Palestine Polytechnic University
Deanship of Graduate Studies and Scientific Research
Master Program of Mathematics

Oscillation of second-order Emden-Fowler neutral delay differential equations

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M.Sc. Thesis

Hebron - Palestine

June, 2026

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M.Sc. Thesis submitted to the Department of Mathematics at Palestine Polytechnic University as a partial fulfillment of the requirement for the degree of Master of Mathematics.

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Dedication

Alhamdulillah, all praise and thanks are due to Allah, who granted me the strength, patience, and guidance to complete this thesis.

To my beloved parents, thank you for your endless love, support, and prayers. Everything I have achieved is because of you.

To my dear sisters, Nagham and Ghina, and my brothers, Mohammad, Osama, Ahmed, and Mahmoud, thank you for always standing by my side and encouraging me.

To my friends and colleagues, thank you for your support, kindness, and the beautiful memories we shared throughout this journey.

To everyone who believed in me and supported me in any way, I am truly grateful.

Finally, I dedicate this work with love and pride to my beloved country, Palestine, whose strength and resilience inspire me every day.

Aseel Akram Yaseen

Acknowledgments

I am deeply grateful to my supervisor, Dr. Ali Zein, for his continuous support, guidance, and encouragement throughout this research. His knowledge and advice have greatly helped me in completing this thesis.

My sincere thanks also go to Dr. Mohammad Adam, the internal committee member, and Dr. Hasan Almanasreh, the external committee member, for their valuable comments and helpful suggestions, which improved the quality of this work.

I would also like to thank the professors and faculty members of the Department of Mathematics at Palestine Polytechnic University for their support and for providing a positive learning environment. May Allah bless and protect you always.

Thank you all for your support.

Aseel Akram Yaseen

Abstract

This thesis is concerned with oscillation results for a class of second-order neutral delay differential equations of Emden–Fowler type. Recent oscillation criteria are presented for both the canonical and non-canonical cases, and illustrative examples are provided to demonstrate the applicability of the results.

In general, oscillation theorems for both cases are derived by reducing the second-order neutral delay differential equations to suitable first-order delay differential inequalities and then applying well-established oscillation results for first-order equations. Moreover, Riccati-type transformations are employed as an additional analytical tool to derive further oscillation criteria through various methodological approaches.

Keywords: Oscillation theory, Neutral delay differential equations, Second-order equations, Emden–Fowler equations, Canonical case, Non-canonical case, Riccati transformation.

Contents

Chapter 1

Introduction

1.1 Overview

Differential equations are fundamental tools in mathematics, widely used to model natural and engineering systems. Among them, *delay differential equations* (DDEs) form a significant class, in which the rate of change of a system depends not only on its current state but also on its past states. Such delays naturally arise in population dynamics, control theory, and electronic circuits [12, 14].

A particularly important subclass of DDEs is the *neutral delay differential equations* (NDDEs), where the highest-order derivative of the unknown function appears both with and without delay.

NDDEs arise naturally in various applied fields. Such equations occur in a variety of applications, including electrical networks with lossless transmission lines, vibration phenomena, and certain problems in variational mechanics with time delay; see [12, 14]. These equations are therefore fundamental in modeling many real-world phenomena; see [2, 3, 8, 13].

A solution of a differential equation is defined as **oscillatory** if it has arbitrarily large zeros, or in other words, if it is neither eventually positive nor eventually negative. A solution that does not satisfy this condition is called nonoscillatory.

Oscillation theory for neutral delay differential equations differs from that of ordinary and retarded delay differential equations. This is due to the presence of the neutral term, which involves the highest-order derivative with delay and may influence the asymptotic behavior of solutions; see [9, 14].

Early works on differential equations with deviating arguments date back to the 18th century with contributions from Bernoulli [18]. However, the systematic development of oscillation theory for NDDEs began in the mid-20th century with seminal contributions by Myshkis (1950) and Kamenskii (1958), who classified equations with delayed arguments into retarded, neutral, and advanced types [18].

The complex characteristics have motivated extensive research to establish criteria for oscillation in neutral differential equations and to understand the conditions under which solutions exhibit oscillations. The last few decades have a significant and growing research interest in this field, and the oscillation of neutral equations has become an important area of research, with constant interest in obtaining new sufficient conditions for oscillation.

In this thesis, we are concerned with the oscillatory behavior of a class of second-order neutral delay differential equations of Emden-Fowler type

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)' + q(t)\left(x^{\beta}(\sigma(t))\right) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where

$$z(t) = x(t) + p(t)\, x(\tau(t)).$$

Here, α and β are a quotient of odd positive integers, $r(t) > 0$, $q(t) \geq 0$, $p(t) \geq 0$ are continuous, and $\sigma(t), \tau(t) \leq t$ are delay functions satisfying

$$\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty.$$

Equation (1.1) will serve as the central model throughout this work.

This thesis aims to provide a comprehensive and systematic examination of the oscillation theory for Equation(1.1). Following the standard classification in the literature, the subsequent chapters are organized around the two primary cases of the differential operator: the **canonical case** and the **non-canonical case**.

Chapter Two is dedicated to study the **canonical case**, defined by the condition

$$\int^{\infty} r^{-1/\alpha}(s)\, ds = \infty.$$

This chapter will review the primary analytical techniques used in this context, specifically the **method of reduction from second-order to first-order equations** and the **Riccati transformation method**.

Subsequently, **Chapter Three** addresses the **non-canonical case**, where the integral

$$\int^{\infty} r^{-1/\alpha}(s)\, ds < \infty.$$

This chapter will review the specialized approaches required for this case.

1.2 Basic concepts and definitions

Delay Differential Equations (DDEs)

Definition 1.1. A delay differential equation (DDE) is a functional differential equation in which the derivative of an unknown function at a given time depends on both its present and past values.

The general form of a **first-order** DDE is

$$x'(t) = f(t, x(t), x(\tau(t))), \quad \tau(t) < t.$$

The general form of a **second-order** DDE can be written as:

$$x''(t) = f(t, x(t), x(\tau(t)), x'(t), x'(\sigma(t))),$$

where $\tau(t) \leq t$ and $\sigma(t) \leq t$.

Neutral Delay Differential Equations (NDDEs)

Definition 1.2. A neutral delay differential equation is a subclass of DDEs in which the highest-order derivative of the unknown function appears with and without delay.

A representative form, studied throughout this work, is

$$(r(z'(t))^{\alpha})'(t) + q(t)x^{\alpha}(\sigma(t)) = 0, \quad t \geq t_0 > 0,$$

with $z(t) = x(t) + p(t)x(\tau(t))$. Such equations arise in models where present dynamics depend not only on delayed states but also on delayed derivatives [4, 9]. Throughout this study, we suppose also the following conditions:

- α is a ratio of odd positive integers.
- $r(t),\ p(t),\ q(t),\ \tau(t),\ \sigma(t)$ are continuous functions.
- p and q are nonnegative, $p \leq p_0$, where p_0 is a positive real number, and q does not vanish identically on any half-line $[t_q, \infty)$, $t_q \geq t_0$.
- $\tau(t) \leq t$ and $\sigma(t) \leq t$ are delay functions and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$.

Solution of a differential equation

Consider Equation (1.1). A solution is a real-valued continuous function $x(t)$ defined on $[t_0, \infty)$ that possesses the required derivatives and satisfies the equation for all $t \geq t_0$.

Oscillatory and non-oscillatory solutions

A solution $x(t)$ is said to be **oscillatory** if it has arbitrarily large zeros, or equivalently, if it is neither eventually positive nor eventually negative see e.g. [11, 22]. Otherwise the solution is called **nonoscillatory**. The equation itself is called oscillatory if all of its solutions oscillate.

1.3 Fundamental lemmas and inequalities

This section presents the essential lemmas, that are needed in the subsequent chapters of the thesis.

Lemma 1.1. *(Jensen's inequality) [24] Let f be a concave function on an interval If $x, y \in I$ and $t \in [0,1]$, then*

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y).$$

Lemma 1.2. *[6] Assume $F \geq 0$, $G \geq 0$, and $\alpha \geq 1$. Then, the following inequality holds:*

$$(F+G)^{\alpha} \leq 2^{\alpha-1}(F^{\alpha} + G^{\alpha}).$$

Proof. We consider the function $h(u) = u^{\alpha}$ for $u \geq 0$ and $\alpha \geq 1$. The second derivative of $h(u)$ is

$$h''(u) = \alpha(\alpha-1)u^{\alpha-2}.$$

Since $\alpha \geq 1$ and $u \geq 0$, we have $h''(u) \geq 0$, which proves that $h(u)$ is a **convex function** on $[0, \infty)$.

By the definition of convexity (or Jensen's Inequality for two points), for any non-negative numbers F and G, we have:

$$h\left(\frac{F+G}{2}\right) \leq \frac{h(F)+h(G)}{2}.$$

Substituting $h(u) = u^\alpha$ into this inequality yields:

$$\left(\frac{F+G}{2}\right)^\alpha \leq \frac{F^\alpha + G^\alpha}{2}.$$

or

$$(F+G)^\alpha \leq \frac{2^\alpha}{2}\left(F^\alpha + G^\alpha\right).$$

Thus,

$$(F+G)^\alpha \leq 2^{\alpha-1}(F^\alpha + G^\alpha).$$

□

Lemma 1.3. *[6] Assume $F \geq 0$, $G \geq 0$, and $0 \leq \alpha \leq 1$. Then, the following inequality holds:*

$$(F+G)^\alpha \leq F^\alpha + G^\alpha$$

Proof. Let $F, G > 0$, otherwise the inequality is trivial. Define for $x > 0$

$$g(x) = (1+x)^\alpha - 1 - x^\alpha$$

we have $g(0) = 0$ and for $0 < \alpha < 1$

$$g'(x) = \alpha\Big((1+x)^{\alpha-1} - x^{\alpha-1}\Big).$$

Since $\alpha - 1 < 0$, the function

$$f(u) = u^{\alpha-1}$$

is strictly decreasing on $(0, \infty)$

$$(1+x)^{\alpha-1} < x^{\alpha-1} \quad \Rightarrow \quad g'(x) < 0,$$

thus g is decreasing on $[0, \infty)$, so $g(x) \leq 0$ for all $x > 0$, i.e.

$$(1+x)^\alpha \leq 1 + x^\alpha \qquad .$$

Now substitute $x := \dfrac{G}{F} \geq 0$. This gives

$$\left(1 + \frac{G}{F}\right)^\alpha \leq 1 + \left(\frac{G}{F}\right)^\alpha,$$

finally multiply both sides by $F^\alpha > 0$ to recover

$$(F+G)^\alpha \leq F^\alpha + G^\alpha.$$

The cases $\alpha = 0$ and $\alpha = 1$ are trivial. □

Lemma 1.4. *[6] If $x(t)$ is a positive solution of equation* (1.1)*and*

$$\int_{t_1}^{\infty} r^{-1/\alpha}(s)\, ds = \infty,$$

then $z(t) = x(t) + p(t)x(\tau(t))$ satisfies $z(t) > 0$, $z'(t) > 0$, and $(r(t)(z'(t))^\alpha)' < 0$ eventually.

Proof. Assume that $x(t) > 0$ is a solution of (1.1). Then (1.1) implies

$$(r(t)\,[z'(t)]^\alpha)' = -q(t)x^\beta(\sigma(t)) < 0.$$

Therefore, $r(t)[z'(t)]^\alpha$ is decreasing and thus either $z'(t) > 0$ or $z'(t) < 0$ eventually, let say for $t \geq t_1$. If $z'(t) < 0$, then there exists a constant $c > 0$ such that:

$$(z'(t))^\alpha \leq -c/r(t) \quad \Longrightarrow \quad z'(t) \leq -\left(\frac{c}{r(t)}\right)^{1/\alpha} < 0.$$

Integrating this inequality from t_1 to t:

$$z(t) \leq z(t_1) - c^{1/\alpha} \int_{t_1}^{t} r^{-1/\alpha}(s) ds.$$

As $t \to \infty$, the integral term goes to ∞.
Hence

$$\lim_{t\to\infty} z(t) = -\infty.$$

This contradicts the fact that $z(t)$ must be positive, and we conclude that $z'(t) > 0$. □

Lemma 1.5. *Consider the first-order delay differential equation*

$$x'(t) + q(t)\, x(\sigma(t)) = 0, \qquad t \geq t_0, \tag{1.2}$$

where $q, \sigma \in C([t_0, \infty), \mathbb{R}^+)$ *with* $\mathbb{R}^+ = [0, \infty)$, $\sigma(t) \leq t$ *for all* $t \geq t_0$, *and*

$$\lim_{t\to\infty} \sigma(t) = \infty.$$

Every solution of equation (1.2) *is oscillatory if one of the following conditions holds:*

(i) ***(Ladas et al. [17])*** *If* σ *is nondecreasing and*

$$A := \limsup_{t\to\infty} \int_{\sigma(t)}^{t} q(s)\, ds > 1. \tag{1.3}$$

(ii) ***(Koplatadze and Chanturija [16])*** *If*

$$B := \liminf_{t\to\infty} \int_{\sigma(t)}^{t} q(s)\, ds > \frac{1}{e}. \tag{1.4}$$

Lemma 1.6. *[25] Consider a first-order delay differential inequality:*

$$y'(t) + q(t) y(\sigma(t)) \leq 0, \tag{1.5}$$

where $q \in C([t_0, \infty))$, $q(t) > 0$, $\sigma \in C([t_0, \infty))$, $\sigma(t) < t$, *and* $\sigma(t) \to \infty$ *as* $t \to \infty$. *If the inequality* (1.5) *has a positive solution, then the associated equation*

$$y'(t) + q(t)y(\sigma(t)) = 0$$

also admits a positive solution.

Lemma 1.7. *[20] Assume that* $\Psi(s) := ks - ls^{1+\frac{1}{\alpha}}$, *where* k *and* l *are real constants,* $k, l > 0$. *Then we have*

$$\Psi(s) \leq \max_{s \in \mathbb{R}} \Psi(s) = \frac{1}{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} k^{\alpha+1} l^{-\alpha}.$$

Proof. To find the maximum value, let's compute the first derivative of $\Psi(s)$ with respect to s and set it equal to zero:

$$\Psi'(s) = k - l\left(1 + \frac{1}{\alpha}\right)s^{1/\alpha} = 0.$$

Then, the critical point occurs at

$$s^* = \left(\frac{k}{l}\right)^{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha}.$$

This critical point corresponds to a maximum since

$$\Psi''(s^*) = -l\left(\frac{\alpha+1}{\alpha}\right)\left(\frac{1}{\alpha}\right)(s^*)^{\frac{1-\alpha}{\alpha}},$$

substitute the value of s^* into the second derivative:

$$\Psi''(s^*) = -\frac{l(\alpha+1)}{\alpha^2}\left[\left(\frac{k}{l}\right)^{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\right]^{\frac{1-\alpha}{\alpha}}.$$

Then we obtain

$$\Psi''(s^*) = -\frac{l(\alpha+1)}{\alpha^2}\left(\frac{k}{l}\right)^{1-\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{1-\alpha} < 0.$$

Now Substitute the expressions for s^* back into the original function $\Psi(s) = ks - ls^{1+\frac{1}{\alpha}}$:

$$\Psi(s^*) = ks^* - ls^* s^{*1/\alpha}.$$

Substitute s^* from the first step:

$$\Psi(s^*) = ks^* - ls^* \left[\frac{k}{l}\left(\frac{\alpha}{\alpha+1}\right)\right].$$

Simplify the expression:

$$\Psi(s^*) = ks^* - ks^* \left(\frac{\alpha}{\alpha+1}\right),$$

or

$$\Psi(s^*) = \frac{ks^*}{\alpha+1}.$$

Then,

$$\Psi(s^*) = \frac{k}{\alpha+1}\left(\frac{k}{l}\right)^{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha} = \frac{1}{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} k^{\alpha+1} l^{-\alpha}.$$

Thus,

$$\Psi(s) \leq \max_{s\in\mathbb{R}} \Psi(s) = \frac{1}{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} k^{\alpha+1} l^{-\alpha}.$$

□

Lemma 1.8. *(Young's inequality) [24] Let $a, b \geq 0$ and let $p, q > 1$ satisfy $\frac{1}{p}+\frac{1}{q} = 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.6}$$

Moreover, equality holds if and only if $a^p = b^q$.

Proof. If $a = 0$ or $b = 0$ it is trivial. Assume $a, b > 0$ and set

$$x := a^p, \qquad y := b^q, \qquad t := \frac{1}{p} \in (0, 1).$$

By applying Jensen's inequality Lemma 1.1 to the concave function $f(x) = \ln$ is concave, we have

$$\ln\Big(tx + (1-t)y\Big) \geq t\ln x + (1-t)\ln y.$$

Exponentiating gives

$$tx + (1-t)y \geq x^t y^{1-t}.$$

Using $x^t = a^{pt} = a$ and $y^{1-t} = b^{q(1-t)} = b$, we get

$$tx + (1-t)y = \frac{a^p}{p} + \frac{b^q}{q} \geq x^t y^{1-t} = ab.$$

□

Lemma 1.9 ([5]). *Let $\alpha \geq 1$ be a ratio of two odd numbers. Then*

$$A^{\frac{\alpha+1}{\alpha}} - (A-B)^{\frac{\alpha+1}{\alpha}} \leq \frac{B^{1/\alpha}}{\alpha}\Big[(1+\alpha)A - B\Big], \qquad AB \geq 0, \tag{1.7}$$

and

$$-C\, v^{\frac{\alpha+1}{\alpha}} + Dv \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \, \frac{D^{\alpha+1}}{C^\alpha}, \qquad C > 0. \tag{1.8}$$

Proof. Set

$$p := \frac{\alpha+1}{\alpha} = 1 + \frac{1}{\alpha} > 1.$$

Since α is a ratio of two odd integers, the real powers $x^{1/\alpha}$ and x^p are well-defined for all $x \in \mathbb{R}$ (odd roots), and in particular $x^p \geq 0$ for all $x \in \mathbb{R}$.

(i) Proof of (1.7)**.** If $B = 0$, the inequality is an equality. Assume $B \neq 0$. From $AB \geq 0$, A and B have the same sign, hence

$$u := \frac{A}{B} \geq 0.$$

Then since $p = 1 + \dfrac{1}{\alpha}$

$$\frac{B^{1/\alpha}}{\alpha}\Big[(1+\alpha)A - B\Big] = \frac{1}{\alpha}\, B^{1/\alpha}\Big[(1+\alpha)Bu - B\Big] = \frac{1}{\alpha}\, B^p\Big[(1+\alpha)u - 1\Big].$$

Because $B^p > 0$, (1.7) is equivalent to

$$u^p - (u-1)^p \leq \frac{1}{\alpha}\Big[(1+\alpha)u - 1\Big], \qquad u \geq 0. \tag{1.9}$$

Define

$$F(u) := \frac{1}{\alpha}\Big[(1+\alpha)u - 1\Big] - u^p + (u-1)^p, \qquad u \geq 0.$$

We prove that $F(u) \geq 0$ for all $u \geq 0$.

If $u = 0$ or $u = 1$, the claim is trivial since $F(0) = 1 - \dfrac{1}{\alpha} \geq 0$ and $F(1) = 0$. Now, consider the following cases:

Case 1: For $0 < u < 1$,

$$F'(u) = \frac{1+\alpha}{\alpha} - pu^{1/\alpha} + p(u-1)^{1/\alpha} = p\Big(1 - u^{1/\alpha} - (1-u)^{1/\alpha}\Big).$$

Let $q := \dfrac{1}{\alpha} \in (0,1]$ and define $g(t) := t^q + (1-t)^q - 1$ on $[0,1]$. Since $g(t)$ is concave on $[0,1]$, and $g(0) = g(1) = 0$, we have

$$g(t) \geq (1-t)g(0) + tg(1) = 0 \quad (0 \leq t \leq 1),$$

i.e. $t^q + (1-t)^q \geq 1$. Therefore $F'(u) = p(1 - u^q - (1-u)^q) \leq 0$ on $(0,1)$. Moreover,

$$F(0) = 1 - \frac{1}{\alpha} \geq 0, \qquad F(1) = 0.$$

Hence $F(u) \geq 0$ for all $u \in [0,1]$.

Case 2: For $u > 1$,

$$F''(u) = p(p-1)\Big((u-1)^{p-2} - u^{p-2}\Big).$$

Since

$$p - 2 = \frac{1-\alpha}{\alpha} \leq 0,$$

the function $x \mapsto x^{p-2}$ is nonincreasing on $(0,\infty)$, hence $(u-1)^{p-2} \geq u^{p-2}$ and so $F''(u) \geq 0$. Therefore F is convex on $[1,\infty)$. Also,

$$F(1) = 0, \qquad F'(1) = \frac{1+\alpha}{\alpha} - p = 0.$$

A convex function with $F'(1) = 0$ has a global minimum at $u = 1$, hence $F(u) \geq 0$ for all $u \geq 1$.

Combining both cases yields $F(u) \geq 0$ for all $u \geq 0$, i.e. (1.9). This proves (1.7).

(ii) Proof of (1.8) Let $p = \dfrac{\alpha+1}{\alpha}$ and $p' = \alpha + 1$ so that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.
By Young's inequality (Lemma 1.8), for all $a, b \geq 0$,

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Choose

$$a := \left(\frac{(\alpha+1)C}{\alpha}\right)^{1/p} v, \qquad b := D\left(\frac{\alpha}{(\alpha+1)C}\right)^{1/p}.$$

Then $ab = Dv$ and

$$\frac{a^p}{p} = \frac{1}{p} \cdot \frac{(\alpha+1)C}{\alpha} v^p = Cv^p,$$

because $p = \dfrac{\alpha+1}{\alpha}$. Hence

$$Dv \leq Cv^p + \frac{b^{p'}}{p'}.$$

Rearranging gives

$$-Cv^p + Dv \leq \frac{b^{p'}}{p'}.$$

Now compute (note that $\dfrac{p'}{p} = \alpha$):

$$b^{p'} = D^{\alpha+1}\left(\frac{\alpha}{(\alpha+1)C}\right)^{p'/p} = D^{\alpha+1}\left(\frac{\alpha}{(\alpha+1)C}\right)^{\alpha}.$$

Therefore

$$\frac{b^{p'}}{p'} = \frac{1}{\alpha+1} D^{\alpha+1} \left(\frac{\alpha}{(\alpha+1)C} \right)^{\alpha} = \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{D^{\alpha+1}}{C^{\alpha}},$$

which is exactly (1.8). □

Chapter 2

Oscillation of second-order neutral delay differential equations in canonical case

In this chapter we consider some oscillation results for a class of second-order neutral delay differential equations in the canonical case.

The canonical case is the most fundamental structural assumption in the oscillation analysis of NDEs. Historically, the concept evolved from the classification framework pioneered by **Myshkis** [23], which linked the growth rate of the principal coefficient to the asymptotic behavior of solutions, and was later formalized by **Hale** [15]. Mathematically, the divergence condition, serves as the essential discriminator because it guarantees the fundamental monotonic property, that the neutral term $z(t)$ for any eventually positive solution must satisfy $z'(t) > 0$ see Lemma 1.4.

During the past two decades, the canonical case has become central to oscillation analysis, serving as a structural assumption in the development of different criteria for second-order neutral equations [1, 11, 19].

Within this canonical framework, Bohner and Saker [9] extended oscillation criteria to half-linear equations, formulating results that generalized classical second-order cases. Grace et al. [11] advanced this line by refining the conditions on delays and coefficients, thereby improved earlier theorems. Moaaz et al. [22] continued this progression by relaxing restrictions on the neutral term and considering wider ranges of the parameters α and β. In parallel, Yang, Bazighifan, and Cesarano employed variants of the Riccati transformation, allowing alternative representations of oscillation criteria.

The chapter is organised into three main sections. In sections 2.1 and 2.2, we present two complementary reduction techniques that transform the second-order equation into a first-order neutral delay differential inequality, simplifying the analysis. The essential difference between these two approaches lies in the treatment of the delay term: Section 2.1 performs the reduction without explicitly incorporating $\tau(t)$, whereas Section 2.2 accounts for the delay effect in the transformation. Finally, Section 2.3 introduces the Riccati transformation method, which has been widely used to obtain precise oscillation conditions.

2.1 Reduction from second-order to first-order DDEs I

Consider the second-order neutral delay differential equation

$$(r(t)(z'(t))^{\alpha})' + q(t)\,x^{\beta}(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{2.1}$$

Define

$$z(t) := x(t) + p(t)x(\tau(t)),$$

under the canonical case

$$\int_{t_0}^{\infty} r^{-1/\alpha}(s)\,ds = \infty, \tag{2.2}$$

and assume that:

(H1) α and β are quotients of odd positive integers

(H2) $r(t) > 0$, $p(t), q(t) \geq 0$ are continuous functions and $q(t)$ is not identically zero.

(H3) Delay functions $\tau(t), \sigma(t)$ are continuous, where $\tau(t) \leq t$, $\sigma(t) < t$, and

$$\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty.$$

The methodology relies on reducing to the second-order NDDE to a first-order neutral delay differential inequality (NDDI), then well-known oscillation criteria for first order equation is used.

Improved reduction for half-linear type (Grace et al., 2018)

This methodology utilizes a **Riccati transformation technique** to reduce the second-order equation to first-order order inequality. In this work ($\alpha = \beta$), the general NDDE (2.1) takes the simplified form:

$$(r(t)(z'(t))^{\alpha})' + q(t)\, x^{\alpha}(\sigma(t)) = 0. \tag{2.3}$$

We assume that conditions **(H1)–(H3)** and $0 \leq p(t) < 1$ are satisfied.

The auxiliary functions used in this framework are:

$$\mathbf{Q_G}(\mathbf{t}) = \Big(1 - p(\sigma(t))\Big)^{\alpha} q(t), \quad \mathbf{R}(\mathbf{t}) = \int_{t_1}^{t} r^{-1/\alpha}(s)\, ds,$$

and

$$\tilde{\mathbf{R}}(\mathbf{t}) = R(t) + \frac{1}{\alpha}\int_{t_1}^{t} R(s) R^{\alpha}(\sigma(s)) Q_G(s)\, ds.$$

Theorem 2.1 ([11])**.** *Let the canonical case* (2.2) *be satisfied. If* σ *is nondecreasing and either*

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} Q_G(s)\tilde{R}^{\alpha}(\sigma(s))\,ds > 1 \tag{2.4}$$

or

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} Q_G(s)\tilde{R}^{\alpha}(\sigma(s))\,ds > \frac{1}{e} \tag{2.5}$$

then every solution of (2.1) *is oscillatory.*

Proof. To the contrary assume that (2.1) has a nonoscillatory solution $x(t)$ on $[t_0,\infty)$. Without loss of generality, we may assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$.

In view if (2.2) by Lemma 1.4 $z'(t) \geq 0$. Then by the definition of $z(t)$, we obtain, for $t \geq t_1$:

$$x(t) = z(t) - p(t)x(\tau(t)) \geq z(t) - p(t)z(\tau(t)) \geq (1 - p(t))z(t).$$

Evaluating this inequality at the delayed argument $\sigma(t)$, and raising both sides to the power α

$$x^{\alpha}(\sigma(t)) \geq (1 - p(\sigma(t)))^{\alpha} z^{\alpha}(\sigma(t)). \tag{2.6}$$

Now, from (2.1)

$$(r(t)(z'(t))^{\alpha})' = -q(t)\,x^{\alpha}(\sigma(t)). \tag{2.7}$$

Multiplying the inequality (2.6) by $-q(t)$, we get

$$-q(t)\,x^{\alpha}(\sigma(t)) \leq -q(t)(1 - p(\sigma(t)))^{\alpha} z^{\alpha}(\sigma(t)). \tag{2.8}$$

Substituting the right-hand side of the equality (2.7) into the left-hand side of the inequality (2.8), we have

$$(r(t)(z'(t))^{\alpha})' \leq -q(t)(1-p(\sigma(t)))^{\alpha} z^{\alpha}(\sigma(t)).$$

Substituting the definition of the auxiliary function $Q_G(t) := (1-p(\sigma(t)))^{\alpha} q(t)$, we obtain the first-order differential inequality:

$$(r(t)(z'(t))^{\alpha})' \leq -Q_G(t) z^{\alpha}(\sigma(t)). \tag{2.9}$$

On the other hand, it follows from the decreasing of $r^{1/\alpha}(t)z'(t)$ that

$$\begin{aligned} z(t) &= z(t_1) + \int_{t_1}^{t} \frac{1}{r^{1/\alpha}(s)} r^{1/\alpha}(s) z'(s) ds \\ &\geq R(t) r^{1/\alpha}(t) z'(t). \end{aligned} \tag{2.10}$$

Then

$$(z(t) - R(t) r^{1/\alpha}(t) z'(t))' = -R(t)(r^{1/\alpha}(t) z'(t))'. \tag{2.11}$$

Applying the chain rule, we have

$$R(t)(r(t)(z'(t))^{\alpha})' = \alpha R(t)(r^{1/\alpha}(t) z'(t))^{\alpha-1}(r^{1/\alpha}(t) z'(t))'. \tag{2.12}$$

By virtue of (2.12), the latter equality yields

$$-R(t)(r^{1/\alpha}(t) z'(t))' \geq \frac{1}{\alpha} R(t)(r^{1/\alpha}(t) z'(t))^{1-\alpha} Q_G(t) z^{\alpha}(\sigma(t)). \tag{2.13}$$

Combining (2.11) and (2.13), we obtain

$$(z(t) - R(t) r^{1/\alpha}(t) z'(t))' \geq \frac{1}{\alpha} R(t)(r^{1/\alpha}(t) z'(t))^{1-\alpha} Q_G(t) z^{\alpha}(\sigma(t)). \tag{2.14}$$

Integrating (2.14) from t_1 to t, we have

$$z(t) \geq R(t)r^{1/\alpha}(t)z'(t) + \frac{1}{\alpha}\int_{t_1}^{t}(r^{1/\alpha}(s)z'(s))^{1-\alpha}R(s)Q_G(s)z^{\alpha}(\sigma(s))ds. \tag{2.15}$$

Taking (2.10) and the monotonicity of $r^{1/\alpha}(t)z'(t)$ into account, we arrive at

$$z(t) \geq r^{1/\alpha}(t)z'(t)\left(R(t) + \frac{1}{\alpha}\int_{t_1}^{t} R(s)R^{\alpha}(\sigma(s))Q_G(s)ds\right).$$

Thus, we conclude that, using the definition of $\tilde{R}(t)$:

$$z(\sigma(t)) \geq r^{1/\alpha}(\sigma(t))z'(\sigma(t))\tilde{R}(\sigma(t)). \tag{2.16}$$

Using (2.16) in (2.12), one can see that $y(t) = r(t)(z'(t))^{\alpha}$ is a positive solution of the first-order delay differential inequality

$$y'(t) + Q_G(t)\tilde{R}^{\alpha}(\sigma(t))y(\sigma(t)) \leq 0 \tag{2.17}$$

in view of Lemma 1.6, the associated delay differential equation

$$y'(t) + Q_G(t)\tilde{R}^{\alpha}(\sigma(t))y(\sigma(t)) = 0 \tag{2.18}$$

also has a positive solution. However, by applying Lemma 1.5 to (2.18) with

$$q_1(t) := Q_G(t)\,\tilde{R}^{\alpha}(\sigma(t)),$$

we see that (2.4) implies

$$\limsup_{t\to\infty}\int_{\sigma(t)}^{t} q_1(s)\,ds > 1,$$

and (2.5) implies

$$\liminf_{t\to\infty}\int_{\sigma(t)}^{t} q_1(s)\,ds > \frac{1}{e}.$$

Since σ is nondecreasing, Lemma 1.5 guarantees that every solution of (2.18) is oscillatory. This contradicts the existence of a positive solution of (2.18). Hence (2.1) cannot have positive solutions. □

Example 2.2. *[11] Consider the second-order half-linear neutral differential equation:*

$$\left((z'(t))^{\alpha}\right)' + \frac{q_0}{t^{\alpha+1}} x^{\alpha}(\lambda t) = 0, \quad t \geq 1,$$

where $z(t) = x(t) + p_0 x(\tau(t))$.

$$r(t) = 1, \quad q(t) = \frac{q_0}{t^{\alpha+1}}, \quad p(t) = p_0, \quad \sigma(t) = \lambda t$$

$$Q(t) = q(t)[1 - p(\sigma(t))]^{\alpha} = \frac{q_0(1-p_0)^{\alpha}}{t^{\alpha+1}}$$

$$R(t) = \int_{t_1}^{t} r^{-1/\alpha}(s)\, ds = t - t_1 \approx t \quad (\textit{for large } t)$$

Using the iterative formula from Theorem 2.1:

$$\tilde{R}(t) = R(t) + \frac{1}{\alpha} \int_{t_1}^{t} R(s) R^{\alpha}(\sigma(s)) Q(s)\, ds$$

Substituting the identified functions:

$$\tilde{R}(t) \approx t + \frac{1}{\alpha} \int_{t_1}^{t} s \cdot (\lambda s)^{\alpha} \cdot \frac{q_0(1-p_0)^{\alpha}}{s^{\alpha+1}}\, ds$$

$$\tilde{R}(t) \approx t + \frac{q_0(1-p_0)^{\alpha}\lambda^{\alpha}}{\alpha} \int_{t_1}^{t} 1\, ds = t\left(\frac{\alpha + q_0(1-p_0)^{\alpha}\lambda^{\alpha}}{\alpha}\right)$$

The oscillation criterion is

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)\tilde{R}^{\alpha}(\sigma(s))\, ds > \frac{1}{e}$$

.

Let ρ be this integral:

$$\rho = \int_{\lambda t}^{t} \frac{q_0(1-p_0)^\alpha}{s^{\alpha+1}} \left[\lambda s \left(\frac{\alpha + q_0(1-p_0)^\alpha \lambda^\alpha}{\alpha}\right)\right]^\alpha ds.$$

Then,

$$\rho = q_0(1-p_0)^\alpha \lambda^\alpha \frac{(\alpha + q_0(1-p_0)^\alpha \lambda^\alpha)^\alpha}{\alpha^\alpha} \int_{\lambda t}^{t} \frac{1}{s}\, ds$$

$$\rho = (1-p_0)^\alpha q_0 \lambda^\alpha \frac{(\alpha + (1-p_0)^\alpha q_0 \lambda^\alpha)^\alpha}{\alpha^\alpha} \ln \frac{1}{\lambda}.$$

Let $\alpha = 1/3$, $\lambda = 0.9$, and $p_0 = 0$. The condition for oscillation is given by $\rho > 1/e$, where:

$$\rho = (1-p_0)^\alpha q_0 \lambda^\alpha \frac{(\alpha + (1-p_0)^\alpha q_0 \lambda^\alpha)^\alpha}{\alpha^\alpha} \ln \frac{1}{\lambda}.$$

By substituting the given values ($\alpha = 1/3, \lambda = 0.9, p_0 = 0$), the condition becomes:

$$q_0(0.9)^{1/3} \frac{\left(1/3 + q_0(0.9)^{1/3}\right)^{1/3}}{(1/3)^{1/3}} \ln \frac{1}{0.9} > \frac{1}{e}.$$

Solving this inequality numerically, we find that every solution of the equation oscillates if:

$$q_0 > 1.92916.$$

In comparison, the earlier criterion from [6] is unable to ensure oscillation unless

$$q_0 > 3.61643.$$

2.2 Reduction from second-order to first-order DDEs II

Consider the second-order neutral delay differential equation

$$(r(t)(z'(t))^{\alpha})' + q(t)\, x^{\beta}(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{2.19}$$

Define

$$z(t) := x(t) + p(t)x(\tau(t)),$$

under the canonical case

$$\int_{t_0}^{\infty} r^{-1/\alpha}(s)\, ds = \infty. \tag{2.20}$$

Besides conditions **(H1)–(H3)** we further assume that:

(H4) $\tau'(t) > \tau_0 > 0$, $\tau \circ \sigma = \sigma \circ \tau$ and $0 \leq p(t) \leq p_0 < \infty$ for all t.

The following auxiliary functions are used in the reduction theorems:

$$\begin{aligned} w(t) &= r(t)(z'(t))^{\alpha}, \\ R(t) &= \int_{t_1}^{t} r^{-1/\alpha}(s)\, ds, \\ Q(t) &= \min\{q(t), q(\tau(t))\} \end{aligned}$$

and

$$Q^{*}(t) = Q(t)\left(\int_{t_1}^{\sigma(t)} r^{-1/\alpha}(s)\, ds\right)^{\beta}.$$

Case I: Sublinear equations $(0 < \beta \leq 1)$

Theorem 2.3. *[6] Assume that $0 < \beta \leq 1$. If the first-order neutral delay differential inequality*

$$\left(w(t) + \frac{p_0^{\beta}}{\tau_0}\, w(\tau(t))\right)' + Q^{*}(t)\, w^{\beta/\alpha}(\sigma(t)) \leq 0 \tag{2.21}$$

has no positive solution, then the equation (2.19) *is oscillatory.*

Proof. Assume to the contrary, that x be a positive solution of (2.19). Then $z(t) = x(t) + p(t)x(\tau(t))$ satisfies

$$0 = (r(t)(z'(t))^{\alpha})' + q(t)x^{\beta}(\sigma(t)), \tag{2.22}$$

which in view of (H_2), (H_3) and (H_4) yields

$$\begin{aligned} 0 &= \frac{{p_0}^{\beta}}{\tau'(t)}\left(r(\tau(t))[z'(\tau(t))]^{\alpha}\right)' + {p_0}^{\beta}q(\tau(t))x^{\beta}(\sigma(\tau(t))) \\ &\geq \frac{{p_0}^{\beta}}{\tau_0}\left(r(\tau(t))[z'(\tau(t))]^{\alpha}\right)' + {p_0}^{\beta}q(\tau(t))x^{\beta}(\sigma(\tau(t))). \end{aligned} \tag{2.23}$$

Combining (2.22) and (2.23), we get

$$(r(t)(z'(t))^{\alpha})' + \frac{{p_0}^{\beta}}{\tau_0}\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)' + Q(t)\left[x^{\beta}(\sigma(t)) + {p_0}^{\beta}q(\tau(t))x^{\beta}(\sigma(\tau(t)))\right] \leq 0. \tag{2.24}$$

On the other hand, by Lemma 1.3, we have

$$z^{\beta}(\sigma(t)) = (x(\sigma(t)) + p(\sigma(t))x(\tau(\sigma(t))))^{\beta} \leq x^{\beta}(\sigma(t)) + {p_0}^{\beta}x^{\beta}(\sigma(\tau(t))). \tag{2.25}$$

Using (2.25) in (2.24), we have

$$\left(r(t)(z'(t))^{\alpha} + \frac{{p_0}^{\beta}}{\tau_0}r(\tau(t))(z'(\tau(t)))^{\alpha}\right)' + Q(t)z^{\beta}(\sigma(t)) \leq 0. \tag{2.26}$$

It follows from Lemma 1.4 that $w(t) = r(t)(z'(t))^{\alpha} > 0$ is decreasing and so

$$\begin{aligned} z(t) &\geq \int_{t_1}^{t} \left[r(s)(z'(s))^{\alpha}\right]^{1/\alpha} r^{-1/\alpha}(s)ds \\ &\geq w^{1/\alpha}(t)\int_{t_1}^{t} r^{-1/\alpha}(s)ds. \end{aligned} \tag{2.27}$$

Therefore, using (2.27) in (2.26), we see that w is a positive solution of

$$\left(w(t)+\frac{{p_0}^{\beta}}{\tau_0}w(\tau(t))\right)'+Q(t)\left(\int_{t_1}^{\sigma(t)}r^{-1/\alpha}(s)ds\right)^{\beta}w^{\beta/\alpha}(\sigma(t))\leq 0. \tag{2.28}$$

This contradicts contradicts the fundamental assumption of this theorem which states that inequality (2.3) has no positive solutions. □

Theorem 2.4. *[6] Assume $0<\beta\leq 1$ and $\sigma(t)\leq\tau(t)\leq t$. Let $C_{\tau,p}=\dfrac{\tau_0^{\beta/\alpha}}{(\tau_0+p_0^{\beta})^{\beta/\alpha}}$. If the delay differential equation*

$$y'(t)+C_{\tau,p}\,Q^{*}(t)\,y^{\beta/\alpha}(\tau^{-1}(\sigma(t)))=0 \tag{2.29}$$

is oscillatory, then (2.19) is oscillatory.

Proof. Assume that x is a positive solution of (2.19). Define

$$y(t)=w(t)+\frac{p_0^{\beta}}{\tau_0}w(\tau(t)), \tag{2.30}$$

where $w(t)=r(t)(z'(t))^{\alpha}$ and $z(t)=x(t)+p(t)x(\tau(t))$.

From the properties of $z(t)$ using Lemma 1.4 and hypotheses (H_2)–(H_4), since $w(t)$ is nonincreasing and $\tau(t)\leq t$, replacing $w(t)$ by $w(\tau(t))$ in

$$y(t)=w(t)+\frac{p_0^{\beta}}{\tau_0}w(\tau(t))$$

gives

$$y(t)\leq w(\tau(t))\left(1+\frac{p_0^{\beta}}{\tau_0}\right). \tag{2.31}$$

Substituting $t=\tau^{-1}(\sigma(t))$

$$w^{\beta/\alpha}(\sigma(t))\geq\frac{\tau_0^{\beta/\alpha}}{(\tau_0+p_0^{\beta})^{\beta/\alpha}}\,y^{\beta/\alpha}(\tau^{-1}(\sigma(t))). \tag{2.32}$$

Using (2.32) in the transformed inequality (2.21), we deduce that $y(t)$ satisfies

$$y'(t) + \frac{\tau_0^{\beta/\alpha}}{(\tau_0 + p_0^\beta)^{\beta/\alpha}} Q^*(t)\, y^{\beta/\alpha}(\tau^{-1}(\sigma(t))) \le 0. \tag{2.33}$$

By Lemma 1.6, the associated equation

$$y'(t) + C_{\tau,p} Q^*(t) y^{\beta/\alpha}(\tau^{-1}(\sigma(t))) = 0$$

admits a positive solution, which contradicts the assumption that this equation is oscillatory. Therefore, the original equation (2.19) must be oscillatory. □

Case II: Superlinear equations ($\beta \ge 1$)

Theorem 2.5. *[6] Assume that $\beta \ge 1$. If the first order delay differential inequality*

$$\left(w(t) + \frac{p_0^\beta}{\tau_0} w(\tau(t)) \right)' + 2^{1-\beta} Q^*(t)\, w^{\beta/\alpha}(\sigma(t)) \le 0 \tag{2.34}$$

has no positive solution, then (2.19) is oscillatory.

Proof. Assume to the contrary that x is a positive solution of (2.19). From (2.19) and hypotheses (H_2)–(H_4), we have

$$0 = \left(r(t)(z'(t))^\alpha \right)' + q(t) x^\beta(\sigma(t)). \tag{2.35}$$

Similarly,

$$\begin{aligned} 0 &= \frac{p_0^\beta}{\tau'(t)} \left(r(\tau(t))(z'(\tau(t)))^\alpha \right)' + p_0^\beta q(\tau(t)) x^\beta(\sigma(\tau(t))) \\ &\ge \frac{p_0^\beta}{\tau_0} \left(r(\tau(t))(z'(\tau(t)))^\alpha \right)' + p_0^\beta q(\tau(t)) x^\beta(\sigma(\tau(t))). \end{aligned} \tag{2.36}$$

Combining (2.35) and (2.36), we obtain

$$\Big(r(t)(z'(t))^{\alpha}\Big)'+\frac{p_0^{\beta}}{\tau_0}\Big(r(\tau(t))(z'(\tau(t)))^{\alpha}\Big)'+Q(t)\Big[x^{\beta}(\sigma(t))+p_0^{\beta}x^{\beta}(\sigma(\tau(t)))\Big]\leq 0. \quad (2.37)$$

By Lemma 1.3,

$$z^{\beta}(\sigma(t))=\Big(x(\sigma(t))+p(\sigma(t))x(\tau(\sigma(t)))\Big)^{\beta}\leq 2^{\beta-1}\Big[x^{\beta}(\sigma(t))+p_0^{\beta}x^{\beta}(\sigma(\tau(t)))\Big]. \quad (2.38)$$

Using (2.38) in (2.37), we get

$$\Big(r(t)(z'(t))^{\alpha}+\frac{p_0^{\beta}}{\tau_0}r(\tau(t))(z'(\tau(t)))^{\alpha}\Big)'+2^{1-\beta}Q(t)z^{\beta}(\sigma(t))\leq 0. \quad (2.39)$$

By Lemma 1.4, $w(t)=r(t)(z'(t))^{\alpha}>0$ is decreasing, and

$$\begin{aligned} z(t) &\geq \int_{t_1}^{t}\Big[r(s)(z'(s))^{\alpha}\Big]^{1/\alpha}r^{-1/\alpha}(s)\,ds \\ &\geq w^{1/\alpha}(t)\int_{t_1}^{t}r^{-1/\alpha}(s)\,ds. \end{aligned} \quad (2.40)$$

Therefore, using (2.40) in (2.39), we see that w satisfies

$$\Big(w(t)+\frac{p_0^{\beta}}{\tau_0}w(\tau(t))\Big)'+2^{1-\beta}Q^*(t)w^{\beta/\alpha}(\sigma(t))\leq 0, \quad (2.41)$$

where

$$Q^*(t)=Q(t)\Big(\int_{t_1}^{\sigma(t)}r^{-1/\alpha}(s)\,ds\Big)^{\beta}.$$

This contradicts the assumption that the inequality (2.43) has no positive solution. Hence, the original equation (2.19) is oscillatory. □

Theorem 2.6. *[6] Assume $\beta\geq 1$ and $\sigma(t)\leq\tau(t)\leq t$. Let $C_{\tau,p}=\dfrac{\tau_0^{\beta/\alpha}}{(\tau_0+p_0^{\beta})^{\beta/\alpha}}$. If the delay differential equation*

$$y'(t)+C_{\tau,p}\,2^{1-\beta}Q^*(t)\,y^{\beta/\alpha}(\tau^{-1}(\sigma(t)))=0 \quad (2.42)$$

is oscillatory, then (2.19) *is oscillatory.*

Proof. Assume for contradiction, that (2.19) admits an eventually positive solution $x(t)$.

Following the proof as Theorem 2.5, $w(t)$ satisfies the neutral delay differential inequality

$$\left(w(t) + \frac{p_0^\beta}{\tau_0} w(\tau(t))\right)' + 2^{1-\beta} Q^*(t) w^{\beta/\alpha}(\sigma(t)) \leq 0, \tag{2.43}$$

where

$$Q^*(t) = Q(t) \left(\int_{t_1}^{\sigma(t)} r^{-1/\alpha}(s)\, ds \right)^\beta.$$

By substituting $y(t) = w(t) + \frac{p_0^\beta}{\tau_0} w(\tau(t))$ given in (2.30) and employing the lower bound for $w^{\beta/\alpha}(\sigma(t))$ obtained in (2.32) into the inequality (2.43), we deduce that:

$$y'(t) + C_{\tau,p}\, 2^{1-\beta} Q^*(t)\, y^{\beta/\alpha}(\tau^{-1}(\sigma(t))) \leq 0. \tag{2.44}$$

where

$$C_{\tau,p} = \frac{\tau_0^{\beta/\alpha}}{(\tau_0 + p_0^\beta)^{\beta/\alpha}}.$$

By the comparison principle, the associated delay differential equation

$$y'(t) + C_{\tau,p}\, 2^{1-\beta} Q^*(t)\, y^{\beta/\alpha}(\tau^{-1}(\sigma(t))) = 0$$

must have an eventually positive solution, which contradicts the assumption that this equation is oscillatory. Therefore, (2.19) is oscillatory. □

The following example demonstrate the application of the above theorem.

Example 2.7. *Consider the second-order neutral differential equation of Emden-Fowler type:*

$$(x(t) + 0.5x(0.95t))'' + \frac{q_0}{t^2} x(0.9t) = 0, \quad t \geq t_0 > 0. \tag{2.45}$$

We have the following parameters:

- $r(t) = 1$, $\alpha = \beta = 1$, *and* $p(t) = p_0 = 0.5$.
- *The delay functions are* $\tau(t) = 0.95t$ *and* $\sigma(t) = 0.9t$, *satisfying the condition* $\tau \circ \sigma = \sigma \circ \tau$. *Therefore, assumption* (H_4) *is hold.*
- *The inverse of the neutral delay is* $\tau^{-1}(t) = \frac{t}{0.95}$, *which leads to the composite delay* $\tau^{-1}(\sigma(t)) = \frac{0.9}{0.95}t \approx 0.947t$.

Since $\int_{t_0}^{\infty} r^{-1}(s)ds = \infty$, *we are investigating the canonical case. To apply Theorem 2.6, we define the constant* $C_{\tau,p}$ *as:*

$$C_{\tau,p} = \frac{\tau_0^{\beta/\alpha}}{(\tau_0 + p_0^{\beta})^{\beta/\alpha}} = \frac{0.95}{0.95 + 0.5} = \frac{0.95}{1.45} \approx 0.6552. \tag{2.46}$$

The integral coefficient $Q^*(t)$ *is determined by:*

$$Q^*(t) = Q(t)\left(\int_{t_1}^{\sigma(t)} r^{-1}(s)ds\right)^{\beta} = \frac{q_0}{t^2}(0.9t) = \frac{0.9q_0}{t}. \tag{2.47}$$

According to the comparison principle in Theorem 2.6, the oscillation of the original equation is guaranteed if the first-order delay differential equation

$$y'(t) + C_{\tau,p}2^{1-\beta}Q^*(t)y^{\beta/\alpha}(\tau^{-1}(\sigma(t))) = 0 \tag{2.48}$$

is oscillatory. Substituting our specific values, we obtain:

$$y'(t) + \frac{0.5897q_0}{t}y(0.947t) = 0. \tag{2.49}$$

Applying the well-known oscillation condition for Euler-type delay equations, $k \ln \frac{1}{\lambda} > \frac{1}{e}$, *where* $k = 0.5897q_0$ *and* $\lambda = 0.947$, *we have:*

$$0.5897q_0 \ln\left(\frac{1}{0.947}\right) > 0.3679. \tag{2.50}$$

Evaluating the logarithmic term $\ln(1.056) \approx 0.05446$*, the inequality becomes:*

$$0.03211q_0 > 0.3679 \implies q_0 > 11.457. \tag{2.51}$$

Consequently, every solution of the studied equation is oscillatory provided that $q_0 > 11.457$*.*

2.3 The Riccati transformation method

Based on the canonical condition. In this section, we consider the oscillation analysis of second-order Emden-Fowler neutral delay differential equations (NDEs) defined by the form [22]:

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)' + q(t)x^{\alpha}(\sigma(t)) = 0, \quad t \geq t_0, \tag{2.52}$$

where $z(t) = x(t) + p(t)x(\tau(t))$.

We assume the following conditions throughout this section:

(H1) α is a ratio of odd positive integers and $r(t) > 0$.

(H2) p, q are nonnegative functions on $[t_0, \infty)$, with $p(t) \leq p_0$ for some $p_0 > 0$, and q does not vanish identically on any half-line $[t_q, \infty)$.

(H3) $\tau(t) \leq t$ and $\sigma(t) \leq t$ are delay functions satisfying $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$.

The Riccati transformation method plays a fundamental role in oscillation theory. It provides a systematic approach to reduce second-order differential equations to first-order differential inequalities, thereby facilitating the derivation of oscillation criteria.

Although several oscillation criteria have been established for the canonical case, these approaches exhibit fundamental limitations. Moaaz, Ramos, and Awrejcewicz

[22] identified two major weaknesses in prior analyses:

(i) the neglect of delay functions, meaning previous criteria were not influenced by the neutral delay function τ.

(ii) the lack of precision in the ordinary case, where traditional conditions failed to provide the exact requirement for oscillation, such as for the Euler-type equation

$$x''(t) + q_0/t^2 x(t) = 0.$$

To overcome these limitations, a **nonstandard Riccati substitution** is introduced by Moaaz and his colleagues [22]. This transformation incorporates a testing function $\varphi(t)$ and is defined as:

$$w(t) := \varphi(t) \cdot \left(\frac{\mathcal{L}(z(t))}{(z(\sigma(t)))^\alpha} + \frac{p_0^\alpha}{\tau_0} \frac{\mathcal{L}(z(\tau(t)))}{(z(\sigma(t)))^\alpha} \right), \tag{2.53}$$

where $\mathcal{L}(z(t)) := r(t)(z'(t))^\alpha$.

And define:

$$Q(t) := \min\{q(t),\, q(\tau(t))\},$$

$$\rho(t) := \int_{t_0}^{t} r^{-1/\alpha}(\eta)\, d\eta \to \infty \quad \text{as} \quad t \to \infty, \tag{2.54}$$

and assume that

$$\tau \circ \sigma = \sigma \circ \tau, \quad \tau'(t) \geq \tau_0 > 0, \quad \sigma(t) \leq \tau(t), \tag{2.55}$$

Theorem 2.8. *[22] Assume that $\alpha \geq 1$, $\sigma'(t) > 0$, (2.54) and (2.55) are hold. If there exists a positive differentiable function $\varphi : [t_0, \infty) \to (0, \infty)$ such that*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(2^{1-\alpha} \varphi(\eta)\, Q(\eta) - \left(1 + \frac{p_0^\alpha}{\tau_0}\right) \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(\sigma(\eta)) \left(\varphi'(\eta)\right)^{\alpha+1}}{\left(\sigma'(\eta)\, \varphi(\eta)\right)^{\alpha}} \right) d\eta = \infty, \tag{2.56}$$

then every solution of the neutral delay differential equation (2.52) is oscillatory.

Proof. We proceed by contradiction. Suppose that equation (2.52) has an eventually positive solution on $[t_0, \infty)$. The case where it has an eventually negative solution can be treated similarly. Then, there exists $t_1 \geq t_0$ such that $x(t)$, $x(\tau(t))$, and $x(\sigma(t))$ are positive for all $t \geq t_1$. From Lemma 1.4, $z(t) > 0$, $z'(t) > 0$, and $(r(t)(z'(t))^\alpha)' < 0$ holds.

Define the operator:

$$\mathcal{L}(z(t)) := r(t)\Big(z'(t)\Big)^\alpha.$$

It follows from (2.52) that

$$q(\tau(t))\, x^\alpha(\sigma(\tau(t))) = -\frac{1}{\tau'(t)}\Big(r(\tau(t))\Big(z'(\tau(t))\Big)^\alpha\Big)' \leq -\frac{1}{\tau_0}\frac{d}{dt}\mathcal{L}(z(\tau(t))). \tag{2.57}$$

Combining (2.52) and (2.57), we obtain:

$$\begin{aligned}\frac{d}{dt}\Big(\mathcal{L}(z(t)) + \frac{p_0^\alpha}{\tau_0}\mathcal{L}(z(\tau(t)))\Big) &\leq -q(t)x^\alpha(\sigma(t)) - p_0^\alpha q(\tau(t))x^\alpha(\sigma(\tau(t))) \\ &\leq -Q(t)\Big(x^\alpha(\sigma(t)) + p_0^\alpha x^\alpha(\tau(\sigma(t)))\Big),\end{aligned} \tag{2.58}$$

where $Q(t) := \min\{q(t), q(\tau(t))\}$.

Using the inequality in Lemma 1.2:

$$(F+G)^\alpha \leq 2^{\alpha-1}(F^\alpha + G^\alpha), \tag{2.59}$$

with $F = x$ and $G = p_0 \cdot (x \circ \tau)$, we get:

$$z^\alpha \leq (x + p_0 \cdot (x \circ \tau))^\alpha \leq 2^{\alpha-1}\Big(x^\alpha + (p_0 \cdot (x \circ \tau))^\alpha\Big).$$

Substituting into (2.58), we obtain:

$$\frac{d}{dt}\Big(\mathcal{L}(z(t)) + \frac{p_0^\alpha}{\tau_0}\mathcal{L}(z(\tau(t)))\Big) \leq -2^{1-\alpha}Q(t)z^\alpha(\sigma(t)). \tag{2.60}$$

Next, define the nonstandard Riccati substitution:

$$w(t) := \varphi(t) \cdot \left(\frac{\mathcal{L}(z(t))}{(z \circ \sigma)^{\alpha}} + \frac{p_0^{\alpha}}{\tau_0} \frac{\mathcal{L}(z \circ \tau)}{(z \circ \sigma)^{\alpha}} \right).$$

Then $w(t) > 0$ for $t \geq t_1$, and after differentiation

$$\begin{aligned} w' = {} & \varphi \cdot \frac{1}{(z \circ \sigma)^{\alpha}} \left(\mathcal{L}(z) + \frac{p_0^{\alpha}}{\tau_0} \mathcal{L}(z \circ \tau) \right)' \\ & + \left(\varphi' \cdot \frac{\mathcal{L}(z)}{(z \circ \sigma)^{\alpha}} - \alpha \varphi \cdot \frac{\mathcal{L}(z)}{(z \circ \sigma)^{\alpha+1}} \cdot (z \circ \sigma)' \right) \\ & + \frac{p_0^{\alpha}}{\tau_0} \left(\varphi' \cdot \frac{\mathcal{L}(z \circ \tau)}{(z \circ \sigma)^{\alpha}} - \alpha \varphi \cdot \frac{\mathcal{L}(z \circ \tau)}{(z \circ \sigma)^{\alpha+1}} \cdot (z \circ \sigma)' \right). \end{aligned} \tag{2.61}$$

Since $(r(z')^{\alpha})' \leq 0$ and $\sigma(t) \leq \tau(t) \leq t$, we have:

$$r^{1/\alpha}(\sigma(t)) z'(\sigma(t)) \geq r^{1/\alpha}(\tau(t)) z'(\tau(t)) \geq r^{1/\alpha}(t) z'(t). \tag{2.62}$$

From (2.60)–(2.62), we have:

$$\begin{aligned} w' \leq {} & -2^{1-\alpha} \varphi \cdot Q \\ & + \left(r\varphi' \cdot \left(\frac{z'}{z \circ \sigma} \right)^{\alpha} - \alpha \varphi \cdot \mathcal{L}(z) \frac{(z \circ \sigma)'}{(z \circ \sigma)^{\alpha+1}} \cdot \frac{r^{1/\alpha} \cdot z' \cdot \sigma'}{(r \circ \sigma)^{1/\alpha}} \right) \\ & + \frac{p_0^{\alpha}}{\tau_0} \left((r \circ \tau) \cdot \varphi' \cdot \left(\frac{z' \circ \tau}{z \circ \sigma} \right)^{\alpha} - \alpha \varphi \cdot \mathcal{L}(z \circ \tau) \frac{(z \circ \sigma)'}{(z \circ \sigma)^{\alpha+1}} \cdot \frac{(r \circ \tau)^{1/\alpha}}{(r \circ \sigma)^{1/\alpha}} \cdot (z' \circ \tau) \cdot \sigma' \right), \end{aligned} \tag{2.63}$$

and so

$$\begin{aligned} w' \leq {} & -2^{1-\alpha} \varphi \cdot Q \\ & + \left(r \cdot \varphi' \cdot \left(\frac{z'}{z \circ \sigma} \right)^{\alpha} - \alpha \cdot \varphi \frac{r^{1+1/\alpha} \cdot \sigma'}{(r \circ \sigma)^{1/\alpha}} \cdot \left(\frac{z'}{z \circ \sigma} \right)^{1+\alpha} \right) \\ & + \frac{p_0^{\alpha}}{\tau_0} \left((r \circ \tau) \cdot \varphi' \cdot \left(\frac{z' \circ \tau}{z \circ \sigma} \right)^{\alpha} - \alpha \varphi \cdot \frac{(r \circ \tau)^{1+1/\alpha} \cdot \sigma'}{(r \circ \sigma)^{1/\alpha}} \cdot \left(\frac{z' \circ \tau}{z \circ \sigma} \right)^{1+\alpha} \right). \end{aligned} \tag{2.64}$$

Using the inequality in Lemma 1.7:

$$As - Bs^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} A^{\alpha+1} B^{-\alpha}, \tag{2.65}$$

we estimate each bracketed term. Thus

$$r \cdot \varphi' \cdot \left(\frac{z'}{z \circ \sigma}\right)^\alpha - \alpha\varphi \cdot \frac{r^{1+1/\alpha} \cdot \sigma'}{(r \circ \sigma)^{1/\alpha}} \cdot \left(\frac{z'}{z \circ \sigma}\right)^{1+\alpha} \leq \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(r \circ \sigma)(\varphi')^{\alpha+1}}{(\sigma' \cdot \varphi)^\alpha}, \tag{2.66}$$

and similarly

$$(r \circ \tau) \cdot \varphi' \cdot \left(\frac{z' \circ \tau}{z \circ \sigma}\right)^\alpha - \alpha\varphi \cdot \frac{(r \circ \tau)^{1+1/\alpha} \cdot \sigma'}{(r \circ \sigma)^{1/\alpha}} \cdot \left(\frac{z' \circ \tau}{z \circ \sigma}\right)^{1+\alpha} \leq \frac{1}{(\alpha+1)^{\alpha+1}} \frac{(r \circ \sigma)(\varphi')^{\alpha+1}}{(\sigma' \cdot \varphi)^\alpha}. \tag{2.67}$$

Therefore, (2.64) becomes:

$$w' \leq -2^{1-\alpha}\varphi \cdot Q + \frac{1}{(\alpha+1)^{\alpha+1}} \left(1 + \frac{p_0^\alpha}{\tau_0}\right) \frac{(r \circ \sigma)(\varphi')^{\alpha+1}}{(\sigma' \cdot \varphi)^\alpha}. \tag{2.68}$$

Integrating (2.68) from t_1 to t, we find:

$$\int_{t_1}^t \left(2^{1-\alpha}\varphi(\eta)Q(\eta) - \frac{1}{(\alpha+1)^{\alpha+1}} \left(1 + \frac{p_0^\alpha}{\tau_0}\right) r(\sigma(\eta)) \frac{(\varphi'(\eta))^{\alpha+1}}{(\sigma'(\eta)\varphi(\eta))^\alpha}\right) d\eta \leq w(t_1), \tag{2.69}$$

which contradicts (2.56). This contradiction completes the proof. □

Theorem 2.9. *[22] Assume that* $0 < \alpha \leq 1$, $\sigma'(t) > 0$, (2.54)*and* (2.55)*hold. If there exists a positive differentiable function* $\varphi : [t_0, \infty) \to (0, \infty)$ *such that*

$$\limsup_{t\to\infty} \int_{t_1}^t \left(\varphi(\eta)\, Q(\eta) - \left(1 + \frac{p_0^\alpha}{\tau_0}\right) \frac{1}{(\alpha+1)^{\alpha+1}} \frac{r(\sigma(\eta)) \left(\varphi'(\eta)\right)^{\alpha+1}}{\left(\sigma'(\eta)\, \varphi(\eta)\right)^\alpha}\right) d\eta = \infty, \tag{2.70}$$

then every solution of the neutral delay differential equation (2.52) *is oscillatory.*

Proof. We proceed as in the proof of Theorem 2.8, to get (2.58). Using Lemma 1.3, for $0 < \alpha \leq 1$:

$$(F+G)^{\alpha} \leq F^{\alpha} + G^{\alpha}, \tag{2.71}$$

with $F = x$ and $G = p_0 \cdot (x \circ \tau)$, we get:

$$z^{\alpha} \leq (x + p_0 \cdot (x \circ \tau))^{\alpha} \leq x^{\alpha} + (p_0 \cdot (x \circ \tau))^{\alpha}.$$

Then $w(t) > 0$ for $t \geq t_1$, and differentiating w gives:

$$\begin{aligned} w' = \varphi \cdot \frac{1}{(z \circ \sigma)^{\alpha}} \Big(\mathcal{L}(z) + \frac{p_0^{\alpha}}{\tau_0} \mathcal{L}(z \circ \tau) \Big)' \\ + \left(\varphi' \cdot \frac{\mathcal{L}(z)}{(z \circ \sigma)^{\alpha}} - \alpha \varphi \cdot \frac{\mathcal{L}(z)}{(z \circ \sigma)^{\alpha+1}} \cdot (z \circ \sigma)' \right) \\ + \frac{p_0^{\alpha}}{\tau_0} \left(\varphi' \cdot \frac{\mathcal{L}(z \circ \tau)}{(z \circ \sigma)^{\alpha}} - \alpha \varphi \cdot \frac{\mathcal{L}(z \circ \tau)}{(z \circ \sigma)^{\alpha+1}} \cdot (z \circ \sigma)' \right). \end{aligned} \tag{2.72}$$

From (2.60), (2.72), and (2.62), we obtain:

$$\begin{aligned} w' &\leq -\varphi \cdot Q \\ &+ \left(r \cdot \varphi' \cdot \left(\frac{z'}{z \circ \sigma} \right)^{\alpha} - \alpha \varphi \cdot \frac{r^{1+1/\alpha} \sigma'}{(r \circ \sigma)^{1/\alpha}} \cdot \left(\frac{z'}{z \circ \sigma} \right)^{1+\alpha} \right) \\ &+ \frac{p_0^{\alpha}}{\tau_0} \left((r \circ \tau) \cdot \varphi' \cdot \left(\frac{z' \circ \tau}{z \circ \sigma} \right)^{\alpha} - \alpha \varphi \cdot \frac{(r \circ \tau)^{1+1/\alpha} \sigma'}{(r \circ \sigma)^{1/\alpha}} \cdot \left(\frac{z' \circ \tau}{z \circ \sigma} \right)^{1+\alpha} \right). \end{aligned} \tag{2.73}$$

Using Lemma 1.7:

$$As - Bs^{\frac{\alpha+1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} A^{\alpha+1} B^{-\alpha},$$

to the bracketed terms in (2.73) by setting

$$A = r\varphi', \quad B = \alpha \varphi \frac{r^{1+1/\alpha} \sigma'}{(r \circ \sigma)^{1/\alpha}}, \quad \text{and} \quad s = \frac{z'}{z \circ \sigma},$$

it follows that each bracketed term satisfies:

$$r\varphi'\left(\frac{z'}{z\circ\sigma}\right)^{\alpha}-\alpha\varphi\frac{r^{1+1/\alpha}\sigma'}{(r\circ\sigma)^{1/\alpha}}\left(\frac{z'}{z\circ\sigma}\right)^{\alpha+1}\leq\frac{1}{(\alpha+1)^{\alpha+1}}\frac{(r\circ\sigma)(\varphi')^{\alpha+1}}{(\sigma'\varphi)^{\alpha}}. \tag{2.74}$$

A similar estimate holds for the term involving $(z'\circ\tau)$. Substituting these into (2.73), we obtain:

$$w'\leq-\varphi\cdot Q+\frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)\frac{(r\circ\sigma)(\varphi')^{\alpha+1}}{(\sigma'\cdot\varphi)^{\alpha}}. \tag{2.75}$$

Integrating (2.75) from t_1 to t, we find:

$$\int_{t_1}^{t}\left(\varphi(\eta)Q(\eta)-\frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{p_0^{\alpha}}{\tau_0}\right)r(\sigma(\eta))\frac{(\varphi'(\eta))^{\alpha+1}}{(\sigma'(\eta)\varphi(\eta))^{\alpha}}\right)d\eta\leq w(t_1),$$

which contradicts (2.70). This contradiction completes the proof. □

Example 2.10. *[22] Consider the neutral delay differential equation*

$$\left(\left((x(t)+p_0x(\lambda t))'\right)^{\alpha}\right)'+\frac{q_0}{t^{\alpha+1}}x^{\alpha}(\mu t)=0,\quad t\geq1 \tag{2.76}$$

where $\alpha>1$, $\lambda\in(0,1]$, $\mu\in(0,\lambda]$, $p_0>0$, and $q_0>0$.

By choosing the testing function $\varphi(t)=t^{\alpha}$, we have $\varphi'(t)=\alpha t^{\alpha-1}$. Substituting into the integral criterion of Theorem 2.8:

$$\int_{t_1}^{t}\left(2^{1-\alpha}\varphi(\eta)Q(\eta)-\left(1+\frac{p_0^{\alpha}}{\lambda}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(\sigma(\eta))(\varphi'(\eta))^{\alpha+1}}{(\sigma'(\eta)\varphi(\eta))^{\alpha}}\right)d\eta,$$

where $Q(\eta)=\dfrac{q_0}{\eta^{\alpha+1}}$, $r(\sigma(\eta))=1$, and $\sigma'(\eta)=\mu$. Then:

$$2^{1-\alpha}\varphi(\eta)Q(\eta)=2^{1-\alpha}\cdot\eta^{\alpha}\cdot\frac{q_0}{\eta^{\alpha+1}}=\frac{2^{1-\alpha}q_0}{\eta}.$$

$$\frac{(1+\frac{p_0^\alpha}{\lambda})}{(\alpha+1)^{\alpha+1}} \cdot \frac{r(\sigma(\eta))(\varphi'(\eta))^{\alpha+1}}{(\sigma'(\eta)\varphi(\eta))^\alpha} = \frac{(1+\frac{p_0^\alpha}{\lambda})}{(\alpha+1)^{\alpha+1}} \cdot \frac{1\cdot(\alpha\eta^{\alpha-1})^{\alpha+1}}{(\mu\eta^\alpha)^\alpha}$$

Thus, the integrand simplifies to:

$$I(\eta) = \frac{2^{1-\alpha}q_0}{\eta} - \frac{(1+\frac{p_0^\alpha}{\lambda})\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}\mu^\alpha} \cdot \frac{1}{\eta}.$$

Integrating:

$$\int_{t_1}^{t} I(\eta)d\eta = \left[\left(2^{1-\alpha}q_0 - \frac{(1+\frac{p_0^\alpha}{\lambda})\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}\mu^\alpha}\right)\ln\eta\right]_{t_1}^{t}.$$

For divergence, the coefficient of $\ln t$ *must be positive, giving:*

$$q_0 > \left(1+\frac{p_0^\alpha}{\lambda}\right)\frac{2^{\alpha-1}\alpha^{\alpha+1}}{\mu^\alpha(\alpha+1)^{\alpha+1}}.$$

So, all solutions are oscillatory if

$$q_0 > \left(1+\frac{p_0^\alpha}{\lambda}\right)\frac{2^{\alpha-1}\alpha^{\alpha+1}}{\mu^\alpha(\alpha+1)^{\alpha+1}}.$$

In the ordinary case ($p_0 = 0, \alpha = 1$)*, the equation reduces to the Euler-type form*

$$x''(t) + \frac{q_0}{t^2}x(t) = 0,$$

solving it using the direct substitution method:

Assume a solution of the form:

$$x(t) = t^m.$$

Calculate the derivatives:

$$x'(t) = mt^{m-1}, \quad x''(t) = m(m-1)t^{m-2}.$$

Substitute into the original equation:

$$m(m-1)t^{m-2} + \frac{q_0}{t^2}t^m = 0.$$

Simplifying the second term using $t^m/t^2 = t^{m-2}$*, we obtain:*

$$m(m-1)t^{m-2} + q_0 t^{m-2} = 0.$$

Derive the characteristic equation (for $t \neq 0$*):*

$$m^2 - m + q_0 = 0.$$

Apply the oscillation condition. A solution is oscillatory if the roots of the characteristic equation are ***complex****. This occurs when the discriminant* Δ *is negative:*

$$\Delta = b^2 - 4ac < 0.$$

Substituting $a = 1, b = -1, c = q_0$*, we get:*

$$(-1)^2 - 4(1)(q_0) < 0 \implies 1 - 4q_0 < 0 \implies q_0 > \frac{1}{4}.$$

And the criterion simplifies to $q_0 > \dfrac{1}{4}$ *, which is the exact oscillation requirement.*

Example 2.11. *[22] Consider the neutral delay differential equation (NDE) of the form*

$$\left(t^{1/4}\Big(\big(x(t) + 2x(\lambda t)\big)'\Big)^{1/3}\right)' + \frac{q_0}{t^{13/12}}x^{1/3}(\mu t) = 0, \quad t \geq 1,$$

where $\lambda \in (0, 1]$*,* $\mu \in (0, 1]$ *with* $\mu \leq \lambda$*, and* $q_0 > 0$*. Here*

$$\alpha = 1/3, \quad r(t) = t^{1/4}, \quad p_0 = 2, \quad \sigma(t) = \mu t, \quad Q(t) = \frac{q_0}{t^{13/12}}.$$

By selecting the testing function $\varphi(t) = t^{1/12}$, we have:

$$\varphi'(t) = \frac{1}{12}t^{-11/12}.$$

Substituting these into the integral criterion of Theorem2.9:

$$\int_{t_1}^{t}\left(\varphi(\eta)Q(\eta) - \left(1+\frac{p_0^\alpha}{\lambda}\right)\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(\sigma(\eta))(\varphi'(\eta))^{\alpha+1}}{(\sigma'(\eta)\varphi(\eta))^\alpha}\right)d\eta.$$

$$\varphi(\eta)Q(\eta) = \eta^{1/12}\cdot\frac{q_0}{\eta^{13/12}} = \frac{q_0}{\eta},$$

and

$$r(\sigma(\eta)) = (\mu\eta)^{1/4} = \mu^{1/4}\eta^{1/4}.$$

Then,

$$(\varphi'(\eta))^{\alpha+1} = \left(\frac{1}{12}\eta^{-11/12}\right)^{4/3} = \frac{1}{12^{4/3}}\eta^{-44/36} = \frac{1}{12^{4/3}}\eta^{-11/9}.$$

$$(\sigma'(\eta)\varphi(\eta))^\alpha = (\mu\cdot\eta^{1/12})^{1/3} = \mu^{1/3}\eta^{1/36}.$$

Combining the powers of η in the second term:

$$\frac{\eta^{1/4}\cdot\eta^{-11/9}}{\eta^{1/36}} = \eta^{(1/4-11/9-1/36)} = \eta^{(9/36-44/36-1/36)} = \eta^{-36/36} = \frac{1}{\eta}.$$

Simplifying the constant (c) of the second term:

$$c = \frac{(1+\frac{2^{1/3}}{\lambda})}{(4/3)^{4/3}}\cdot\frac{\mu^{1/4}}{\mu^{1/3}\cdot 12^{4/3}} = \frac{(\lambda+2^{1/3})}{\lambda(4/3)^{4/3}\cdot\mu^{1/12}\cdot 12^{4/3}}$$

$$c = \frac{\lambda+2^{1/3}}{\lambda\mu^{1/12}\cdot(\frac{4}{3}\cdot 12)^{4/3}} = \frac{\lambda+2^{1/3}}{\lambda\mu^{1/12}\cdot 16^{4/3}} = \frac{\lambda+2^{1/3}}{64\lambda\mu^{1/12}\cdot 2^{1/3}}$$

$$c = \frac{2^{2/3}\lambda+2}{64\lambda\sqrt[12]{\mu}}.$$

The integral simplifies to:

$$\int_{t_1}^{t} \left(q_0 - \frac{2^{2/3}\lambda + 2}{64\lambda \sqrt[12]{\mu}} \right) \frac{1}{\eta} d\eta.$$

As $t \to \infty$, the integral diverges if the coefficient is positive, yielding the oscillation condition:

$$q_0 > \frac{1}{64\lambda \sqrt[12]{\mu}} (2^{2/3}\lambda + 2).$$

This example illustrates the strength of the new criterion for cases where $p_0 \geq 1$, which previous approaches could not handle. For instance, when $q_0 = 2$, the condition gives $2 > 0.095673$, confirming oscillation.

Remark 2.12. *The results in [22] incorporate all relevant delay functions, including $\tau(t)$ and $\sigma(t)$, and are valid for both $p(t) \geq 1$ and $p(t) < 1$. This represents a clear extension over earlier results (e.g., C2–C4 in [22]), which were typically limited to the case $p(t) < 1$ and often neglected the effect of $\tau(t)$. The results in [6] similarly test oscillation for $p \geq 1$, offering a broader applicability than most previous studies.*

In particular, the new criteria provide precise results in the ordinary case when the delays are absent, i.e., $\tau(t) = \sigma(t) = t$. For example, the Euler-type NDDE

$$x''(t) + q_0 t^{-2} x(t) = 0 \tag{2.77}$$

is oscillatory if and only if $q_0 > \frac{1}{4}$ according to criterion (3.2) in [22], a condition that previous criteria (C1–C4) fail to capture accurately.

Chapter 3

Oscillation of second-order neutral delay differential equations in the non-canonical case

This chapter is devoted to the study of oscillation and asymptotic properties of a class of second-order neutral delay differential equations of Emden-Fowler type. The analysis is restricted to the **non-canonical case**, which is characterized by a specific integrability condition on the leading coefficient, $r(t)$.

In particular, the equation is said to be in the non-canonical case if the following condition holds

$$R(t_0) := \int_{t_0}^{\infty} r^{-1/\alpha}(s)\, ds < \infty.$$

As reported in [11, 21], this condition allows the existence of eventually positive solutions that may converge to zero as $t \to \infty$. Consequently, the qualitative analysis of solutions requires a careful treatment of the possible behavior of the derivative of the associated neutral function.

Several approaches have been proposed in the literature to investigate oscillation in the non-canonical case. In particular, comparison techniques and reduction arguments were employed in [7, 10]. Moreover, Riccati-type transformations involving the function $R(t)$ were considered in [9, 11] to derive oscillation criteria under the non-canonical assumption.

3.1 Reduction to first-order DDEs

This section presents a reduction technique that transforms the oscillation problem of the second–order neutral delay equation into the oscillation of an associated first–order delay differential equation. In particular, we consider the neutral Emden–Fowler type equation

$$\Big(r(t)\left(z'(t)\right)^{\alpha}\Big)' + q(t)\, f\Big(x(\sigma(t))\Big) = 0, \qquad t \geq t_0, \tag{3.1}$$

where

$$z(t) = x(t) + p(t)\, x(\tau(t)). \tag{3.2}$$

The hypotheses will be assumed throughout this section:

(H1) α is a ratio of positive odd integers.

(H2) $r \in C^1\Big([t_0,\infty),(0,\infty)\Big)$ and p, q, τ, σ are continuous on $[t_0, \infty)$.

(H3)

$$\tau(t) \leq t, \qquad \sigma(t) \leq t, \qquad \lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty.$$

(H4)

$$q(t) \geq 0, \qquad 0 \leq p(t) < \min\left\{\frac{R(t)}{R(\tau(t))}, 1\right\}, \qquad t \geq t_0,$$

where R is defined in **(H5)**.

(H5)

$$R(t_0) := \int_{t_0}^{\infty} r(s)^{-1/\alpha}\, ds < \infty, \qquad \text{and hence} \qquad R(t) = \int_{t}^{\infty} r(s)^{-1/\alpha}\, ds.$$

(H6) $f \in C(\mathbb{R}, \mathbb{R})$ and there exists a constant $k > 0$ such that

$$f(u) > k\, u^{\beta} \qquad \text{for all } u \neq 0, \text{ where } \beta \quad \text{is a ratio of positive odd integers .}$$

To formulate the reduction, we introduce the following auxiliary quantities. For $t \geq t_0$, define

$$Q(t) := q(t)\left(1 - p(t)\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^{\beta}, \tag{3.3}$$

and for some $t_1 \geq t_0$ set

$$\widetilde{Q}(t) := \left(\frac{k}{r(t)} \int_{t_1}^{t} Q(s)\, ds\right)^{1/\alpha}, \qquad t \geq t_1, \tag{3.4}$$

where $k > 0$ is the constant appearing in **(H6)**.

Lemma 3.1. *[21] Let x be a positive solution of (3.1) on $[t_0, \infty)$ and let z be defined by (3.2). Assume that there exists $t_1 \geq t_0$ such that*

$$\int_{t_1}^{\infty} Q(\nu)\, d\nu = \infty, \tag{3.5}$$

then z is decreasing and $r(t)\big(z'(t)\big)^{\alpha}$ is nonincreasing, eventually.

Proof. Let x be a positive solution of (3.1). Then there exists $t_1 \geq t_0$ such that

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad x(\sigma(t)) > 0 \quad \text{for all } t \geq t_1.$$

From the definition of z in (3.2), it follows that

$$z(t) = x(t) + p(t)x(\tau(t)) \geq x(t) > 0 \quad \text{for } t \geq t_1.$$

Moreover, by equation (3.1) and hypothesis **(H6)**, we have

$$\left(r(t)\big(z'(t)\big)^{\alpha}\right)' = -q(t)f\big(x(\sigma(t))\big) \le -k\,q(t)\,x^{\beta}(\sigma(t)) \le 0,$$

which implies that $r(t)\big(z'(t)\big)^{\alpha}$ is nonincreasing on $[t_1, \infty)$.

Suppose, on the contrary, that $z'(t) > 0$ for all $t \ge t_2 \ge t_1$. Then z is increasing. From (3.2) we have

$$x(t) = z(t) - p(t)x(\tau(t)) \ge z(t) - p(t)z(\tau(t)),$$

where we used the fact that $x(\tau(t)) \le z(\tau(t))$.

Since z is increasing and R is decreasing in the non-canonical case, and using the restriction on $p(t)$ in **(H4)**, we obtain

$$x(\sigma(t)) \ge z(\sigma(t)) \left(1 - p(t)\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right).$$

Substituting this estimate into (3.1) and using **(H6)**, we obtain

$$\left(r(t)\big(z'(t)\big)^{\alpha}\right)' \le -k\,Q(t)\,z^{\beta}(\sigma(t)). \tag{3.6}$$

Integrating (3.6) from t_2 to t gives

$$r(t)\big(z'(t)\big)^{\alpha} \le r(t_2)\big(z'(t_2)\big)^{\alpha} - k\int_{t_2}^{t} Q(s)\,z^{\beta}(\sigma(s))\,ds.$$

Since z is increasing, we have $z(\sigma(s)) \ge z(\sigma(t_2)) > 0$. Letting $t \to \infty$ and using (3.5), the right-hand side tends to $-\infty$, which is impossible since $z'(t) > 0$ and $r(t) > 0$.

Hence $z'(t) < 0$ eventually, that is, z is decreasing. Combined with the fact that $r(t)\big(z'(t)\big)^{\alpha}$ is nonincreasing, the proof is complete. □

Theorem 3.2. *[21] Assume **(H1)**–**(H6)**. Let Q and $\widetilde{Q}$ be defined by* (3.3) *and*

(3.4), *respectively. Suppose that there exists* $t_1 \geq t_0$ *such that* (3.5) *holds. Assume moreover that* σ *is nondecreasing on* $[t_1, \infty)$*. If the first–order delay differential equation*

$$y'(t) + \widetilde{Q}(t)\, y^{\beta/\alpha}\big(\sigma(t)\big) = 0, \qquad t \geq t_1, \tag{3.7}$$

is oscillatory, then every solution of (3.1) *is oscillatory.*

Proof. Assume, to the contrary, that (3.1) has a nonoscillatory solution x. Without loss of generality, we may assume that x is eventually positive. Hence, there exists $t_1 \geq t_0$ such that

$$x(t) > 0, \qquad x\big(\tau(t)\big) > 0, \qquad x\big(\sigma(t)\big) > 0, \qquad \forall\, t \geq t_1.$$

Let z be defined by (3.2). Then $z(t) \geq x(t) > 0$ for all $t \geq t_1$.

By Lemma 3.1, condition (3.5) implies that there exists $t_2 \geq t_1$ such that for all $t \geq t_2$,

$$z'(t) < 0 \quad \text{and} \quad r(t)\big(z'(t)\big)^{\alpha} \ \text{is nonincreasing.} \tag{3.8}$$

From (3.1) and **(H6)**, for $t \geq t_2$,

$$\Big(r(t)\big(z'(t)\big)^{\alpha}\Big)' = -\,q(t)\, f\big(x(\sigma(t))\big) \leq -\,k\, q(t)\, x^{\beta}\big(\sigma(t)\big). \tag{3.9}$$

Since $z(t) = x(t) + p(t)x(\tau(t))$, we have

$$x(t) = z(t) - p(t)\, x(\tau(t)).$$

this together with **(H4)** yields

$$x\big(\sigma(t)\big) \ \geq \ \left(1 - p(t)\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right) z\big(\sigma(t)\big), \ \text{for } t \geq t_2 \tag{3.10}$$

Substituting (3.10) into (3.9) and using the definition of Q in (3.3), we obtain, for $t \geq t_2$,

$$\Big(r(t)\big(z'(t)\big)^{\alpha}\Big)' \leq -\,k\,Q(t)\,z^{\beta}\big(\sigma(t)\big). \tag{3.11}$$

Integrating (3.11) from t_2 to t gives

$$r(t)\big(z'(t)\big)^{\alpha} \leq r(t_2)\big(z'(t_2)\big)^{\alpha} - k\int_{t_2}^{t} Q(s)\,z^{\beta}\big(\sigma(s)\big)\,ds. \tag{3.12}$$

Since $r(t)\big(z'(t)\big)^{\alpha} < 0$, we get

$$r(t)\big(z'(t)\big)^{\alpha} \leq -\,k\int_{t_2}^{t} Q(s)\,z^{\beta}\big(\sigma(s)\big)\,ds, \qquad t \geq t_3. \tag{3.13}$$

Since z is decreasing on $[t_2,\infty)$ and σ is nondecreasing, for $s \in [t_2, t]$ we have $\sigma(s) \leq \sigma(t)$ and hence $z(\sigma(s)) \geq z(\sigma(t))$. Consequently,

$$\int_{t_2}^{t} Q(s)\,z^{\beta}\big(\sigma(s)\big)\,ds \geq z^{\beta}\big(\sigma(t)\big)\int_{t_2}^{t} Q(s)\,ds.$$

Combining this with (3.13) yields, for $t \geq t_3$,

$$r(t)\big(z'(t)\big)^{\alpha} \leq -\,k\,z^{\beta}\big(\sigma(t)\big)\int_{t_2}^{t} Q(s)\,ds. \tag{3.14}$$

Since $z'(t) < 0$ for $t \geq t_3$, (3.14) implies

$$-\,z'(t) \;\geq\; \left(\frac{k}{r(t)}\int_{t_2}^{t} Q(s)\,ds\right)^{1/\alpha}\; z^{\beta/\alpha}\big(\sigma(t)\big).$$

Thus, by the definition of $\widetilde{Q}$ in (3.4), we obtain for all $t \geq t_3$,

$$z'(t) + \widetilde{Q}(t)\,z^{\beta/\alpha}\big(\sigma(t)\big) \leq 0. \tag{3.15}$$

By Lemma1.6, the existence of an eventually positive solution of (3.15) implies that (3.7) admits an eventually positive solution. This contradicts the oscillation of

(3.7). Hence every solution of (3.1) oscillates. □

Corollary 3.3. *[21] Assume the hypotheses of Theorem 3.2. If* $\alpha = \beta$ *and*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} \widetilde{Q}(s)\, ds > \frac{1}{e}, \tag{3.16}$$

then every solution of (3.1) *is oscillatory.*

Proof. If $\alpha = \beta$, then (3.7) becomes the linear delay equation

$$y'(t) + \widetilde{Q}(t)\, y(\sigma(t)) = 0.$$

By Lemma1.5 (ii), the condition (3.16) guarantees that (3.7) is oscillatory. Therefore, Theorem 3.2 yields that every solution of (3.1) is oscillatory. □

Example 3.4. *Consider the NDDE*

$$\Big(t^2\, z'(t)\Big)' + t^3\, x\Big(\frac{t}{2}\Big) = 0, \qquad t \geq 1, \tag{3.17}$$

where

$$z(t) = x(t) + \frac{1}{4}\, x\Big(\frac{t}{2}\Big). \tag{3.18}$$

Here we take $\alpha = \beta = 1$ *and* $f(u) = u$ *(thus* $k = 1$*), with*

$$r(t) = t^2, \qquad q(t) = t^3, \qquad \sigma(t) = \tau(t) = \frac{t}{2}, \qquad p(t) \equiv \frac{1}{4}.$$

Clearly, $r(t) > 0$ *and* $q(t) \geq 0$*, while* $\tau(t) \leq t$*,* $\sigma(t) \leq t$*,* $\tau(t) \to \infty$ *and* $\sigma(t) \to \infty$ *as* $t \to \infty$*, and* σ *is nondecreasing. Moreover,*

$$R(t) = \int_t^{\infty} r(s)^{-1}\, ds = \int_t^{\infty} s^{-2}\, ds = \frac{1}{t} < \infty,$$

hence we are in the non–canonical case. Also,

$$\frac{R(t)}{R(\tau(t))} = \frac{(1/t)}{(1/(t/2))} = \frac{1}{2},$$

so

$$0 \le p(t) \equiv \frac{1}{4} < \min\left\{\frac{1}{2}, 1\right\} = \frac{1}{2},$$

*and therefore **(H1)**–**(H6)** hold.*

Since $\tau(\sigma(t)) = t/4$ and $\sigma(t) = t/2$, we have

$$\frac{R(\tau(\sigma(t)))}{R(\sigma(t))} = \frac{R(t/4)}{R(t/2)} = \frac{(4/t)}{(2/t)} = 2.$$

Hence,

$$Q(t) = q(t)\left(1 - p(t)\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right) = t^3\left(1 - \frac{1}{4}\cdot 2\right) = \frac{1}{2}\,t^3.$$

In particular,

$$\int_{t_1}^{\infty} Q(s)\,ds = \int_{t_1}^{\infty} \frac{1}{2}\,s^3\,ds = \infty,$$

so condition (3.5) holds for any $t_1 \ge 1$.

By definition,

$$\widetilde{Q}(t) = \frac{1}{r(t)}\int_{t_1}^{t} Q(s)\,ds = \frac{1}{t^2}\int_{t_1}^{t} \frac{1}{2}\,s^3\,ds = \frac{1}{t^2}\cdot\frac{t^4 - t_1^4}{8} = \frac{t^2}{8} - \frac{t_1^4}{8t^2}.$$

Therefore, for all sufficiently large t,

$$\widetilde{Q}(t) \ge \frac{t^2}{16}.$$

Consider the corresponding first–order delay equation (here $\alpha = \beta = 1$)

$$y'(t) + \widetilde{Q}(t)\,y\left(\frac{t}{2}\right) = 0, \qquad t \ge t_1. \tag{3.19}$$

Assume that (3.19) admits an eventually positive solution y. Then there exists $T \ge t_1$

such that $y(t) > 0$ *and* $y(t/2) > 0$ *for all* $t \geq T$. *Hence, for* $t \geq T$,

$$y'(t) = -\widetilde{Q}(t)\, y\Big(\frac{t}{2}\Big) \leq -\frac{t^2}{16}\, y\Big(\frac{t}{2}\Big) < 0,$$

so y *is decreasing on* $[T, \infty)$. *In particular, for* $t \geq 2T$ *we have* $y(t/2) \geq y(t)$, *and thus*

$$y'(t) \leq -\frac{t^2}{16}\, y(t), \qquad t \geq 2T.$$

Dividing by $y(t) > 0$ *and integrating from* $2T$ *to* t *yields*

$$\ln \frac{y(t)}{y(2T)} \leq -\int_{2T}^{t} \frac{s^2}{16}\, ds = -\frac{1}{16} \cdot \frac{t^3 - (2T)^3}{3} \xrightarrow[t\to\infty]{} -\infty,$$

which implies $y(t) \to 0$ *super–exponentially. In particular, the inequality forces* y *to become arbitrarily small, contradicting the existence of an eventually positive solution of* (3.19) *in the sense required by the oscillation theory (no solution can stay of one sign for all large* t*). Therefore,* (3.19) *is oscillatory.*

Since (3.19) *is oscillatory, Theorem 3.2 implies that every solution of* (3.17) *is oscillatory.*

Example 3.5. *Let* $t_0 = 1$ *and consider*

$$\Big(t^2 z'(t)\Big)' + 4t\, x\Big(\frac{t}{2}\Big) = 0, \qquad t \geq 1, \tag{3.20}$$

where

$$z(t) = x(t) + \frac{1}{4}\, x\Big(\frac{t}{2}\Big). \tag{3.21}$$

Here we take $\alpha = \beta = 1$ *and* $f(u) = u$ *(so* $k = 1$*).*

We have $r(t) = t^2 > 0$ *and* $q(t) = 4t \geq 0$. *The delay arguments are*

$$\tau(t) = \sigma(t) = \frac{t}{2} \leq t, \qquad \lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty,$$

and σ is nondecreasing on $[1, \infty)$. Moreover,

$$R(t) = \int_t^\infty r(s)^{-1}\, ds = \int_t^\infty s^{-2}\, ds = \frac{1}{t} < \infty,$$

so the non–canonical condition holds. Also,

$$\frac{R(t)}{R(\tau(t))} = \frac{(1/t)}{(1/(t/2))} = \frac{1}{2},$$

hence

$$0 \le p(t) \equiv \frac{1}{4} < \min\left\{\frac{R(t)}{R(\tau(t))}, 1\right\} = \frac{1}{2},$$

*so **(H4)** is satisfied.*

Since $\tau(\sigma(t)) = t/4$ and $\sigma(t) = t/2$, we obtain

$$\frac{R(\tau(\sigma(t)))}{R(\sigma(t))} = \frac{R(t/4)}{R(t/2)} = \frac{(4/t)}{(2/t)} = 2.$$

Therefore,

$$Q(t) = q(t)\left(1 - p(t)\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right) = 4t\left(1 - \frac{1}{4}\cdot 2\right) = 2t.$$

In particular,

$$\int_{t_1}^\infty Q(s)\, ds = \int_{t_1}^\infty 2s\, ds = \infty,$$

so condition (3.5) holds for any $t_1 \ge 1$. Moreover,

$$\widetilde{Q}(t) = \frac{1}{r(t)}\int_{t_1}^t Q(s)\, ds = \frac{1}{t^2}\int_{t_1}^t 2s\, ds = \frac{t^2 - t_1^2}{t^2} = 1 - \left(\frac{t_1}{t}\right)^2, \qquad t \ge t_1.$$

***Applying Corollary 3.3.** Since $\alpha = \beta = 1$, we check condition (3.16):*

$$\int_{\sigma(t)}^t \widetilde{Q}(s)\, ds = \int_{t/2}^t \left(1 - \frac{t_1^2}{s^2}\right) ds = \frac{t}{2} - t_1^2\left(\frac{1}{t/2} - \frac{1}{t}\right) = \frac{t}{2} - \frac{t_1^2}{t} \xrightarrow[t\to\infty]{} \infty.$$

Hence,

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} \widetilde{Q}(s)\, ds = \infty > \frac{1}{e}.$$

By Corollary 3.3 (and thus Theorem 3.2), every solution of (3.20) *is oscillatory.*

3.2 Riccati substitution technique

In this section, the Riccati substitution technique introduced by Agarwal et al. [5] is described in the context of oscillation theory for second-order neutral delay differential equations. The technique is based on transforming the second-order equation into an appropriate first-order differential inequality, which allows the derivation of oscillation criteria.

In this section, we consider the oscillation of second-order Emden-Fowler neutral delay differential equations of the form

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)' + q(t)x^{\alpha}(\sigma(t)) = 0, \quad t \geq t_0, \tag{3.22}$$

where

$$z(t) = x(t) + p(t)x(\tau(t)),$$

and the following conditions are assumed to hold:

(H1) The exponent $\alpha > 0$ is a quotient of odd positive integers.

(H2) The function $r \in C([t_0, \infty), (0, \infty))$ satisfies the non-canonical condition

$$R(t_0) := \int_{t_0}^{\infty} r^{-1/\alpha}(s)\, ds < \infty. \tag{3.23}$$

(H3) The delay functions $\sigma, \tau \in C^1([t_0, \infty), \mathbb{R})$ satisfy

$$\tau(t) \leq t, \quad \sigma(t) \leq t, \quad \sigma'(t) > 0,$$

and
$$\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty.$$

(H4) The coefficient functions $q, p \in C([t_0, \infty), [0, \infty))$ where $0 \le p(t) < 1$, and the function q does not vanish identically on any half-line $[t^*, \infty)$ with $t^* \ge t_0$.

Theorem 3.6 ([5]). *Suppose that conditions* (H1)–(H4) *are satisfied where* $\alpha > 1$. *If there exist functions* $\rho, \delta \in C^1([t_0, \infty), (0, \infty))$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\rho(s)Q(s) - \frac{((\rho'(s))_+)^{\alpha+1} r(\sigma(s))}{(\alpha+1)^{\alpha+1} \rho^\alpha(s)(\sigma'(s))^\alpha} \right] ds = \infty, \tag{3.24}$$

where

$$Q(t) := q(t)\big(1 - p(\sigma(t))\big)^\alpha \quad \textit{and} \quad (\rho'(t))_+ := \max\{0, \rho'(t)\},$$

as well as

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left(\Psi(s) - \frac{\delta(s) r(s) ((\Phi(s))_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \right) ds = \infty, \tag{3.25}$$

where

$$\Psi(t) := \delta(t) \left[q(t) \left(1 - p(\sigma(t)) \frac{R(\tau(\sigma(t)))}{R(\sigma(t))} \right)^\alpha + \frac{1-\alpha}{r^{1/\alpha}(t) R^{\alpha+1}(t)} \right],$$
$$\Phi(t) := \frac{\delta'(t)}{\delta(t)} + \frac{1+\alpha}{r^{1/\alpha}(t) R(t)}, \quad \textit{and} \quad (\Phi(t))_+ := \max\{0, \Phi(t)\}.$$

Then every solution of equation (3.22) *is oscillatory.*

Proof. Assume, to the contrary, that equation (3.22) has a non-oscillatory solution. Without loss of generality assume that there exists $t_1 \ge t_0$ such that

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad x(\sigma(t)) > 0 \quad \text{for all } t \ge t_1.$$

Consequently, the associated neutral function $z(t) = x(t) + p(t)x(\tau(t))$ is positive for all $t \ge t_1$.

From equation (3.22), we have

$$\left(r(t)(z'(t))^{\alpha}\right)' = -q(t)x^{\alpha}(\sigma(t)) \leq 0, \quad t \geq t_1,$$

Hence, the function $r(t)(z'(t))^{\alpha}$ is non-increasing on $[t_1, \infty)$. As a consequence, the derivative $z'(t)$ is eventually of one sign. Therefore, there exists $t_2 \geq t_1$ such that either

$$z'(t) > 0 \quad \text{for all } t \geq t_2,$$

or

$$z'(t) < 0 \quad \text{for all } t \geq t_2.$$

Case I: $z'(t) > 0$ for all $t \geq t_2$.

In this case, $z(t)$ is an increasing function. Although the inequality

$$x(t) \geq (1 - p(t))z(t)$$

does not hold in general in the non-canonical case, as pointed out in Agarwal [5], it follows from the monotonicity of $z(t)$ that any eventually positive solution satisfying $z'(t) > 0$ fulfills

$$\begin{aligned} z(t) &= x(t) + p(t)x(\tau(t)) \\ x(t) &= z(t) - p(t)x(\tau(t)) \\ x(t) &\geq z(t) - p(t)z(\tau(t)) \\ &\geq (1 - p(t))z(t), \end{aligned}$$

or

$$x(\sigma(t)) \geq (1 - p(\sigma(t)))z(\sigma(t)). \tag{3.26}$$

for all sufficiently large t.

Substituting (3.26) into equation (3.22) yields

$$\begin{aligned}\Big(r(t)(z'(t))^{\alpha}\Big)' &= -q(t)x^{\alpha}(\sigma(t))\\ &\le -q(t)\Big(1-p(\sigma(t))\Big)^{\alpha}z^{\alpha}(\sigma(t))\\ &= -Q(t)z^{\alpha}(\sigma(t)),\end{aligned}\tag{3.27}$$

where $Q(t) = q(t)\Big(1-p(\sigma(t))\Big)^{\alpha}$.

Define the Riccati-type transformation

$$w(t) := \rho(t)\frac{r(t)(z'(t))^{\alpha}}{z^{\alpha}(\sigma(t))}, \quad t \ge t_2. \tag{3.28}$$

Since $z'(t) > 0$, it follows that $w(t) > 0$ for all $t \ge t_2$.

Differentiating the function (3.28) one obtains

$$\begin{aligned}w'(t) = \rho'(t)&\frac{r(t)(z'(t))^{\alpha}}{z^{\alpha}(\sigma(t))}\\ &+\rho(t)\frac{(r(t)(z'(t))^{\alpha})'}{z^{\alpha}(\sigma(t))}\\ &-\alpha\rho(t)r(t)(z'(t))^{\alpha}\frac{z'(\sigma(t))\sigma'(t)}{z^{\alpha+1}(\sigma(t))}.\end{aligned}\tag{3.29}$$

Using inequality (3.27) in (3.29), and noting that

$$\frac{r(t)(z'(t))^{\alpha}}{z^{\alpha}(\sigma(t))} = \frac{w(t)}{\rho(t)},$$

one obtains

$$w'(t) \le \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)Q(t) - \alpha\rho(t)r(t)(z'(t))^{\alpha}\frac{z'(\sigma(t))\sigma'(t)}{z^{\alpha+1}(\sigma(t))}. \tag{3.30}$$

Since $r(t)(z'(t))^\alpha$ is non-increasing and $\sigma(t) \le t$, it follows that

$$r(t)(z'(t))^\alpha \le r(\sigma(t))(z'(\sigma(t)))^\alpha,$$

which implies

$$z'(\sigma(t)) \ge \frac{r^{1/\alpha}(t)}{r^{1/\alpha}(\sigma(t))} z'(t).$$

Substituting this estimate into (3.30) yields

$$w'(t) \le \frac{(\rho'(t))_+}{\rho(t)} w(t) - \rho(t)Q(t) - \frac{\alpha\sigma'(t)}{(\rho(t)r(\sigma(t)))^{1/\alpha}} w^{\frac{\alpha+1}{\alpha}}(t).$$

$$w'(t) \le -\rho(t)Q(t) + \left[\frac{(\rho'(t))_+}{\rho(t)} w(t) - \frac{\alpha\sigma'(t)}{(\rho(t)r(\sigma(t)))^{1/\alpha}} w^{\frac{\alpha+1}{\alpha}}(t)\right]. \tag{3.31}$$

Applying Lemma 1.9, specifically inequality (1.8), to the last two terms in (3.31), where

$$v = w(t), \qquad D = \frac{(\rho'(t))_+}{\rho(t)}, \qquad C = \frac{\alpha\sigma'(t)}{(\rho(t)r(\sigma(t)))^{1/\alpha}}.$$

in (3.31) gives

$$w'(t) \le -\left[\rho(t)Q(t) - \frac{((\rho'(t))_+)^{\alpha+1} r(\sigma(t))}{(\alpha+1)^{\alpha+1}\rho^\alpha(t)(\sigma'(t))^\alpha}\right].$$

Integrating from t_2 to t yields

$$w(t) - w(t_2) \le -\int_{t_2}^{t} \left[\rho(s)Q(s) - \frac{((\rho'(s))_+)^{\alpha+1} r(\sigma(s))}{(\alpha+1)^{\alpha+1}\rho^\alpha(s)(\sigma'(s))^\alpha}\right] ds.$$

$$\int_{t_2}^{t} \left[\rho(s)Q(s) - \frac{((\rho'(s))_+)^{\alpha+1} r(\sigma(s))}{(\alpha+1)^{\alpha+1}\rho^\alpha(s)(\sigma'(s))^\alpha}\right] ds \le w(t_2),$$

which contradicts condition (3.24) as $t \to \infty$. Thus, case I is impossible.

Case II: $z'(t) < 0$

Since $r(t)(z'(t))^\alpha$ is a non-increasing function on $[t_2, \infty)$, it follows that for any $s \geq t \geq t_2$,

$$r^{1/\alpha}(s)z'(s) \leq r^{1/\alpha}(t)z'(t).$$

Dividing both sides by $r^{1/\alpha}(s) > 0$ and integrating from t to $u > t$, we obtain

$$z(u) - z(t) = \int_t^u z'(s)\,ds \leq r^{1/\alpha}(t)z'(t)\int_t^u r^{-1/\alpha}(s)\,ds,$$

or

$$z(t) \geq -r^{1/\alpha}(t)z'(t)\int_t^u r^{-1/\alpha}(s)\,ds + z(u)$$

Letting $u \to \infty$ and using the non-canonical condition (3.23), we arrive at the inequality

$$z(t) \geq -R(t)r^{1/\alpha}(t)z'(t). \tag{3.32}$$

Inequality (3.32) implies that the function $z(t)/R(t)$ is non-decreasing. Indeed,

$$\left(\frac{z(t)}{R(t)}\right)' = \frac{z'(t)R(t) + z(t)r^{-1/\alpha}(t)}{R^2(t)} \geq 0.$$

Since $\tau(t) \leq t$, it follows that

$$z(\tau(t)) \leq z(t)\frac{R(\tau(t))}{R(t)}.$$

In view of this inequality, we obtain

$$\begin{aligned} x(t) &= z(t) - p(t)x(\tau(t)) \\ &\geq z(t) - p(t)z(\tau(t)) \\ &\geq z(t)\left(1 - p(t)\frac{R(\tau(t))}{R(t)}\right), \quad t \geq t_2. \end{aligned} \tag{3.33}$$

Replacing t by $\sigma(t)$ yields

$$x(\sigma(t)) \geq z(\sigma(t)) \left(1 - p(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right). \tag{3.34}$$

Substituting (3.34) into equation (3.22), we obtain

$$\left(r(t)(z'(t))^{\alpha}\right)' \leq -q(t) \left(1 - p(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^{\alpha} z^{\alpha}(\sigma(t)). \tag{3.35}$$

Define a generalized Riccati function

$$w(t) := \delta(t) \left[\frac{r(t)(z'(t))^{\alpha}}{z^{\alpha}(t)} + \frac{1}{R^{\alpha}(t)}\right], \quad t \geq t_2. \tag{3.36}$$

It is clear that $w(t) \geq 0$ for all $t \geq t_2$.

Differentiating (3.36) with respect to t, we obtain

$$w'(t) = \frac{\delta'(t)}{\delta(t)}w(t) + \delta(t)\frac{(r(t)(z'(t))^{\alpha})'}{z^{\alpha}(t)} - \alpha\delta(t)r(t)\frac{(z'(t))^{\alpha+1}}{z^{\alpha+1}(t)} + \frac{\alpha\delta(t)}{r^{1/\alpha}(t)R^{\alpha+1}(t)}. \tag{3.37}$$

Using inequality (3.35) in (3.37), and rewriting the nonlinear term in terms of $w(t)$ via

$$\frac{r(t)(z'(t))^{\alpha}}{z^{\alpha}(t)} = \frac{w(t)}{\delta(t)} - \frac{1}{R^{\alpha}(t)},$$

$$\frac{(z'(t))^{\alpha+1}}{z^{\alpha+1}(t)} = \left(\frac{w(t)}{\delta(t)r(t)} - \frac{1}{r(t)R^{\alpha}(t)}\right)^{\frac{\alpha+1}{\alpha}}.$$

Then

$$w'(t) = \frac{\delta'(t)}{\delta(t)}\, w(t) + \delta(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{z^{\alpha}(t)} - \alpha\, \delta(t)\, r(t)^{-1/\alpha} \left(\frac{1}{R^{\alpha}(t)} - \frac{w(t)}{\delta(t)}\right)^{\frac{\alpha+1}{\alpha}} + \frac{\alpha\, \delta(t)}{r^{1/\alpha}(t)\, R^{\alpha+1}(t)}. \tag{3.38}$$

To estimate this nonlinear expression, we apply inequality (1.7) from Lemma 1.9 with

$$A := \frac{w(t)}{\delta(t)r(t)}, \qquad B := \frac{1}{r(t)R^\alpha(t)}.$$

We obtain

$$\left(\frac{w(t)}{\delta(t)r(t)} - \frac{1}{r(t)R^\alpha(t)}\right)^{\frac{\alpha+1}{\alpha}} \geq \left(\frac{w(t)}{\delta(t)r(t)}\right)^{\frac{\alpha+1}{\alpha}} - \frac{1}{\alpha r^{1/\alpha}(t)R(t)}\left[(1+\alpha)\frac{w(t)}{\delta(t)r(t)} - \frac{1}{r(t)R^\alpha(t)}\right]. \tag{3.39}$$

Substituting (3.39) into (3.38), and using inequality (3.35), we arrive at

$$\begin{aligned} w'(t) \leq & -\delta(t)q(t)\left(1 - p(\sigma(t))\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^\alpha + \frac{\delta'(t)}{\delta(t)}w(t) \\ & + \frac{1+\alpha}{r^{1/\alpha}(t)R(t)}w(t) - \frac{\alpha}{(\delta(t)r(t))^{1/\alpha}}w^{\frac{\alpha+1}{\alpha}}(t) - \frac{(1-\alpha)\delta(t)}{r^{1/\alpha}(t)R^{\alpha+1}(t)}. \end{aligned} \tag{3.40}$$

That is

$$w'(t) \leq -\Psi(t) + (\Phi(t))_+ w(t) - \frac{\alpha}{(\delta(t)r(t))^{1/\alpha}}w^{\frac{\alpha+1}{\alpha}}(t), \quad t \geq t_2. \tag{3.41}$$

To apply Lemma 1.9, specifically inequality (1.8), the last two terms in (3.41) are written in the form

$$Dv - Cv^{\frac{\alpha+1}{\alpha}},$$

where

$$v := w(t), \qquad D := (\Phi(t))_+, \qquad C := \frac{\alpha}{(\delta(t)r(t))^{1/\alpha}}.$$

Since $v \geq 0$ and $C > 0$, all assumptions of Lemma 1.9 are satisfied. Hence,

$$(\Phi(t))_+ w(t) - \frac{\alpha}{(\delta(t)r(t))^{1/\alpha}}w^{\frac{\alpha+1}{\alpha}}(t) \leq \frac{\delta(t)r(t)((\Phi(t))_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}}.$$

Combining this estimate with (3.41) yields

$$w'(t) \le -\left[\Psi(t) - \frac{\delta(t)r(t)((\Phi(t))_+)^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right]. \tag{3.42}$$

Integrating (3.42) from t_2 to t gives

$$\int_{t_2}^{t}\left[\Psi(s) - \frac{\delta(s)r(s)(\Phi^+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right] ds \le w(t_2) - w(t).$$

$$\int_{t_2}^{t}\left[\Psi(s) - \frac{\delta(s)r(s)(\Phi^+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right] ds \le w(t_2) < \infty.$$

Taking the limit superior as $t \to \infty$, we obtain:

$$\limsup_{t\to\infty} \int_{t_2}^{t}\left[\Psi(s) - \frac{\delta(s)r(s)(\Phi^+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\right] ds \le w(t_2) < \infty.$$

Which contradicts condition (3.25).

Therefore, Case II cannot occur, and equation (3.22) is oscillatory. □

Example 3.7. *Consider the second-order neutral delay differential equation*

$$\left((t+1)^4 \left(z'(t)\right)^3\right)' + x^3(t-1) = 0, \quad t \ge 1, \tag{3.43}$$

where

$$z(t) = x(t) + \frac{1}{4}x(t/2),$$

and

$$\alpha = 3, \quad r(t) = (t+1)^4, \quad q(t) = 1, \quad p(t) = \frac{1}{4}, \quad \sigma(t) = t-1, \quad \tau(t) = \frac{t}{2}.$$

Verification of assumptions (H1)–(H4).

(H1) The exponent $\alpha = 3$ is a quotient of odd positive integers.

(H2) The equation is in the non-canonical case since

$$R(t) := \int_t^\infty r^{-1/\alpha}(s)\,ds = \int_t^\infty (s+1)^{-4/3}\,ds = 3(t+1)^{-1/3} < \infty, \quad t \geq 1.$$

(H3) The delay functions satisfy

$$\tau(t) \leq t, \qquad \sigma(t) \leq t, \qquad \sigma'(t) = 1 > 0,$$

and

$$\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty.$$

(H4) The coefficient functions satisfy

$$0 \leq p(t) = \frac{1}{4} < 1,$$

and the function $q(t) = 1$ does not vanish identically on any half-line $[t^*, \infty)$.

Verification of the integral conditions. *Choose the weight functions*

$$\rho(t) = 1, \qquad \delta(t) = 1.$$

Condition (3.24)***:*** *Since $q(t) = 1$ and $p(\sigma(t)) = 1/4$, we have*

$$Q(t) = q(t)\Big(1 - p(\sigma(t))\Big)^3 = \left(1 - \frac{1}{4}\right)^3 = \frac{27}{64}.$$

Moreover, $\rho'(t) = 0$, and hence $(\rho'(t))_+ = 0$. Therefore,

$$\int_1^t \left[\rho(s)Q(s) - \frac{((\rho'(s))_+)^4 r(\sigma(s))}{4^4(\rho(s))^3(\sigma'(s))^3}\right] ds = \int_1^t \frac{27}{64}\,ds = \frac{27}{64}(t-1),$$

which implies

$$\lim_{t\to\infty}\int_1^t\left[\rho(s)Q(s)-\frac{((\rho'(s))_+)^4 r(\sigma(s))}{4^4(\rho(s))^3(\sigma'(s))^3}\right]ds=\infty.$$

Condition* (3.25)*: *Since* $\delta'(t)=0$*, we obtain*

$$\Phi(t)=\frac{1+\alpha}{r^{1/\alpha}(t)R(t)}=\frac{4}{3(t+1)},\qquad t\geq 1.$$

Hence $(\Phi(t))_+=\Phi(t)$ *and*

$$\frac{\delta(t)r(t)(\Phi(t))^4}{4^4}=\frac{(t+1)^4\left(\frac{4}{3(t+1)}\right)^4}{4^4}=\frac{1}{81}.$$

Furthermore,

$$\Psi(t)=\left(1-\frac{1}{4}\frac{R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^3+\frac{1-\alpha}{r^{1/\alpha}(t)R^4(t)}.$$

A direct computation yields

$$\Psi(t)=\left(1-\frac{1}{4}\,2^{1/3}\left(\frac{t}{t+1}\right)^{1/3}\right)^3-\frac{2}{81}.$$

Set

$$F(t):=\Psi(t)-\frac{\delta(t)r(t)\Big((\Phi(t))_+\Big)^{\alpha+1}}{(\alpha+1)^{\alpha+1}},$$

we obtain

$$F(t)=\left(1-\frac{1}{4}\,2^{1/3}\left(\frac{t}{t+1}\right)^{1/3}\right)^3-\frac{1}{27}.$$

Since $\left(\dfrac{t}{t+1}\right)^{1/3}\to 1$ *as* $t\to\infty$*, it follows that*

$$\lim_{t\to\infty}F(t)=\left(1-\frac{1}{4}\,2^{1/3}\right)^3-\frac{1}{27}=:L.$$

A direct computation shows that $L > 0$*. Hence, there exist* $T \geq t_0$ *and* $c := L/2 > 0$ *such that* $F(t) \geq c$ *for all* $t \geq T$*.*

Consequently, there exists a constant $c > 0$ *such that*

$$\Psi(t) - \frac{r(t)(\Phi(t))^4}{4^4} \geq c, \quad t \geq 1.$$

Therefore,

$$\int_1^t \left[\Psi(s) - \frac{r(s)(\Phi(s))^4}{4^4}\right] ds \geq c(t-1) \to \infty, \text{ as } t \to \infty$$

and condition (3.25) *is satisfied.*

Since assumptions (H1)–(H4) are satisfied and both integral conditions (3.24) *and* (3.25) *diverge to infinity, it follows from Theorem 3.6 that every solution of* (3.43) *is oscillatory.*

3.3 Bohner et al. technique

In this section, we consider the second-order neutral delay differential equation

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)' + q(t)x^{\alpha}(\sigma(t)) = 0, \quad t \geq t_0, \tag{3.44}$$

where

$$z(t) = x(t) + p(t)x(\tau(t)).$$

Throughout this section, equation (3.44) is assumed to satisfy the same assumptions *(H1)–(H4)* stated in the previous section, in addition to

(H5) The inequality $p(t) < \dfrac{R(t)}{R(\tau(t))}$ holds for all sufficiently large t. .

Equation (3.44), studied by Bohner et al. [9], has the same form as equation (3.22) and is analyzed under the identical assumptions (H1)–(H5). However, the analytical technique differs. In particular, the method used in [9] allows the exponent α to be any positive real number, and explicitly incorporates the effect of the delay function $\sigma(t)$ through the behavior of the ratio $z(t)/R(t)$. This leads to alternative oscillation criteria.

Theorem 3.8. *[9] Assume that conditions* (H1)–(H5) *are satisfied and that the equation is in the non-canonical case If*

$$\int_{t_1}^{\infty}\left(\frac{1}{r(t)}\int_{t_1}^{t}Q(s)R^{\alpha}(\sigma(s))\,ds\right)^{1/\alpha}dt=\infty, \tag{3.45}$$

where

$$Q(t):=q(t)\left(1-\frac{p(\sigma(t))R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^{\alpha} \tag{3.46}$$

then every solution of equation (3.44) *is oscillatory.*

Proof. Suppose to the contrary that x is a positive solution of (3.44) on $[t_0,\infty)$. Then there exists $t_1\geq t_0$ such that $x(\tau(t))>0$ and $x(\sigma(t))>0$ for all $t\geq t_1$. Obviously, for all $t\geq t_1$, we have $z(t)\geq x(t)>0$ and $r(t)\big(z'(t)\big)^{\alpha}$ is nonincreasing since

$$\Big(r(t)(z'(t))^{\alpha}\Big)'=-q(t)x^{\alpha}(\sigma(t))\leq 0. \tag{3.47}$$

Therefore, z' is either eventually negative or eventually positive.

Case 1: $z'(t)<0$ for all $t\geq t_1$. Then

$$\begin{aligned}z(t)&\geq-\int_{t}^{\infty}r^{-1/\alpha}(s)\,r^{1/\alpha}(s)\,z'(s)\,ds\\&\geq-r^{1/\alpha}(t)\,z'(t)\int_{t}^{\infty}r^{-1/\alpha}(s)\,ds.\\&=-R(t)\,r^{1/\alpha}(t)\,z'(t),\end{aligned} \tag{3.48}$$

using $R'(t)=-r^{-1/\alpha}(t)$, we obtain

$$\left(\frac{z(t)}{R(t)}\right)'=\frac{z'(t)R(t)-z(t)R'(t)}{R(t)^2}=\frac{z'(t)R(t)+z(t)r^{-1/\alpha}(t)}{R(t)^2}.$$

From (3.48) it follows that

$$z(t) \geq -R(t)r^{1/\alpha}(t)z'(t) \quad \Longrightarrow \quad z(t)r^{-1/\alpha}(t) \geq -R(t)z'(t),$$

and hence

$$\left(\frac{z(t)}{R(t)}\right)' \geq 0.$$

Using the definition of z, we get

$$x(t) = z(t) - p(t)x(\tau(t)) \geq z(t) - p(t)z(\tau(t)) \geq z(t)\left(1 - \frac{p(t)R(\tau(t))}{R(t)}\right),$$

and hence from (3.47) we obtain

$$\left(r(t)(z'(t))^\alpha\right)' \leq -q(t)\left(1 - \frac{p(\sigma(t))R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^\alpha z^\alpha(\sigma(t)) = -Q(t)\, z^\alpha(\sigma(t)). \tag{3.49}$$

Taking into account the monotonicity of $r(t)(z'(t))^\alpha$, we have

$$-r(t)(z'(t))^\alpha \geq -r(t_1)(z'(t_1))^\alpha =: \gamma > 0 \quad \text{for all } t \geq t_1,$$

then (3.48) implies that

$$z(t) \geq \gamma^{1/\alpha}R(t) \quad \text{for all } t \geq t_1. \tag{3.50}$$

Combining (3.49) with (3.50) yields

$$\left(r(t)(z'(t))^\alpha\right)' \leq -\gamma\, Q(t)\, R^\alpha(\sigma(t)) \quad \text{for all } t \geq t_1. \tag{3.51}$$

Integrating (3.51) from t_1 to t, we obtain

$$r(t)(z'(t))^\alpha \leq r(t_1)(z'(t_1))^\alpha - \gamma \int_{t_1}^{t} Q(s)R^\alpha(\sigma(s))\, ds \leq -\gamma \int_{t_1}^{t} Q(s)R^\alpha(\sigma(s))\, ds. \tag{3.52}$$

Integrating (3.52) from t_1 to t

$$z(t) \leq z(t_1) - \gamma^{1/\alpha} \int_{t_1}^{t} \left(\frac{1}{r(s)} \int_{t_1}^{s} Q(u)R^\alpha(\sigma(u))\, du \right)^{1/\alpha} ds \to -\infty \quad .$$

As $t \to \infty$, the condition (3.45) contradicts the positivity of $z(t)$.

Case 2: $z'(t) > 0$ for all $t \geq t_1$. Since $z(t) = x(t) + p(t)x(\tau(t))$, we have

$$x(t) = z(t) - p(t)x(\tau(t)).$$

Moreover, $z(\tau(t)) \geq x(\tau(t))$ for all t, hence

$$x(t) \geq z(t) - p(t)z(\tau(t)).$$

Since $z'(t) > 0$ and $\tau(t) \leq t$, then $z(\tau(t)) \leq z(t)$, and therefore

$$x(t) \geq (1 - p(t))z(t).$$

Consequently,

$$\left(r(t)(z'(t))^\alpha\right)' = -q(t)x^\alpha(\sigma(t)) \leq -q(t)(1 - p(\sigma(t)))^\alpha z^\alpha(\sigma(t)). \tag{3.53}$$

Since $\tau(\sigma(t)) \leq \sigma(t)$ and R is decreasing, we have

$$\frac{R(\tau(\sigma(t)))}{R(\sigma(t))} \geq 1,$$

and thus, for all sufficiently large t (say $t \geq t_2 \geq t_1$),

$$1 - p(\sigma(t)) \geq 1 - \frac{p(\sigma(t))R(\tau(\sigma(t)))}{R(\sigma(t))}. \tag{3.54}$$

By the monotonicity of $u \mapsto u^\alpha$ on $[0, \infty)$, it follows that

$$q(t)(1 - p(\sigma(t)))^\alpha \geq q(t)\left(1 - \frac{p(\sigma(t))R(\tau(\sigma(t)))}{R(\sigma(t))}\right)^\alpha = Q(t).$$

Note that for $s \geq t_1$ we have $R(\sigma(s)) \leq R(\sigma(t_1))$ since R is decreasing. Therefore,

$$\int_{t_1}^t Q(s)R^\alpha(\sigma(s))\,ds \leq R^\alpha(\sigma(t_1))\int_{t_1}^t Q(s)\,ds.$$

In view of (3.45) it follows that

$$\int_{t_1}^t Q(s)\,ds \to \infty \quad \text{as } t \to \infty. \tag{3.55}$$

Integrating (3.53) from t_2 to t, we obtain

$$r(t)(z'(t))^\alpha \leq r(t_2)(z'(t_2))^\alpha - \int_{t_2}^t q(s)(1 - p(\sigma(s)))^\alpha z^\alpha(\sigma(s))\,ds.$$

Since $z'(t) > 0$ on $[t_2, \infty)$, the function z is increasing; and since $\sigma'(t) > 0$, we have $\sigma(s) \geq \sigma(t_2)$ for all $s \geq t_2$. Hence $z(\sigma(s)) \geq z(\sigma(t_2))$ and $z^\alpha(\sigma(s)) \geq z^\alpha(\sigma(t_2))$ for $s \geq t_2$, which yields

$$\int_{t_2}^t q(s)(1 - p(\sigma(s)))^\alpha z^\alpha(\sigma(s))\,ds \geq z^\alpha(\sigma(t_2))\int_{t_2}^t q(s)(1 - p(\sigma(s)))^\alpha\,ds.$$

Consequently,

$$\begin{aligned} r(t)(z'(t))^\alpha &\leq r(t_2)(z'(t_2))^\alpha - z^\alpha(\sigma(t_2))\int_{t_2}^t q(s)(1 - p(\sigma(s)))^\alpha\,ds \\ &\leq r(t_2)(z'(t_2))^\alpha - z^\alpha(\sigma(t_2))\int_{t_2}^t Q(s)\,ds. \end{aligned} \tag{3.56}$$

In view of (3.55), the right-hand side of (3.56) tends to $-\infty$ as $t \to \infty$. This contradicts $r(t)(z'(t))^{\alpha} > 0$ (since $r(t) > 0$ and $z'(t) > 0$ in Case 2). The proof is complete. $\square$

Example 3.9. *Consider the second-order neutral delay differential equation*

$$\left(t^2\, z'(t)\right)' + t\, x\left(\frac{t}{2}\right) = 0, \quad t \geq 1, \tag{3.57}$$

where

$$z(t) = x(t) + \frac{1}{4}\, x\left(\frac{t}{2}\right),$$

and

$$\alpha = 1, \qquad r(t) = t^2, \qquad q(t) = t, \qquad p(t) = \frac{1}{4}, \qquad \sigma(t) = \frac{t}{2}, \qquad \tau(t) = \frac{t}{2}.$$

Verification of assumptions (H1)–(H5).

(H1) $\alpha = 1 > 0$ *is a quotient of odd positive integers.*

(H2) The equation is in the non-canonical case since

$$R(t) := \int_t^{\infty} r^{-1/\alpha}(s)\, ds = \int_t^{\infty} s^{-2}\, ds = \frac{1}{t} < \infty, \quad t \geq 1.$$

(H3) The delay functions satisfy

$$\tau(t) = \frac{t}{2} \leq t, \qquad \sigma(t) = \frac{t}{2} \leq t, \qquad \sigma'(t) = \frac{1}{2} > 0,$$

and

$$\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty.$$

(H4) The coefficient functions satisfy

$$0 \leq p(t) = \frac{1}{4} < 1,$$

and $q(t) = t$ does not vanish identically on any half-line $[t^*, \infty)$.

(H5) Since $R(t) = 1/t$ and $\tau(t) = t/2$, we have

$$\frac{R(t)}{R(\tau(t))} = \frac{1/t}{1/(t/2)} = \frac{1}{2}.$$

Hence

$$p(t) = \frac{1}{4} < \frac{1}{2} = \frac{R(t)}{R(\tau(t))},$$

so assumption (H5) holds for all $t \geq 1$.

Verification of the integral condition* (3.45) *in Theorem 3.8. *For $\alpha = 1$,*

$$Q(t) = q(t)\left(1 - \frac{p(\sigma(t))R(\tau(\sigma(t)))}{R(\sigma(t))}\right).$$

Here $p(\sigma(t)) = \frac{1}{4}$, $\sigma(t) = \frac{t}{2}$, $\tau(\sigma(t)) = \frac{t}{4}$, and

$$R(\sigma(t)) = R\Big(\frac{t}{2}\Big) = \frac{2}{t}, \qquad R(\tau(\sigma(t))) = R\Big(\frac{t}{4}\Big) = \frac{4}{t},$$

so

$$Q(t) = t\left(1 - \frac{1}{4}\cdot\frac{4/t}{2/t}\right) = t\left(1 - \frac{1}{4}\cdot 2\right) = \frac{t}{2}.$$

Moreover,

$$R(\sigma(s)) = R\Big(\frac{s}{2}\Big) = \frac{2}{s},$$

and hence

$$\int_{t_1}^{t} Q(s)R(\sigma(s))\,ds = \int_{t_1}^{t} \frac{s}{2}\cdot\frac{2}{s}\,ds = \int_{t_1}^{t} 1\,ds = t - t_1.$$

Therefore,

$$\int_{t_1}^{\infty}\left(\frac{1}{r(t)}\int_{t_1}^{t} Q(s)R(\sigma(s))\,ds\right)dt = \int_{t_1}^{\infty}\frac{t-t_1}{t^2}\,dt = \infty.$$

Thus, condition (3.45) holds, and by Theorem 3.8 every solution of (3.57) is oscillatory.

Conclusion

This thesis presents a systematic review of oscillation criteria for second-order Emden–Fowler neutral delay differential equations. It brings together and organizes important results from the literature into a unified framework based on the canonical and non-canonical cases.

The study is based on well-known methods, including the reduction to first-order delay differential inequalities and Riccati-type transformations. In the canonical case, recent results improve classical criteria by incorporating the effect of delay functions and providing more accurate oscillation conditions. In the non-canonical case, additional techniques are used to describe the different asymptotic behavior of solutions.

The presented examples illustrate the applicability of these results and show that modern criteria are generally more effective and less restrictive than earlier ones, especially when the neutral coefficient $p_0 \geq 1$.

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