4)}\right) du. \tag{3.8}$$

From (8.447.3) and (8.447.1) in Gradshteyn et al. (2007) we have

$$K_0(z) = -ln\left(\frac{z}{2}\right) I_0(z) + \sum_{k=0}^{\infty} \frac{z^{2k}}{2^{2k}(k!)^2}\psi(k+1), \tag{3.9}$$

$$I_0(z) = \sum_{r=0}^{\infty} \frac{z^{2r}}{2^{2r}(r!)^2}. \tag{3.10}$$

Substituting (3.9) and (3.10) in (3.8), we get

$$F_Z(z) = \frac{-1}{\sigma^2\pi\sqrt{\theta^2-4}} \int_{-\infty}^z \exp\left\{\frac{2u}{\sigma^2(\theta^2-4)}\right\} ln\left(\frac{-\theta u}{2\sigma^2(\theta^2-4)}\right) \sum_{r=0}^{\infty} \frac{\left(\frac{-\theta u}{2\sigma^2(\theta^2-4)}\right)^{2r}}{(r!)^2} du$$
$$+ \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}} \int_{-\infty}^z \exp\left\{\frac{2u}{\sigma^2(\theta^2-4)}\right\} \sum_{k=0}^{\infty} \frac{\left(\frac{-\theta u}{2\sigma^2(\theta^2-4)}\right)^{2k}}{2^{2k}(k!)^2}\psi(k+1) du.$$

Changing the order of integration and summation, we get

$$
\begin{aligned}
F_Z(z) &= \frac{-1}{\sigma^2\pi\sqrt{\theta^2-4}} \sum_{r=0}^{\infty} \frac{\left(\frac{-\theta}{2\sigma^2(\theta^2-4)}\right)^{2r}}{(r!)^2} \int_{-\infty}^{z} u^{2r} \exp\left\{\frac{2u}{\sigma^2(\theta^2-4)}\right\} ln\left(\frac{-\theta u}{2\sigma^2(\theta^2-4)}\right) du \\
&+ \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}} \sum_{k=0}^{\infty} \frac{\left(\frac{-\theta}{2\sigma^2(\theta^2-4)}\right)^{2k}}{2^{2k}(k!)^2} \Psi(k+1) \int_{-\infty}^{z} u^{2k} \exp\left\{\frac{2u}{\sigma^2(\theta^2-4)}\right\} du \\
&= \sum_{r=0}^{\infty} -\frac{\sqrt{\theta^2-4}2^{-2r-1}\sigma^{4r}\left(2-\frac{\theta^2}{2}\right)^{2r}\left(-\frac{\theta}{\sigma^2(\theta^2-4)}\right)^{2r}}{\pi(r!)^2} G_{2,3}^{3,0}\left(-\frac{2u}{\sigma^2(\theta^2-4)} \middle| \begin{matrix} 1,1 \\ 0,0,2r+1 \end{matrix}\right) \\
&+ \sum_{r=0}^{\infty} -\frac{\sqrt{\theta^2-4}2^{-2r-1}\sigma^{4r}\left(2-\frac{\theta^2}{2}\right)^{2r}\left(-\frac{\theta}{\sigma^2(\theta^2-4)}\right)^{2r}}{\pi(r!)^2} \left(2\log(\sigma)-\log\left(-\frac{2\sigma^4(\theta^2-4)}{\theta u}\right)\right) \\
&\times \Gamma\left(2r+1,-\frac{2u}{\sigma^2(\theta^2-4)}\right) \\
&+ \sum_{k=0}^{\infty} -\frac{2^{-2k}u^{2k+1}\psi(k+1)\left(-\frac{\theta}{\sigma^2(\theta^2-4)}\right)^{2k} E_{-2k}\left(-\frac{2u}{\sigma^2(\theta^2-4)}\right)}{\pi\sigma^2\sqrt{\theta^2-4}(k!)^2},
\end{aligned}
$$

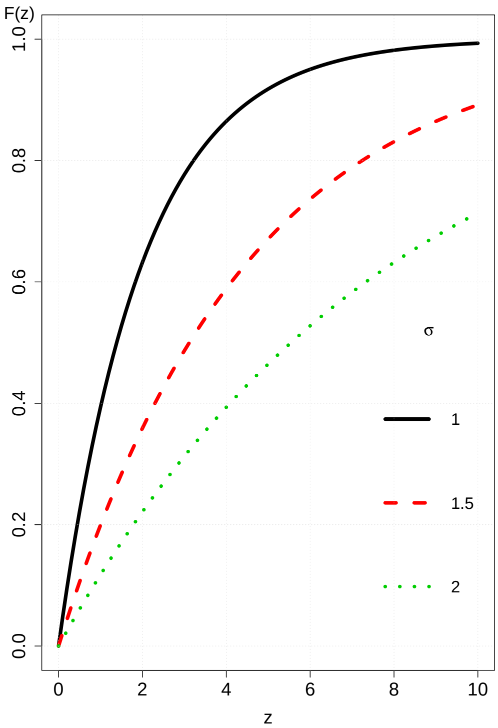
where $E_n(\cdot)$ is the exponential integral, $\psi(\cdot)$ is the digamma function, and $G_{p,q}^{m,n}(\cdot|\cdot)$ is the Meijer G-function.

(ii) Similarly, for $Z > 0$,

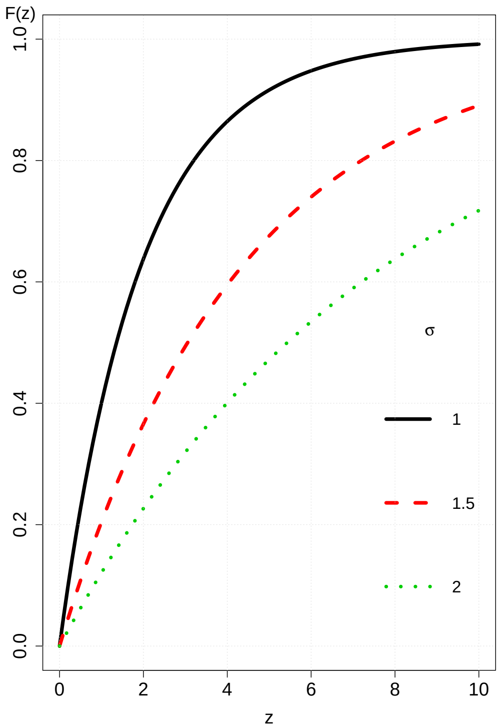
$$
\begin{aligned}
F_Z(z) = \sum_{r=0}^{\infty} \Bigg[& \frac{-1}{\pi(r!)^2} 2^{-4(r-1)}\sigma^{4r}\left(\theta^2-4\right)^{2r-\frac{1}{2}}(-u)^{-2r}u^{2r}\left(\frac{\theta}{\sigma^2(\theta^2-4)}\right)^{2r} \\
& \times \left(\theta^2-4\right) G_{2,3}^{3,0}\left(-\frac{2u}{\sigma^2(\theta^2-4)} \middle| \begin{matrix} 1,1 \\ 0,0,2r+1 \end{matrix}\right)\Bigg] + \sum_{r=0}^{\infty}\Bigg[\frac{-1}{\pi(r!)^2} 2^{-4r-1}\sigma^{4r} \\
& \times \left(\theta^2-4\right)^{2r-\frac{1}{2}}(-u)^{-2r}u^{2r}\left(\theta^2-4\right)\log\left(\frac{\theta u}{2\sigma^2(\theta^2-4)}\right) \\
& \times \Gamma\left(2r+1,-\frac{2u}{\sigma^2(\theta^2-4)}\right) - 2\left(\theta^2 r\Gamma(2r) - 2\Gamma(2r+1)\right) \\
& \times \left(\psi^{(0)}(2r+1) - \log\left(-\frac{4u}{\theta}\right) + \log(u)\right)\Bigg] \\
& + \sum_{k=0}^{\infty}\Bigg[\frac{2^{-4k-1}\psi^{(0)}(K+1)\left(-\frac{1}{\sigma^2(\theta^2-4)}\right)^{-2k-1}\left(\frac{\theta}{\sigma^2(\theta^2-4)}\right)^{2k}}{\pi\sigma^2\sqrt{\theta^2-4}(k!)^2} \\
& \times \left(\Gamma(2k+1) - \Gamma\left(2k+1,-\frac{2u}{\sigma^2(\theta^2-4)}\right)\right)\Bigg].
\end{aligned}
$$

The CDF curves of Z for distinct value of θ and σ^2 are shown in Figures 3.5 and 3.6.

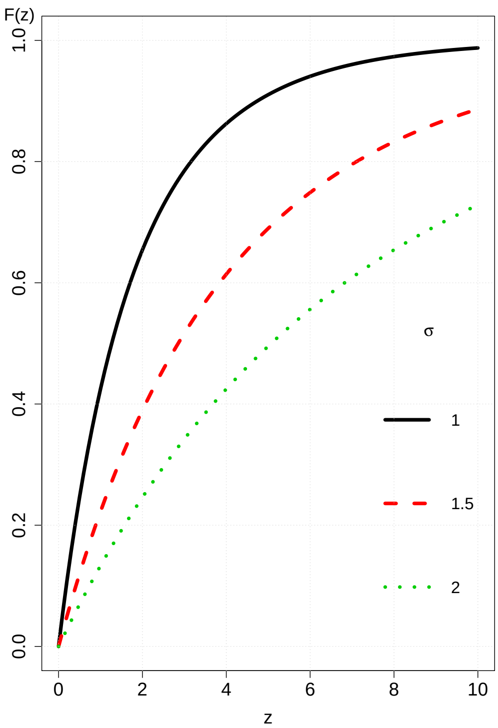
It is worth to observe that when $\theta = 0$, the Z distribution in (1.1) reduces to the



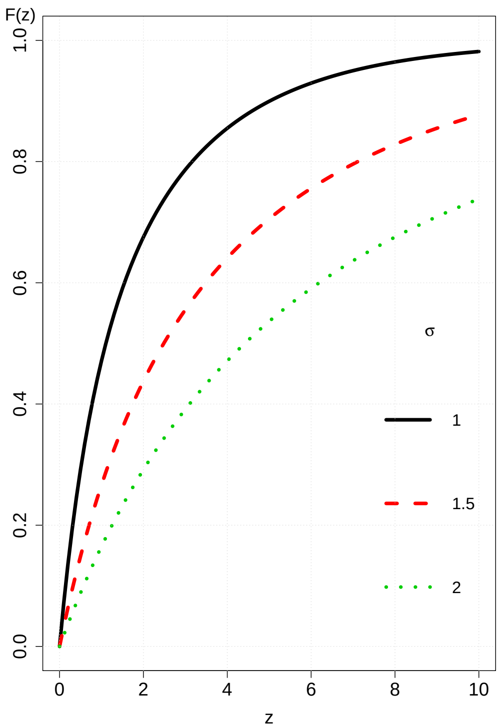
(a) When $\theta = 0$



(b) When $\theta = 0.5$

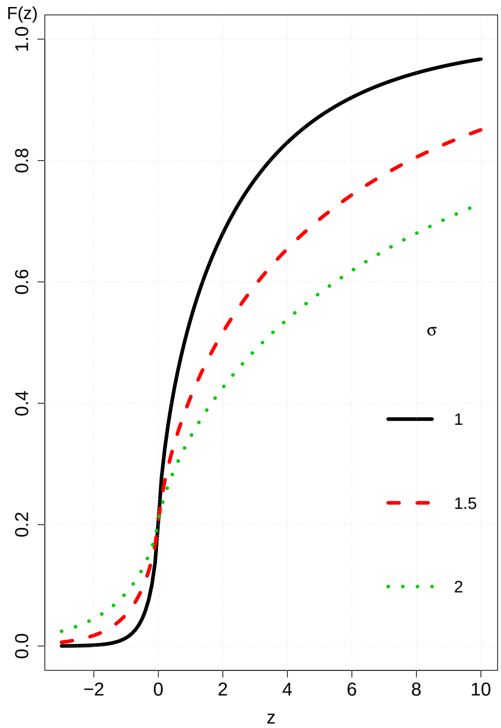


(c) When $\theta = 1$

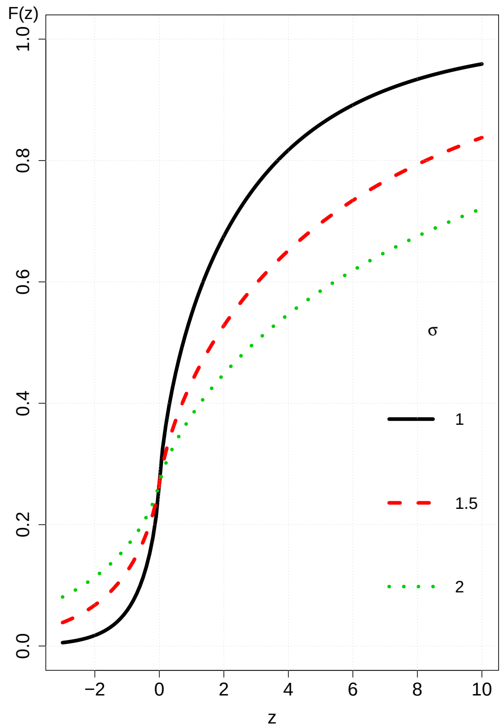


(d) When $\theta = 1.5$

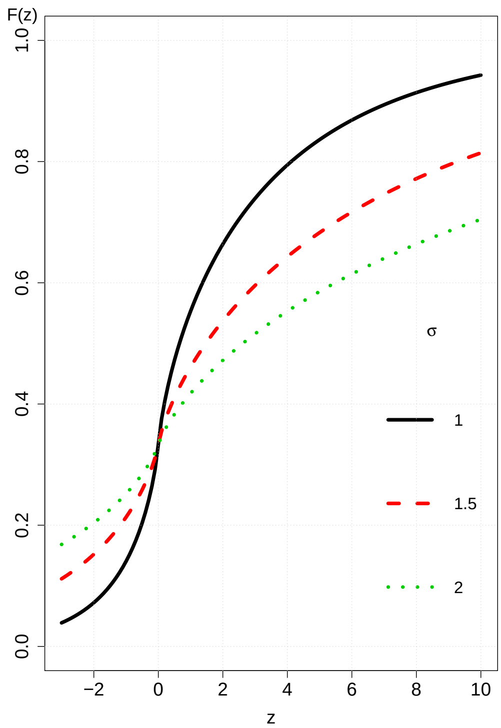
Figure 3.5: Plot of $F_Z(z)$ when $0 \leq \theta < 2$



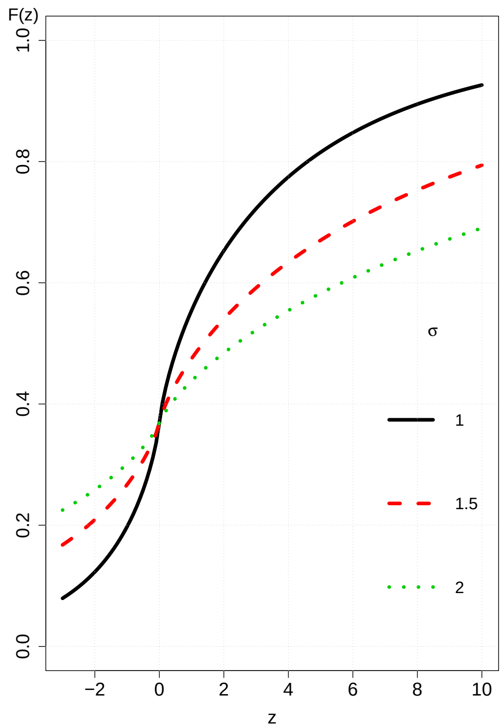
(a) When $\theta = 2.5$



(b) When $\theta = 3$



(c) When $\theta = 4$



(d) When $\theta = 5$

Figure 3.6: Plot of $F_Z(z)$ when $\theta > 2$

exponential distribution with rate parameter $\lambda = \frac{1}{2\sigma^2}$, and when $\theta = \sigma^2 = 1$, the Z distribution reduces to the following density function:

$$f_Z(z) = \frac{1}{\sqrt{3}} \exp\left\{\frac{-2z}{3}\right\} I_0\left(\frac{z}{3}\right).$$

Finally, when $\theta = 2$, the Z distribution in (1.1) reduces to the chi-square distribution with 1 degree of freedom. This result coincides with Theorem 2.1.4.

$$F_Z(z) = \frac{-1}{\sigma^2\pi\sqrt{\theta^2-4}} \int_{-\infty}^{z} \exp\left\{\frac{2u}{\sigma^2(\theta^2-4)}\right\} ln\left(\frac{-\theta u}{2\sigma^2(\theta^2-4)}\right) \sum_{r=0}^{\infty} \frac{\left(\frac{-\theta u}{2\sigma^2(\theta^2-4)}\right)^{2r}}{(r!)^2} du$$
$$+ \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}} \int_{-\infty}^{z} \exp\left\{\frac{2u}{\sigma^2(\theta^2-4)}\right\} \sum_{k=0}^{\infty} \frac{\left(\frac{-\theta u}{2\sigma^2(\theta^2-4)}\right)^{2k}}{2^{2k}(k!)^2} \psi(k+1) du.$$

Changing the order of integration and summation, we get

$$F_Z(z) = \frac{-1}{\sigma^2\pi\sqrt{\theta^2-4}} \sum_{r=0}^{\infty} \frac{\left(\frac{-\theta}{2\sigma^2(\theta^2-4)}\right)^{2r}}{(r!)^2} \int_{-\infty}^{z} u^{2r} \exp\left\{\frac{2u}{\sigma^2(\theta^2-4)}\right\} ln\left(\frac{-\theta u}{2\sigma^2(\theta^2-4)}\right) du$$
$$+ \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}} \sum_{k=0}^{\infty} \frac{\left(\frac{-\theta}{2\sigma^2(\theta^2-4)}\right)^{2k}}{2^{2k}(k!)^2} \Psi(k+1) \int_{-\infty}^{z} u^{2k} \exp\left\{\frac{2u}{\sigma^2(\theta^2-4)}\right\} du$$
$$= \sum_{r=0}^{\infty} -\frac{\sqrt{\theta^2-4}2^{-2r-1}\sigma^{4r}\left(2-\frac{\theta^2}{2}\right)^{2r}\left(-\frac{\theta}{\sigma^2(\theta^2-4)}\right)^{2r}}{\pi(r!)^2} G_{2,3}^{3,0}\left(-\frac{2u}{\sigma^2\left(\theta^2-4\right)}\middle| \begin{array}{c} 1,1 \\ 0,0,2r+1 \end{array}\right)$$
$$+ \sum_{r=0}^{\infty} -\frac{\sqrt{\theta^2-4}2^{-2r-1}\sigma^{4r}\left(2-\frac{\theta^2}{2}\right)^{2r}\left(-\frac{\theta}{\sigma^2(\theta^2-4)}\right)^{2r}}{\pi(r!)^2} \left(2\log(\sigma) - \log\left(-\frac{2\sigma^4\left(\theta^2-4\right)}{\theta u}\right)\right)$$
$$\times \Gamma\left(2r+1, -\frac{2u}{\sigma^2\left(\theta^2-4\right)}\right)$$
$$+ \sum_{k=0}^{\infty} -\frac{2^{-2k}u^{2k+1}\psi(k+1)\left(-\frac{\theta}{\sigma^2(\theta^2-4)}\right)^{2k} E_{-2k}\left(-\frac{2u}{\sigma^2(\theta^2-4)}\right)}{\pi\sigma^2\sqrt{\theta^2-4}(k!)^2}.$$

3.4 Some Statistical Properties of Z

In this section, some statistical properties of Z will be derived.

3.4.1 Limit Behaviour

Definition 3.4.1. *(Vertical Cusp)*

A function f has a vertical cusp at $x = a$ *when the one-sided derivatives are both infinite, but one is positive and the other is negative. For example, if*

$$\lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} = +\infty, \text{ and } \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} = -\infty,$$

then the graph of f has a vertical cusp that slopes up on the left side and down on the right side.

As with vertical tangents, vertical cusps can sometimes be detected for a continuous function by examining the limit of the derivative. For example, if

$$\lim_{x \to a^-} f'(x) = -\infty, \text{ and } \lim_{x \to a^+} f'(x) = +\infty,$$

then the graph of f will have a vertical cusp that slopes down on the left side and up on the right side. This corresponds to a vertical asymptote on the graph of the derivative that goes to ∞ on the left and $-\infty$ on the right.
It is clear that the graph of the function when $\theta > 2$ has a vertical cusp (see Definition 3.4.1 above) that slopes up on the left side and down on the right side at $z = 0$, see Figures 3.4 and 3.6.

$$\begin{aligned}
\lim_{z \to 0+} f'(z) = \lim_{z \to 0+} \frac{1}{\pi\sigma^2\sqrt{\theta^2 - 4}} & \left(-\frac{2 \exp\left\{-\frac{2z}{\sigma^2(4-\theta^2)}\right\} K_0\left(\frac{z\theta}{\sigma^2(\theta^2-4)}\right)}{\sigma^2\left(4-\theta^2\right)} \right) \\
& + \frac{1}{\pi\sigma^2\sqrt{\theta^2 - 4}} \left(\frac{\theta \exp\left\{-\frac{2z}{\sigma^2(4-\theta^2)}\right\} K_1\left(-\frac{z\theta}{\sigma^2(\theta^2-4)}\right)}{\sigma^2\left(\theta^2-4\right)} \right) \\
& + \frac{1}{\pi\sigma^2\sqrt{\theta^2 - 4}} \left(-\frac{2 \exp\left\{-\frac{2z}{\sigma^2(4-\theta^2)}\right\} K_0\left(-\frac{z\theta}{\sigma^2(\theta^2-4)}\right)}{\sigma^2\left(4-\theta^2\right)} \right) \\
& + \frac{1}{\pi\sigma^2\sqrt{\theta^2 - 4}} \left(-\frac{\theta \exp\left\{-\frac{2z}{\sigma^2(4-\theta^2)}\right\} K_1\left(\frac{z\theta}{\sigma^2(\theta^2-4)}\right)}{\sigma^2\left(\theta^2-4\right)} \right) \\
& = -\infty.
\end{aligned}$$

Note that, (8.486.18) in Gradshteyn et al. (2007)) is used. Similarly, $\lim_{z \to 0-} f'(z) = \infty$.

3.4.2 Mean

To evaluate the mean or the expectation of Z, we have two cases:

Case 1: For $0 < \theta < 2$,

$$E(Z) = \int_0^{\infty} \frac{1}{\sigma^2\sqrt{4-\theta^2}} z \exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\} I_0\left(\frac{\theta z}{\sigma^2(4-\theta^2)}\right) dz.$$

Using (8.447.1) in Gradshteyn et al. (2007) our integral becomes

$$E(Z) = \int_0^{\infty} \frac{\exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\}}{\sigma^2\sqrt{4-\theta^2}} \sum_{n=0}^{\infty} \frac{\theta^{2n}}{4^n(n!)^2(\sigma^2)^{2n}(4-\theta^2)^{2n}} z^{\{2n+1\}} dz.$$

Switching the order of summation and integration we get

$$E(Z) = \sum_{n=0}^{\infty} \frac{\theta^{2n}}{4^n(n!)^2(\sigma^2)\sqrt{4-\theta^2}(\sigma^2)^{2n}(4-\theta^2)^{2n}} \int_0^{\infty} \exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\} z^{\{2n+1\}} dz.$$

Now, let $u = \frac{2z}{\sigma^2(4-\theta^2)}$. Then $du = \frac{2}{\sigma^2(4-\theta^2)} dz$ and imploying (3.326.2) in Gradshteyn et al. (2007) we get

$$\begin{aligned}
E(Z) &= \sum_{n=0}^{\infty} \frac{\theta^{2n}(\sigma^2)^{2n+1}(4-\theta^2)^{2n}\sigma^2(4-\theta^2)^2}{2^{2n+1}4^n(n!)^2\sigma^2\sqrt{4-\theta^2}(\sigma^2)^{2n}(4-\theta^2)^{2n}} \int_0^{\infty} \exp\{-u\}\, u^{\{2n+1\}} du \\
&= \sum_{n=0}^{\infty} \frac{\theta^{2n}(4-\theta^2)^{\frac{3}{2}}\sigma^2}{2^{2n+1}4^n(n!)^2} \Gamma[2n+2] \\
&= 2\sigma^2,
\end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function.

Case 2: For $\theta > 2$,

(i) For $Z > 0$,

$$E(Z) = \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}} \int_0^{\infty} z \exp\left\{\frac{2z}{\sigma^2(\theta^2-4)}\right\} K_0\left(\frac{\theta z}{\sigma^2(\theta^2-4)}\right) dz.$$

Employing (8.432.8) in Gradshteyn et al. (2007) we get

$$\begin{aligned}
E(Z) &= \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}} \int_0^{\infty} z \exp\left\{\frac{2z}{\sigma^2(\theta^2-4)}\right\} \exp\left\{\frac{-\theta z}{\sigma^2(\theta^2-4)}\right\} \int_0^{\infty} \frac{\exp\left\{\frac{-zt}{2\sigma^2}\right\}}{\sqrt{t^2+\frac{4t\theta}{\theta^2-4}}} dt dz \\
&= \frac{1}{\sigma^2\pi} \int_0^{\infty} \frac{1}{\sqrt{(\theta^2-4)t^2+4t\theta}} \int_0^{\infty} z \exp\left\{\frac{-z(2+t(\theta+2))}{2\sigma^2(\theta+2)}\right\} dz dt.
\end{aligned}$$

Using (3.351.3) in Gradshteyn et al. (2007) the integral becomes

$$\begin{aligned} E(Z) &= \frac{1}{\sigma^2\pi}\int_0^\infty \frac{1}{\sqrt{(\theta^2-4)t^2+4t\theta}}\frac{4\sigma^4(\theta+2)^2}{(2+t(\theta+2))^2}dt \\ &= \frac{4\sigma^2(\theta+2)^2}{\pi}\int_0^\infty \frac{1}{\sqrt{(\theta^2-4)t^2+4t\theta}}\frac{1}{(2+t(\theta+2))^2}dt. \end{aligned}$$

We set $\sqrt{t(\theta+2)} = \sqrt{2}\tan(x)$. Then $dt = \frac{4\sec^2(x)\tan(x)}{(\theta+2)}dx$ and $t^2 = \frac{4\tan^4(x)}{(\theta+2)^2}$, so that by substitution we have

$$\begin{aligned} E(Z) &= \frac{4\sigma^2(\theta+2)^2}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(\theta^2-4)\frac{4\tan^4(x)}{(\theta+2)^2}+\frac{8\theta\tan^2(x)}{\theta+2}}}\frac{1}{(2\sec^2(x))^2}\frac{4\tan(x)\sec^2(x)}{(\theta+2)}dx \\ &= \frac{2\sigma^2(\theta+2)}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\frac{(\theta-2)}{(\theta+2)}\tan^2(x)+\frac{2\theta}{(\theta+2)}}}\frac{1}{\sec^2(x)}dx \\ &= \frac{2\sigma^2(\theta+2)^{\frac{3}{2}}}{\pi}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(\theta-2)\tan^2(x)+2\theta}}\frac{1}{\sec^2(x)}dx. \end{aligned}$$

Now, take the indefinite integral for $0 < x < \frac{\pi}{2}$.

$$\begin{aligned} v &= \int \frac{1}{\sqrt{(\theta-2)\tan^2(x)+2\theta}}\frac{1}{\sec^2(x)}dx \\ &= \int \frac{\cos^2(x)}{\sqrt{(\theta-2)\frac{\sin^2(x)}{\cos^2(x)}+2\theta\frac{\cos^2(x)}{\cos^2(x)}}}dx \\ &= \int \frac{\cos^3(x)}{\sqrt{(\theta-2)\sin^2(x)+2\theta\cos^2(x)}}dx \\ &= \int \frac{\cos^3(x)}{\sqrt{(\theta-2)\sin^2(x)+2\theta(1-\sin^2(x))}}dx \\ &= \int \frac{\cos^3(x)}{\sqrt{\theta\sin^2(x)-2\sin^2(x)+2\theta-2\theta\sin^2(x)}}dx \\ &= \int \frac{\cos^3(x)}{\sqrt{-\sin^2(x)(\theta+2)+2\theta}}dx \\ &= \int \frac{\cos^3(x)}{\sqrt{2\theta-\sin^2(x)(\theta+2)}}dx. \end{aligned}$$

Using the trigonometric substitution $\sqrt{2\theta}\sin(\alpha) = \sqrt{\theta+2}\sin(x)$, $\alpha \in \left(0, \sin^{-1}\left(\sqrt{\frac{\theta+2}{2\theta}}\right)\right)$.

Then $dx = \sqrt{\frac{2\theta}{\theta+2}\frac{\cos(\alpha)}{\cos(x)}}d\alpha$, $\sin^2(\alpha) = \frac{(\theta+2)}{2\theta}\sin^2(x)$. Then, v becomes

$$\begin{aligned} v &= \frac{1}{\sqrt{2\theta}}\int \frac{\cos^3(x)}{\cos(\alpha)}\sqrt{\frac{2\theta}{\theta+2}\frac{\cos(\alpha)}{\cos(x)}}d\alpha \\ &= \frac{1}{\sqrt{\theta+2}}\int \cos^2(x)d\alpha. \end{aligned}$$

Substitute $\cos^2(x) = 1 - \frac{2\theta}{\theta+2}\sin^2(\alpha)$, we get

$$\begin{aligned} v &= \frac{1}{\sqrt{\theta+2}}\int \left(1 - \frac{2\theta}{\theta+2}\sin^2(\alpha)\right) d\alpha \\ &= \frac{1}{\sqrt{\theta+2}}\int 1 - \frac{2\theta}{\theta+2}\left(\frac{1-\cos(2\alpha)}{2}\right) d\alpha \\ &= \frac{1}{\sqrt{\theta+2}}\int \frac{2}{\theta+2} + \frac{\theta\cos(2\alpha)}{\theta+2}d\alpha \\ &= \frac{1}{\sqrt{\theta+2}}\left(\frac{2\alpha}{\theta+2} + \frac{\theta\sin(\alpha)\cos(\alpha)}{\theta+2}\right). \end{aligned}$$

Substitute α in terms of x, we get

$$\begin{aligned} v = &\frac{1}{\sqrt{\theta+2}}\Big[\frac{2}{\theta+2}\sin^{-1}\left(\frac{\sqrt{\theta+2}}{2\theta}\sin(x)\right) \\ &+ \frac{\theta}{\theta+2}\sqrt{\frac{\theta+2}{2\theta}}\sin(x)\frac{\sqrt{2\theta-(\theta+2)\sin^2(x)}}{\sqrt{2\theta}}\Big] \\ &= \frac{2}{(\theta+2)^{\frac{3}{2}}}\sin^{-1}\left(\frac{\sqrt{\theta+2}}{2\theta}\sin(x)\right) \\ &+ \frac{1}{2(\sqrt{\theta+2})^2}\sin(x)\sqrt{2\theta-(\theta+2)\sin(x)}\Bigg|_0^{\frac{\pi}{2}} \\ &= \frac{2}{(\theta+2)^{\frac{3}{2}}}\left(\sqrt{\theta^2-4} + 4\sin^{-1}\left(\sqrt{\frac{1}{2}+\frac{1}{\theta}}\right)\right). \end{aligned}$$

Now, we have

$$\begin{aligned}
E(Z) &= \frac{2\sigma^2(\theta+2)^{\frac{3}{2}}}{\pi}\left(\frac{2}{(\theta+2)^{\frac{3}{2}}}\left(\sqrt{\theta^2-4}+4\sin^{-1}\left(\sqrt{\frac{1}{2}+\frac{1}{\theta}}\right)\right)\right)\\
&= \frac{\sigma^2}{\pi}\left(\sqrt{\theta^2-4}+4\sin^{-1}\left(\sqrt{\frac{1}{2}+\frac{1}{\theta}}\right)\right).
\end{aligned}$$

To simplify, employing (1.624.1), (1.626.2), and (1.623.1) in Gradshteyn et al. (2007), and (1.6.3) in Thomas (2014) we obtain

$$\begin{aligned}
\sin^{-1}\left(\sqrt{\frac{1}{2}+\frac{1}{\theta}}\right) &= \cos^{-1}\left(1-\left(\frac{1}{2}+\frac{1}{\theta}\right)\right)\\
&= \cos^{-1}\left(\sqrt{\frac{1}{2}-\frac{1}{\theta}}\right)\\
&= \frac{1}{2}\cos^{-1}\left(2\left(\frac{1}{2}-\frac{1}{\theta}\right)-1\right)\\
&= \frac{1}{2}\cos^{-1}\left(\frac{-2}{\theta}\right)\\
&= \frac{1}{2}\left(\pi-\cos^{-1}\left(\frac{2}{\theta}\right)\right)\\
&= \frac{1}{2}\left(\pi-\left(\frac{\pi}{2}-\sin^{-1}\left(\frac{2}{\theta}\right)\right)\right)\\
&= \frac{\pi}{4}+\frac{1}{2}\sin^{-1}\left(\frac{2}{\theta}\right).
\end{aligned}$$

The expectation of Z is

$$E(Z) = \frac{\sigma^2}{\pi}\left(\pi+\sqrt{\theta^2-4}+2\sin^{-1}\left(\frac{2}{\theta}\right)\right). \tag{3.11}$$

(ii) For $Z < 0$, along the same lines as above, it can be shown that

$$E(Z) = -\frac{\sigma^2}{\pi}\left(\sqrt{\theta^2-4}-2\cos^{-1}\left(\frac{2}{\theta}\right)\right). \tag{3.12}$$

Therefore, the expectation of Z is given by

$$E(Z) = \begin{cases} 2\sigma^2 & \text{for } 0 < \theta < 2, \\ \frac{\sigma^2}{\pi}\left(\pi + \sqrt{\theta^2 - 4} + 2\sin^{-1}\left(\frac{2}{\theta}\right)\right) & \text{for } z > 0 \text{ and } \theta > 2, \\ -\frac{\sigma^2}{\pi}\left(\sqrt{\theta^2 - 4} - 2\cos^{-1}\left(\frac{2}{\theta}\right)\right) & \text{for } z < 0 \text{ and } \theta > 2. \end{cases} \tag{3.13}$$

By summing (3.11) and (3.12), we get $E(Z) = 2\sigma^2$. It is obvious to note that the expectation for the two cases is $2\sigma^2$.

3.4.3 Variance

To find the second moment of Z the same steps as Subsection 3.4.2 can be used (See the Appendix). We get

$$E(Z^2) = \sigma^4(8 + \theta^2). \tag{3.14}$$

Using (3.13) and (3.14), the variance of our distribution is given by

$$\begin{aligned} var(Z) &= E(Z^2) - (E(Z))^2 \\ &= \sigma^4(8 + \theta^2) - (2\sigma^2)^2 \\ &= \sigma^4(4 + \theta^2). \end{aligned} \tag{3.15}$$

3.4.4 Moments and Related Measures

In this subsection, a closed form formula for the r^{th} moment of Z will be provided, which can be obtained using

$$E(Z^r) = \int_{-\infty}^{\infty} z^r f_Z(z) dz.$$

We need to consider two cases:

Case 1: For $0 < \theta < 2$,

$$
\begin{aligned}
E(Z^r) &= \int_0^\infty z^r f_Z(z)dz \\
&= \int_0^\infty z^r \frac{1}{\sigma^2\sqrt{4-\theta^2}} \exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\} I_0\left(\frac{\theta z}{\sigma^2(4-\theta^2)}\right) dz \\
&= \int_0^\infty z^r \frac{1}{\sigma^2\sqrt{4-\theta^2}} \exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\} \frac{1}{\pi}\int_0^\pi \exp\left\{\frac{\theta z\cos(\alpha)}{\sigma^2(4-\theta^2)}\right\} d\alpha dz \\
&= \frac{1}{\pi\sigma^2\sqrt{4-\theta^2}} \int_0^\infty \int_0^\pi z^r \exp\left\{-\frac{z(2-\theta\cos(\alpha))}{\sigma^2(4-\theta^2)}\right\} d\alpha dz.
\end{aligned}
$$

Employing (3.351.1) in Gradshteyn et al. (2007) our integral becomes

$$
\begin{aligned}
E(Z^r) &= \frac{1}{\pi\sigma^2\sqrt{4-\theta^2}} \int_0^\pi r!\frac{(\sigma^2(4-\theta^2))^{r+1}}{(2-\theta\cos(\alpha))^{r+1}} d\alpha \\
&= \frac{1}{\pi\sigma^2\sqrt{4-\theta^2}} r!\sigma^{2r+2}(4-\theta^2)^{r+1} \int_0^\pi \frac{1}{(2-\theta\cos(\alpha))^{r+1}} d\alpha \\
&= \frac{1}{\pi}\sigma^{2r}\Gamma(r+1)(4-\theta^2)^{r+\frac{1}{2}} \int_0^\pi \frac{1}{(2-\theta\cos(\alpha))^{r+1}} d\alpha.
\end{aligned}
$$

Employing (1.323.1) in Gradshteyn et al. (2007) we get

$$
\begin{aligned}
E(Z^r) &= \frac{1}{\pi}\sigma^{2r}\Gamma(r+1)(4-\theta^2)^{r+\frac{1}{2}} \int_0^\pi \frac{1}{(2-\theta(2\cos^2(\frac{\alpha}{2})-1))^{r+1}} d\alpha \\
&= \frac{1}{\pi}\sigma^{2r}\Gamma(r+1)(4-\theta^2)^{r+\frac{1}{2}} \int_0^\pi \frac{1}{(\theta+2-2\theta\cos^2(\frac{\alpha}{2}))^{r+1}} d\alpha.
\end{aligned}
$$

Using the substitution $u = \frac{\alpha}{2}$, we get $d\alpha = 2du$. The limits of the integral will be $\alpha = 0 \to u = 0$, $\alpha = \pi \to u = \frac{\pi}{2}$.

$$
E(Z^r) = \frac{1}{\pi}\sigma^{2r}\Gamma(r+1)(4-\theta^2)^{r+\frac{1}{2}} \int_0^{\frac{\pi}{2}} \frac{1}{(\theta+2-2\theta\cos^2(u))^{r+1}} 2du.
$$

Employing (3.682) in Gradshteyn et al. (2007) we get

$$
E(Z^r) = \frac{1}{\pi}\sigma^{2r}\Gamma(r+1)(4-\theta^2)^{r+\frac{1}{2}}(\theta+2)^{-r-1} B\left(\frac{1}{2},\frac{1}{2}\right) {}_2F_1\left(\frac{1}{2}, r+1; 1; \frac{2\theta}{\theta+2}\right).
$$

Using (8.384.1) in Gradshteyn et al. (2007) we get

$$E(Z^r) = \sigma^{2r}\Gamma(r+1)(4-\theta^2)^{r+\frac{1}{2}}(\theta+2)^{-r-1}\,{}_2F_1\left(\frac{1}{2}, r+1; 1; \frac{2\theta}{\theta+2}\right), \tag{3.16}$$

where ${}_2F_1\left(\cdot,\cdot;\cdot;\cdot\right)$ is the Gauss hypergeometric function.

It is clear that if we substitue $r = 1, 2$ in (3.16), then we get (3.13) and (3.14).

Case 2: Similarly for $\theta > 2$, we get

$$\begin{aligned}
E(Z^r) =& \frac{(\theta^2-4)\sigma^{2r}\left(2\theta-\frac{8}{\theta}\right)^r}{2\pi\theta^2\sqrt{\theta^2-4}}\Bigg[\theta\Gamma\left(\frac{r+1}{2}\right)^2 {}_2F_1\left(\frac{r+1}{2}, \frac{r+1}{2}; \frac{1}{2}; \frac{4}{\theta^2}\right) \\
&+ 4\Gamma\left(\frac{r}{2}+1\right)^2 {}_2F_1\left(\frac{r}{2}+1, \frac{r}{2}+1; \frac{3}{2}; \frac{4}{\theta^2}\right)\Bigg] + \frac{(\theta^2-4)2^{r-1}\sigma^r\left(\sigma\left(\frac{4}{\theta}-\theta\right)\right)^r}{\pi\theta^2\sqrt{\theta^2-4}(r+1)^2} \\
&\times\Bigg[\theta(r+1)^2\Gamma\left(\frac{r+1}{2}\right)^2 {}_2F_1\left(\frac{r+1}{2}, \frac{r+1}{2}; \frac{1}{2}; \frac{4}{\theta^2}\right) + \Gamma\left(\frac{r}{2}+1\right)^2 \\
&\times\left(\left(-\theta^2+8r+16\right)\,{}_2F_1\left(\frac{r}{2}+1, \frac{r}{2}+1; \frac{1}{2}; \frac{4}{\theta^2}\right)\right) + \Gamma\left(\frac{r}{2}+1\right)^2 \\
&\times\left(\theta^2-4\right)\,{}_2F_1\left(\frac{r}{2}+1, \frac{r}{2}+1; -\frac{1}{2}; \frac{4}{\theta^2}\right)\Bigg].
\end{aligned}$$

In the following, some other statistical properties of the Z distribution will be derived.

The coefficient of variation

$$\gamma = \frac{\sigma}{\mu} = \frac{\sqrt{\sigma^4(4+\theta^2)}}{2\sigma^2} = \frac{\sqrt{4+\theta^2}}{2}. \tag{3.17}$$

The function is differentiable over its entire domain, so the only critical point is where $\frac{d\gamma}{d\theta} = \frac{\theta}{2\sqrt{\theta^2+4}} = 0$, namely $\theta = 0$. The functions values at $\theta = 0$, $\gamma(0) = 1$. Hence the minimum value of γ is 1 at $\theta = 0$.

Skewness

The skewness can be defined as

$$
\begin{aligned}
\gamma_1 &= \frac{\mu_3}{\sigma^3} \\
&= \frac{E[(Z-\mu)^3]}{\sigma^3} \\
&= \frac{E(Z^3) - 3\mu E(Z^2) + 3\mu^2 E(Z) - \mu^3}{\sigma^3} \\
&= \frac{\sigma^6(8+3\theta^2) - 3(2\sigma^2)\sigma^4(8+\theta^2) + 3(2\sigma^2)^2(2\sigma^2) - (2\sigma^2)^3}{(\sigma^2)^{\frac{3}{2}}} \\
&= \frac{\sigma^6(8+3\theta^2) - 6\sigma^6(8+\theta^2) + 24\sigma^6 - 8\sigma^6}{(\sigma^4(4+\theta^2))^{\frac{3}{2}}} \\
&= \frac{12\theta^2+16}{(4+\theta^2)^{\frac{3}{2}}}. \qquad (3.18)
\end{aligned}
$$

The critical points of γ_1 are $\theta = -2, 0, 2$. Recall that we assume $\theta \geq 0$, so -2 is ignored, we need to check the functions values at $\theta = 0, 2$. Critical point values are $\gamma_1(0) = 2$ and $\gamma_1(2) = 2.828$. The function has a minimum value of 2 at $\theta = 0$ and maximum value of 2.828 at $\theta = 2$.

Note that the skewness of our distribution is always positive, so the distribution is right skewed.

Kurtosis

The kurtosis can be obtained as

$$\begin{aligned}
\gamma_2 &= \frac{\mu_4}{\sigma^4} \\
&= \frac{E[(Z-\mu)^4]}{\sigma^4} \\
&= \frac{E(Z^4) - 4\mu E(Z^3) + 6\mu^2 E(Z^2) - 4\mu^3 E(Z) + \mu^4}{(\sigma^2)^2} \\
&= \frac{1}{(\sigma^4(\theta^2+4))^2}\Big[3\sigma^8(128 + 96\theta^2 + 3\theta^4) - 4(2\sigma^2)(6\sigma^6(8+3\theta^2)) \\
&+ 6(2\sigma^2)^2(\sigma^4(8+\theta^2)) - 4(2\sigma^2)^3(2\sigma^2) + (2\sigma^2)^4\Big] \\
&= \frac{9\theta^4 + 168\theta^2 + 144}{(\theta^2+4)^2}.
\end{aligned}$$

The critical points of γ_2 are $\theta = -2, 0, 2$ (same as γ_1). We need to check the functions values at $\theta = 0, 2$. Critical point values are $\gamma_2(0) = 9$ and $\gamma_2(2) = 15$. The function has a minimum value of 9 at $\theta = 0$, and maximum value of 15 at $\theta = 2$.

It is worth to observe that γ, γ_1, γ_2 are free of σ. Figure 3.7 depicts the graphs of γ, γ_1 and γ_2 for different values of θ.

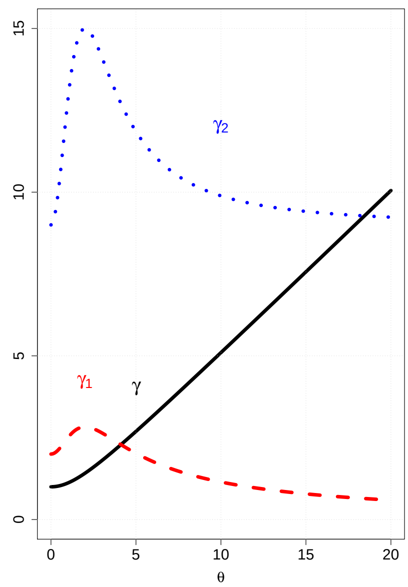


Figure 3.7: The coefficient of variation, skewness and kurtosis of Z as a function of θ

Remark 4. *Note that the kurtosis will be always positive, which indicates that our distribution is peaked and possess thick tails, so it is called a* Leptokurtic distribution. *A leptokurtic distribution has a higher peak and taller (i.e. fatter and heavy) tails than a normal distribution. An extreme positive kurtosis indicates a distribution where more of the values are located in the tails of the distribution rather than around the mean.*

In Table 3.1, the descriptive measures of Z are determined for distinct values of σ and θ. The results show that for the same value of σ, the mean remains fixed (this is obvious and clear from (3.13)), the median decreases, the variance, the skewness, and the kurtosis all are increasing. It is worth to mention that the formulas in (3.13) and (3.15) are conformity with the data given in Table 3.1.

3.4.5 Moment Generating Function

In this subsection, a closed form for the moment generating function (MGF) of Z will be derived.

Case 1: For $0 < \theta < 2$,

$$
\begin{aligned}
M_Z(t) &= E(\exp\{Zt\}) = \int_{-\infty}^{\infty} \exp\{zt\}\, f_Z(z)dz \\
&= \int_{0}^{\infty} \frac{1}{\sigma^2\sqrt{4-\theta^2}} \exp\{zt\} \exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\} I_0\left(\frac{\theta z}{\sigma^2(4-\theta^2)}\right) dz.
\end{aligned}
$$

Using (8.431.5) in Gradshteyn et al. (2007) we get

$$
\begin{aligned}
M_Z(t) &= \frac{1}{\pi\sigma^2\sqrt{4-\theta^2}} \int_{0}^{\infty} \exp\{zt\} \exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\} \int_{0}^{\pi} \exp\left\{\frac{\theta z\cos(\alpha)}{\sigma^2(4-\theta^2)}\right\} d\alpha dz \\
&= \frac{1}{\pi\sigma^2\sqrt{4-\theta^2}} \int_{0}^{\infty} \int_{0}^{\pi} \exp\left\{\frac{zt\sigma^2(4-\theta^2) - 2z + \theta z\cos(\alpha)}{\sigma^2(4-\theta^2)}\right\} d\alpha dz.
\end{aligned}
$$

Table 3.1: Descriptive measures of Z for different values of σ and θ

σ	θ	Mean	Median	Variance	Skewness	Kurtosis
0.5	0.5	0.5	0.343	0.266	2.169	10.329
0.5	1	0.5	0.324	0.313	2.504	12.84
0.5	1.5	0.5	0.284	0.391	2.752	14.530
0.5	3	0.5	0.198	0.813	2.645	14.112
0.5	5	0.5	0.162	1.813	2.023	11.854
1	0.5	2	1.377	4.25	2.169	10.329
1	1	2	1.288	5	2.504	12.84
1	1.5	2	1.125	6.25	2.752	14.530
1	3	2	0.792	13	2.645	14.112
1	5	2	0.677	29	2.023	11.854
1.5	0.5	4.5	3.114	21.516	2.169	10.329
1.5	1	4.5	2.907	25.313	2.504	12.84
1.5	1.5	4.5	2.576	31.641	2.752	14.530
1.5	3	4.5	1.763	65.813	2.645	14.112
1.5	5	4.5	1.494	146.813	2.023	11.854
2.5	0.5	12.5	8.601	166.016	2.169	10.329
2.5	1	12.5	8.144	195.313	2.504	12.84
2.5	1.5	12.5	7.080	244.141	2.752	14.530
2.5	3	12.5	4.976	507.813	2.645	14.112
2.5	5	12.5	4.138	1132.812	2.023	11.8537
3	0.5	18	12.442	344.25	2.169	10.329
3	1	18	11.660	405	2.504	12.84
3	1.5	18	10.228	506.25	2.752	14.530
3	3	18	7.100	1053	2.645	14.112
3	5	18	6.151	2349	2.023	11.854

Using (3.310) in Gradshteyn et al. (2007) we have

$$\begin{aligned} M_Z(t) &= \frac{1}{\pi\sigma^2\sqrt{(4-\theta^2)}}\int_0^{\pi}\frac{\sigma^2(4-\theta^2)}{-t\sigma^2(4-\theta^2)+2-\theta\cos(\alpha)}d\alpha \\ &= \frac{\sqrt{4-\theta^2}}{\pi}\int_0^{\pi}\frac{1}{-t\sigma^2(4-\theta^2)+2-\theta\cos(\alpha)}d\alpha. \end{aligned}$$

Now take the indefinite integral

$$H = \int\frac{1}{-t\sigma^2(4-\theta^2)+2-\theta\cos(\alpha)}d\alpha.$$

We set $u = \tan(\frac{\alpha}{2})$. Then $du = \frac{1}{2}\sec^2(\frac{\alpha}{2})d\alpha$, so that by substitution we have $\sin(\alpha) = \frac{2u}{u^2+1}$, $\cos(\alpha) = \frac{1-u^2}{u^2+1}$ and $d\alpha = \frac{2}{u^2+1}du$.

$$\begin{aligned}
H &= \int \frac{1}{(-t\sigma^2(4-\theta^2)+2-\frac{\theta(1-u^2)}{u^2+1})}\frac{2}{(u^2+1)}du \\
&= \int \frac{2}{\sigma^2\theta^2tu^2+\sigma^2\theta^2t-4\sigma^2tu^2-4\sigma^2t+\theta u^2-\theta+2u^2+2}du \\
&= 2\int \frac{1}{(\sigma^2\theta^2t-4\sigma^2t-\theta+2)\left(\frac{\sigma^2\theta^2tu^2-4\sigma^2tu^2+\theta u^2+2u^2}{\sigma^2\theta^2t-4\sigma^2t-\theta+2}+1\right)}du \\
&= \frac{2}{(\sigma^2\theta^2t-4\sigma^2t-\theta+2)}\int \frac{1}{\left(\frac{u^2(\sigma^2\theta^2t-4\sigma^2t+\theta+2)}{(\sigma^2\theta^2t-4\sigma^2t-\theta+2)}+1\right)}du.
\end{aligned}$$

Now, substitute $s = \frac{u\sqrt{\sigma^2\theta^2t-4\sigma^2t+\theta+2}}{\sqrt{\sigma^2\theta^2t-4\sigma^2t-\theta+2}}$, so $ds = \frac{\sqrt{\sigma^2\theta^2t-4\sigma^2t+\theta+2}}{\sqrt{\sigma^2\theta^2t-4\sigma^2t-\theta+2}}du$.

$$\begin{aligned}
H &= \frac{2}{(\sigma^2\theta^2t-4\sigma^2t-\theta+2)}\int \frac{1}{s^2+1}\frac{\sqrt{\sigma^2\theta^2t-4\sigma^2t-\theta+2}}{\sqrt{\sigma^2\theta^2t-4\sigma^2t+\theta+2}}ds \\
&= \frac{2}{\sqrt{\sigma^2\theta^2t-4\sigma^2t-\theta+2}\sqrt{\sigma^2\theta^2t-4\sigma^2t+\theta+2}}\tan^{-1}(s).
\end{aligned}$$

Now, substitute the values of s,u, so H becomes

$$H = \frac{2}{\sqrt{\theta^2-4}\sqrt{\sigma^4(\theta^2-4)t^2+4\sigma^2t-1}}\tan^{-1}\left(\frac{\tan\left(\frac{\alpha}{2}\right)\sqrt{\sigma^2\theta^2t-4\sigma^2t+\theta+2}}{\sqrt{\sigma^2\theta^2t-4\sigma^2t-\theta+2}}\right).$$

Now, finding the moment generating function by substituting the limits of α from 0 to π produces

$$\begin{aligned}
M_Z(t) &= \frac{\sqrt{4-\theta^2}}{\pi}\frac{2}{\sqrt{\theta^2-4}\sqrt{\sigma^4(\theta^2-4)t^2+4\sigma^2t-1}}\frac{\pi}{2} \\
&= \frac{\sqrt{4-\theta^2}}{\sqrt{\theta^2-4}\sqrt{\sigma^4(\theta^2-4)t^2+4\sigma^2t-1}}, \qquad t \leq \frac{1}{\sigma^2(\theta+2)}.
\end{aligned}$$

Case 2: For $\theta > 2$, similarly as the previous case, we get

$$M_Z(t) = \frac{1}{\sqrt{-\sigma^4(\theta^2-4)t^2-4\sigma^2t+1}}, \qquad \frac{-1}{\sigma^2(-2+\theta)} < t < \frac{1}{\sigma^2(\theta+2)}.$$

3.4.6 Order Statistics

In this subsection, the order statistics of the Z distribution will be derived.

Lemma 3.4.1. *Let $Z_1, Z_2, \ldots, Z_n$ be a random sample (RS) from our distribution and $0 < \theta < 2$. Then we get the following PDF's*

$$f_{Z_{(j)}}(z) = n \begin{pmatrix} n-1 \\ j-1 \end{pmatrix} \frac{1}{\sigma^2\sqrt{4-\theta^2}} \exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\} I_0\left(\frac{\theta z}{\sigma^2(4-\theta^2)}\right)$$
$$\left(\sum_{k=0}^{\infty} \frac{\theta^{2k}\sqrt{4-\theta^2}}{2^{4k+1}(k!)^2}\left(\Gamma(2k+1) - \Gamma\left(2k+1, \frac{2z}{\sigma^2(4-\theta^2)}\right)\right)\right)^{j-1}$$
$$\left(1 - \sum_{k=0}^{\infty} \frac{\theta^{2k}\sqrt{4-\theta^2}}{2^{4k+1}(k!)^2}\left(\Gamma(2k+1) - \Gamma\left(2k+1, \frac{2z}{\sigma^2(4-\theta^2)}\right)\right)\right)^{n-j}.$$

In particular, the 1^{st} o.s PDF is given by

$$f_{Z_{(1)}}(z) = n \frac{1}{\sigma^2\sqrt{4-\theta^2}} \exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\} I_0\left(\frac{\theta z}{\sigma^2(4-\theta^2)}\right)$$
$$\left(1 - \sum_{k=0}^{\infty} \frac{\theta^{2k}\sqrt{4-\theta^2}}{2^{4k+1}(k!)^2}\left(\Gamma(2k+1) - \Gamma\left(2k+1, \frac{2z}{\sigma^2(4-\theta^2)}\right)\right)\right)^{n-1}.$$

and the n^{th} o.s PDF is

$$f_{Z_{(n)}}(z) = n \frac{1}{\sigma^2\sqrt{4-\theta^2}} \exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\} I_0\left(\frac{\theta z}{\sigma^2(4-\theta^2)}\right)$$
$$\left(\sum_{k=0}^{\infty} \frac{\theta^{2k}\sqrt{4-\theta^2}}{2^{4k+1}(k!)^2}\left(\Gamma(2k+1) - \Gamma\left(2k+1, \frac{2z}{\sigma^2(4-\theta^2)}\right)\right)\right)^{n-1}.$$

Remark 5. *Similarly, when $\theta > 2$, it can be easily obtained the PDF's of order statistic.*

3.5 Estimation and Inference

In this section, the estimation of the parameters using MME and MLE will be investigated.

3.5.1 Method of Moments Estimation

In this subsection, the two unknown parameters, σ and θ, will be estimated via the method of moments. In order to use this method we need to define two equations by equating the first two population moments with the sample moments as follows:

$$M_k = \mu_k \ \ for \ \ k = 1, 2.$$

We have the following two cases:

Case 1: For $0 < \theta < 2$,

$$\mu_1 = E(Z) = 2\sigma^2,$$

$$M_1 = \bar{Z} = \frac{1}{n}\sum_{i=1}^{n} z_i.$$

Thus, the MME of σ is the solution of the equation

$$\mu_1 = M_1 \Rightarrow \bar{Z} = 2\sigma^2 \Rightarrow \tilde{\sigma} = \sqrt{\frac{\bar{Z}}{2}}. \tag{3.19}$$

Now, the MME of θ by solving the following equation. Substitute the value of σ using (3.19) we get

$$\mu_2 = M_2 \Rightarrow \sigma^4(8 + \theta^2) = \frac{1}{n}\sum_{i=1}^{n} z_i^2 \Rightarrow \tilde{\theta} = \sqrt{\frac{4n\sum_{i=1}^{n} z_i^2}{(\sum_{i=1}^{n} z_i)^2} - 8}.$$

Remark 6. *This solution is valid when* $\frac{4n\sum z_i^2}{(\sum z_i)^2} > 8$, *otherwise the solution will be outside the domain.*

Remark 7. *Note that the MME's of* θ *and* σ *when* $\theta > 2$ *will be same as the case of* $0 < \theta < 2$.

3.5.2 Maximum Likelihood Estimation

The likelihood function of the parameter vector Θ can be written as

$$L(\Theta; z_1, z_2, \ldots, z_n) = \prod_{i=1}^{n} f(z_i; \theta, \sigma).$$

In the following, we have two cases:

Case 1: For $0 < \theta < 2$,

$$\begin{aligned} L(\Theta; z_1, z_2, \ldots, z_n) &= \prod_{i=1}^{n} f(z_i; \theta, \sigma) \\ &= \prod_{i=1}^{n} \frac{1}{\sigma^2\sqrt{4-\theta^2}} \exp\left\{\frac{-2z_i}{\sigma^2(4-\theta^2)}\right\} I_0\left(\frac{\theta z_i}{\sigma^2(4-\theta^2)}\right) \\ &= \frac{1}{\sigma^{2n}(\sqrt{4-\theta^2})^n} \exp\left\{\frac{-2\sum z_i}{\sigma^2(4-\theta^2)}\right\} \prod_{i=1}^{n} I_0\left(\frac{\theta z_i}{\sigma^2(4-\theta^2)}\right). \end{aligned}$$

Now, set $\sum z_i = n\overline{z}$. The total log-likelihood function for Θ is

$$\begin{aligned} l(\Theta; z_1, z_2, \ldots, z_n) &= \ln\ L(\Theta; z_1, z_2, \ldots, z_n) \\ &= \ln\left(\frac{1}{\sigma^{2n}(\sqrt{4-\theta^2})^n}\right) + \ln\left(\exp\left\{\frac{-2n\overline{z}}{\sigma^2(4-\theta^2)}\right\}\right) \\ &+ \ln\left(\prod_{i=1}^{n} I_0\left(\frac{\theta n\overline{z}}{\sigma^2(4-\theta^2)}\right)\right). \end{aligned}$$

Case 2: For $\theta > 2$, we have two regions.

(i) For $Z > 0$,

$$\begin{aligned} L(\Theta; z_1, z_2, \ldots, z_n) &= \prod_{i=1}^{n} f(z_i; \theta, \sigma) \\ &= \prod_{i=1}^{n} \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}} \exp\left\{\frac{2z_i}{\sigma^2(\theta^2-4)}\right\} K_0\left(\frac{\theta z_i}{\sigma^2(\theta^2-4)}\right) \\ &= \frac{1}{\sigma^{2n}\pi^n(\sqrt{\theta^2-4})^n} \exp\left\{\frac{2\sum z_i}{\sigma^2(\theta^2-4)}\right\} \prod_{i=1}^{n} K_0\left(\frac{\theta z_i}{\sigma^2(\theta^2-4)}\right). \end{aligned}$$

The total log-likelihood function for Θ is

$$\begin{aligned} l(\Theta; z_1, z_2, \ldots, z_n) &= \ln\ L(\Theta; z_1, z_2, \ldots, z_n) \\ &= \ln\left(\frac{1}{\sigma^{2n}\pi^{2n}(\sqrt{\theta^2-4})^n}\right) + \ln\left(\exp\left\{\frac{2n\overline{z}}{\sigma^2(\theta^2-4)}\right\}\right) \\ &+ \ln\left(\prod_{i=1}^{n} K_0\left(\frac{\theta z_i}{\sigma^2(\theta^2-4)}\right)\right) \\ &= -2n\ln(\sigma) - n\ln(\pi) - n\ln(\sqrt{\theta^2-4}) + \frac{2\sum z_i}{\sigma^2(\theta^2-4)} \\ &+ \ln\left(\prod_{i=1}^{n} K_0\left(\frac{\theta z_i}{\sigma^2(\theta^2-4)}\right)\right). \end{aligned}$$

(ii) For $Z < 0$, we proceed similarly to obtain

$$\begin{aligned} l(\Theta; z_1, z_2, \ldots, z_n) &= \ln\ L(\Theta; z_1, z_2, \ldots, z_n) \\ &= -2n\ln(\sigma) - n\ln(\pi) - n\ln(\sqrt{\theta^2-4}) + \frac{2\sum z_i}{\sigma^2(\theta^2-4)} \\ &+ \ln\left(\prod_{i=1}^{n} K_0\left(\frac{-\theta z_i}{\sigma^2(\theta^2-4)}\right)\right). \end{aligned}$$

By summing the log-likelihood function, we get

$$\begin{aligned} l(\Theta; z_1, z_2, \ldots, z_n) &= \ln\ L(\Theta; z_1, z_2, \ldots, z_n) \\ &= -4n\ln(\sigma) - 2n\ln(\pi) - 2n\ln(\sqrt{\theta^2-4}) + \frac{4\sum z_i}{\sigma^2(\theta^2-4)} \\ &+ \ln\left(\prod_{i=1}^{n} K_0\left(\frac{-\theta z_i}{\sigma^2(\theta^2-4)}\right)\right) + \ln\left(\prod_{i=1}^{n} K_0\left(\frac{\theta z_i}{\sigma^2(\theta^2-4)}\right)\right). \end{aligned}$$

The MLEs of σ and θ can be found by differentiating the log-likelihood function for each case with respect to σ and θ and equating these partial derivatives to zero. However, these equations cannot be solved analytically. So statistical software can be used to solve them numerically. Iterative techniques such as Newton-Raphson algorithm can be used to obtain the estimates.

3.5.3 Evaluating the Goodness of an Estimator

Suppose $\hat{\Theta}$ is an estimator used to estimate (σ, θ). Then the bias of $\hat{\Theta}$ is given by

$$Bias(\hat{\Theta}) = E(\hat{\Theta}) - \Theta,$$

and the mean square error of $\hat{\Theta}$ is evaluated as

$$MSE(\hat{\Theta}) = E(\Theta - \hat{\Theta})^2 = Var(\hat{\Theta}) + Bias(\hat{\Theta})^2.$$

The performance of the estimator is evaluated via Monte-Carlo simulation.

Due to the unknown distribution of the derived estimator $\hat{\Theta}$, empirical method is used to construct a confidence interval. An approximate 100(1 - α)% of Θ can be constructed as

$$P\left(-Q_{\frac{\alpha}{2}} \leq \frac{\hat{\Theta} - \Theta}{SE(\Theta)} \leq Q_{\frac{\alpha}{2}}\right) = 1 - \alpha,$$

where Q_α is the α^{th} upper percentile of the empirical distribution of the corresponding estimator, and $SE(\Theta) = \sqrt{MSE(\hat{\Theta})}$ is the empirical standard error. Therefore, the approximate 100(1 - α)% confidence limits for σ and θ of the Z distribution are given, respectively, by

$$P\left(\hat{\sigma} - Q_{\frac{\alpha}{2}} SE(\hat{\sigma}) \leq \sigma \leq \hat{\sigma} + Q_{\frac{\alpha}{2}} SE(\hat{\sigma})\right) = 1 - \alpha,$$
$$P\left(\hat{\theta} - Q_{\frac{\alpha}{2}} SE(\hat{\theta}) \leq \theta \leq \hat{\theta} + Q_{\frac{\alpha}{2}} SE(\hat{\theta})\right) = 1 - \alpha.$$

3.5.4 Simulation Study

In this section, the Monte Carlo simulation will be introduced, and it will be used to evaluate the properties of the MMEs and MLEs. The simulation will be carried out using R software.

Monte Carlo Simulation

Gentle (2010) said that the basis of a Monte Carlo simulation is that the probability of varying outcomes cannot be determined because of random variable interference. Therefore, a Monte Carlo simulation focuses on constantly repeating random samples to achieve certain results. A Monte Carlo simulation takes the variable that has uncertainty and assigns it a random value. The model is then run and a result is provided. This process is repeated again and again while assigning the variable in question with many different values. Once the simulation is completed, the results are averaged together to provide an estimate. A Monte Carlo simulation can be used to treat a range of problems in virtually every field such as finance, engineering, supply chain, and science. It is also referred to as a multiple probability simulation.

As it is shown in Figure 3.8, the CDF curve of our distribution is approximately identical to the Monte Carlo simulation curve, which ascertain our derivations. The Monte Carlo simulation curve graphed by generating a 10000 random sample from the distribution in R, for $\sigma^2 = 1$, and $\theta = (0, 3)$.

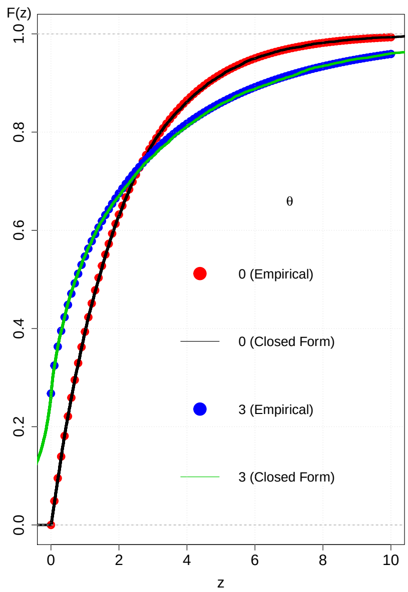


Figure 3.8: Plot of $F_Z(z)$ and $\hat{F}_Z(z)$

MME and MLE Simulation

To evaluate the properties of the MMEs and MLEs, a simulation study will be carried out in this section. Sample of sizes 50, 150, and 300 from Z distribution will be considered. 2000 random samples will be selected for each set up with the following combinations that included the two cases of θ:

$$(\sigma, \theta) = \{(0.5, 1.5), (1.5, 3)\}.$$

This simulation study will be carried out using the R software (R Core Team, 2020).

Table 3.2: Mean of MMEs of all parameters, bias, and MSE, for distinct values of n and distinct setting of (σ, θ)

Sample size	(σ,θ)	(0.5,1.5) σ	(0.5,1.5) θ	(1.5,3) σ	(1.5,3) θ
50	Estimate	0.498	1.236	1.488	2.875
	Bias	-0.002	-0.264	-0.012	-0.125
	MSE	0.002	0.572	0.034	0.592
	C.I	(0.480,0.523)	(1.235,3.182)	(1.280,1.829)	(1.671,6.389)
	Width	0.043	1.947	0.550	4.718
100	Estimate	0.499	1.335	1.492	2.925
	Bias	-0.001	-0.165	-0.008	-0.075
	MSE	0.0009	0.294	0.018	0.288
	C.I	(0.485,0.516)	(1.333,2.608)	(1.326,1.726)	(1.851,5.124)
	Width	0.031	1.275	0.400	3.274
150	Estimate	0.499	1.380	1.495	2.951
	Bias	-0.0007	-0.120	-0.005	-0.049
	MSE	0.0006	0.195	0.012	0.209
	C.I	(0.488,0.513)	(1.166,2.350)	(1.356,1.678)	(1.964,4.753)
	Width	0.026	1.185	0.322	2.789

Table 3.2 and Table 3.3 indicate that, as the sample size increases, the estimate becomes closer to the actual value, the bias decreases, the MSE decreases, and the width of the C.I becomes narrower. Note that $\hat{\sigma}$ is an unbiased estimator for σ, while $\hat{\theta}$ is a biased estimator.

Table 3.3: Mean of MLEs of all parameters, bias, and MSE, for distinct values of n and distinct setting of (σ, θ)

Sample size	(σ, θ)	(0.5,1.5) σ	θ	(1.5,3) σ	θ
50	Estimate	0.498	1.297	1.488	3.058
	Bias	-0.002	-0.104	-0.012	0.058
	MSE	0.002	0.350	0.034	0.223
	C.I	(0.480,0.524)	(1.297,2.450)	(1.280,1.829)	(1.930,5.039)
	Width	0.043	1.154	0.550	3.109
100	Estimate	0.499	1.396	1.492	3.031
	Bias	-0.001	-0.104	-0.008	1.531
	MSE	0.0009	0.156	0.018	2.430
	C.I	(0.485,0.516)	(1.395,2.134)	(1.326,1.726)	(-0.956,8.812)
	Width	0.031	0.739	0.400	9.768
150	Estimate	0.499	1.425	1.495	3.023
	Bias	-0.0007	-0.075	-0.005	0.023
	MSE	0.0006	0.089	0.012	0.059
	C.I	(0.488,0.513)	(1.225, 1.965)	(1.356,1.678)	(2.381,3.899)
	Width	0.026	0.740	0.322	1.518

Chapter 4

Distribution of Function of Dependent Normal Random Variables

4.1 Introduction

In this chapter, another distribution presents an engineering problem specifically in wireless communications under the effect of I/Q imbalance will be studied. Let $U = X_1^2 + X_2^2 - 2\rho X_1 X_2$, where X_1 and X_2 are dependent and identically distributed (iid) normal random variables with zero means, variances σ^2, and correlation coefficient ρ. In the sequel, the probability density function of U will be derived in terms of the Bessel functions. This chapter is organized as follows: Section 4.2 describes the derivation of the PDF and CDF of U. Section 4.3 presents some statistical properties of the distribution derived as the mean, variance, moments, moment generating function. In Section 4.4 the estimation of the parameters U distribution will be derived. Also, Section 4.4 provides a simulation study. An application will be introduced in Section 4.6.

4.2 The Probability Density Function of U

Our purpose in this section is to derive the PDF of U, where $U = Y_1 + Y_2$ and $Y_1 = X_1^2 + X_2^2$ and $Y_2 = -2\rho X_1 X_2$. First, we need to find the JPDF of Y_1 and Y_2, then the PDF of U is to be obtained.

Let (X_1, X_2) be a bivariate normal random variable with zero mean and variance σ^2 with correlation coefficient ρ, $|\rho| \leq 1$. Recall,

$$U = X_1^2 + X_2^2 - 2\rho X_1 X_2, -1 < \rho < 1. \tag{4.1}$$

Let us start with the following transformation:

$$\begin{aligned} Y_1 + \frac{Y_2}{\rho} &= X_1^2 + X_2^2 - \frac{2\rho X_1 X_2}{\rho} = (X_1 - X_2)^2, \\ Y_1 - \frac{Y_2}{\rho} &= X_1^2 + X_2^2 + \frac{2\rho X_1 X_2}{\rho} = (X_1 + X_2)^2. \end{aligned}$$

Note that $E(X_1 + X_2) = E(X_1) + E(X_2) = 0$ and $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$. Recall that

$$\rho = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)Var(X_2)}} \rightarrow Cov(X_1, X_2) = \rho\sigma^2. \tag{4.2}$$

Hence $Var(X_1 + X_2) = 2\sigma^2 + 2\rho\sigma^2 = 2\sigma^2(1 + \rho)$. Note that (see Theorem 2.1.2), $(X_1 \pm X_2) \sim N(0, 2\sigma^2(1 \pm \rho))$. Accordingly, let

$$\begin{aligned} W_1 &:= \frac{1}{2\sigma^2(1-\rho)} \left(Y_1 + \frac{Y_2}{\rho} \right), \\ W_2 &:= \frac{1}{2\sigma^2(1+\rho)} \left(Y_1 - \frac{Y_2}{\rho} \right). \end{aligned}$$

W_1 and W_2 are iid χ^2 RV's each with 1 degree of freedom in the case of bivariate normal random variable with the same variance. Thus, the JPDF of W_1 and W_2 is given by:

$$g(w_1, w_2) = g_{W_1}(w_1) g_{W_2}(w_2) = \frac{1}{2\pi} w_1^{\frac{-1}{2}} w_2^{\frac{-1}{2}} \exp\left\{ \frac{-1}{2}(w_1 + w_2) \right\}. \tag{4.3}$$

To find the JPDF of Y_1 and Y_2, transformation technique (see Definition 2.1.15) is used. Hence

$$f(y_1, y_2) = g(w_1, w_2)|J|,$$

where

$$J = \begin{vmatrix} \frac{dw_1}{dy_1} & \frac{dw_1}{dy_2} \\ \frac{dw_2}{dy_1} & \frac{dw_2}{dy_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sigma^2(1-\rho)} & \frac{1}{2\rho\sigma^2(1-\rho)} \\ \frac{1}{2\sigma^2(1+\rho)} & \frac{-1}{2\rho\sigma^2(1+\rho)} \end{vmatrix} = \frac{-2}{4\rho\sigma^4(1-\rho^2)} = \frac{-1}{2\rho\sigma^4(1-\rho^2)}.$$

Thus, $|J| = \frac{1}{2\rho\sigma^4(1-\rho^2)}$, for $-1 < \rho < 1$. Hence the JPDF of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \frac{1}{2\pi} w_1^{\frac{-1}{2}} w_2^{\frac{-1}{2}} \exp\left\{\frac{-1}{2}(w_1 + w_2)\right\} \frac{1}{2\rho\sigma^4(1-\rho^2)}, \qquad \text{for } -1 < \rho < 1.$$

Substitute w_1 and w_2, we get

$$f(y_1, y_2) = \frac{\exp\left\{\frac{y_1+y_2}{2\sigma^2(\rho^2-1)}\right\}}{2\sigma^2\pi\sqrt{1-\rho^2}\sqrt{\rho^2 y_1^2 - y_2^2}}; \qquad \text{for } -1 < \rho < 1.$$

We have mentioned before that W_1 and W_2 follow a chi-square distribution, so both of them are positive,

$$\begin{aligned} Y_1 &= X_1^2 + X_2^2 > 0, \\ W_1 > 0 \Rightarrow Y_1 &+ \frac{Y_2}{\rho} > 0, \\ W_2 > 0 \Rightarrow Y_1 &- \frac{Y_2}{\rho} > 0. \end{aligned}$$

For $-1 < \rho < 0 \Rightarrow \rho Y_1 < Y_2 < -\rho Y_1$, and for $0 < \rho < 1 \Rightarrow -\rho Y_1 < Y_2 < \rho Y_1$.

$f(y_1, y_2)$ is a JPDF, and has to be non-negative and the integral over its support is one. In the following, we check these two properties:

(i) It is clear that $f(y_1, y_2)$ is non-negative over its support.

(ii) For $-1 < \rho < 0$, $\int_{\forall y_1} \int_{\forall y_2} f(y_1, y_2) dy_2 dy_1 = 1$.

$$\int_0^\infty \int_{\rho y_1}^{-\rho y_1} f(y_1, y_2) dy_2 dy_1 = \int_0^\infty \int_{\rho y_1}^{-\rho y_1} \frac{\exp\left\{\frac{y_1+y_2}{2\sigma^2(\rho^2-1)}\right\}}{2\sigma^2\pi\sqrt{1-\rho^2}\sqrt{\rho^2 y_1^2 - y_2^2}} dy_2 dy_1.$$

As in Chapter 3, it can be shown that

$$\begin{aligned} L &= \int_{\rho y_1}^{-\rho y_1} \frac{\exp\left\{\frac{y_1+y_2}{2\sigma^2(\rho^2-1)}\right\}}{2\sigma^2\pi\sqrt{1-\rho^2}\sqrt{\rho^2 y_1^2 - y_2^2}} dy_2 \\ &= \frac{\exp\left\{\frac{y_1}{2\sigma^2(\rho^2-1)}\right\} I_0\left(\frac{\rho y_1}{2\sigma^2(\rho^2-1)}\right)}{2\sigma^2\sqrt{1-\rho^2}}, \end{aligned} \tag{4.4}$$

and

$$\int_0^\infty \frac{\exp\left\{\frac{y_1}{2\sigma^2(\rho^2-1)}\right\} I_0\left(\frac{\rho y_1}{2\sigma^2(\rho^2-1)}\right)}{2\sigma^2\sqrt{1-\rho^2}} dy_1 = 1.$$

Similarly, for $0 < \rho < 1$,

$$\int_0^\infty \int_{-\rho y_1}^{\rho y_1} f(y_1, y_2) dy_2 dy_1 = \int_0^\infty \int_{-\rho y_1}^{\rho y_1} \frac{\exp\left\{\frac{y_1+y_2}{2\sigma^2(\rho^2-1)}\right\}}{2\sigma^2\pi\sqrt{1-\rho^2}\sqrt{\rho^2 y_1^2 - y_2^2}} dy_2 dy_1.$$

The inner integral

$$\begin{aligned} R &= \int_{-\rho y_1}^{\rho y_1} \frac{\exp\left\{\frac{y_1+y_2}{2\sigma^2(\rho^2-1)}\right\}}{2\sigma^2\pi\sqrt{1-\rho^2}\sqrt{\rho^2 y_1^2 - y_2^2}} dy_2 \\ &= \frac{\exp\left\{\frac{y_1}{2\sigma^2(\rho^2-1)}\right\} I_0\left(\frac{\rho y1}{2\sigma^2(\rho^2-1)}\right)}{2\sigma^2\sqrt{1-\rho^2}}. \end{aligned} \tag{4.5}$$

$$\int_0^\infty \frac{\exp\left\{\frac{y_1}{2\sigma^2(\rho^2-1)}\right\} I_0\left(\frac{\rho y_1}{2\sigma^2(\rho^2-1)}\right)}{2\sigma^2\sqrt{1-\rho^2}} dy_1 = 1.$$

It's worth to note that the inner integrals L (4.4) and R (4.5) are the PDF of Y_1, which are the same. So, for $|\rho| \leq 1$ the PDF of Y_1 is

$$f_{Y_1}(y_1) = \frac{1}{2\sigma^2\sqrt{1-\rho^2}} \exp\left\{\frac{y_1}{2\sigma^2(\rho^2-1)}\right\} I_0\left(\frac{\rho y_1}{2\sigma^2(\rho^2-1)}\right), \qquad 0 < y_1 < \infty.$$

Before we derive the PDF of U, it is of interest to investigate the PDF of Y_2. To find the marginal PDF of Y_2, we need to integrate the JPDF $f(y_1, y_2)$ with respect to Y_1. Case

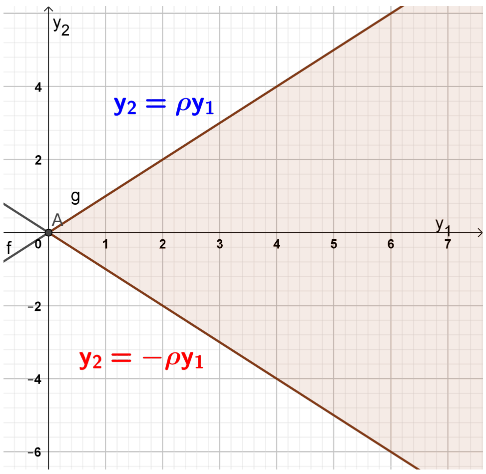


Figure 4.1: Plot of Y_2 vs Y_1

1: For $Y_2 > 0$,

$$f_{Y_2}(y_2) = \frac{1}{2\pi\sigma^2\rho\sqrt{1-\rho^2}} \exp\left\{\frac{y_2}{2\sigma^2(\rho^2-1)}\right\} K_0\left(\frac{y_2}{2\sigma^2(\rho-\rho^3)}\right).$$

Case 2: For $Y_2 < 0$,

$$f_{Y_2}(y_2) = \frac{1}{2\pi\sigma^2\rho\sqrt{1-\rho^2}} \exp\left\{\frac{y_2}{2\sigma^2(\rho^2-1)}\right\} K_0\left(\frac{-y_2}{2\sigma^2(\rho-\rho^3)}\right).$$

Hence, the PDF of Y_2, $y_2 \in \mathbb{R}$ and $-1 < \rho < 1$, is

$$f_{Y_2}(y_2) = \frac{1}{2\pi\sigma^2\rho\sqrt{1-\rho^2}} \exp\left\{\frac{y_2}{2\sigma^2(\rho^2-1)}\right\} K_0\left(\frac{|y_2|}{2\sigma^2(\rho-\rho^3)}\right).$$

Using the transformation method that was explained before, the probability density function of U is given by

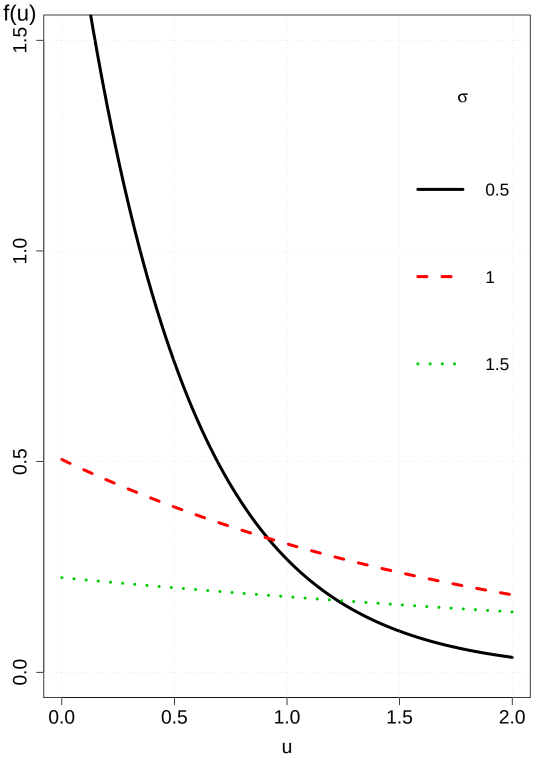
$$f_U(u) = \frac{\exp\left\{-\frac{u}{2\sigma^2(1-\rho^2)}\right\}}{2\sigma^2(1-\rho^2)}, \qquad u > 0.$$

It's worth to mention that the distribution of U is an exponential with rate parameter $\lambda = \frac{1}{2\sigma^2(1-\rho^2)}$, and the cumulative distribution function is given as

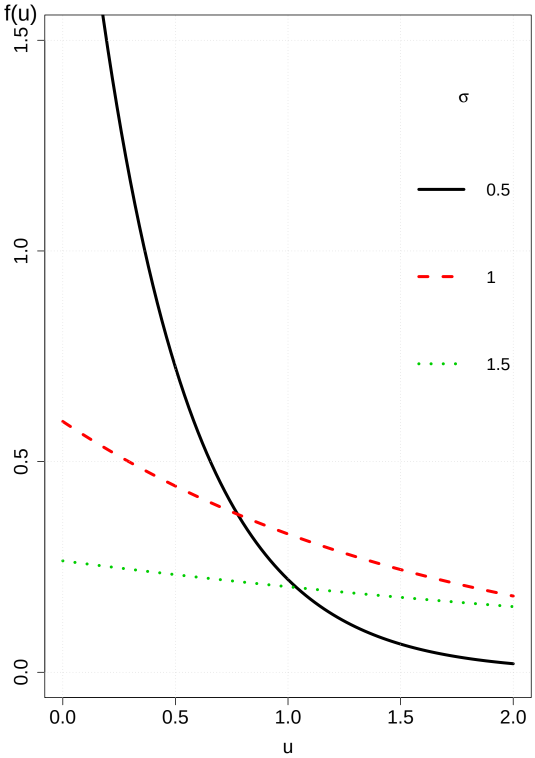
$$F_U(u) = 1 - \exp\left\{-\frac{u}{2\sigma^2(1-\rho^2)}\right\}, \qquad u > 0.$$

Recall that $U = X_1^2 + X_2^2 - 2\rho X_1 X_2$ is a combination of normal RV's. As discussed in Section 2.3 in Mohsenipour (2012), the RV U is distributed as a linear combination of independent central chi-square variable when $\mu = 0$. The PDF of a linear combination of chi-squares is derived in Moschopoulos et al. (1984). It is obvious that our PDF of U is the same as in Moschopoulos et al. (1984) by putting $j = 0, s = 1$.

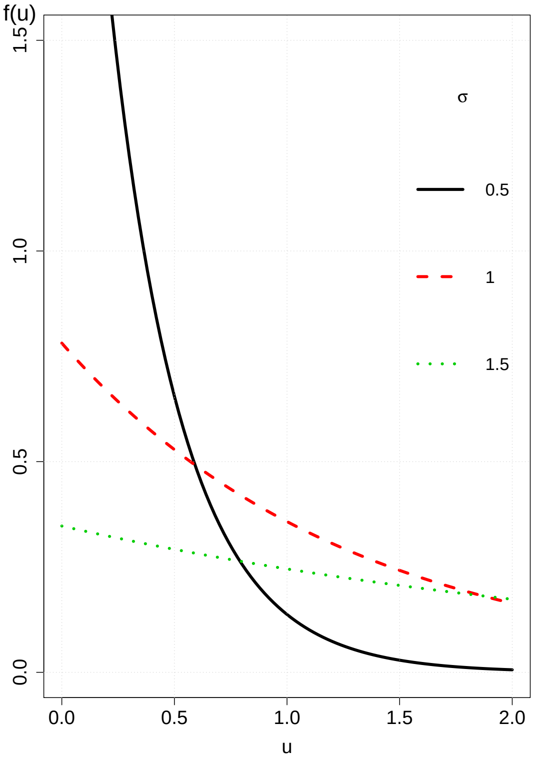
The PDF and CDF curves of U for distinct values of ρ and σ^2 are shown in Figure 4.2 and Figure 4.3. It is obvious from Figure 4.2 that the PDF is always convex and is stretched to the right as σ decreases in value. Figure 4.3 shows that the CDF value at a specific Z increases as ρ increases.



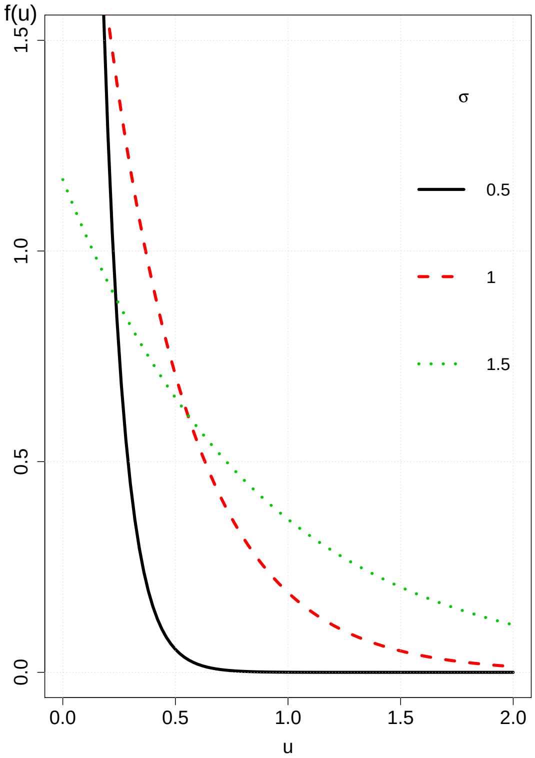
(a) When $\rho = 0.1$



(b) When $\rho = 0.4$

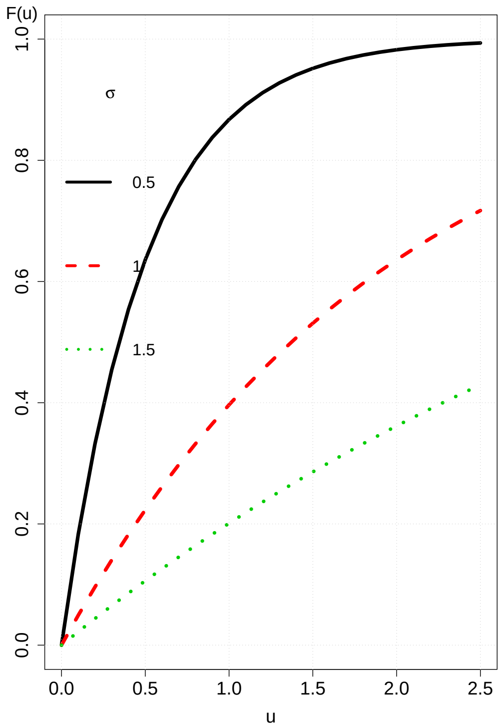


(c) When $\rho = 0.6$

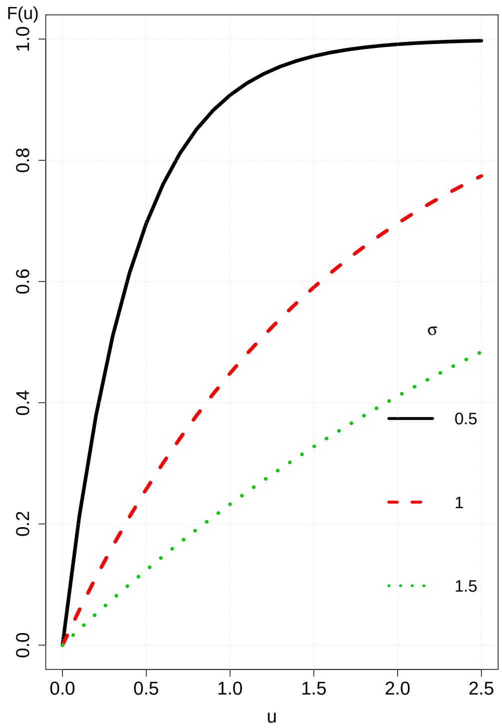


(d) When $\rho = 0.9$

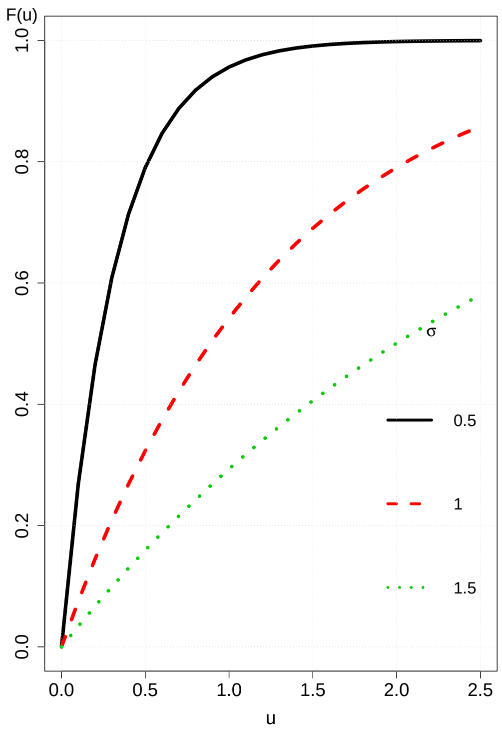
Figure 4.2: Plot of $f_U(u)$ for different values of ρ



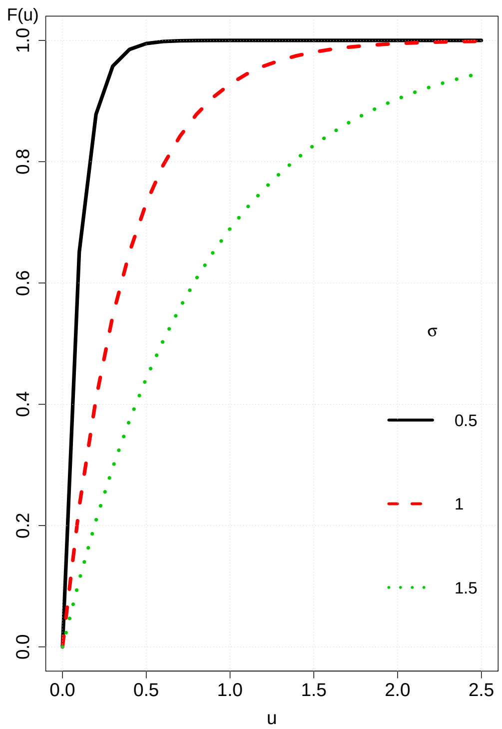
(a) When $\rho = 0.1$



(b) When $\rho = 0.4$



(c) When $\rho = 0.6$



(d) When $\rho = 0.9$

Figure 4.3: Plot of $F_U(u)$ for different values of ρ

4.3 Some Statistical Properties of the Distribution

The mean or expected value of an exponentially distributed random variable U with rate parameter λ is given by

$$\mu = E(U) = \frac{1}{\lambda} = 2\sigma^2(1-\rho^2). \tag{4.6}$$

The variance of U is given by

$$Var(U) = \frac{1}{\lambda^2} = (2\sigma^2(1-\rho^2))^2 = 4\sigma^4(1-\rho^2)^2. \tag{4.7}$$

The moments of U, for $n \in \mathbb{N}$ are given by

$$E(U^n) = \frac{n!}{\lambda^n} = n!(2\sigma^2(1-\rho^2))^n. \tag{4.8}$$

The median of U is given by

$$M = \frac{ln(2)}{\lambda} = ln(2)2\sigma^2(1-\rho^2). \tag{4.9}$$

The MGF of U is given by

$$M_U(t) = \frac{1}{1-2\sigma^2(1-\rho^2)t}. \tag{4.10}$$

4.4 Estimation and Inference

In this section, the estimation of the parameters using MME and MLE will be investigated.

4.4.1 Maximum Likelihood Estimation

The likelihood function of U for the parameter vector $\Theta = (\sigma^2, \rho)$ can be written as

$$\begin{aligned} L(\Theta; u_1, u_2, \ldots, u_n) &= \prod_{i=1}^{n} f(u_i; \sigma^2, \rho) \\ &= \prod_{i=1}^{n} \frac{\exp\left\{-\frac{u_i}{2\sigma^2(1-\rho^2)}\right\}}{2\sigma^2(1-\rho^2)} \\ &= \frac{1}{(2\sigma^2(1-\rho^2))^n} \exp\left\{-\frac{\sum_{i=1}^{n} u_i}{2\sigma^2(1-\rho^2)}\right\}. \end{aligned}$$

Now, set $\sum_{i=1}^{n} u_i = n\overline{u}$. The total log-likelihood function for (σ^2, ρ) will be

$$\begin{aligned} l(\Theta; u_1, u_2, \ldots, u_n) &= -\ln(2^n(\sigma^2)^n(1-\rho^2)^n) - \frac{n\overline{u}}{2\sigma^2(1-\rho^2)} \\ &= -n\ln(2) - n\ln(\sigma^2) - n\ln(1-\rho^2) - \frac{n\overline{u}}{2\sigma^2(1-\rho^2)}. \end{aligned}$$

Differentiating the log likelihood function with respect to σ^2 and ρ results in

$$\begin{aligned} \frac{d}{d\sigma^2} l(\Theta; u_1, u_2, \ldots, u_n) &= -\frac{n}{\sigma^2} + \frac{n\overline{u}}{2\sigma^4(1-\rho^2)} \\ \frac{d}{d\rho} l(\Theta; u_1, u_2, \ldots, u_n) &= \frac{2n\rho}{1-\rho^2} - \frac{4n\rho\sigma^2\overline{u}}{4\sigma^4(1-\rho^2)^2}. \end{aligned}$$

Solve these equations, by equalling them to zero, setting $\rho = \hat{\rho}$ and $\sigma^2 = \hat{\sigma}^2$, and then solving for $\hat{\rho}$ and $\hat{\sigma}^2$, we get

$$\begin{aligned} \hat{\sigma}^2 &= \frac{\sum_{i=1}^{n} u_i}{2(1-\rho^2)}. \\ \hat{\rho} &= \pm\sqrt{1 - \frac{\sum_{i=1}^{n} u_i}{2\sigma^2}}. \end{aligned}$$

Remark 8. *The Moment of Method estimation for the exponential distribution is the same as MLE.*

4.5 Simulation Study

As it is shown in Figure 4.4, the CDF curve of our distribution is approximately identical to the Monte Carlo simulation curve, which ascertain our derivations. The Monte Carlo simulation curve is graphed by generating a 10000 random sample from the distribution in R, for $\sigma^2 = 1$, and $\rho = (0, 0.8)$.

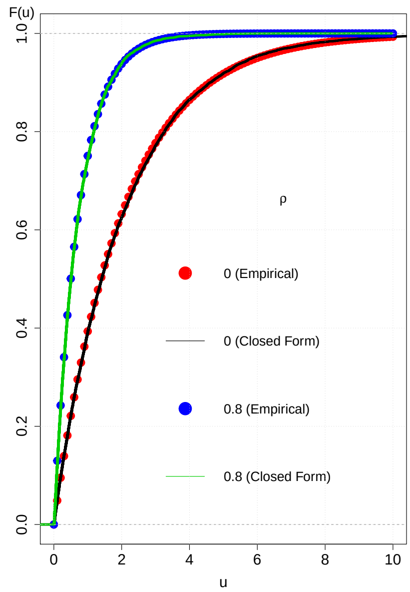


Figure 4.4: Plot of $F_U(u)$ and $\hat{F}_U(u)$

MLE Simulation

To study the properties of the MLEs, a simulation study is carried out in this section. Sample of sizes 20, 50, and 100 from U distribution are considered. 10000 random samples are selected for each setup with the following combinations that included different cases for (ρ, σ^2):

$$(\rho, \sigma^2) = \{(0.2, 0.5), (0.2, 1.5), (0.8, 0.5), (0.8, 1.5)\}.$$

This simulation study is carried out using R software.

Remark 9. *The results in Tables (4.1 and 4.2) indicate that:*

(i) Everything else being fixed, increasing the sample size, the bias decreases.

(ii) Everything else being fixed, increasing the sample size, the MSE decreases.

(iii) Everything else being fixed, increasing the sample size, the confidence interval will be narrower, so the estimator being more efficient.

Table 4.1: Mean of MLEs of all parameters, bias, and MSE, for distinct values of n and distinct setting of (ρ, σ)

Sample size	(ρ, σ)	(0.2,0.5)		(0.2,1.5)	
		ρ	σ	ρ	σ
20	Estimate	0.196	0.514	0.194	1.538
	Bias	0.004	-0.264	0.006	0.712
	MSE	0.050	0.074	0.050	0.547
	C.I	(0.255,0.330)	(-0.020,1.049)	(0.252,0.327)	(0.088,2.988)
	Width	0.075	1.069	0.075	2.900
50	Estimate	0.197	0.504	0.198	1.512
	Bias	0.003	-0.254	0.002	0.738
	MSE	0.019	0.066	0.019	0.559
	C.I	(0.208,0.260)	(0.0003,1.009)	(0.209,0.260)	(0.047,2.977)
	Width	0.0525	1.009	0.051	2.930
100	Estimate	0.2000	0.5019	0.200	1.508
	Bias	$-5.2 * 10^{-5}$	-0.252	0.0005	0.742
	MSE	0.009	0.064	0.009	0.557
	C.I	(0.198,0.235)	(0.005,0.999)	(0.199,0.237)	(0.045,2.971)
	Width	0.037	0.993	0.038	2.927

4.6 Application: The Effect of I/Q Imbalance in Advanced Wireless Communication System

Alsmadi (2020) mentioned that modern digital communication systems require considerable digital signal processing and advanced analog circuity. On a high level, the communication system can be represented using some basic stages: starting from a bitstream at the transmitter to a physical channel and ending with a bitstream at the receiver. This can be illustrated in Figure 4.5. In an In-phase/Quadrature (I/Q) based RF receiver, analog to digital converter (ADC) is required to convert the signal from an analog signal to a digital one. In addition, the received signal is demodulated (down-converted) to retrieve the baseband transmitted signal.

The direct-conversion transceivers have a simple, low cost and flexible structure with less analog components and low implementation. Since direct-conversion transceivers convert the RF signal directly to the baseband instead of using an Intermediate Frequency (IF), it does not need either external IF or image rejection filters. Besides these

Table 4.2: Mean of MLEs of all parameters, bias, and MSE, for distinct values of n and distinct setting of (ρ, σ)

Sample size	(ρ, σ)	(0.8,0.5) ρ	σ	(0.8,1.5) ρ	σ
	Estimate	0.790	0.518	0.792	1.563
	Bias	0.010	-0.268	0.008	0.687
20	MSE	0.009	0.086	0.008	0.600
	C.I	(0.738,0.875)	(-0.057,1.094)	(0.741,0.876)	(0.044,3.081)
	Width	0.137	1.151	0.135	3.037
	Estimate	0.797	0.508	0.797	1.525
	Bias	0.003	-0.258	0.003	0.725
50	MSE	0.003	0.071	0.003	0.570
	C.I	(0.761,0.844)	(-0.016,1.031)	(0.761,0.844)	(0.034,3.007)
	Width	0.083	1.047	0.0842	2.973
	Estimate	0.798	0.503	0.799	1.512
	Bias	0.002	-0.253	0.001	0.738
100	MSE	0.001	0.067	0.001	0.565
	C.I	(0.772,0.830)	(-0.002,1.009)	(0.772,0.831)	(0.038,2.986)
	Width	0.058	1.011	0.0582	2.948

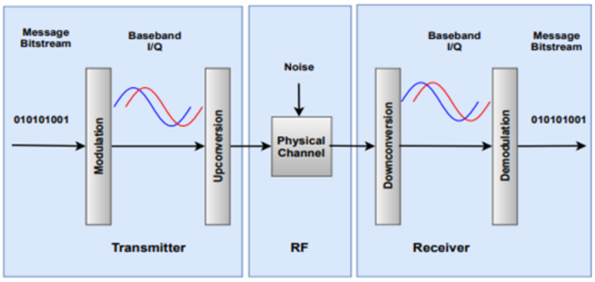


Figure 4.5: Block diagram of a typical communication system (Alsmadi, 2020)

advantages, it is appropriate for higher levels of integration because no need to use IF filters. As such, direct-conversion transceiver have attracted remarkable attention in next generation wireless communication studies, which require flexible and software reconfigurable transceivers that are capable to satisfy demands of desired quality of service. Consequently, due to the imperfections: 1) the phase difference between the I and Q parts of the transmitter and/or receiver signals might not be exactly 90 degrees which is called phase imbalance, 2) small variations might be between the amplitude of the I and Q parts of the signal at the transmitter and/or receiver, which is called amplitude imbalance. Then, I/Q imbalance can dramatically affect the systems performance by changing the transmitted signal at the transmitter or corrupting the received signal at the receiver.

Three main reasons make the effects of I/Q imbalance critical for the next-generation communication systems. First, the I/Q imbalance effects become harsher with larger signal constellation since the probability of incorrect detection increases when the distances between the constellation becomes smaller. Second, fifth generation (5G) and IoT require a massive number of connected devices. This can be supported by manufacturing low-cost transceivers which are expected to come with poor quality. Third, new technologies such as mmWave might not always able to meet the standard requirements for acceptable levels of I/Q imbalance. Figure 4.6 show the effect of the phase imbalance on the 16-QAM constellation diagram appears as a rotation of the constellation diagram in the I/Q plane. Figure 4.7 illustrates the effects of the I/Q applied to the 4-QAM and 64-QAM signal constellation diagrams. As this figure shows the constellation impaired received symbols overlap with the baseband transmitted symbols when using 64-QAM. This means it may be impractical to obtain the baseband symbols when using the any receiver to detect signals with higher modulation orders. In our case, we have only the

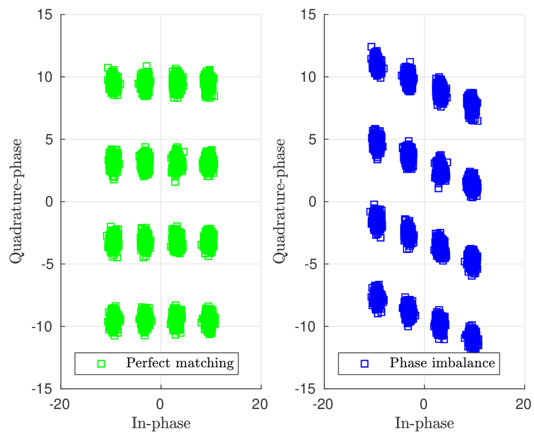


Figure 4.6: The result of phase imbalance for specific values (Alsmadi, 2020)

phase imbalance that affect on the receiver performance as shown in Figure 4.8. Denoting ρ as the phase imbalance, where $\rho = \sin(\phi)$ as ϕ is the phase difference between 0° and

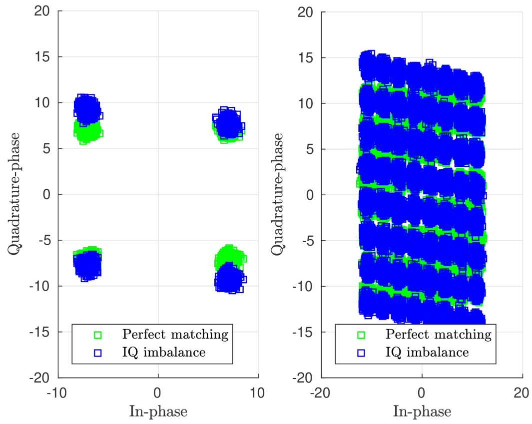


Figure 4.7: The result of transmitter and receiver I/Q imbalances applied to the 4-QAM and 64-QAM signal constellation diagrams for specific values (Alsmadi, 2020)

90°.

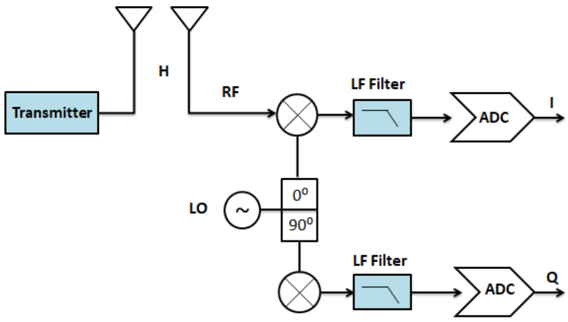


Figure 4.8: Wireless communication system under the effect of I/Q imbalance.

4.6.1 Optimal ML Receiver

In this subsection, an optimal ML receiver is proposed for the presented wireless communication system, which has I/Q imbalance at the receiver. The joint PDF of the real part, y^I and the imaginary part, y^Q, of the received signal can be written as Canbilen et al. (2018) and Balakrishnan et al. (1985)

$$f_{y^I,y^Q}(y^I, y^Q|x_i) = \frac{1}{2\pi\sqrt{1-\rho^2}}$$
$$\exp\left\{\frac{-1}{2(1-\rho^2)}\left((y^I - \sqrt{E}\chi_i^I)^2 + (y^Q - \sqrt{E}\chi_i^Q)^2 - 2\rho(y^I - \sqrt{E}\chi_i^I)(y^Q - \sqrt{E}\chi_i^Q)\right)\right\}.$$

The primary task of the ML receiver is to decide which x_i transmitted among M hypotheses. Assuming that the channel inputs are equally likely, the optimal receiver will be designed based on maximizing the following statement

$$\hat{x}_i = arg \;\; max_{i=1,\ldots,M} \left\{ f_{y^I,y^Q}(y^I, y^Q|x_i) \right\}.$$

Now, to study the behaviour of the proposed optimal receiver, an average pairwise error probability (APEP) well be derived. Considering perfect amplitude and phase mismatch, the channel gain coefficient h^I and h^Q have to be zero means, same variances and being correlated. Benefiting from the Beckmann fading channels to choose the suitable channel for our case depends on the mean and variance. More details about Beckmann fading channels can be found in Section 2.4 by Alsmadi (2020). h can be modeled as $N \sim (0, \sigma^2)$.

Average Pairwise Error Probability

Considering the optimal ML detection, the APEP can be calculated from 4.19 in Alsmadi (2020) under the assumption of that x_i has been transmitted and detected erroneously at the receiver, since we only consider the phase imbalance as

$$\gamma = (h_i^I)^2 + (h_i^Q)^2 - 2\rho h_i^I h_i^Q. \tag{4.11}$$

It is obvious that γ has the same PDF as U distribution, since h_i^I, h_i^Q are identical correlated with zero mean and equal variances.

The APEP can be defined by using the expected value of PEP expression as given in 4.26 in Alsmadi (2020), we derived earlier the PDF of γ, substitute it in this expression

$$\begin{aligned} APEP &= E\{Q(\sqrt{E_s\gamma})\} \\ &= \int_{-\infty}^{\infty} Q(\sqrt{E\gamma}) f_\gamma(\gamma) d\gamma. \end{aligned}$$

Note that we derived a closed formula for the PDF, so it more easy to know the relationship of the variables. As it mentioned in Alsmadi (2020), Canbilen et al. (2018), and Balakrishnan et al. (1985) that there is no closed formula for the PDF, so they use alternative solution by deriving a general formula for the MGF, then by some numerical methods and simulating using Matlab software they almost know the effect of each variables.

We can also find the following properties for the wireless communication channel.

Average Bit Error Rate

$$\begin{aligned} P_e &= \frac{q^p}{2\Gamma(p)} \int_0^\infty \exp\{-q\gamma\} \gamma^{p-1} F_\gamma(\gamma) d\gamma \\ &= \frac{q^p}{2\Gamma(p)} \int_0^\infty \exp\{-q\gamma\} \gamma^{p-1} \left(1 - \exp\left\{-\frac{\gamma}{2\sigma^2(1-\rho^2)}\right\}\right) d\gamma \\ &= \frac{q^p}{2\Gamma(p)} \int_0^\infty \exp\{-q\gamma\} \gamma^{p-1} d\gamma - \int_0^\infty \exp\left\{-\gamma\left(q + \frac{1}{2\sigma^2(1-\rho^2)}\right)\right\} \gamma^{p-1} d\gamma. \end{aligned}$$

Employing (3.351.3) in Gradshteyn et al. (2007) we get

$$\begin{aligned} P_e &= \frac{q^p}{2\Gamma(p)} \left((p-1)!q^{-p} - (p-1)!\left(q + \frac{1}{2\sigma^2(1-\rho^2)}\right)\right) \\ &= \frac{1}{2} - \frac{q^p}{2}\left(q + \frac{1}{2\sigma^2(1-\rho^2)}\right)^{-p}. \end{aligned}$$

Capacity

$$\begin{aligned} C &= \frac{1}{ln(2)} \int_0^\infty ln(1+\gamma) f_\gamma(\gamma) d\gamma \\ &= \frac{1}{ln(2)} \int_0^\infty ln(1+\gamma) \frac{\exp\left\{-\frac{\gamma}{2\sigma^2(1-\rho^2)}\right\}}{2\sigma^2(1-\rho^2)} d\gamma \\ &= \frac{1}{2ln(2)\sigma^2(1-\rho^2)} \int_0^\infty ln(1+\gamma) \exp\left\{-\frac{\gamma}{2\sigma^2(1-\rho^2)}\right\} d\gamma. \end{aligned}$$

Using (4.337.1) in Gradshteyn et al. (2007) we get

$$C = -\frac{1}{ln(2)} \exp\left\{-\frac{1}{2\sigma^2(1-\rho^2)}\right\} E_i\left(\frac{-1}{2\sigma^2(1-\rho^2)}\right).$$

Chapter 5

Conclusion and Future Work

5.1 Conclusion

The main objective of this thesis consists in obtaining accurate distribution for two types of combinations of normal random variables. The methodology used was rewrite the functions as the sum of chi-square random variable and other variable involves the product of two normal random variables, taking in our consideration the independent and dependent case. Then using a transformation technique to get a two independent and identical chi-square random variable, which the joint probability density function is well known. The probability density function, cumulative distribution function as well as all statistical properties of the two distributions derived. Simulating our data using Monte Carlo stimulation. The method of moment, and maximum likelihood estimation used to estimate the distribution parameters. Finally, an application introduced for the dependent case.

5.2 Future Work

First, I am planning to extend the density methodology that was used to random vectors and matrices. This will entail making use of multivariate base densities, which would be adjusted by linear combinations of multivariate orthogonal polynomials on the basis of the joint moments of the variables involved. This approach would allow for much more flexibility than that univariate random variable, and its applicable in many fields in engineering and physics. Second, I want to obtain a closed-form approximation for the PDF and CDF of Z by approximating the Bessel functions involved in the exact expressions of the PDF and CDF of Z. This closed-form approximation will make the derivation more practical and easily handling with them. Third, extend the distribution for any mean and variance, and for more distribution like gamma, nakagami, etc. Fourth, searching or designing a suitable application for the Z distribution.

Appendix A

Proof of the Second Moment

To evaluate the second moment of Z, we have two cases:

Case 1: For $0 < \theta < 2$,

$$E(Z^2) = \int_0^\infty \frac{1}{\sigma^2\sqrt{4-\theta^2}} z^2 \exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\} I_0\left(\frac{\theta z}{\sigma^2(4-\theta^2)}\right) dz.$$

Using (8.447.1) in Gradshteyn et al. (2007) our integral becomes

$$E(Z^2) = \int_0^\infty \frac{\exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\}}{\sigma^2\sqrt{4-\theta^2}} \sum_{n=0}^{\infty} \frac{\theta^{2n}}{4^n(n!)^2(\sigma^2)^{2n}(4-\theta^2)^{2n}} z^{\{2n+2\}} dz.$$

Switching the order of summation and integration we get

$$E(Z^2) = \sum_{n=0}^{\infty} \frac{\theta^{2n}}{4^n(n!)^2(\sigma^2)\sqrt{4-\theta^2}(\sigma^2)^{2n}(4-\theta^2)^{2n}} \int_0^\infty \exp\left\{\frac{-2z}{\sigma^2(4-\theta^2)}\right\} z^{\{2n+2\}} dz.$$

Now, let $u = \frac{2z}{\sigma^2(4-\theta^2)}$. Then $du = \frac{2}{\sigma^2(4-\theta^2)}dz$ and imploying (3.326.2) in Gradshteyn et al. (2007) we get

$$\begin{aligned}
E(Z^2) &= \sum_{n=0}^{\infty} \frac{\theta^{2n}(\sigma^2)^{2n+1}(4-\theta^2)^{2n}\sigma^4(4-\theta^2)^3}{2^{2n+1}4^n(n!)^2\sigma^2\sqrt{4-\theta^2}(\sigma^2)^{2n}(4-\theta^2)^{2n}} \int_0^\infty \exp\{-u\}\, u^{\{2n+2\}} du \\
&= \sum_{n=0}^{\infty} \frac{\theta^{2n}(4-\theta^2)^{\frac{5}{2}}\sigma^4}{2^{2n+3}4^n(n!)^2}\Gamma(2n+3) \\
&= \sigma^4(8+\theta^2),
\end{aligned}$$

where $\Gamma(\cdot)$ is the Gamma function.

Case 2: For $\theta > 2$,

(i) For $Z > 0$,

$$E(Z) = \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}} \int_0^\infty z^2 \exp\left\{\frac{2z}{\sigma^2(\theta^2-2)}\right\} K_0\left(\frac{\theta z}{\sigma^2(\theta^2-4)}\right) dz.$$

Employing (8.432.8) in Gradshteyn et al. (2007) we get

$$\begin{aligned}
E(Z) &= \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}} \int_0^\infty z^2 \exp\left\{\frac{2z}{\sigma^2(\theta^2-2)}\right\} \exp\left\{\frac{-\theta z}{\sigma^2(\theta^2-2)}\right\} \int_0^\infty \frac{\exp\left\{\frac{-zt}{2\sigma^2}\right\}}{\sqrt{t^2+\frac{4t\theta}{\theta^2-4}}} dt dz \\
&= \frac{1}{\sigma^2\pi\sqrt{\theta^2-4}} \int_0^\infty z^2 \exp\left\{\frac{-2z}{2\sigma^2(\theta^2-2)}\right\} \int_0^\infty \frac{\exp\left\{\frac{-zt(\theta+2)}{2\sigma^2(\theta+2)}\right\}}{\sqrt{t^2(\theta^2-4)+4t\theta}} \sqrt{\theta^2-4} dt dz \\
&= \frac{1}{\sigma^2\pi} \int_0^\infty \frac{1}{\sqrt{(\theta^2-4)t^2+4t\theta}} \int_0^\infty z^2 \exp\left\{\frac{-z(2+t(\theta+2))}{2\sigma^2(\theta+2)}\right\} dz dt.
\end{aligned}$$

Using (3.351.3) in Gradshteyn et al. (2007) the integral becomes

$$\begin{aligned}
E(Z) &= \frac{1}{\sigma^2\pi} \int_0^\infty \frac{1}{\sqrt{(\theta^2-4)t^2+4t\theta}} \frac{2!8\sigma^6(\theta+2)^3}{(2+t(\theta+2))^3} dt \\
&= \frac{16\sigma^4(\theta+2)^3}{\pi} \int_0^\infty \frac{1}{\sqrt{(\theta^2-4)t^2+4t\theta}} \frac{1}{(2+t(\theta+2))^3} dt.
\end{aligned}$$

We set $\sqrt{t(\theta+2)} = \sqrt{2}\tan(x)$. Then $dt = \frac{4\sec^2(x)\tan(x)}{(\theta+2)}dx$ and $t^2 = \frac{4\tan^4(x)}{(\theta+2)^2}$, so that by substitution we have

$$\begin{aligned}
E(Z) &= \frac{16\sigma^4(\theta+2)^3}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(\theta^2-4)\frac{4\tan^4(x)}{(\theta+2)^2} + \frac{8\theta\tan^2(x)}{\theta+2}}} \frac{1}{(8\sec^2(x))^3} \frac{4\tan(x)\sec^2(x)}{(\theta+2)} dx \\
&= \frac{4\sigma^4(\theta+2)^2}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\frac{(\theta-2)}{(\theta+2)}\tan^2(x) + \frac{2\theta}{(\theta+2)}}} \frac{1}{\sec^4(x)} dx \\
&= \frac{4\sigma^4(\theta+2)^{\frac{5}{2}}}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(\theta-2)\tan^2(x)+2\theta}} \frac{1}{\sec^4(x)} dx.
\end{aligned}$$

Now, take the indefinite integral for $0 < x < \frac{\pi}{2}$.

$$\begin{aligned}
v &= \int \frac{1}{\sqrt{(\theta-2)\tan^2(x)+2\theta}} \frac{1}{\sec^4(x)} dx \\
&= \int \frac{\cos^4(x)}{\sqrt{(\theta-2)\frac{\sin^2(x)}{\cos^2(x)}+2\theta\frac{\cos^2(x)}{\cos^2(x)}}} dx \\
&= \int \frac{\cos^5(x)}{\sqrt{(\theta-2)\sin^2(x)+2\theta\cos^2(x)}} dx \\
&= \int \frac{\cos^5(x)}{\sqrt{(\theta-2)\sin^2(x)+2\theta(1-\sin^2(x))}} dx \\
&= \int \frac{\cos^5(x)}{\sqrt{\theta\sin^2(x)-2\sin^2(x)+2\theta-2\theta\sin^2(x)}} dx \\
&= \int \frac{\cos^5(x)}{\sqrt{-\sin^2(x)(\theta+2)+2\theta}} dx \\
&= \int \frac{\cos^5(x)}{\sqrt{2\theta-\sin^2(x)(\theta+2)}} dx.
\end{aligned}$$

Using the trigonometric substitution $\sqrt{2\theta}\sin(\alpha) = \sqrt{\theta+2}\sin(x)$, $\alpha \in \left(0, \sin^{-1}\left(\sqrt{\frac{\theta+2}{2\theta}}\right)\right)$. Then $dx = \sqrt{\frac{2\theta}{\theta+2}}\frac{\cos(\alpha)}{\cos(x)}d\alpha$, $\sin^2(\alpha) = \frac{(\theta+2)}{2\theta}\sin^2(x)$. Then, v becomes

$$\begin{aligned}
v &= \frac{1}{\sqrt{2\theta}} \int \frac{\cos^5(x)}{\cos(\alpha)} \sqrt{\frac{2\theta}{\theta+2}} \frac{\cos(\alpha)}{\cos(x)} d\alpha \\
&= \frac{1}{\sqrt{\theta+2}} \int \cos^4(x) d\alpha.
\end{aligned}$$

Substitute $\cos^2(x) = 1 - \frac{2\theta}{\theta+2}\sin^2(\alpha)$ and imploying (1.321.1) and (1.321.3), we get

$$\begin{aligned}
v &= \frac{1}{\sqrt{\theta+2}} \int \left(1 - \frac{2\theta}{\theta+2}\sin^2(\alpha)\right)^2 d\alpha \\
&= \frac{1}{\sqrt{\theta+2}} \int \left(1 - \frac{4\theta}{\theta+2}\sin^2(\alpha) + \frac{4\theta^2}{(\theta+2)^2}\sin^4(\alpha)\right) d\alpha \\
&= \frac{1}{\sqrt{\theta+2}} \int \Bigg[1 - \frac{4\theta}{\theta+2}\left(\frac{1-\cos(2\alpha)}{2}\right) + \frac{4\theta^2}{8(\theta+2)^2} \\
&\times (\cos(4\alpha) - 4\cos(2\alpha) + 3)\Bigg] d\alpha.
\end{aligned}$$

Unifying denominators, grouping and simplifying produces

$$\begin{aligned}
v &= \frac{1}{\sqrt{\theta+2}} \int \left(\frac{\theta^2+8}{2(\theta+2)^2} + \frac{4\theta\cos(2\alpha)}{(\theta+2)^2} + \frac{\theta^2\cos(4\alpha)}{2(\theta+2)^2} \right) d\alpha \\
&= \frac{1}{\sqrt{\theta+2}} \left(\frac{\theta^2+8}{2(\theta+2)^2}\alpha + \frac{2\theta\sin(2\alpha)}{(\theta+2)^2} + \frac{\theta^2\sin(4\alpha)}{8(\theta+2)^2} \right).
\end{aligned}$$

Substitute α in terms of x, we get

$$\begin{aligned}
v &= \frac{1}{\sqrt{\theta+2}} \Big[\frac{\theta^2+8}{2(\theta+2)^2} \sin^{-1}\left(\sqrt{\frac{\theta+2}{2\theta}} \sin(x) \right) + \frac{4\theta}{(\theta+2)^2} \sqrt{\frac{\theta+2}{2\theta}} \sin(x) \\
&\times \frac{\sqrt{2\theta - (\theta+2)\sin^2(x)}}{\sqrt{2\theta}} \Big] \\
&+ \frac{4\theta^2}{8(\theta+2)^2} \sqrt{\frac{\theta+2}{2\theta}} \sin(x) \frac{\sqrt{2\theta-(\theta+2)\sin^2(x)}}{\sqrt{2\theta}} \left(1 - \frac{\theta+2}{\theta}\sin^2(x) \right) \Bigg|_0^{\frac{\pi}{2}} \\
&= \frac{1}{\sqrt{\theta+2}} \left(\frac{\theta^2+8}{2(\theta+2)^2} \sin^{-1}\left(\sqrt{\frac{\theta+2}{2\theta}} \right) + \frac{\sqrt{\theta^2-4}}{(\theta+2)^2} - \frac{\sqrt{\theta^2-4}}{2(\theta+2)^2} \right).
\end{aligned}$$

Now, we have

$$\begin{aligned}
E(Z) &= \frac{4\sigma^4(\theta+2)^{\frac{5}{2}}}{\pi} \left(\frac{1}{(\theta+2)^{\frac{5}{2}}} \left(\frac{3\sqrt{\theta^2-4}}{2} + \frac{\theta^2+8}{2} \sin^{-1}\left(\sqrt{\frac{1}{2}+\frac{1}{\theta}} \right) \right) \right) \\
&= \frac{\sigma^4}{\pi} \left(6\sqrt{\theta^2-4} + (\theta^2+8)\sin^{-1}\left(\sqrt{\frac{1}{2}+\frac{1}{\theta}} \right) \right).
\end{aligned}$$

To simplify, employing (1.624.1),(1.626.2), and (1.623.1) in Gradshteyn et al. (2007), and (1.6.3) in Thomas (2014) we obtain

$$\begin{aligned}
\sin^{-1}\left(\sqrt{\frac{1}{2}+\frac{1}{\theta}}\right) &= \cos^{-1}\left(1-\left(\frac{1}{2}+\frac{1}{\theta}\right)\right) \\
&= \cos^{-1}\left(\sqrt{\frac{1}{2}-\frac{1}{\theta}}\right) \\
&= \frac{1}{2}\cos^{-1}\left(2\left(\frac{1}{2}-\frac{1}{\theta}\right)-1\right) \\
&= \frac{1}{2}\cos^{-1}\left(\frac{-2}{\theta}\right) \\
&= \frac{1}{2}\left(\pi-\cos^{-1}\left(\frac{2}{\theta}\right)\right) \\
&= \frac{1}{2}\left(\pi-\left(\frac{\pi}{2}-\sin^{-1}\left(\frac{2}{\theta}\right)\right)\right) \\
&= \frac{\pi}{4}+\frac{1}{2}\sin^{-1}\left(\frac{2}{\theta}\right).
\end{aligned}$$

The expectation of Z is

$$E(Z) = \frac{\sigma^4}{\pi}\left(\frac{\pi(\theta^2+8)}{2}+6\sqrt{\theta^2-4}+(\theta^2+8)\sin^{-1}\left(\frac{2}{\theta}\right)\right). \tag{A.1}$$

(ii) For $Z<0$, along the same lines as above, it can be shown that

$$E(Z) = \frac{\sigma^4}{\pi\sqrt{\theta^2-4}}\left(-6(\theta^2-4)+\sqrt{\theta^2-4}(8+\theta^2)\cos^{-1}\left(\frac{2}{\theta}\right)\right). \tag{A.2}$$

Therefore, the expectation of Z is given by

$$E(Z) = \begin{cases} \sigma^4(8+\theta^2) & \text{for } 0<\theta<2, \\ \frac{\sigma^4}{\pi}\left(\frac{\pi(\theta^2+8)}{2}+6\sqrt{\theta^2-4}+(\theta^2+8)\sin^{-1}\left(\frac{2}{\theta}\right)\right) & \text{for } z>0 \text{ and } \theta>2, \\ \frac{\sigma^4}{\pi\sqrt{\theta^2-4}}\left(-6(\theta^2-4)+\sqrt{\theta^2-4}(8+\theta^2)\cos^{-1}\left(\frac{2}{\theta}\right)\right) & \text{for } z<0 \text{ and } \theta>2. \end{cases} \tag{A.3}$$

By summing (A.1) and (A.2), we get $E(Z^2)=\sigma^4(8+\theta^2)$. It is obvious to note that the second moment for the two cases is $\sigma^4(8+\theta^2)$.

Bibliography

M. Abramowitz and I. Stegun, Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables, Dover Publications, 10th ed., Washington, D.C., 1965.

S. Acitas, T. Arslan, and B. Senoglu, Slash Maxwell distribution: definition, modified maximum likelihood estimation and applications*, Gazi University Journal of Science, Vol. 33, No. 1, pp. 249-263, 2020.*

M. Ahsanullah, M. Shakil, B. M. Golam Kibria, and F. George, Distribution of the product of Bessel distribution of second kind and gamma distribution properties and characterization*, International Journal of Mathematical Analysis, Vol. 9, No. 21, pp. 1031-1048, 2015.*

M. Alsmadi, The Effects of I/Q Imbalance and Improper Gaussian Noise on Different Wireless Communication Systems*. Ph.D. Thesis, Lakehead University, 2020.*

G. B. Arfken, H. J. Weber, and F. E. Harris, Mathematical Methods for Physicists, *7th ed., University of Florida, USA: Academic Press, 2012.*

L. A. Aroian, The probability function of the product of two normally distributed variables, *The Annals of Mathematical Statistics, Vol. 18, No. 2, pp. 265 - 271, 1947.*

L. A. Aroian, V. S. Taneja, and L. W.Cornwell, Mathematical forms of the distribution of the product of two normal variables*, Communications in Statistics - Theory and Methodology, Vol. 7, No. 2, pp. 165 - 172, 1978.*

N. Balakrishnan, NL. Johnson, and S. Kotz, Continuous Univariate Distributions, Wiley, 2nd ed., New York, 1994.

P. S. Bithas, N. C. Sagias, T. A. Tsiftsis, and G. K. Karagiannidis, Distributions involving correlated generalized gamma variables*. In: Proc. lnt. Conf. on Applied Stochastic Models and Data Analysis, 2007.*

A. E. Canbilen, M. M. Alsmadi, E. Basar, and S. S. Ikki, Spatial modulation in the presence of I/Q imbalance: optimal detector amp; performance analysis, IEEE Commun. Lett., Vol. 22, No. 8, pp. 1572 - 1575, 2018.

G. Cui, X. Yu, S. Iommelli and L. Kong, Exact distribution for the product of two correlated gaussian random variables*, IEEE Signal Processing Letters, Vol. 23, No. 11, pp. 1662 - 1666, 2016.*

J. H. Curtiss, A note on the theory of moment generating functions*, The Annals of Mathematical Statistics, Vol. 13, No. 4, pp. 430 - 434, 1942.*

R. Hogg, E. Tanis, and D. Zimmerman, Probability and Statistical Inference, University of Iowa, 9th ed., 2015.

R. E. Gaunt, A note on the distribution of the product of zero-mean correlated normal random variables*, Statistica Neerlandica, Vol. 73, No. 2, pp. 176 - 179, 2018.*

J. E. Gentle, Computational Statistics,*. In: International Encyclopedia of Education, 3rd ed., pp. 93-97, 2010.*

G. Z. Georgiev, Statistical Methods in Online A/B Testing: Statistics for Data-Driven Business Decisions and Risk Management in E-commerce, 2019.

I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products*, 7th ed., England: University of Newcastle upon Tyne, 2007.*

C. C. Graig, On the frequency function of XY*, The Annals of Mathematical Statistics, Vol. 7, No. 1, pp. 1 - 15, 1936.*

D. M. Lane, M. Hebl, R. Guerra, D. Osherson, and H. Zimmer, History of the Normal Distribution*, Online ed., Rice University, 1809.*

A. S. Macías and A. Oliveira, An approach to distribution of the product of two normal variables*. In: Discussiones Mathematicae. Probability and Statistics, Vol. 32, No. 1/2, pp. 87 - 99, 2012.*

A. A. Mohsenipour, On the Distribution of Quadratic Expressions in Various Types of Random Vectors*. Ph.D. Thesis. The University of Western Ontario, 2012.*

P. G. Moschopoulos and W. B. Canada, The distribution function of a linear combination of chi-squares*, University of Texas at Dallas, Vol. 10, No. 4/5, pp. 383-386, 1984.*

S. Nadarajah, Economic models based on pareto and gamma random variables*, Spanish Economic Review, Vol. 9, No. 4, pp. 1435 - 5477, 2007.*

S. Nadarajah, Sum, Product and ratio of Pareto and gamma variables*, Journal of Statistical Computation and Simulation, Vol. 80, No. 10, pp. 1071 - 1082, 2010.*

S. Nadarajah, Exact distribution of the product of m gamma and n Pareto random variables*, Journal of Computational and Applied Mathematics, Vol. 235, No. 15, pp. 4496 - 4512, 2011.*

S. Nadarajah and T. K. Pogťany, On the distribution of the product of correlated normal random variables*, Comptes Rendus de l'Académie des Sciences - Series I, Vol. 354, No. 2, pp. 201 - 204, 2016.*

H. Pishro-Nik, Introduction to Probability, Statistics, and Random Processes. Available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

R Core Team, R: A Language and Environment for Statistical Computing, *R Foundation for Statistical Computing, https://www.R-project.org/, 2020.*

V. K. Rohatgi, An Introduction to Probability Theory Mathematical Studies*, 1st ed., Wiley, New York, 1976.*

J. Salo, H. M. El-Sallabi, and P. Vainikainen, The distribution of the product of independent Rayleigh random variables*, IEEE Transactions on Antennas and Propagation, Vol. 54, No. 2, pp. 639-643, 2006.*

G. B. Thomas, Thomas Calculus Early Transcendentals*, Massachusetts Institute of Technology, 13th ed., 2014.*

D. D. Wackerly, W. Mendenhall, and R. L. Scheaffer, Mathematical Statistics with Applications*, 7th ed., Thomoson Higher Education, 2008.*

R. Ware and F. Lad, Approximating the distribution for sums of products of normal variables*. Tech. rep. The University of Queensland, 2003.*