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Norm estimates for random multilinear Hankel forms

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Abstract

We obtain sharp estimates for average norms of multilinear Hankel forms on complex Hilbert space whose entries are independent Rademacher functions.

Keywords: Norm estimates; Multilinear Hankel forms; Rademacher functions

1. Introduction

The computation of the norm of a Hilbert space operator is generally a difficult problem. However, when an appropriate degree of randomness is introduced, the situation becomes, on average, much more tractable. To be specific, let us consider a linear operator $T : l_2^{N+1} \to l_2^{N+1}$ on the standard $(N+1)$-dimensional complex Hilbert space with usual (!) orthonormal basis $\{e_0, \ldots, e_N\}$. This operator then has a matrix representation $T = (t_{km})_{k,m=0}^{N}$ and its norm is given by

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$$\|T\| := \sup\left\{|(Tx)(y)| : \|x\| \leqslant 1, \|y\| \leqslant 1\right\}$$
$$= \sup\left\{\left|\sum_{k,m=0}^{N} t_{km}x_k y_m\right| : \sum_{k=0}^{N}|x_k|^2 \leqslant 1, \sum_{m=0}^{N}|y_m|^2 \leqslant 1\right\}.$$

We shall work with operators T whose entries t_{km} are independent identically distributed random variables taking values ± 1 with equal probability. (A useful model for these random variables is the collection of Rademacher functions.) Such operators have been studied extensively by several authors, including Bennett et al. [2], and have been used to provide important insights in the geometry of Banach spaces. The reason for their importance is that, on average, such operators have norm close to the minimum possible. The main estimate [2,4] is that

$$AN^{1/2} \leqslant E(\|T\|) \leqslant BN^{1/2},$$

where E denotes mathematical expectation and A, B are constants independent of N. In fact [4], *all* linear operators on l_2^{N+1} whose entries are ± 1 have norm at least $(N+1)^{1/2}$.

It is natural to ask how this average behavior is affected by the introduction of algebraic structure into the matrix representation of the operator. A very common class that arises in this sort of way is the class of Hankel operators, A Hankel operator T on l_2 has infinite matrix representation

$$T = (t_{k+m})_{k,m=0}^{\infty},$$

and so the entries are constant on “counter-diagonals”, A famous theorem of Nehari (see [6]) establishes a link with function theory; it asserts that

$$\|T\| = \inf\left\{\|f\|_\infty : \hat{f}(n) = t_n,\ n = 0, 1, 2, \dots\right\}.$$

Here, f is an L^∞ function defined on $[0, 2\pi]$, $\|f\|_\infty$ is the supremum norm, and $\hat{f}(n)$ is the nth Fourier coefficient.

When we work with Hankel operators T on l_2^{N+1} whose *distinct* entries are independent identically distributed random vaxiables taking values ± 1 with equal probability, we are able to exploit Nehari's theorem in conjunction with probabilistic estimates introduced by Salem and Zygmund [8] to show that

$$A(N\log N)^{1/2} \leqslant E(\|T\|) \leqslant B(N\log N)^{1/2},$$

where A, B are constants independent of N.

In this case it is interesting to note that the expected value of $\|T\|$ deviates by a logarithmic factor from the lowest possible norm of a Hankel operator on l_2^{N+1} with ± 1 entries. To see this, it is enough to consider the operator $S = (S_{k+m})_{k,m=0}^{2^p}$ where the entries $S_0, \dots, S_{2^{p+1}-1}$ are the coefficients of the $(p+1)$st Rudin–Shapiro polynomial [3] and $S_{2^{p+1}} = 0$. Since the norm of this Rudin–Shapiro polynomial is bounded above by $10(2^p)^{1/2}$, an application of Nehari's theorem shows that $\|S\| \leqslant 10(2^p)^{1/2}$.

The results we have mentioned on linear operators are a special aspect of a much more general theory; the clue to this is given by the fact that a linear operator $l_2^{N+1} \to l_2^{N+1}$ can be canonically and isometrically identified with a bilinear operator $l_2^{N+1} \times l_2^{N+1} \to \mathbb{C}$.

An r-linear form $T : l_2^{N+1} \times \cdots \times l_2^{N+1} \to \mathbb{C}$ may be specified by

$$T(e_{k_1}, \ldots, e_{k_r}) = t_{k_1 \cdots k_r} \quad (0 \leqslant k_1, \ldots, k_r \leqslant N),$$

and its norm is given by

$$\|T\| = \sup\left\{|T(x_1, \ldots, x_r)| : \|x_1\| \leqslant 1, \ldots, \|x_r\| \leqslant 1\right\}.$$

Multilinear forms whose entries $t_{k_1 \cdots k_r}$ are independent identically distributed random variables taking values ± 1 with equal probability have been studied by Mantero and Tonge [4,5] and Varopoulos [9]. They proved effective in working with many variable von Neumann inequalities. It is remarkable that the r-linear norm estimates [4] are exactly the same as the bilinear norm estimates:

$$AN^{1/2} \leqslant E(\|T\|) \leqslant BN^{1/2},$$

where A and B depend on r, but are independent of N.

Our object in this paper is to obtain the corresponding estimates for norms of random multilinear Hankel forms T given by

$$T(e_{k_1}, \ldots, e_{k_r}) = t_{k_1 + \cdots + k_r} \quad (0 \leqslant k_1, \ldots, k_r \leqslant N).$$

For $r > 2$ the Hankel situation is quite different from the general case.

Theorem 1.1. *If $T : l_2^{N+1} \times \cdots \times l_2^{N+1} \to \mathbb{C}$ is an r-linear Hankel form whose distinct entries $t_{k_1+\cdots+k_r}$ are independent identically distributed random variables taking values ± 1 with equal probability, then*

$$A(N^{r-1} \log N)^{1/2} \leqslant E(\|T\|) \leqslant B(N^{r-1} \log N)^{1/2},$$

where A and B are constants depending on r, but independent of N.

We shall also find that all r-linear Hankel forms T on l_2^{N+1} whose entries are ± 1s satisfy $\|T\| \geqslant CN^{(r-1)/2}$ where C is a constant independent of N. The Rudin–Shapiro example would show that this global lower bound cannot be improved.

2. The upper estimate

We begin by introducing a link with Hardy spaces. For $p > 0$, the Hardy space H^p is the subspace of $L_p[0, 2\pi]$ consisting of those functions f with $\hat{f}(n) = 0$ for

$n < 0$. Our interest in these spaces stems from their adaptability to "fractional integration".

Theorem 2.1 (Zygmund [10, Theorem 9.15]). *Let* $0 < p < q < \infty$ *and set* $\beta = \frac{1}{p} - \frac{1}{q}$. *There is a constant* $k \geqslant 0$ *such that if* $g \in H_p$ *and* $g_\beta(z) = \sum_{n=1}^{\infty} \hat{g}(n)(\mathrm{i}n)^{-\beta} z^n$ *then* $g_\beta \in H^q$ *with* $\|g_\beta\|_q \leqslant k\|g\|_p$.

This theorem enables us to obtain information on the distribution of $\|T\|$.

Proposition 2.1. *There is a constant* A, *independent of* N, *such that for any* $\lambda > 0$ *and* $\tau > 0$, *we have*

$$P\left(\|T\| \geqslant A\left(\lambda N^{r-1} + \frac{2}{\lambda}\log N^3\tau\right)\right) \leqslant \frac{1}{\tau}.$$

Proof. For $1 \leqslant j \leqslant r$, take $x^{(j)}$ in the sphere of l_2^{N+1} and set

$$k_j(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n=0}^{N} x_n^{(j)} \mathrm{e}^{\mathrm{i}n\theta}.$$

Then $\|k_j\|_{H^2} = \|x^{(j)}\|$ and so, by a general form of Hölder's inequality, $k_1 \cdots k_r \in H^{2/r}$ with $\|k_1 \cdots k_r\|_{2/r} \leqslant 1$.

Consequently

$$\begin{aligned}
\|T\| &= \sup\left\{\left|\sum_{i_1,\ldots,i_r=0}^{N} t_{i_1+\cdots+i_r} x_{i_1}^{(1)} \cdots x_{i_r}^{(r)}\right| : \|x^{(j)}\| \leqslant 1,\ 1 \leqslant j \leqslant r\right\}\\
&= \sup\left\{\left|\sum_{p=0}^{rN} t_p \sum_{i_2,\ldots,i_r=0}^{N} x_{p-(i_2+\cdots+i_r)}^{(1)} x_{i_2}^{(2)} \cdots x_{i_r}^{(r)}\right| : \|x^{(j)}\| \leqslant 1,\ 1 \leqslant j \leqslant r\right\}\\
&\leqslant \sup\left\{\left|\sum_{p=0}^{rN} t_p \widehat{k_1 \cdots k_r}(p)\right| : \|k_1\|_{H^2} \leqslant 1, \ldots, \|k_r\|_{H^2} \leqslant 1\right\}\\
&\leqslant \sup\left\{\left|\sum_{p=0}^{rN} t_p \hat{g}(p)\right| : \|g\|_{H^{2/r}} \leqslant 1\right\}.
\end{aligned}$$

This enables us to make use of Theorem 2.1 with $p = 2/r$ and $q = 1$, and so $\beta = (r-2)/2$. Let $f(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{p=0}^{rN} t_p\, \mathrm{e}^{-\mathrm{i}p\theta}$ and

$$f^{(r-2)/2}(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{p=0}^{rN} (\mathrm{i}p)^{(r-2)/2} t_p\, \mathrm{e}^{-\mathrm{i}p\theta}.$$

Then, using the orthogonality of the functions $\mathrm{e}^{\mathrm{i}n\theta}$, we find

$$\begin{aligned}\|T\| &\leqslant \sup\left\{\left|\sum_{p=0}^{rN}\left[(\mathrm{i}p)^{(r-2)/2}t_p\right]\left[\hat{g}(p)/(ip)^{(r-2)/2}\right]\right| : \|g\|_{H^{2/r}} \leqslant 1\right\}\\ &\leqslant \sup\left\{\left|\int_0^{2\pi} f^{(r-2)/2}(\mathrm{e}^{\mathrm{i}\theta})g_{(r-2)/2}(\mathrm{e}^{\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi}\right| : \|g\|_{H^{2/r}} \leqslant 1\right\}\\ &\leqslant k\sup\left\{\left|\int_0^{2\pi} f^{(r-2)/2}(\mathrm{e}^{\mathrm{i}\theta})h(\mathrm{e}^{\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi}\right| : \|h\|_{H^1} \leqslant 1\right\}\\ &= k\|f^{(r-2)/2}\|_{(H^1)^*}\\ &\leqslant k\left\|\sum_{p=0}^{rN} t_p(\mathrm{i}p)^{(r-2)/2}\,\mathrm{e}^{\mathrm{i}pt}\right\|_\infty .\end{aligned}$$

By the proof of [3, Theorem 1, p. 68], we have

$$P\left(\left\|\sum_{s=0}^{rN} t_s(\mathrm{i}s)^{(r-2)/2}\,\mathrm{e}^{\mathrm{i}st}\right\|_\infty \geqslant \lambda q + \frac{2}{\lambda}\log(4\pi(rN)^2\tau)\right) \leqslant \frac{1}{\tau},$$

with $q = \sum_{s=0}^{rN}\left\|(\mathrm{i}s)^{(r-2)/2}\,\mathrm{e}^{\mathrm{i}s\theta}\right\|_2^2 = \sum_{s=0}^{rN} s^{(r-2)} \leqslant BN^{r-1}$, where B is a constant independent of N. The result follows at once. □

We can now derive our upper estimate. Notice that

$$\begin{aligned}E(\|T\|) &= \int_0^\infty P(\|T\| \geqslant z)\,\mathrm{d}z\\ &\leqslant 7A(N^{r-1}\log N)^{1/2} + \int_{7A(N^{r-1}\log N)^{1/2}}^\infty P(\|T\| \geqslant z)\,\mathrm{d}z\\ &= 7A(N^{r-1}\log N)^{1/2} + \int_0^\infty P\big(\|T\| \geqslant z + 7A(N^{r-1}\log N)^{1/2}\big)\,\mathrm{d}z.\end{aligned}$$

However, setting $\lambda = (N^{-(r-1)}\log N)^{1/2}$ in Proposition 2.1 we get

$$P\big(\|T\| \geqslant 7A(N^{r-1}\log N)^{1/2} + 2AN^{(r-1)/2}\log\tau\big) \leqslant 1/\tau$$

for any $\tau > 0$, and so it follows that

$$\begin{aligned}E(\|T\|) &\leqslant 7A(N^{r-1}\log N)^{1/2} + \int_0^\infty \exp\big(-z/2AN^{(r-1)/2}\big)\,\mathrm{d}z\\ &\leqslant 9A(N^{r-1}\log N)^{1/2}.\end{aligned}$$

3. The lower estimate

Our method consists of choosing appropriate vectors to estimate the norm from below, and then extracting the necessary information by using techniques introduced by Salem and Zygmund [8], Some of these techniques have also been used effectively by Bennett [1] in the study of uncomplemented Hilbert subspaces of $L_p[0,1]$ for $p > 2$.

First, observe that for each $0 \leqslant x \leqslant 2\pi$,

$$\begin{aligned}\|T\| &\geqslant N^{-r/2}\left|\sum_{k_1,\dots,k_r=0}^{N} t_{k_1+\cdots+k_r}\,\mathrm{e}^{\mathrm{i}k_1x}\cdots\mathrm{e}^{\mathrm{i}k_rx}\right|\\ &= N^{-r/2}\left|\sum_{k=0}^{rN}\left(\sum_{k_1+\cdots+k_r=k}1\right)t_k\,\mathrm{e}^{\mathrm{i}kx}\right|\\ &= N^{-r/2}\left|\sum_{k=0}^{rN}\binom{r+k-1}{r-1}t_k\,\mathrm{e}^{\mathrm{i}kx}\right|.\end{aligned}$$

The last equality is due to a standard counting technique which may be found, for example in [7]. We can now assert that

$$\|T\| \geqslant N^{-r/2}\left\|\sum_{k=0}^{rN}\binom{r+k-1}{r-1}t_k\,\mathrm{e}^{\mathrm{i}kx}\right\|_\infty, \tag{$*$}$$

and we proceed to estimate the expected value of the supremun norm.

In order to apply the methods of Salem and Zygmund, we need a preliminary remark.

Lemma 3.1. *Set* $a_k = \binom{r+k-1}{r-1}$ *for* $0 \leqslant k \leqslant rN$, *and define* $R_N = \sum_{k=0}^{rN} a_k^2$ *and* $Q_N = \sum_{k=0}^{rN} a_k^4$. *Then there is a constant* C, *independent of* N, *for which* $Q_N \leqslant CN^{-1}R_N^2$.

Proof. It is clear that for each k,

$$\frac{k^{r-1}}{(r-1)!} \leqslant a_k \leqslant \frac{(r+k)^{r-1}}{(r-1)!}.$$

Consequently

$$Q_N = \sum_{k=0}^{rN} a_k^4 \leqslant \sum_{k=0}^{rN}\left[\frac{(r+k)^{r-1}}{(r-1)!}\right]^4 \leqslant \sum_{k=0}^{r(N+1)}\left[\frac{k^{r-1}}{(r-1)!}\right]^4 \leqslant AN^{4r-3},$$

where A is a constant depending on r, but not on N.

On the other hand

$$R_N = \sum_{k=0}^{rN} a_k^2 \geqslant \sum_{k=0}^{rN} \left[\frac{k^{r-1}}{(r-1)!}\right]^2 \geqslant BN^{2r-1}, \qquad (**)$$

where B is another constant depending on r, but not on N. Assembling this information, we find

$$Q_N \leqslant (AB^{-2})N^{-1}R_N^2. \qquad \square$$

Recall [4] that *any* r-linear form $T : l_2^{N+1} \times \cdots \times l_2^{N+1} \to \mathbb{C}$, whose entries $t_{k_1,\dots,k_r} = T(e_{k_1},\dots,e_{k_r})$ have values ± 1, must satisfy $\|T\| \geqslant N^{1/2}$. The extra structure imposed by the Hankel condition, namely that $t_{k_1,\dots,k_r} = t_{k_1+\cdots+k_r}$, forces this lower bound up when $r > 2$. We can see this by combining $(*)$ and $(**)$:

$$\begin{aligned}\|T\| &\geqslant N^{-r/2}\left\|\sum_{k=0}^{rN} a_k t_k\, \mathrm{e}^{\mathrm{i}kx}\right\|_\infty \geqslant N^{-r/2}\left\|\sum_{k=0}^{rN} a_k t_k\, \mathrm{e}^{\mathrm{i}kx}\right\|_2 \\ &= N^{-r/2}R_N^{1/2} \geqslant BN^{(r-1)/2}.\end{aligned}$$

The estimates of Lemma 3.1 allow us to use the techniques, but not the statement, of Salem and Zygmund [8, Theorem 4.5.1]. To aid the reader, we indicate briefly how to extract the information, we need. For simplicity we suppose that the random variable t_k is the $(k+1)$st Rademacher function, which we (unconventionally!) label r_k.

Lemma 3.2 (Salem and Zygmund). *Let* $P_N(x,w) = \sum_{k=0}^{rN} a_k r_k(w)\,\mathrm{e}^{\mathrm{i}kx}$. *Set* $R_N = \sum_{k=0}^{rN} a_k^2$ *and* $Q_N = \sum_{k=0}^{rN} a_k^4$. *Now select* $0 < \theta < 1$ *and* $\lambda < 1$. *Finally, when*

$$I_N(w) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\lambda P_N(x,w)}\,\mathrm{d}w,$$

denote by E_N *the set of all points* $w \in [0, 2\pi]$ *where*

$$I_N(w) \geqslant (rN)^{\theta^2-1}\,\mathrm{e}^{\frac{1}{4}\lambda^2 R_N - \lambda^4 Q_N}.$$

Then the measure of E_N *satisfies*

$$|E_N| \geqslant \left(1-(rN)^{\theta^2-1}\right)^2 \mathrm{e}^{-2\lambda^4 Q_N}\left(1+\frac{5}{36}\frac{Q_N}{R_N^2}\,\mathrm{e}^{\frac{3}{2}\lambda^2 R_N}\right)^{-1}.$$

The proof may be found in [8, pp. 273–275].

We can now achieve our goal rapidly. Using the a_k's of Lemma 3.1 and choosing $\lambda^2 = (2\theta^2 \log rN)/(3R_N)$ we can follow the route of Salem and Zygmund and use Lemma 3.1 to derive

$$|E_N| \geqslant 1 - CN^{\theta^2-1},$$

where here and later C is a constant depending on r, but not on N. Incorporating the lower bound for $I_N(w)$ on E_N, we find, just as in Salem and Zygmund, that for $w \in E_N$

$$\sup_x |P_N(x, w)| \geqslant C(R_N \log rN)^{1/2} \geqslant CN^{r-\frac{1}{2}}(\log N)^{1/2}$$

provided that θ is chosen sufficiently close to 1.

Since $|E_N| \geqslant \frac{1}{2}$ for large N, it follows at once that

$$E(\|P_N\|_\infty) \geqslant CN^{r-\frac{1}{2}}(\log N)^{1/2}$$

for large N, and so for all N. Consequently, we can use (∗) to conclude that

$$E(\|T\|) \geqslant C(N^{r-1} \log N)^{1/2}.$$

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