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Conformal Mappings with Applications to Computational Geometry

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We the members of the committee certify the thesis of Ala'a Badawi and that, in our opinion, it is fully adequate in scope and quality as a thesis for the degree of Master of Science.

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DEDICATION AND THANKS

This thesis is sincerely dedicated

To

My loving Parents

For raising me to be the person I am today. Thank you for everything. I love you.

And

To

My dear husband, Talal.A very special thank for your practical and emotional support

And

To

My siblings (Mahdi, Majdi, Mohannad, Deema,LamaMaha, Samah, and Izzat) for their kind help, moral supports, and encouragement.

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Ala'a R. Badawi

ABSTRACT

Conformal mapping is extremely important in complex analysis as well as in many areas of physics and engineering.

Our main goal in this thesis is to present a successful application of conformal transformations in solving boundary value problems. conformal transformation has been used to generate orthogonal curvilinear coordinates for solving various problems in simply connected regions. Consequently, if one is solving an elliptic boundary value problem, an appropriate conformal mapping could be used simultaneously to fit the boundary curves with coordinate lines and simplify the original partial differential equation.

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Chapter 1

Introduction

Over the span of several decades, many advances have been made in the area of mathematics known as *conformal mapping*, a key theoretical tool used by mathematicians, engineers, and scientists to translate information from complicated shapes to simpler shapes; so that it is easier to analyze. In general, a conformal transformation is a map of one region upon another that is one-to-one and continuous and such that angles are preserved; in particular, maps that preserve the form (or shape) of small-scale features are also called *conformal.* Conformal means *same form* or *same shape* and *invariance* as used in physics means the preservation of some physically observable characteristic. The most well known conformal relationship is scale invariance, which relates systems of different scale.

Conformal mappings are an old mathematical topic and they have been a part of mathematics for centuries. In the 19th century, great strides were made in conformal mapping theory by using the idea of complex numbers. The culmination of this work was the Riemann Mapping Theorem, considered by many to be the greatest accomplishment of 19th century mathematics. In 1851, Georg Friedrich Bernhard Riemann (1826-1866) proved that any planar region with no holes in it can be conformally mapped to a disk. Unfortunately, the theorem provided no way to actually construct the map whose existence was proved. The theorem was a *nonconstructive theorem*, *i.e.*, it proved the existence of conformal mapping of a simply-connected planar region by demonstrating that nonexistence would lead to a logical contradiction.

A conformal map of a region in the complex plane is an analytic (smooth) function whose derivative never vanishes within the region. More precisely, a conformal map transforms any pair of curves intersecting at a point in the region so that the image curves intersect at the same angle. Conformal maps are both mathematically interesting and practically useful in applied sciences. Geometrically, because of the feature of the preservation of angles, a conformal map of a square grid in the plane results in a curvilinear orthogonal grid. Traditionally, conformal mapping is strongly founded in complex analysis with emphasis on the theoretical analysis and some physical applications. The field of conformal transformation includes a whole wealth of mappings in mathematical sciences most important to complex theory, the theory of differential geometry, and differential equations. They naturally arise in mathematical physics; and, therefore, conformal mapping holds the importance as an enabling mathematical tool for physicists who, in turn, are trying to understand the basic forces of nature. Classical references on this topic include the work of Ahlfors [1], Conway [2], Greene and Krantz [3], and Dettman [4].

Conformal maps are extremely powerful tools for analyzing a wide range of mathematical and scientific problems. This theoretical tool has a long history of use in diverse applications in a large number of fields including modeling airflow patterns over intricate wing shapes in aeronautics. Furthermore, the transformation by a conformal map of a solution to Laplace's equation is still a solution in the image region, so that conformal maps can be used in heat transfer, electrostatics, steady fluid flows, and other physical applications in which Laplace's equation arises. The usefulness of conformal mapping for applied problems stems from the fact that the Laplacian operator transforms in a simple way under a conformal map. The literature abounds with methods for constructing conformal mappings, as in Nahari [8], Cohen [9], and Bieberbach [10]. Conformal mappings have already stepped beyond the realm of complex analysis with strong manifestation in differential equations and computational geometries, see Kreyszig [23], Thorpe [24], and Struik [25].

Recently, conformal mapping methods have attracted researcher in computational geometry and grid generation fields for the purpose of constructing domain discretization applied to a multitude computational engineering problems. Researchers have utilized conformal mappings in grid generation methods a mean to an end application in computational sciences. As a mater of fact, the successful elliptic methods applied to computational grid generation were those borrowed most heavily from the properties of conformal mappings. The main impetus for the development of this type of elliptic methods is the inherit geometric properties of conformal mapping. Experience in the field of computational simulation has shown that geometric quality in terms of smoothness and orthogonality of coordinate grid lines effects the accuracy of numerical solutions. Conformal transformation possess a commanding rule in control of the geometric characteristics of the computational model. Details on use of conformal maps to generate orthogonal coordinates by Khamayseh *et al.* [29], Ryskin and Leal [30], Ives [31], and Godunov and Prokopov [32].

Currently, there are new extensions and enchantments made to computational conformal mapping in areas of image processing and medicine. For example, Stephenson [26] based his computational methodology of *circle-packing* on conformal mapping which is currently being used in neuroscience to visualize the complicated structure of the human brain imaging. New methods using circle-packing are ideally suited to *unfold* and *flatten* the cortical surface of the brain. Computational methods in the area of circle-packing are used to create approximations to discrete conformal maps of the brain. Conformal computational techniques of the brain maps are becoming more popular in the field of medicine, particularly among neuroscientists who are interested in depression, schizophrenia, and Alzheimer's diseases.

The geometric motivation for constructing conformal mappings lies also in their application to the generation of curvilinear coordinate systems for two-dimensional regions. Conformal mappings may be used to reduce any second order linear elliptic partial differential equation to canonical form, *i.e.*, the principal part of the differential operator reduces to the Laplacian. Consequently, if one is solving an elliptic boundary value problem, an appropriate conformal mapping could be used simultaneously to fit the boundary curves with coordinate lines and simplify the original partial differential equation. The equation, in canonical form, could possibly be solved more efficiently or by methods which would not be applicable to the original equation in cartesian coordinates. Examples and study cases are given in Mathews and Howell [5] and Kwok [6].

In the research, there are three principal themes of this investigation which are structured as follows: (1) The first chapter is concerned with preliminary study of the analytic concept and properties of conformal mappings form the theory of complex analysis. The notion of conformal mapping on Riemannian surfaces will be presented in conjunction with geometrical representations of such surfaces in the two-dimensional space. We also shed more light on the strong connection between holomorphic functions and conformal mappings and establish the fact that one map leads to the other and *visa versa.* Furthermore, much of the theory on the necessary and sufficient conditions is discussed for the map to achieve conformality.

(2) The next chapter is centered around the investigation of conformal mappings form a geometric point view with the intention of examining basic geometric features such as local shape invariance. In addition to the investigation of fundamental analytic properties, the presented research explores geometric characteristics of conformal mappings. For practical purposes, we consider the proposition of classical conformal mappings as geometrical maps by examining the underlying geometric properties. We show that conformal mappings have the characteristics know geometrically as circle-to-circle mapping. A class of conformal maps called Möbiuos Transformations is closely examine as geometric maps for a wide range of regions. Conformal characteristics have been exhibited by several numerical

examples throughout the course of this study. In particular, the two dominant geometric features, namely the preservation of angles and shapes are highlighted and supported by numerical results.

(3) Finally, chapter three consists of several initiatives to overcome pitfalls of the conformal mapping. Conformal maps often times viewed as rigid and extremely hard to construct for arbitrary domains. It is a well known fact that conformal mappings are powerful mathematical tool in computational geometry with quality control over angles and shapes of geometric elements. However, conformal mappings lack the necessary feature of control over the spacing between coordinate curves. To resolve this deficiency, we introduced the concept of *quasiconformal* mapping which is considered a generalization to conformal mapping. Since we are interested in the core geometric properties of conformal mapping, our version of quasiconformal mapping is *mild* departure from the relationship of conformality, enough to account for grid lines spacing. We conclude this chapter with a comprehensive development of geometric boundary value problem based on conformal mapping and harmonic function theory. This boundary value problem is best suited for numerical construction in the retrieval of conformal and/or quasiconformal mappings, while preserving the true shape of the overall domain.

Chapter 2

Conformal Mapping on Riemannian Surfaces

In classical differentiable geometry, a Riemannian manifold $(\mathcal{M}, \mathrm{g})$ (with Riemannian metric g) is a real differentiable manifold $\mathcal{M}$ in which each tangent space is equipped with an inner product g in a manner that varies smoothly from point to point. This allows one to define various metrics such as angles, lengths of curves, areas of cells, curvature, or differential operators such as gradients of functions, and divergence of vector fields. Even though Riemann manifold are n-dimensional, we restrict our attention to the two-dimensional space of $\mathbb{R}^2 = \mathbb{C}$; the complex plane. In this case, Riemann surfaces can be thought of as deformable geometric regions in the complex plane $\mathbb{C}$: locally linear at every point, they look like patches of the complex plane, but the global topology can be quite different. For example, they can look like a sphere or a torus or a couple of curved sheets glued together with curvilinear boundaries.

The main emphasis of this study is that one can define and construct conformal maps between Riemannian surfaces with well-defined coordinate transformation. These surfaces are referred to as boundary-curved (or curvilinear) surfaces and nowadays considered the natural setting for studying analytic and computational geometry. However, the interesting geometric facts about Riemannian surfaces that are deformable and flexible enough with smooth curvatures, *i.e.*, infinitely differentiable. Furthermore, they have the benefit of allowing for geometric modeling of surfaces with varying boundaries and shapes. They often provide the intuition and motivation for the generalizations of higher dimensional curves and manifolds. In two-dimensional space, examples include planes, regions bounded by piecewise polygons or parametric conics (circles, ellipsis, *etc.*), quadric surfaces (sphere, torus, *etc.*) or general spline surfaces. For this part of Riemannian geometry the books by Jost [27] and Farkas and Kra [28] are informative references.

2.1 The Concept of Conformality

In mathematics, the word *conformal* refers to the transformation of a surface or a region onto another surface in which inherent metric structures remain unchanged. For example, in complex analysis, a conformal map is defined as a function which preserves angles. In addition, conformal means *same form* or *same shape* which means the preservation and invariance of some physically quantitative characteristics. The most well known conformal relationship is scale invariance, which relates transformations of different scale.

A conformal mapping, also called a conformal map, conformal transformation, angle-preserving transformation, or biholomorphic (bijective and holomorphic) map, is a transformation that preserves local angles. However, the concept of conformality strongly depends on the metric properties of the base space. In Riemannian geometry, the concept of conformality we use in our approach deals with the scale invariance of metrics such as inner products, norms and angles which implies conformal invariance.

In addition, the concept of conformality has a distinctive relevance to geometric mappings across domains of interest. Conformal maps have the property of preserving orientation of boundary and coordinate curves, sometimes called sense-preserving transformations. Furthermore, one of the most fundamental characteristic of conformal transformation is the property of local shape preservation. Indeed, this type of transformation leads to the mapping of tiny disks in the domain onto another uniformly scaled disks in the image domain. The primary focus of this study is to closely investigate the characteristics of conformal mappings and demonstrate their geometric applications.

We next examine basic analytic properties of conformal maps and establish the close relationship to holomorphic functions. In fact, conformal maps on surfaces are basically the same as complex-analytic maps. The phrase *holomorphic* at a point ζ in the complex plane $\mathbb{C}$ means not just differentiable at ζ, but complex-differentiable everywhere within some open disk centered at ζ in the complex plane. This is a much stronger condition than real differentiability and implies that the function is infinitely differentiable, *i.e.*, C^∞, and can be described by its Taylor series.

2.2 Analytic Properties of Conformal Maps

Formally, a complex map $w = \psi(z)$ is called conformal (or angle-preserving) at ζ in the complex plane $\mathbb{C}$, if it preserves oriented angles between intersecting curves through ζ, as well as their orientation, *i.e.*, direction. Conformal maps preserve both angles and the shapes of infinitesimally small figures, but not necessarily their size. An important family of examples of conformal maps comes from complex analysis. In this section we examine the close relationship between the analyticity of a complex function and the conformality of such mapping. The term analytic function is often used interchangeably with *holomorphic* function, although the former term is also used in the broader sense of a function (real, complex, or of more general type), that is equal to its Taylor series in a neighborhood of each point in the domain. The fact that the class of analytic functions coincides with the class of holomorphic functions is a major theorem in complex analysis. In this study we use the term *holomorphic* instead of *analytic* function.

Definition 2.1. *Analyticity– Let* $\mathbf{U}$ *be a non-empty open subset of* $\mathbb{C}$ *and the function* $\psi : \mathbf{U} \to \mathbb{C}$, *we say that* ψ *is holomorphic at* ζ *if*

$$\lim_{z\to\zeta} \frac{\psi(z) - \psi(\zeta)}{z - \zeta}$$

exists. This limit is unique and denoted by $\psi'(\zeta)$ *as the derivative of* ψ *at* ζ. *If* ψ *is holomorphic at every point of* $\mathbf{U}$, *we say it is a holomorphic function on* $\mathbf{U}$.

In other words, a complex-valued function ψ is holomorphic (or analytic) at a point $\zeta \in \mathbb{C}$ *if and only if* it is differentiable at every point within some open disk centered at ζ, which can be expanded as a convergent power series. Thus, holomorphic functions are infinitely differentiable at any point in a domain and that its Taylor series converges to it in some neighbourhood of this point. This concept of differentiability shares several properties with real differentiability: it is linear and obeys the product, quotient and chain rules. On the other hand, the following noteworthy fact is established in the theory of holomorphic functions: If a function ψ is differentiable in a domain Ω, it is holomorphic in this domain (for a single point this statement is not true: $\psi(z) = |z|^2 = z\bar{z}$ is differentiable at $\zeta = 0$, but is nowhere holomorphic). Thus, the concepts of complex differentiability and holomorphy of a function ψ in a domain Ω are identical; together with property satisfaction of the CauchyRiemann equations.

We now present the notion of mathematical nature of conformality when mapping different domains in the complex plane $\mathbb{C}$. Let $w = \psi(z)$ be a holomorphic function that maps a domain Ω_z in the complex z–plane onto a domain Ω_w in the the complex w–plane that satisfy the following two conditions: (1) Every curve

$(C_z : z(t)\,(0 \leq t \leq 1))$ that starts at any point $z_0 \in \Omega_z$ and possesses a tangent there has an image curve $(C_w : w(t) = \psi(z(t))\,(0 \leq t \leq 1))$ that also possess a tangent at the image point $w_0 = \psi(z_0)$. (2) The angle between any two curves C_z^1 and C_z^2 possessing tangent at z_0 is equal to the angle between their image curves C_w^1 and C_w^2, where the direction of the angle is also taken into account as shown in Figure 2.1. Then the mapping from Ω_z onto Ω_w is said to be conformal. We will present a proof that $\psi(z)$ is necessarily a function analytic in Ω_z and $\psi'(z_0) \neq 0$.

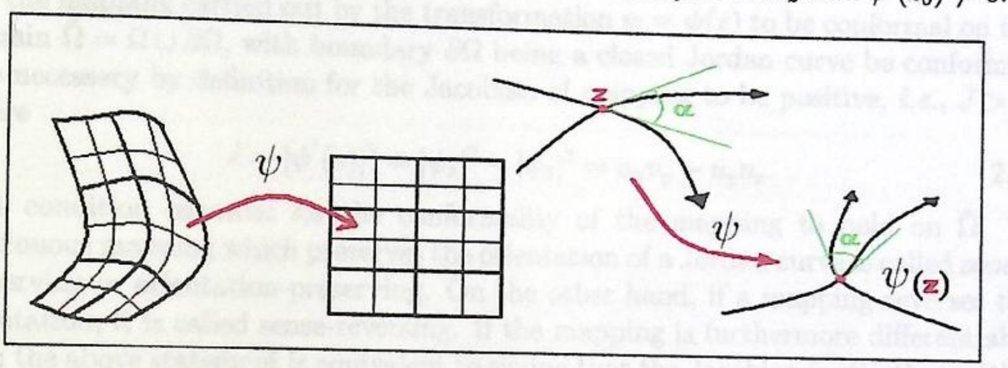

Figure 2.1: Conformal mapping from the $z-$plane to the $w-$plane.

It is worth noting that throughout the course of this study we assume– without loss of generality– the candidate mapping is *homeomorphism*; if not otherwise, classified as holomorphic or conformal.

Definition 2.2. *Homeomorphism– A mapping ψ between two topological spaces Ω_z and Ω_w is called a homeomorphism if the mapping is continuous, bijective, and the inverse mapping ψ^{-1} is continuous, i.e., Jacobian conditions of ψ do not vanish.*

In mathematical analysis, the notion of homeomorphism is often interchangeable with *diffeomorphism* in which the two concepts have the same properties and intuitions. The definition of diffeomorphism is a differentiable isomorphism between smooth manifolds and has a differentiable inverse, *i.e.,* $\psi'(z) \neq 0$.

Definition 2.3. *Conformality– We say ψ is* conformal *at a point $\zeta \in \Omega$ if for each pair of smooth curves γ_1 and γ_2 passing through ζ, say at an arc-length parameter $t = \tau$, the angle between their tangent vectors $\gamma_1'(\tau)$ and $\gamma_2'(\tau)$ equals the angle between their tangent vectors $(\psi \circ \gamma_1)'(\tau)$ and $(\psi \circ \gamma_2)'(\tau)$, their images under ψ. If ψ is conformal at every point in Ω, we say ψ is conformal.*

In the study of conformal mappings we need to define partial derivatives in terms of real and complex parts. Let $\psi : \Omega \to \psi(\Omega)$ be a holomorphic map, where $\psi(z) = u(z) + iv(z)$ and $z = x + iy$. Then $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ and $dz = dx + idy$. The total derivative of ψ with respect to z and $\bar{z}$ is computed as follows.

$$
\begin{gathered}
du = u_x dx + u_y dy,\ dv = v_x dx + v_y dy, \\
\psi_x = u_x + iv_x,\ \psi_y = u_y + iv_y, \\
\psi_z = \frac{1}{2}(\psi_x - i\psi_y),\ \psi_{\bar{z}} = \frac{1}{2}(\psi_x + i\psi_y),\ d\psi = \psi_z dz + \psi_{\bar{z}} d\bar{z}.
\end{gathered}
\tag{2.1}
$$

For the mapping carried out by the transformation $w = \psi(z)$ to be conformal on the domain $\bar{\Omega} = \Omega \cup \partial\Omega$, with boundary $\partial\Omega$ being a closed Jordan curve be conformal, it is necessary by definition for the Jacobian of mapping to be positive, *i.e.*, $J > 0$, where

$$
J = |\psi'(z)|^2 = |\psi_z|^2 - |\psi_{\bar{z}}|^2 = u_x v_y - u_y v_x. \tag{2.2}
$$

This condition essential for the conformality of the mapping to hold on $\bar{\Omega}$. A continuous mapping which preserves the orientation of a Jordan curve is called sense-preserving or orientation-preserving. On the other hand, if a mapping reverses the orientation, it is called sense-reversing. If the mapping is furthermore differentiable then the above statement is equivalent to saying that the Jacobian is strictly positive at every point of the domain. In case that the mapping ψ is it still preserves angles, yet reverses their orientation, then the mapping is no longer conformal and it is referred to as antiholomorphic, *i.e.*, the conjugate to a holomorphic function.

In our applications, conformal mappings in geometric representation and boundary value problems are required to have boundary behavior of non-self-intersecting curves, *i.e.*, simply-closed curves. The conformality condition is required not only in the domain Ω but also extended to the boundary $\partial\Omega$. It should be specially mentioned that the demand of the preservation of traversal orientation of the boundary has been added to the conditions of conformality. If, however, we look at the simple example shown in Figure 2.2, in this case the image of the boundary is non-Jordan curve (self-intersecting), the demand can not be excluded. The function $\psi(z) = z + z^2$ is analytic inside the disk $\mathbb{D} = \{z : |z| < 1\}$ maps the circumference $|z| = 1$ onto a self-intersecting closed curve. Through the discourse of this study, we only consider domains with boundaries $\partial\Omega$ are Jordan curves, and where it is analytic closed curve.

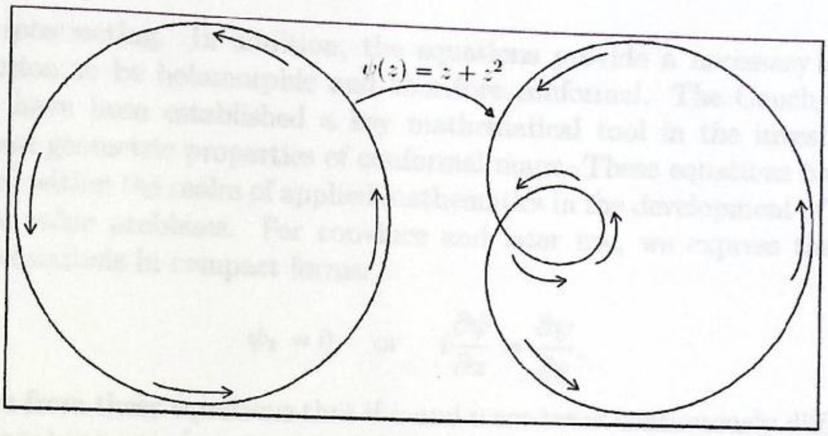

Figure 2.2: Self-intersecting conformal map when extended to boundary.

For a later reference, in this section we define elemental length and area of a conformal transformation. The property of *confomality* applies to small neighborhoods of a point ζ which is being mapped from the z-plane onto the w-plane by means of transformation $w = \psi(z)$. Suppose that $w = \psi(z)$ is holomorpic at ζ for which $\psi'(\zeta) \neq 0$. Since ψ is differentiable at ζ, then at neighboring points we can express the map in the form of a Taylor expansion,

$$\psi(z) = \psi(\zeta) + \psi'(\zeta)(z - \zeta) + \lambda(z)(z - \zeta), \tag{2.3}$$

where λ tends to zero as z tends to ζ. The transformation of incremental lengths and areas are given

$$ds = |dw| = |\psi'(z)||dz|,$$

and

$$dw d\bar{w} = \psi'(z)\overline{\psi'(z)} dz d\bar{z}, \tag{2.4}$$

or

$$dA = dudv = |\psi'(z)|^2 dxdy.$$

The relationship between real differentiability and complex differentiability is the following. The complex function $\psi(z) = u(z) + iv(z)$ is holomorphic *if and only if* u and v are continuously differentiable, *i.e.*, C^1, and their partial derivatives satisfy the Cauchy-Riemann equations:

$$u_x = v_y \text{ and } u_y = -v_x. \tag{2.5}$$

The Cauchy-Riemann equations have a remarkable significance in the study of complex analysis theory and applications. This system of equations ensure the Jacobian of the mapping is positive and in turn the mapping sense-preserving and none-self intersecting. In addition, the equations provide a necessary conditions for a function to be holomorphic and therefore conformal. The Cauchy-Riemann equations have been established a key mathematical tool in the investigation of analytic and geometric properties of conformal maps. These equations have special importance within the realm of applied mathematics in the development of boundary and initial value problems. For convince and later use, we express the Cauchy-Riemann equations in compact forms:

$$\psi_{\bar{z}} = 0 \quad \text{or} \quad i\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y}. \tag{2.6}$$

We observe from these equations that if u and v are twice continuously differentiable then they are harmonic functions since they satisfy Laplace equation $\nabla^2 u = \nabla^2 v = 0$, where the Laplace operator is defined $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The equations can therefore be seen as the conditions on a given pair of harmonic functions to become as real and imaginary parts of a holomorphic function. For a given harmonic function u a corresponding harmonic function v is called a harmonic conjugate. The converse is not necessarily true. A simple converse is that if u and v have continuous first partial derivatives (C^1*-differentiable*) and satisfy the Cauchy-Riemann equations, then ψ is holomorphic. A more satisfying converse, if ψ is continuous, u and v have first partial derivatives, and they satisfy the Cauchy-Riemann equations, then ψ is holomorphic.

Theorem 2.4. *Suppose that $\psi : \Omega \to \mathbb{C}$ is a homeomorphism defined by $\psi(z) = u(z) + iv(z)$. Then ψ is holomorphic onto Ω if and only if the function components u and v are continuously differentiable and satisfy the Cauchy-Riemann equations.*

Proof. Consider the mapping $w = \psi(z) = u(z) + iv(z)$ has continuous partial derivatives in a neighborhood of a point $z_0 \in \Omega$, $\psi_x = u_x + iv_x$, $\psi_y = u_y + iv_y$ and the Jacobian $J = u_x v_y - u_y v_x > 0$. The continuity condition of u and v and all partial derivatives implies that all directional derivations exist as well. Then, first-order Taylor expansion of u and v at z_0,

$$\begin{aligned} u(z) - u(z_0) &= u_x(z_0)(x - x_0) + u_y(z_0)(y - y_0) + \lambda_1(z)(z - z_0), \\ v(z) - v(z_0) &= v_x(z_0)(x - x_0) + v_y(z_0)(y - y_0) + \lambda_2(z)(z - z_0), \end{aligned} \tag{2.7}$$

where $\lambda_k(z) \to 0$, for $k = 1, 2$ as $z \to z_0$. We rewrite the system of equations 2.7 in complex form

$$\begin{aligned} \psi(z) - \psi(z_0) = &\, \psi_x(z_0)(x - x_0) + \psi_y(z_0)(y - y_0) \\ &+ \lambda(z)(z - z_0), \end{aligned} \tag{2.8}$$

where $\lambda(z) = \lambda_1(z) + \lambda_2(z)$. Now suppose $\mathbf{r} \equiv \{z = z_0 + re^{i\theta}\}$ is the ray emanating from z_0; we seek the derivative of $\psi(z)$ at z_0 along $\mathbf{r}$. When z is on $\mathbf{r}$, we have

$$\frac{\psi(z) - \psi(z_0)}{z - z_0} = \frac{\psi_x r \cos\theta + \psi_y r \sin\theta}{re^{i\theta}} + \lambda(z), \tag{2.9}$$

where ψ_x and ψ_y denote $\psi_x(z_0)$ and $\psi_y(z_0)$, respectively. By letting $r \to 0$, we obtain

$$\frac{d\psi}{dz} = e^{-i\theta}(\psi_x \cos\theta + \psi_y \sin\theta). \tag{2.10}$$

If the function ψ is holomorphic then the above derivative is independent of direction, then letting $\theta = 0$ and $\theta = \frac{\pi}{2}$, we get $\frac{d\psi}{dz} = \psi_x = -i\psi_y$, and hence the Cauchy-Riemann differential equations,

$$u_x = v_y, \quad u_y = -v_x.$$

Conversely, if the real-valued components u and v of ψ are continuously differentiable at the point z_0 and Cauchy-Riemann equations hold, then from equation 2.7 we have

$$\frac{\psi(z) - \psi(z_0)}{z - z_0} - [u_x(z_0) + iv_x(z_0)] = \lambda(z),$$

where $\lambda(z) \to 0$ as $z \to z_0$. Hence, we obtain the existence of

$$\psi'(z_0) = \lim_{z \to z_0} \frac{\psi(z) - \psi(z_0)}{z - z_0},$$

the derivative of $\psi(z)$ at z_0 which is independent of direction. Nevertheless, for any $z \in \Omega$, this complex derivative is expressed as

$$\psi'(z) = u_x + iv_x = v_y - iu_y. \tag{2.11}$$

□

The previous result asserts the natural relationship and dependency between holomorphic functions and conformal mappings. An analytic function is conformal at any point where it has a nonzero derivative. Conversely, any conformal mapping of a complex variable which has continuous partial derivatives is analytic. The conformal property may also be described in terms of the Jacobian derivative matrix of a coordinate transformation. If the Jacobian matrix of the transformation is everywhere a scalar times a rotation matrix, then the transformation is conformal.

It is worthwhile to clarify this conformal property which falls from the realm of linear algebra. Suppose that $\psi(z) = u(z) + iv(z)$ maps an open subset of $\mathbb{C}$ into $\mathbb{C}$, can also be regarded as a function from an open subset of $\mathbb{R}^2$ into $\mathbb{R}^2$, is holomorphic at ζ for which $\psi'(\zeta) \neq 0$. Then the transformation $w = \psi(z)$ has the linear approximation

$$l(z) = \alpha + \beta(z - \zeta), \tag{2.12}$$

where $\alpha = \psi(\zeta)$ and $\beta = \psi'(\zeta)$. The effect of the linear mapping l is a rotation of the plane through the angle $\theta = \arg \psi'(\zeta)$, followed by a magnification of a factor $\lambda = |\psi'(\zeta)|$, followed by a rigid translation by the vector $\alpha - \beta\zeta$. Consequently, the Jacobian matrix $\mathcal{J} = \frac{\partial(u,v)}{\partial(x,y)}\big|_{z=\zeta}$ of the mapping is structurally given

$$\mathcal{J}(\zeta) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}. \tag{2.13}$$

However, the Cauchy-Riemann equations for ψ means exactly that the tangent matrix in 2.13 has the form $\mathcal{J} = r\mathcal{A}$, where the matrix $\mathcal{A}$

$$\mathcal{A} = \begin{pmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{pmatrix},$$

where $a = u_x$, $b = v_x$, and $r = \sqrt{a^2 + b^2}$. The matrix $\mathcal{A}$ is an orthogonal matrix for which its transpose is the same as its inverse, *i.e.*, $\mathcal{A}^T = \mathcal{A}^{-1}$. Notice that the matrix $\mathcal{A}$ constitutes a vector $\mathbf{V} = (a, b)$ and its orthogonal rotation $\mathbf{V}^{\perp} = \mathbf{V}e^{-i\frac{\pi}{2}} = (-b, a)$. The vector $\mathbf{V}$ can be uniquely written as the linear combination $\mathbf{V} = a\mathbf{e}_1 + b\mathbf{e}_2$, where the vectors $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$ are the standard basis of $\mathbb{R}^2$. Indeed, the right triangle formed by $\mathbf{V}$, $a\mathbf{e}_1$, and $b\mathbf{e}_2$ with the angle $\theta \in \mathbb{R}$ between the vectors $\mathbf{V}$ and $a\mathbf{e}_1$ has

$$\cos\theta = \frac{a}{\sqrt{a^2 + b^2}},$$

$$\sin\theta = \frac{b}{\sqrt{a^2 + b^2}}.$$

Thus, the Jacobian matrix $\mathcal{J} = \lambda\mathcal{R}_\theta$, where $\mathcal{R}_\theta$ is the rotation matrix

$$\mathcal{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.14}$$

Therefore, the product of $\lambda\mathcal{R}_\theta$ represents the same operation on $\mathbb{R}^2$ as does multiplication on $\mathbb{C}$ by the complex number $\lambda e^{i\theta}$. Geometrically, this expresses the conformal nature by a combination of rotation and enlargement for any holomorphic function at a point where its derivative is not zero. At this end, we summarize the prior discussion in the following result:

Corollary 2.5. *Let $\psi : \Omega \to \mathbb{C}$ a holomorphic function on Ω and let $\lambda = |\psi'(z)| > 0$. Then the Jacobian matrix $\mathcal{J} = \lambda \mathcal{R}_\theta$, where $\mathcal{R}_\theta$ is the rotation matrix.*

We now present a main theorem in this chapter which precisely states the connection between holomorphic functions and angle-preserving maps.

Theorem 2.6. *Let $\psi : \Omega \to \mathbb{C}$ be a holomorphic function and that $\psi'(z) \neq 0$, then ψ is conformal. Conversely, if $\psi : \Omega \to \mathbb{C}$ is conformal, then ψ is holomorphic on Ω.*

Proof. Suppose that ψ is holomorphic at ζ, and γ_1 and γ_2 are two smooth curves in Ω passing through ζ at τ. By the chain rule,

$$(\psi \circ \gamma_1)'(\tau) = \psi'(\gamma_1(\tau))\,\gamma_1'(\tau),$$

and

$$(\psi \circ \gamma_2)'(\tau) = \psi'(\gamma_2(\tau))\,\gamma_2'(\tau).$$

Hence,

$$\arg(\psi \circ \gamma_1)'(\tau) = \arg\psi'(\gamma_1(\tau)) + \arg\gamma_1'(\tau),$$

and similarly,

$$\arg(\psi \circ \gamma_2)'(\tau) = \arg\psi'(\gamma_2(\tau)) + \arg\gamma_2'(\tau).$$

Since the curves γ_1 and γ_2 pass through ζ at τ, it follows that

$$\arg\psi'(\gamma_1(\tau)) = \arg\psi'(\gamma_2(\tau)).$$

Therefore,

$$\arg(\psi \circ \gamma_1)'(\tau) - \arg(\psi \circ \gamma_2)'(\tau) = \arg\gamma_1'(\tau) - \arg\gamma_2'(\tau).$$

This means that the angle between the tangent vectors $\gamma_1'(\tau)$ and $\gamma_2'(\tau)$ equals to the angle between $(\psi \circ \gamma_1)'(\tau)$ and $(\psi \circ \gamma_2)'(\tau)$. Since the two curves are arbitrary, as is ζ in Ω, implies that map is angle-preserving and so the function ψ is conformal.

Conversely, suppose ψ is conformal on Ω. By definition, the mapping is angle preserving map. Therefore, from the previous discussion we know the Jacobian of mapping at ζ is a scalar times a proper rotation matrix by angle θ, *i.e.*, $\mathcal{J}(\zeta) = \lambda\mathcal{R}_\theta$, where $\mathcal{R}_\theta$ and $\mathcal{J}$ are the rotation and Jacobian matrices, respectively. Notice that, since $\mathcal{R}_\theta \mathcal{R}_\theta^T = \mathcal{I}$ is the identity matrix, then we have $\mathcal{J}\mathcal{J}^T = \lambda^2\mathcal{I}$ and of course

$$\mathcal{J}\mathcal{J}^T = \begin{pmatrix} u_x^2 + u_y^2 & u_x v_x + u_y v_y \\ u_x v_x + u_y v_y & v_x^2 + v_y^2 \end{pmatrix}.$$

Thus, preservation of angles is fulfilled if and only if $u_x^2 + u_y^2 = v_x^2 + v_y^2 \neq 0$ and $u_x v_x + u_y v_y = 0$. We now combine these two equations to give the complex equation

$$(u_x + iv_x)^2 + (u_y + iv_y)^2 = 0,$$

This quadratic equation can be solved as

$$u_x + iv_x = \pm i(u_y + iv_y).$$

For the angles and orientation to be preserved, the sign $\pm$ needs to be chosen such that the Jacobian $J = u_x v_y - u_y v_x > 0$. Therefore, by choosing the negative sign will make $J = u_x^2 + u_y^2 = v_x^2 + v_y^2 > 0$. Thus, the Cauchy-Riemann equations are satisfied at each point in Ω; and hence, the map ψ is holomorphic. $\square$

We conclude this section by highlighting some the basic properties of conformal maps. Any conformal map is infinitely differentiable and invertible, *i.e.*, C^∞ and $J > 0$. The inverse map ψ^{-1} of a conformal map is conformal and the composition $\phi \circ \psi$ of two conformal maps ψ and ϕ is conformal as well. We now showcase two examples of conformal maps presented in Figure 2.3 depicting the preservation of angles between coordinate lines. The family of coordinates lines in the domain $\Omega = [0.1, 0.5] \times [0.1, 0.5]$ are perpendicular lines and in turn the coordinate curves in the map image $\psi(\Omega)$ intersect orthogonally.

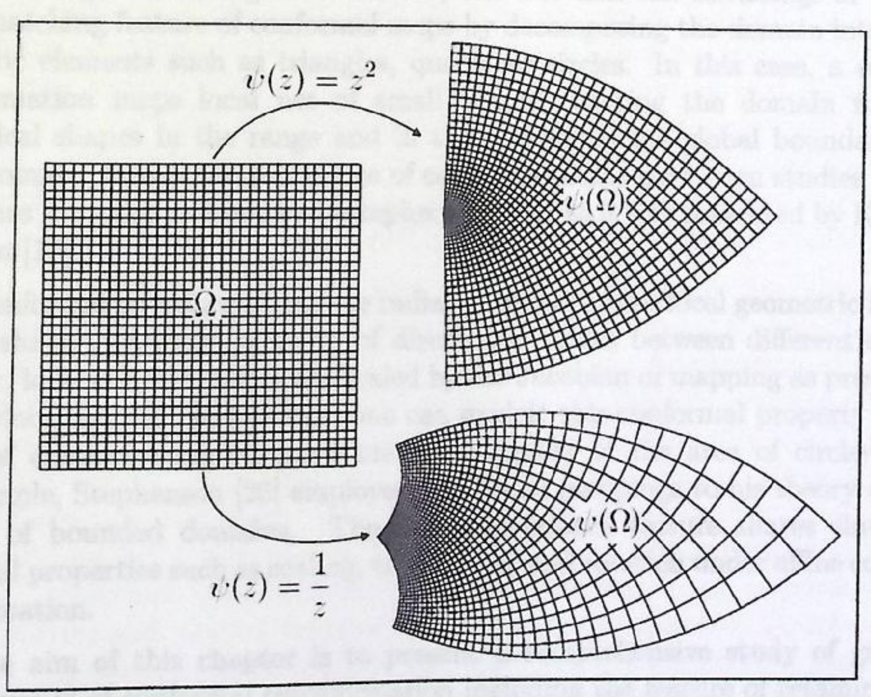

Figure 2.3: Conformal mappings of simply connected domains.

Chapter 3

Geometric Characteristics of Conformal Transformation

In theoretical and computational geometry, conformal mappings play a fundamental rule in the study of boundary-curved surfaces and manifolds. In essence, conformal transformations are viewed as spatial maps from one geometric region to another. A well-known property pertains to conformal maps is local shape preservation up to scale-invariance. Generally, this property can not be extended to the overall shape of the geometric region. However, one can take full advantage of the local shape matching feature of conformal maps by decomposing the domain into smaller geometric elements such as triangles, quads, or circles. In this case, a conformal transformation maps local net of small shapes covering the domain to similar topological shapes in the range and in turn preserves the global boundary shape of the domain. Geometric properties of conformal maps have been studies for quite sometimes from the viewpoint of complex analysis, as it was presented by Krantz [7] and Wen [18].

Conformal transformations are radial in nature at the local geometric level and possess shape invariance property of discretized circles between different domains. However, local circular shapes are scaled by the Jacobian of mapping as presented in the previous chapter. In practice, one can exploit this conformal property through means of effective computational methods applied to the area of circle-packing. For example, Stephenson [26] employed conformal mappings to his theory of circle-packing of bounded domains. The shape invariance feature shares similar key conformal properties such as scaling, translation, and rotation under affine conformal transformation.

The aim of this chapter is to present a comprehensive study of geometric characteristics of conformal transformation including the feature of retaining shape similarity across mapped domains. Defining a useful shape similarity maps is in itself a difficult problem, since the similarity measure should be translation-invariant, scale-invariant, and at least partially rotation-invariant. From geometric research it is known that conformal maps are useful for measuring and retaining shape similarity. We also present a section on the class of bilinear conformal transformation supported by a closed-form explicit formulation. The affine conformal properties such as invariance translation, rotation, and scaling, the mapping uses composition of conformal maps to guarantee translation- and rotation-invariance, and normalizing the arc length of domain boundaries guarantees scale-invariance.

3.1 Local Shape Invariance

An important merit of conformal geometric maps, including harmonic maps and conformal maps, is to reduce the shape complexity of the global region into a simpler canonical domain (auxiliary domain) enabling theoretical or numerical analysis to be carried out easily. However, for this transformation to be valid certain inherent geometric structured must be invariant under conformal coordinate transformation between domains. In fact, the concept of conformal invariants is of particular importance to the development of certain Riemannian metrics on underlying surfaces. Conformal maps are often viewed as isotropic (or isometric) maps; in the sense they preserve infinitesimal shapes (conformality) as well as angles between level curves, of mapped geometric regions. The premise of this fundamental property allow for local discretized geometric shapes to preserve basic conformal structures. For example, conformality of this type of transformation leads to the mapping of circles to circles, squares to squares, *etc.*, however, preserving the shape but not necessarily the size the geometric element as illustrated in Figure 3.1.

The motivation for constructing conformal mappings lies also in their application to the generation of curvilinear coordinate systems for two-dimensional regions. Indeed, the notion of conformality is key to the development of conformal geometry and boundary-conformal gird generation. In practice, conformal mapping has been widely recognized among the grid generation community to efficient tools to construct smooth orthogonal coordinates for solving various physical problem in simply connected regions. The potential of this approach was explored by Ryskin and Leal [30], Godunov and Prokopov [32], and Khamayseh and Mastin [33]. Generally, conformal transformation between manifolds is an important aspect in mathematics and physics because it allows more complicated structures to be expressed and understood in terms of the relatively well-understood properties of simpler spaces. Indeed, the *Riemann theorem* of general shapes and domains is a prime example of this influence.

Theorem 3.1. *Riemann Mapping Theorem (1851): Every simply connected Riemann surface can be mapped conformally onto the sphere, the plane, or the unit disk, and the resulting map is unique up to Möbius transformations.*

The Riemann mapping theorem, one of the profound results of complex analysis, has enormous applications in conformal differential geometry of the plane. This theorem alternatively states that if Ω is a simply connected open subset of the complex plane $\mathbb{C}$ which is not all of $\mathbb{C}$, then there exists a unique biholomorphic (bijective and holomorphic) mapping $\psi : \Omega \to \mathbb{C}$, such that for a given $\zeta \in \Omega$, we have

1. $\psi(\zeta) = 0$ and $\psi'(\zeta) > 0$ (real and positive);
2. ψ is injective (one-to-one);
3. $\psi(\Omega) = \{z \in \mathbb{C} : |z| < 1\}$ (open unit disk $\mathbb{D}$)

The fact that ψ is biholomorphic implies that it is a conformal map and therefore angle-preserving. Intuitively, such a map preserves the shape of any sufficiently small figure, while possibly rotating and stretching (but not reflecting) it. As a consequence of this theorem, any simply connected region $\Omega \subset \mathbb{C}$ is conformally equivalent to unit disk $\mathbb{D}$. Furthermore, the theorem guarantees that any two simply connected regions (none of which is $\mathbb{C}$) can be mapped conformally onto each other, or what is equivalent, onto the unit disk.

A third formulation of the Riemann mapping theorem which extend the conformal mapping to the boundary of mapped domains,is that, given three boundary points $z_1, z_2, z_3 \in \partial(\Omega)$, then there exist a unique conformal map $w = \psi(z) : \Omega \to \mathbb{D}$ such that $w_1 = \psi(z_1)$, $w_2 = \psi(z_2)$, and $w_3 = \psi(z_3)$, where w_1, w_2, w_3 the image boundary points on $\partial(\mathbb{D})$. This assertion is due to the fact that any three distinct none-collinear points in space uniquely define a circle, the circumscribed circle of a triangle. This particular version of Riemann mapping theorem has geometric significance that it allows for the construction of conformal maps in closed-form at the local domain partitions. In addition, this theorem pertains the geometric notion of mapping local triangles to triangles. Indeed, the fact that every triangle possesses a unique circumscribed circle which passes through its vertices, is therefore also, circles are mapped to circles as illustrated in Figure 3.1.

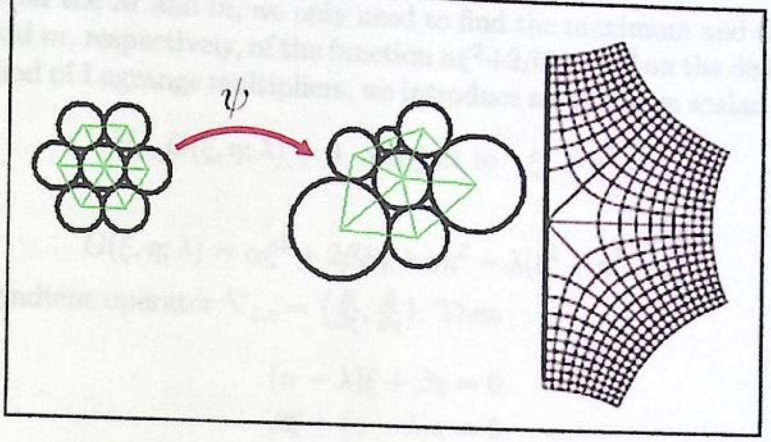

Figure 3.1: Isotropic mapping under conformal transformation.

We now begin with a closer look to examine geometric properties of conformal mappings by assuming the mapping $w = \psi(z) = u(z) + iv(z)$ to be homeomorphism of class $C^1(\Omega)$. For the general case, from the differentiability equation 2.10, we have

$$\begin{aligned} \left|\frac{d\psi}{dz}\right|^2 &= |\psi_x \cos\theta + \psi_y \sin\theta|^2 \\ &= (u_x \cos\theta + u_y \sin\theta)^2 + (v_x \cos\theta + v_y \sin\theta)^2 \\ &= \alpha \cos^2\theta + 2\beta \cos\theta \sin\theta + \gamma \sin^2\theta = \Phi(\theta), \end{aligned} \tag{3.1}$$

where the metrics α, β, and γ are defined

$$\begin{aligned} \alpha &= |\psi_x|^2 = u_x^2 + v_x^2, \\ \beta &= \psi_x \cdot \psi_y = u_x u_y + v_x v_y, \\ \gamma &= |\psi_y|^2 = u_y^2 + v_y^2. \end{aligned} \tag{3.2}$$

The function $\Phi(\theta)$ has a maximum value M and a minimum value m on the segment $0 \le \theta \le 2\pi$, and for all values of θ we have the inequality

$$m \le \left|\frac{d\psi}{dz}\right|^2 \le M. \tag{3.3}$$

In order to find the M and m, we only need to find the maximum and the minimum values, M and m, respectively, of the function $\alpha\xi^2+2\beta\xi\eta+\gamma\eta^2$ on the disk $\xi^2+\eta^2=1$. By the method of Lagrange multipliers, we introduce an unknown scalar, λ, and solve

$$\nabla_{\xi,\eta}G(\xi,\eta;\lambda)=0 \quad \text{subject to} \quad \xi^2+\eta^2=1$$

with

$$G(\xi,\eta;\lambda)=\alpha\xi^2+2\beta\xi\eta+\gamma\eta^2-\lambda(\xi^2+\eta^2-1),$$

where the gradient operator $\nabla_{\xi,\eta}=(\frac{\partial}{\partial\xi},\frac{\partial}{\partial\eta})$. Then

$$\begin{aligned}(\alpha-\lambda)\xi+\beta\eta&=0,\\ \beta\xi+(\gamma-\lambda)\eta&=0,\end{aligned} \tag{3.4}$$

whence,

$$\begin{vmatrix} \alpha \ \lambda & \beta \\ \beta & \gamma-\lambda \end{vmatrix}=0, \tag{3.5}$$

the determinant of the eigenvalue matrix of the above system 3.4 simply leads to solving the quadratic equation $\lambda^2-(\alpha+\gamma)\lambda+\alpha\gamma-\beta^2=0$. Hence, M and m are two roots (eigenvalues) of equation 3.5, and solving this equation yields:

$$\begin{aligned} M&=\frac{1}{2}\Big\{\alpha+\gamma+[(\alpha+\gamma)^2-4J^2]^{1/2}\Big\},\\ m&=\frac{1}{2}\Big\{\alpha+\gamma-[(\alpha+\gamma)^2-4J^2]^{1/2}\Big\},\\ J^2&=(u_xv_y-u_yv_x)^2=\alpha\gamma-\beta^2>0.\end{aligned} \tag{3.6}$$

Let $\mathcal{D}\equiv\max|\frac{d\psi}{dz}|/\min|\frac{d\psi}{dz}|=(M/m)^{1/2}$, and call it *dilation* of $w=\psi(z)$ at $z_0\in\Omega$. From the system of equations in 3.6, we have

$$M+m=\alpha+\gamma, \quad mM=J^2. \tag{3.7}$$

Now, if our transformation $w=\psi(z)$ is conformal on Ω, then Riemann mapping theorem asserts, the mapping ψ maps the circle $\mathcal{C}_z=\{z\in\Omega:|z-z_0|=r\}$ for all $z=z_0+re^{i\theta}\in\mathcal{C}_z$ to the circle $\mathcal{C}_w=\{w\in\psi(\Omega):|w-w_0|=\rho\}$ for all $w=w_0+\rho e^{i\theta}\in\mathcal{C}_w$. Therefore, then the minimum and the maximum of $\frac{d\psi}{dz}$ on $\mathcal{C}_z$ and thus $M=m$. In this case, the dilation $\mathcal{D}=1$ which means no stretching in the mapping, *i.e.*, circle is only mapped to scaled-invariant circle. Furthermore, we know from 3.7 that if $M=m$ and thus we have $(\alpha+\gamma)^2=4J^2$ and $(\alpha-\gamma)^2+4\beta^2=0$; hence

$$\alpha=\gamma, \quad \textit{and} \quad \beta=0.$$

This important result has two-fold geometric significance: first, we deduce that conformal mappings are orthogonal maps, which mean coordinate level curves intersect at right angle. Whereas, the spacings between coordinate curves are essentially uniform the property of spatial uniformity (or map isotropy). We now state these conformal geometric features in the following theorem:

Theorem 3.2. *Let the mapping $\psi : \Omega \to \mathbb{C}$ be a homeomorphism. Then $\psi = u + iv$ is conformal if and only if the real and imaginary parts u and v are C^1-differentiable functions and satisfy the following two conditions in Ω:*
(i) $\psi_x \cdot \psi_y = 0$ (Orthogonality),
(ii) $|\psi_x| = |\psi_y|$ (Uniformity).

The second geometric significance we infer from the previous discussion is the distinctive characteristic of local shape invariance. Consider the principle linear part of equation 2.8

$$w = \psi(z_0) + \psi_x(z_0)(x - x_0) + \psi_y(z_0)(y - y_0), \tag{3.8}$$

which can also be rewritten as

$$\begin{aligned} u - u_0 =& u_x(x - x_0) + u_y(y - y_0), \\ v - v_0 =& v_x(x - x_0) + v_y(y - y_0). \end{aligned} \tag{3.9}$$

where $u_0 = u(z_0)$ and $v_0 = v(z_0)$. The inverse transformation is

$$\begin{aligned} x - x_0 =& \frac{v_y}{J}(u - u_0) - \frac{u_y}{J}(v - v_0), \\ y - y_0 =& -\frac{v_x}{J}(u - u_0) + \frac{u_x}{J}(v - v_0). \end{aligned} \tag{3.10}$$

The affine transformation 3.9 maps the circle C

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \tag{3.11}$$

onto the ellipse $\mathcal{E}$ centered at $w_0 = \psi(z_0)$:

$$\alpha(u - u_0)^2 - 2\beta(u - u_0)(v - v_0) + \gamma(v - v_0)^2 = J^2r^2. \tag{3.12}$$

where α, β, and γ are defined in equation 3.2. In particular, if our mapping ψ is conformal, then by virtue of Theorem 3.2, the coefficients α, β, γ satisfy $\alpha = \gamma$ and $\beta = 0$. Hence, the ellipse $\mathcal{E}$ represented by equation 3.12 is actually a circle $|w - w_0| = \sqrt{J}r$. This circle-to-circle mapping property under conformal transformation is highlighted in the example shown in Figure 3.2. This example illustrates the mapping from the finite strip $\Omega = [-10, 10] \times [0, 1]$ to a crescent shape domain. As expected, the conformal map $\psi(z) = \frac{4i}{z+i}$ maps the circle packed strip Ω to the crescent and preserved the shape of the tile of circle. However, the size of the circles is transformed from uniform to dilated or contracted circles though the circular shape remains unchanged. The following observations carry over geometric properties of conformal maps through inversion and composition.

Corollary 3.3.

(i) If $\psi : \Omega \to \psi(\Omega)$ is a conformal map, then ψ and its inverse ψ^{-1} are circle-preserving maps.

(ii) If $\psi : \Omega \to \psi(\Omega)$ and $\phi : \psi(\Omega) \to \mathbb{C}$ are conformal maps, then the composed mapping $\phi \circ \psi$ is a circle-preserving map.

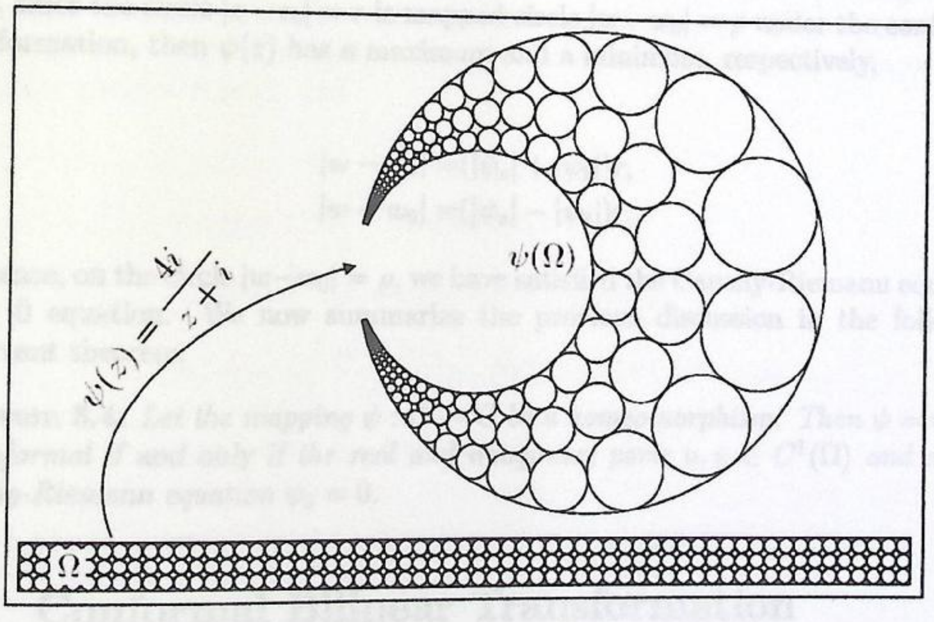

Figure 3.2: Circle-preserving mapping from a finite strip to the crescent.

Now we consider again equation 3.8. From the coordinate maps relation

$$\begin{cases} 2(x-x_0) &= (z-z_0)+\overline{(z-z_0)}, \\ 2i(y-y_0) &= (z-z_0)-\overline{(z-z_0)} \end{cases}, \tag{3.13}$$

consequently, equation 3.8 can be rewritten as

$$w - w_0 = \psi_z(z_0)\,(z-z_0) + \psi_{\bar{z}}(z_0)\,\overline{(z-z_0)}, \tag{3.14}$$

where $w_0 = u_0 + iv_0$, $\psi_z = \frac{1}{2}(\psi_x - i\psi_y)$, and $\psi_{\bar{z}} = \frac{1}{2}(\psi_x + i\psi_y)$.
Again by 3.14, we obtain

$$(|\psi_z| - |\psi_{\bar{z}}|)|z-z_0| \leq |w-w_0| \leq (|\psi_z| + |\psi_{\bar{z}}|)|z-z_0|. \tag{3.15}$$

Thus, since the circle $|z-z_0| = r$ is mapped circle $|w-w_0| = \rho$ under the conformal transformation, then $\psi(z)$ has a maximum and a minimum, respectively,

$$\begin{aligned} |w-w_0| &= (|\psi_z| + |\psi_{\bar{z}}|)r, \\ |w-w_0| &= (|\psi_z| - |\psi_{\bar{z}}|)r, \end{aligned} \tag{3.16}$$

and hence, on the circle $|w-w_0| = \rho$, we have satisfied the Cauchy-Riemann equation $\psi_{\bar{z}} = 0$ equation. We now summarize the previous discussion in the following important theorem.

Theorem 3.4. *Let the mapping $\psi : \Omega \to \mathbb{C}$ be a homeomorphism. Then $\psi = u + iv$ is conformal if and only if the real and imaginary parts $u, v \in C^1(\Omega)$ and satisfy Cauchy-Riemann equation $\psi_{\bar{z}} = 0$.*

3.2 Conformal Bilinear Transformation

One of the challenging issues in practical applications of conformal mapping is the problem of devising such maps on any arbitrary complex domains. Although Riemann mapping theorem guarantees the existence of the desired conformal transformation; it however, provides no direct mechanism of constructing such a transformation for complex domains. In this section we consider the problem of devising conformal maps in closed-form on complex domains. Nonetheless, in certain cases Riemann mapping theorem provides insight into the formulation of conformal maps between specific class of domains. For example, Schwarz-Christoffel transformation is a conformal map which maps the upper half-plane

$\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ conformally onto polygonal domain (or *visa verse*), which in turn can be mapped onto the unit disk. However, for practical purpose the core focus of this study is on constructing conformal maps between finite bounded regions without intermediate transformation steps. A class of well-defined conformal mappings between simply connected domains is the Möbius transformation. The general form of a Möbius transformation is given by

$$w = \psi(z) = \frac{az + b}{cz + d} \tag{3.17}$$

where a, b, c, d are complex numbers satisfying $ad - bc \neq 0$. This transformation can be expressed in a bilinear form $Azw + Bz + Cw + D = 0$, where A, B, C, D are factors of a, b, c, d, and is sometimes referred to as linear fractional. The set of all Möbius transformations forms a group of first-order conformal maps under composition. This group can be given the structure of a complex manifold in such a way that composition and inversion are also conformal maps. Decomposition and elementary properties of a Möbius transformation is equivalent to a sequence of geometric transformations. Let

$$\begin{aligned}
\psi_1(z) &= z + \frac{d}{c} \quad \text{(translation)} \\
\psi_2(z) &= \frac{1}{z} \quad \text{(inversion and reflection)} \\
\psi_3(z) &= -\left(\frac{ad - bc}{c^2}\right) z \quad \text{(dilation and rotation)} \\
\psi_4(z) &= z + \frac{a}{c} \quad \text{(translation)}
\end{aligned} \tag{3.18}$$

then algebraic composition of Möbius transformations is a Möbius transformation, namely

$$\psi_4 \circ \psi_3 \circ \psi_2 \circ \psi_1(z) = \frac{az + b}{cz + d}$$

It is clear from the definition that the composition of two such maps and the inverse of any such map is again an invertible conformal mapping, so the set of such mappings forms a conformal group. This decomposition makes many properties of the Möbius transform obvious. From this decomposition, we also see that Möbius transformation carries over all non-trivial properties of circle inversion. Namely, that any line in the preimage becomes a circle in the postimage, and any circle in the preimage remains a circle in the postimage. This is justified by the fact that Möbius transform is composed of a sequence of isometries plus a inversion.

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The isometries preserves distances, while the inversion does exactly map circles to circles. For example, it is easy to see that the map

$$\psi(z) = \frac{e^{i\theta}(z-a)}{1-\bar{a}z}, \quad \theta \in \mathbb{R} \text{ and } |a| < 1,$$

maps the unit circle $|z| = 1$ to itself and a to 0.

Furthermore, if two points are symmetric with respect to a circle (as in geometric inversion), then, after a Möbius transformation, the two points still be symmetric with respect to the image circle. This fact can easily be seen from the decomposition of Möbius transformation, and the fact that two geometric inversions over the same circle will maintain two symmetric points over the circle. In addition, since a Möbius transform is composed of a sequence of translation, dilation, reflection, and geometric inversion, and all these preserve angles, therefore a Möbius transform also preserves angles.

Corollary 3.5. *The following assertions pertaining to Möbius transformation are equivalent to conformality:*

(i) The Möbius transformations are biholomorphic on $\mathbb{C}$.

(ii) The Möbius transformations map circles to circles.

(ii) The Möbius transformations preserve angles, together with their orientation.

An important practical application of this transformation is the generation of smooth curvilinear coordinates around an airfoil geometry in the plane. In this case, conformal mapping is viewed as mathematical technique to transform (or map) one mathematical problem and solution into another. A typical case study of this application is the mapping represented by curvilinear curves in the Figure 3.3. The primary mapping is $\psi(z) = z + \frac{1}{z}$ which transform points around an airfoil type geometry to points exterior to the disk and *visa-versa.* The shape domain is prescribed by the parametrization exterior to the disk centered at $z_c = -0.132$ with radius $r = 1.132$ and interior of hexagonal polygon. Then the mapping $\psi(z) = z + \frac{1}{z}$ converts the entire flow field around the cylinder into the flow field around the airfoil–Figure 3.3. By knowing the velocity and pressures in the plane containing the cylinder, the conformal map gives the velocity and pressures around the airfoil and therefore one can compute the lift.

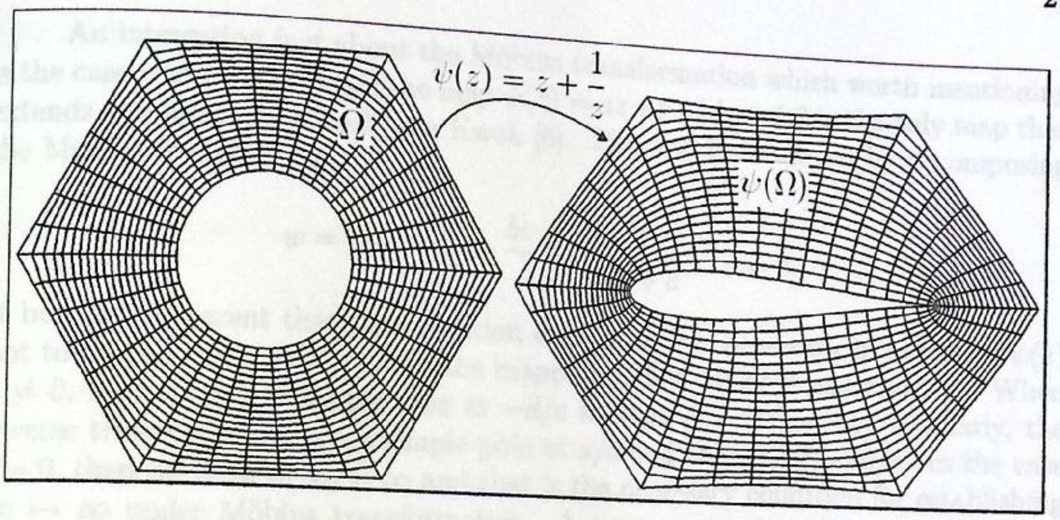

Figure 3.3: Conformal mapping around an airfoil domain.

However, from another geometric prospective the same map $\psi(z) = z + \frac{1}{z}$ converts a domain exterior to an inner circle centered at $z_c = 0$ with radius $r = 0.25$ and interior to an outer triangle to a completely different configuration–Figure 3.4. this means that conformal maps in most explicit forms are domain dependant and yet fully control the image domain.

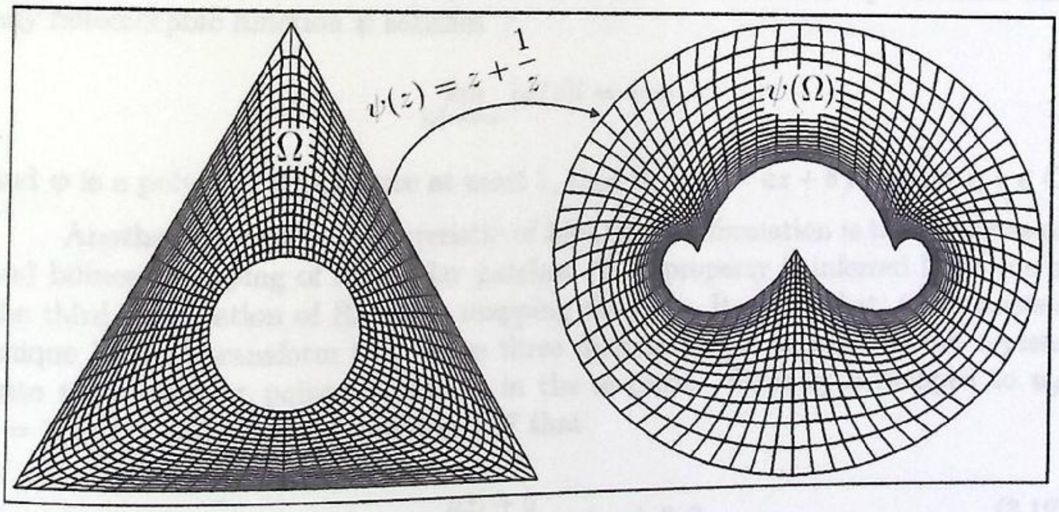

Figure 3.4: Conformal mapping of multi-connected domains.

An interesting fact about the Möbius transformation which worth mentioning is the case when $c = 0$. Then the map $\psi(z) = az + b$ with $a \neq 0$ is the only map that extends the plane onto itself, see Kwok [6]. This can be observed by decomposing the Möbius transform as

$$w = \psi(z) = \frac{a}{c} + \frac{bc - ad}{c} \frac{1}{cz + d}, \quad c \neq 0.$$

It becomes apparent that the condition $ad - bc \neq 0$ is necessary in order for $\psi(z)$ not to be constant. When $c = 0$ the mapping ψ is a linear transformation. When $c \neq 0$, $w = \psi(z)$ has simple pole at $-d/c$ so that $\psi(-d/c) = \infty$. Similarly, the inverse transformation has a simple pole at a/c and that $\psi(\infty) = a/c$. In the case $c = 0$, then we have $\psi(\infty) = \infty$ and that is the necessary condition for establishing $\infty \mapsto \infty$ under Möbius transformation. A more sophisticated coupling between conformal maps and Möbius transformation is presented by Greene and Krantz [3].

Corollary 3.6. *A map $\psi : \mathbb{C} \to \mathbb{C}$ is conformal* if and only if *there are complex numbers a, b with $a \neq 0$ such that $\psi(z) = az + b$, $z \in \mathbb{C}$.*

In other words, a map of the extended complex plane $\mathbb{C}$ (which is conformally equivalent to a sphere) onto itself is conformal *if and only if* it is a Möbius transformation. The proof of this assertion can be motivated by the facts that any holomorphic function ψ satisfies

$$\lim_{|z| \to +\infty} |\psi(z)| = +\infty$$

and ψ is a polynomial of degree at most 1, that is, $\psi(z) = az + b$ for some $a, b \in \mathbb{C}$.

Another geometric characteristic of Möbius transformation is the preservation and bilinear mapping of triangular patches. This property is inferred by virtue of the third formulation of Riemann mapping theorem. It states that, there exists a unique Möbius transform that maps three distinct points z_1, z_2, z_3 in the z–plane onto three distinct points w_1, w_2, w_3 in the w–plane. Since z_j is mapped to w_j, $j = 1, 2, 3$, it follows from equation 3.17 that

$$w_j = \frac{az_j + b}{cz_j + d}, \quad j = 1, 2, 3, \tag{3.19}$$

Also notice that

$$w_k - w_j = \frac{(ad - bc)(z_k - z_j)}{(cz_k + d))(cz_j + d)}. \tag{3.20}$$

This transformation has also the following property known as cross-ratio preservation

$$\frac{(w_4 - w_1)(w_2 - w_3)}{(w_2 - w_1)(w_4 - w_3)} = \frac{(z_4 - z_1)(z_2 - z_3)}{(z_2 - z_1)(z_4 - z_3)} \tag{3.21}$$

is invariant under a Möbius transformation that maps from $z-$ to $w-$planes. Here, (z_k, z_{k+1}) and (w_k, w_{k+1}) are end points of arc-segments of the domain boundaries, $\mathbb{D}$ and Ω respectively.

The formula for a general first-order (bilinear) conformal mappings is conveniently expressed by this implicit formula:

$$\frac{(w - w_1)(w_2 - w_3)}{(w_2 - w_1)(w - w_3)} = \frac{(z - z_1)(z_2 - z_3)}{(z_2 - z_1)(z - z_3)} \tag{3.22}$$

Moreover, this expression interpreted such that choosing three specific points in the domain and their images in the targeted range, the mapping given in 3.22 determines for all the remaining pots z and w. Making the the substitution of the expression 3.20 in the previous equation yields the explicit form of the transformation $w = \psi(z)$

$$\psi(z) = \frac{w_1(z - z_3) - \mu w_3(z - z_1)}{(z - z_3) - \mu(z - z_1)} \tag{3.23}$$

where $\mu = (z_2 - z_3)(w_2 - w_1)/(z_2 - z_1)(w_2 - w_3)$. This explicit formula represents a general bilinear conformal transformation which maps three distinct boundary points, namely $z_1, z_2, z_3 \in \partial(\Omega)$ onto three distinct boundary points w_1, w_2, w_3 in points on $\partial(\mathbb{D})$, respectively.

Geometrically, given arbitrary two multi-connected domains Ω and Ω_r in $\mathbb{C}$ with sampled boundary curves. Then, the boundary points can be connected to make subdivision of triangular patches, namely $\mathbf{T}_k = \{z_j, z_{j+1}, z_{j+2}\} \in \Omega$ and $\mathcal{T}_k = \{w_j, w_{j+1}, w_{j+2}\} \in \Omega_r$ such that $\bigcup \mathbf{T}_k = \Omega$ and $\bigcup \mathcal{T}_k = \Omega_r$, see Figure 3.5. The piecewise mapping ψ in equation 3.23 guarantees the bilinear transformation of Ω onto the range Ω_r maintaining the overall shape of the domains.

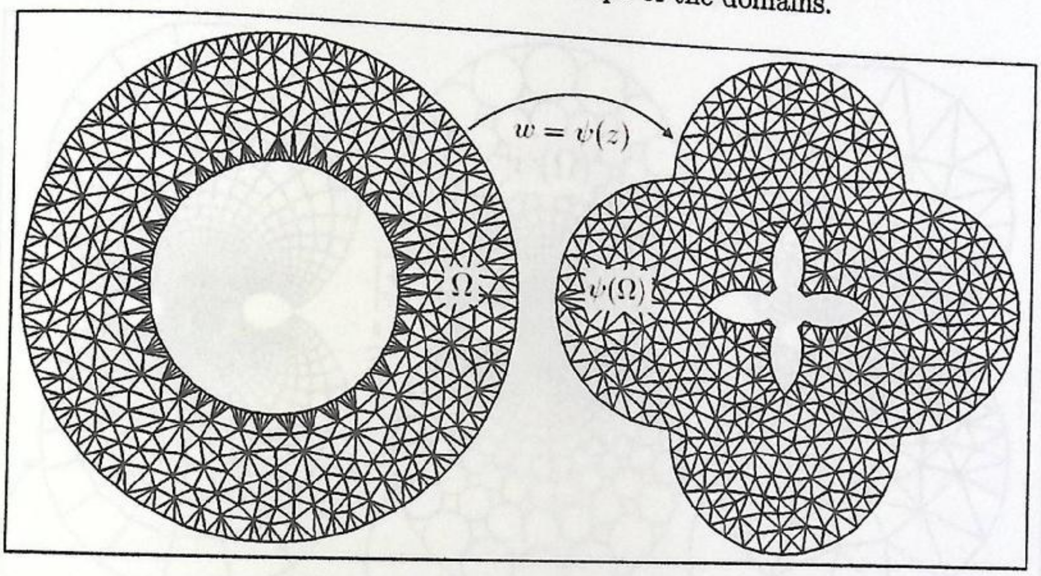

Figure 3.5: Bilinear conformal transformation of triangulated domains.

We conclude this chapter with the presentation of a numerical result portraying the overall picture of geometric characteristics of conformal transformations. The showcase demonstration in Figure 3.6 exhibits the fundamental properties of conformal mappings. We show three different discretizations of the same rectangular domain $\Omega = [-0.5, 0.5] \times [-0.85, 0.85]$ being transformed under the same conformal map $\psi(z) = \frac{z}{z^2+1}$. The decomposition of Ω encapsulates the very basic geometric shapes of quadrilaterals elements (right), triangle elements (middle), and circles (right).

These particular multi-element geometric shapes are often the choice of grid cells needed for numerical methods used to solve governing physical models. As excepted, the conformal map transforms the three different grids of the canonical domain to the same shape in the physical image domain. Furthermore, the angles between the cell-edges and shapes of cells are preserved as well, whereas the size of the cells increased in certain subregion or decreased in other locations.

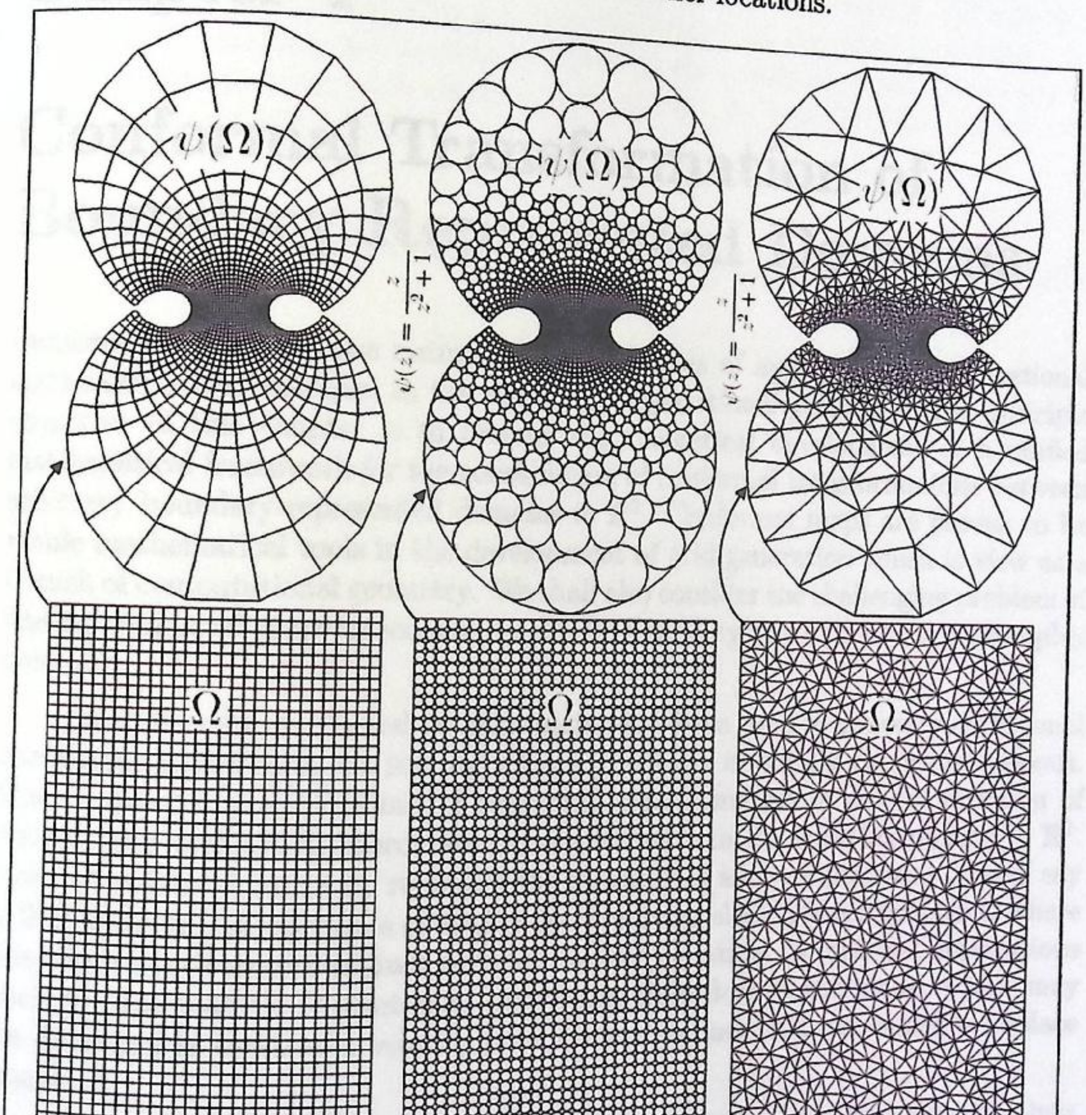

Figure 3.6: Showcase depicting fundamental geometric Conformal properties.

Chapter 4

Conformal Transformation of Boundary Represented Domains

Conformal mappings arise naturally in many areas of applied and computational mathematics, for example, in the study of computational geometry. The principle objective of this chapter is to present the theoretical development of a unified mathematical framework for the construction of conformal transformations between arbitrary boundary-represented domains in $\mathbb{R}^2$. Conformal maps are proven to be viable mathematical tools in the development of grid generation which is view as a branch of computational geometry. We shall also consider the challenging problem of finding conformal maps as solution to elliptic boundary value problems on complex domains.

Traditionally, conformal transformation has been used to generate orthogonal curvilinear coordinates for solving various problems in simply connected regions. One of the prime applications of conformal transformation is the construction of smooth and orthogonal coordinate curvilinear grids in the complex regions in $\mathbb{R}^2$. Such a conformal map may represent the streamlines and equipotential lines of say a fluid motion. Another area of application is solving electrostatics problems, where the electrostatic potential lines satisfies Laplace equation. Additional applications include heat transfer, irrotational, nonviscous fluid flow in the plane, which may be described in terms of a velocity potential streamlines that also satisfies Laplace equation.

The motivation for constructing conformal mappings lies also in their application to the generation of curvilinear coordinate systems for two-dimensional regions. Conformal mappings may be used to reduce a second order elliptic partial differential equation to its canonical form, *i.e.*, the principal part of the differential operator reduces to the Laplacian. Consequently, if one is solving an elliptic boundary value problem, an appropriate conformal mapping could be used simultaneously to fit the boundary curves with coordinate lines and simplify the original partial differential equation. The equation, in canonical form, can be easily investigated for solution existence and uniqueness, and possibly be solved more efficiently by methods which would not be applicable to the original equation in cartesian coordinates, see Wen [18] and Krantz and Gavosto [20]. With the development of a method of constructing a conformal transformation, the selection of the transformation may give a better distribution of grid lines or simplify the transformed equations that need to be solved on the canonical region. This concept of conformal mapping methods have been utilized by Ives [31], Khamayseh and Mastin [33], and Mastin [38].

4.1 Quasiconformal Maps on Rectangular Domains

Conformal maps have been effectively used in the development of elliptic partial differential equations applied to geometric and physical modeling. In earlier chapters, we analyzed several analytic and geometric properties of conformal mappings with emphasis on the existence and analysis of such maps. A class of conformal maps are feasible to be constructed explicitly for a wide range of domains. However, the drawback of this scenario is that the conformal transformation controls the shape of image map and may not produce the desired geometric configuration with the exception of limited cases. A more challenging proposition is, given two arbitrary bounded domains in $\mathbb{R}^2$, is it feasible to construct a conformal map between them? Indeed, this chapter is intended to answer this question.

In this section we briefly introduce the notion of quasiconformal mapping and the rationale as to why it needs to be induced in this study of conformal transformation. To further aid our understanding of the topic at hand and its geometric relevance, we begin with this preliminary argument. There are three fundamental questions in the study of computational geometry and grid generation. The first two are as follows:

(i) Given two regions of the same dimension, is there a one-to-one and onto transformation between the regions?

(ii) More important, if such a transformation exists, can a practical construction be provided for it?

The third question is, (iii) How can a transformation (or a mapping) between the regions (domains) be controlled? Currently none of these questions has been answered satisfactorily. The first two have long attracted the attention of mathematicians and applied scientists for reasons other than computational geometry. Riemann developed the theory of conformal mappings to show that the answer to (i) and (ii) in two-dimensional space is a conformal mapping. Under some rather restrictive conditions, a candidate to (i) and (ii) is a harmonic mapping.

Conformal mapping techniques have long served as instrumental mathematical mean for constructing smooth transformations between planar domains. The existence of the conformal map between simply connected domains is guarantees by the Riemann mapping theorem, this much we know. However, the theorem does not say what this mapping may look like, it is merely an existence theorem. In this study we are primarily concerned with the issue of constructing conformal mappings between arbitrary domains and rectangular domains in $\mathbb{R}^2$ space. The particular reason for this judicious choice of rectangular domains and the associated technical requirement of such approach will be explained at length in a later stage of this chapter.

In two dimensions, conformal mapping techniques have long served as a mean of constructing geometric mappings between planar domains. Here again, consider the conformal mapping of an arbitrary simply connected domain Ω in the z–plane onto a rectangular region Ω_r in the w–plane. The domain Ω is chosen to be represented by oriented smooth boundary curves $\partial\Omega$ which encircle the interior of Ω. In computational geometry this type of geometric domains are referred to as boundary-represented or *B-rep* regions for short. Furthermore, consider the problem of mapping not only the interior of Ω conformally into the interior of Ω_r but also mapping boundary curves and vertices of $\partial\Omega$ onto the corresponding edges and corners of $\partial\Omega_r$, as shown in Figure 4.1.

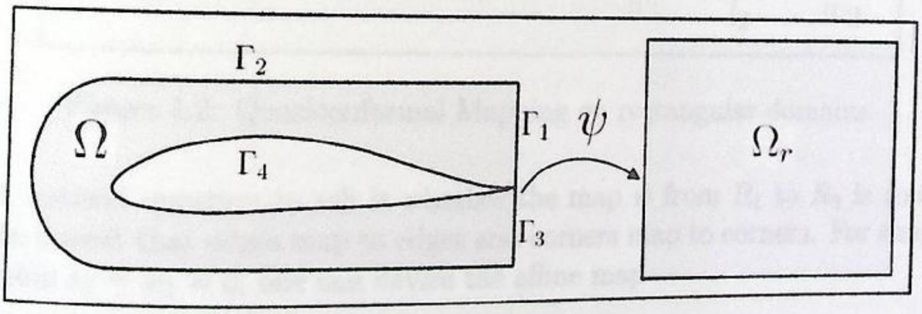

Figure 4.1: Mapping of a curvilinear domain to a rectangular domain.

In this case, the Riemann mapping theorem affirms that a conformal mapping exists between Ω and Ω_r, and it also implies that the mapping is uniquely determined by specifying three points on the boundary $\partial\Omega$. Suppose we wish to indicate four specific points on $\partial\Omega$ to be mapped to the corner vertices of the rectangle Ω_r. If the rectangle is fixed, then the problem is over-determined and no conformal mapping exists. Therefore, the mapping must determine one of the dimensions of the rectangular region Ω_r with length l and hight h. In this case, the mapping $\psi : \Omega \rightarrow \Omega_r$ where $\psi(z) = u + iv$ is still a conformal mapping and satisfies the

Cauchy-Riemann equation $\psi_{\bar{z}} = 0$. The following assertion is a variation of the Riemann mapping theorem.

Theorem 4.1. *Let Ω be a bounded simply connected region in the z-plane whose boundary $\partial\Omega$ is a simple closed curve. Let z_1, z_2, z_3, and z_4 be distinct boundary points ordered by the orientation on $\partial\Omega$. There exists a unique conformal mapping of Ω onto a rectangular region Ω_r in the w-plane such that the points z_i map to boundary points of Ω_r which also ordered by the orientation of the rectangle.*

Now consider the following motivational simple problem, let the mapping $\psi : R_1 \to R_2$ be a homeomorphism with R_1 be the rectangle with vertices z_1, z_2, z_3, z_4 prescribed in counterclockwise and edge hight h_1 and length l_1, and also R_2 be a rectangle with corresponding vertices w_1, w_2, w_3, w_4 and edge lengths h_2 and l_2 such that $\psi(z_k) = w_k$ for $k = 1, 4$, see Figure 4.2.

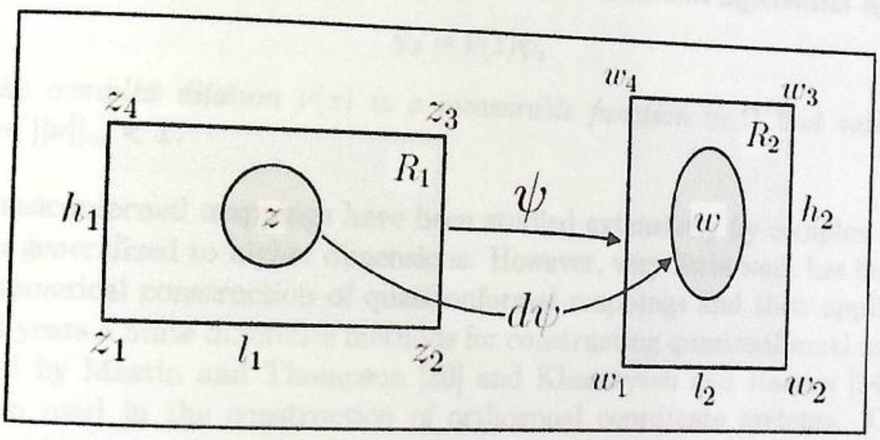

Figure 4.2: Quasiconformal Mapping on rectangular domains.

A natural question to ask is whether the map ψ from R_1 to R_2 is conformal with the caveat that edges map to edges and corners map to corners. For simplicity, we assume $z_1 = w_1 = 0$, one can devise the affine map

$$A_\psi : (x, y) \mapsto \left(\frac{l_2}{l_1}x, \frac{h_2}{h_1}y\right),$$

which maps R_1 onto R_2, but this is not conformal unless the ratios $\frac{h_1}{l_1} = \frac{h_2}{l_2}$. Recall from the previous chapter, there is only one conformal map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ namely $\psi(z) = az + b$, where a, $b \in \mathbb{C}$. This is the only conformal map from R_1 to R_2 under the condition, the domains have equal aspect ratios, *i.e.*, $\frac{h_1}{l_1} = \frac{h_2}{l_2}$. In fact, the affine map A_ψ above can be expressed in the following explicit form:

$$A_\psi(z) = \frac{1}{2}\left(\frac{l_2}{l_1} + \frac{h_2}{h_1}\right)z + \frac{1}{2}\left(\frac{l_2}{l_1} - \frac{h_2}{h_1}\right)\bar{z}.$$

Indeed, it can be easily seen that if $\frac{h_1}{l_1} \neq \frac{h_2}{l_2}$, then this map does not satisfy the Cauchy-Riemann equations and therefore is not conformal. However, the affine map $\mathcal{A}_\psi(z)$ is *nearly* or *quasai* conformal and it possesses most of the standard conformal properties. Though, there one exception that a circle in R_1 is now mapped to an ellipse in R_2 via the linear differential $d\psi = \sqrt{J}dz$ as shown in Figure 4.2. One may ask, what is the map from R_1 to R_2 which is closest to a conformal map and does not require the condition of equal aspect ratios. This map is now referred to as *quasiconformal* and satisfies the extended Cauchy-Riemann equations which are called the *Beltarmi equations.* We now introduce the formal definition of quasiconformal mapping on simply connected domains.

Definition 4.2. *The function $\psi(z) = u + iv$ is a quasiconformal mapping of the region Ω contained in the extended complex plane $\mathbb{C}_\infty$ if ψ maps Ω homeomorphically onto a rectangular region Ω_r and ψ is a solution to Beltrami differential equation*

$$\psi_{\bar{z}} = \nu(z)\psi_z \tag{4.1}$$

where the complex dilation $\nu(z)$ is a measurable function in Ω that satisfies the condition $||\nu||_\infty < 1$.

Quasiconformal mappings have been studied extensively by complex analysts and even generalized to higher dimensions. However, very little work has been done on the numerical construction of quasiconformal mappings and their applications. In recent years a finite difference methods for constructing quasiconformal mappings developed by Mastin and Thompson [36] and Khamayseh and Hansen [34] which have been used in the construction of orthogonal coordinate systems. Classical references on quasiconformal mapping include the work of Ahlfors [12], Krushkal [13], and Lehto and Virtanen [14]. A modern outlook on quasiconformal mappings is presented recently by Fletcher and Markovic [15], Fletcher [16], Lebl [17].

The stand alone existence of quasiconformal mapping was established by Mastin and Thompson [37] in the following result:

Theorem 4.3. *Let Ω be a bounded simply connected region in the z-plane whose boundary $\partial\Omega$ is a simple closed curve. Let z_1, z_2, z_3, and z_4 be distinct boundary points ordered by the orientation on $\partial\Omega$. There exists a unique quasiconformal mapping of Ω onto a rectangular region Ω_r in the w-plane such that the points z_i map to the vertices of the rectangle Ω_r.*

Thus far, we have established that conformal mappings determine the aspect ratio of the rectangular region, *i.e.*, the ratio of hight-to-length. Rather than allow a rectangle with variable width, one can equivalently introduce the parameter μ such that the width of the rectangle is set to unity, its height must be some conformal invariant quantity μ, which is referred to as *conformal module* of Ω.

Frequently in computational grid generation process, we must construct a homeomorphism from a simply connected region Ω onto a region that is not the rectangle Ω_r, but rather onto a square region Ω_s. In this situation conformal mappings can still be utilized to construct such a mapping. We choose the unit square, of course with unity aspect ratio, for several practical reasons: (1) Square regions are the simplest computational domains in $\mathbb{R}^2$, thus facilitating the numerical solution procedure for the governing equations. (2) Any simply connected domain can be mapped quite easily to the unit square under arc-length transformation. (3) The interior discretization of the domain can be constructed by straitforward bilinear interpolation from the edges. (4) Last but not least, what is sought after here is to adopt a unified framework for coordinate mapping from arbitrary domains to one type domain and that chosen to be the square.

Since we are primarily focused on mapping Ω onto a square Ω_s, This process is carried in two steps: First, and without loss of generality, let z_1, z_2, z_3, and z_4 be distinct points on $\partial\Omega$, in counterclockwise order, which will map conformally to the vertices of the rectangle 0, 1, $1+i\mu$, and $i\mu$, respectively. The invariant quantity μ is to be determined by the conformal mapping $\psi : \Omega \to \Omega_r$ which satisfies the Cauchy-Riemann equations:

$$\begin{aligned} u_x &= v_y, \\ -u_y &= v_x. \end{aligned} \tag{4.2}$$

Second, we follow the first step by a bijective transformation from the rectangle to the unit square. Any rectangular domain Ω_r can be mapped onto a unit square region Ω_s if an additional stretching transformation is introduced. This transformation is can be given by the quasiconformal map $\zeta : \Omega_r \to \Omega_s$, where $\zeta = \xi + i\eta$ with the following change of variables;

$$\begin{aligned} \xi &= u, \\ \eta &= \frac{v}{\mu}. \end{aligned} \tag{4.3}$$

We observe that the composite of a conformal and quasiconformal maps yields a quasiconformal map. Thus, we assert that the composite mapping $\zeta : \Omega \to \Omega_s$ where $\zeta(z) = \xi(z) + i\eta(z)$ is no longer conformal map between the rectangle and the square, while mapping corresponding corners to corners since the aspect ratios of the two regions do not match. However, the conformal mapping can be easily retrieved by simply multiplying the η coordinate by μ. On the unit square, using the change of variables 4.3 in Cauchy-Riemann equations 4.2, the functions ξ and η of the composite mapping satisfy the following first-order elliptic system, which now include the module of Ω_r as a parameter

$$
\begin{aligned}
\xi_x &= \mu\eta_y, \\
-\xi_y &= \mu\eta_x.
\end{aligned} \tag{4.4}
$$

This coupled first-order elliptic system is referred to as *Beltrami equations* and the resulting map is then quasiconformal map, see Mastin [38]. The following relation can also be observed from equation 4.3, for every $z \in \Omega$ we can define the conformal module of the mapping ζ

$$
\mu(z) = \sqrt{\frac{\xi_y^2 + \eta_y^2}{\xi_x^2 + \xi_y^2}}, \tag{4.5}
$$

where as the modulo of the domain $\mu := \mu(\Omega)$ can be evaluated as

$$
\mu(\Omega) = \iint_\Omega \mu(z)dxdy. \tag{4.6}
$$

The quasiconformal mapping of the region Ω onto a rectangular region can be obtained by constructing a homeomorphic mapping which satisfies Beltrami system 4.4. This class of quasiconformal mapping is classified as μ–quasiconformal which allows for the mapping of arbitrary regions onto the unit quare Ω_s with corners being mapped to corners and edges to edges. Quasiconformal mappings and conformal mappings are closely related and quasiconformal mappings are considered as a generalization of conformal mappings. In contracts to conformal transformations which map circle-to-circle, quasiconformal maps transform circle-to-ellipse due to the induced stretching (dilation) factor μ in the linear differential $d\zeta = \sqrt{J}dz$. Moreover, setting $\mu = 1$, then the Beltrami equations in 4.4 are reduces to Cauchy-Riemann equations and the mapping is called 1–quasiconformal. In this particular case the mapping $\zeta = \xi + i\eta$ from the region Ω to the rectangle Ω_r is conformal. The close relationship between quasiconformal mappings and conformal mappings can be observed in the next result, see Ahlfors [12].

Corollary 4.4. *The following assertions pertain to quasiconformality.*

(i) A C^1 homeomorphism is conformal if and only if it is 1–quasiconformal.

(ii) The inverse of a quasiconformal map is quasiconformal.

(iii) Composition of a conformal map and a quasiconformal is quasiconformal map.

We now present an extension to Theorem 3.2 which we conclude in this special case of quasiconformal mapping are still orthogonal map. Whereas, the spacing between coordinate curves is controlled and preserved by the spatial anisotropy factor μ. These notions of geometric characteristics of quasiconformal mapping become more apparent later in this chapter. The proof of this theorem follows exactly the same procedure as in Theorem 2.6 and Theorem 3.2.

Theorem 4.5. *Let the mapping $\psi : \Omega \to \Omega_r$ be a homeomorphism with a non-vanishing Jacobian. Then $\psi = \xi + i\eta$ is quasiconformal if and only if the real-valued functions ξ and η are C^1-differentiable and satisfy the following two conditions:*

(i) $\nabla\xi \cdot \nabla\eta = 0$ (Orthogonality),

(ii) $|\nabla\xi| = \mu|\nabla\eta|$ (Anisotropy).

This investigation shows that, with the aid of quasiconformal mappings we are able to overcome the rigidity of conformal mapping and allow for arbitrary complex regions to be mapped to simplistic canonical domain, namely the unit square. This type of mapping carry over essential conformal properties such as control over angle between coordinate lines. In addition, we are able to map corresponding prescribed domain boundaries with just a simple scale composition keeping the map inherently closest to conformal. The advantage of this approach is to eliminate any complications required for further coordinate transformations. Furthermore, the choice of this special case of quasiconformal mapping produces desired geometric characteristics in terms of exerting control over the spacing and angles between intersecting coordinate lines. As a matter of fact, under quasiconformal transformation the coordinate lines intersect orthogonally and the modulo of the domain μ controls the spacing. In conclusion, the knowledge and tools developed for the theoretical quasiconformal mapping can and should be applied to elliptic systems required for solving boundary value problems. It is worthwhile mentioning that much of the discussion on conformal maps in previous chapters can be extended to study the analytic and geometric properties of quasiconformal mapping.

4.2 Boundary Value Problem for Harmonic Maps

It had been shown over the years, there is an intimate connection between partial differential equations, conformal mappings, and harmonic maps. Note that the word *harmonic* is used in two different but related ways in this chapter: (i) there is the classical notion of harmonic function as a solution of the homogeneous Laplace equation, and (2) the notion of harmonic map as a transformation of $\mathbb{R}^n$ to $\mathbb{R}^n$, in our case $n = 2$, for which each component is harmonic. The theory of harmonic maps can be extended to mappings of manifolds of equal dimensions; the components of such maps satisfy the Laplace-Beltrami equation. The theory of harmonic maps has been used in the study of computational geometric maps in the plane, see Mastin [38], Dvinsky [39], Sritharan [40].

In general, there is a close relationship between boundary-value elliptic systems and quasiconformal mappings in two dimensions is prevalent in the work of Wen [18]. Likewise, Renelt [19] based his treatment of general elliptic systems of first order partial differential equations entirely on quasiconformal mapping. Additionally, there are other classical applications of conformal transformation of a surface onto a planar region and the reduction of elliptic partial differential equations to canonical form, proposed by Khamayseh and Mastin [33].

Many of the harmonic mapping theory fundamentals is based on conformal and quasiconformal mappings. This can be seen by considering the quasiconformal mapping of a simply connected region Ω onto the square Ω_s as a solution to the Beltrami equations in 4.4. However, the difficulty with approach is that the first-order elliptic systems given by the Beltrami equations is tightly-coupled and the quantity μ is not known *a priori* since it is domain dependent. We shall show how to overcome this difficulty and derive an alternative way of obtaining quasiconformal mapping by solving a decoupled boundary-value problem. The relation between quasiconformal transformation and elliptic systems can be seen by utilizing the Beltrami equations in 4.4. We easily see that the function $\zeta(x,y) = \xi(x,y) + i\eta(x,y)$ is harmonic and therefore satisfy Laplace equation

$$\nabla^2 \zeta = 0, \tag{4.7}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator. It is interesting to note that the coordinate functions ξ and η are still harmonic even though the Cauchy-Riemann equations are not satisfied. This is an important observation that the harmonic mapping theory can still be applied in the case of quasiconformal mapping, *i.e.*,

$$\begin{aligned} \xi_{xx} + \xi_{yy} &= 0, \\ \eta_{xx} + \eta_{yy} &= 0. \end{aligned} \tag{4.8}$$

To determine completely the solution of the elliptic system 4.8, it is necessary to specify boundary conditions on the solution along $\partial\Omega$. In the study of partial differential equations, there are three essential issues need to be addressed. The basic problems are the solution existence and uniqueness of the elliptic system subject to prescribed boundary conditions. From a practical point of view, an equally important problem is the solvability of the system by means of analytic or computational methods.

we now begin by describing the overall geometric boundary representation of a simply connected domain Ω and boundary conditions needed for solving system 4.8.

Assume the boundary of Ω is subdivided to four arcs, namely Γ_1 from z_1 to z_2, Γ_2 from z_2 to z_3, Γ_3 from z_3 to z_4, and Γ_4 from z_4 to z_1 where $\partial\Omega = \bigcup \Gamma_j$. The boundary conditions of the solution ξ of 4.8 will be investigated first. Clearly ξ is constant on the boundary arc Γ_2 and Γ_4, whereas η must be constant on the boundary arcs Γ_1 and Γ_3 of $\partial\Omega$. Furthermore, to specify boundary conditions of ξ on the arcs Γ_1 and Γ_3, we introduce a new coordinate system, these directions are the *normal* and the *tangent* vectors of a smooth curve. We denote these perpendicular directions by n and s, where s is the arc length parameter. In the new (n, s) coordinate system, we shall assume that the relative orientation of the $n-$ and s–direction are like those of the $x-$ and y–axis, respectively (see Nehari [8]). This is in agreement with our conventions regarding the positive direction of the normal and the positive sense in which the boundary of the region $\partial\Omega$ is described below.

If $\frac{\partial}{\partial n}$ and $\frac{\partial}{\partial s}$ denote the differentiations with respect to the positive n–direction and the positive s–direction, respectively, we have

$$\begin{aligned}
\xi_s &= \nabla\xi \cdot (x_s, y_s),\\
\xi_n &= \nabla\xi \cdot (y_s, -x_s),\\
\eta_s &= \nabla\eta \cdot (x_s, y_s),\\
\eta_n &= \nabla\eta \cdot (y_s, -x_s),
\end{aligned} \tag{4.9}$$

where ∇ is the gradient operator and $\mathbf{s} = (x_s, y_s)$ is the unit tangent vector. The outer unit normal vector $\mathbf{n} = (y_s, -x_s)$ is obtained by rotating the tangent clockwise by $\frac{\pi}{2}$, *i.e.*, $\mathbf{n} = (x_s + iy_s)e^{-i\frac{\pi}{2}} = y_s - ix_s$. Since the derivative of η with respect to the arc-length parameter s must vanish on the arcs Γ_1 and Γ_3, *i.e.*, $\eta_s = 0$, we obtain

$$\eta_x x_s + \eta_y y_s = 0. \tag{4.10}$$

Substitution of η_x and η_y in Beltrami system 4.4 into equation 4.10 yields

$$y_s\xi_x - x_s\xi_y = 0. \tag{4.11}$$

Let us now rewrite the first two equations of the system 4.9as follows

$$\begin{aligned}
x_s\xi_x + y_s\xi_y &= \xi_s,\\
-y_s\xi_x + x_s\xi_y &= \xi_n.
\end{aligned} \tag{4.12}$$

Solving the above system for ξ_x and ξ_y and then substitute into equation 4.11, a straightforward calculation will show that $\eta_s = 0$ implies the normal derivative condition;

$$\frac{\partial\xi}{\partial n} = 0. \tag{4.13}$$

Further, we can also proceed with the same argument based on the constraints in 4.9 and deduce the tangent derivative condition $\xi_s = -\mu\eta_n$ being specified on the boundary curves Γ_1 and Γ_3.

The analogy of previous discussion is carried through in the treatment process of the boundary conditions for the solution η in the harmonic system 4.8. Knowing that the function $\eta = 0$ on the boundary arc Γ_1 and $\eta = 1$ on the boundary arc Γ_3 of $\partial\Omega$, therefore, on the boundary arcs Γ_2 and Γ_4 of $\partial\Omega$, the function ξ is constant, whereas the necessary boundary conditions of the function η are $\frac{\partial\eta}{\partial n} = 0$ and $\eta_s = \frac{1}{\mu}\xi_n$.

In summary, the functions ξ and η of the complex map $\zeta = \xi + i\eta$ must be a solutions of the self-adjoint Laplace equation

$$\nabla^2\zeta = \zeta_{xx} + \zeta_{yy} = 0, \tag{4.14}$$

where the coordinate function ξ satisfies the following mixed (Dirichlet-to-Nuemann) boundary conditions

$$\begin{aligned} \xi &= 0 \quad \text{on} \quad \Gamma_4, \\ \xi &= 1 \quad \text{on} \quad \Gamma_2, \\ \frac{\partial\xi}{\partial n} &= 0 \quad \text{on} \quad \Gamma_1, \Gamma_3, \\ \frac{\partial\xi}{\partial s} &= -\mu\eta_n \quad \text{on} \quad \Gamma_1, \Gamma_3. \end{aligned} \tag{4.15}$$

Similarly, the function η satisfies the following boundary conditions

$$\begin{aligned} \eta &= 0 \quad \text{on} \quad \Gamma_1, \\ \eta &= 1 \quad \text{on} \quad \Gamma_3, \\ \frac{\partial\eta}{\partial n} &= 0 \quad \text{on} \quad \Gamma_2, \Gamma_4, \\ \frac{\partial\eta}{\partial s} &= \frac{1}{\mu}\xi_n \quad \text{on} \quad \Gamma_2, \Gamma_4. \end{aligned} \tag{4.16}$$

In general, the formulation of the boundary conditions are, however, an essential element to the solution existence and uniqueness of the Laplace equation. In particular, if the solution exits, we shall show that the solution to the boundary value problem 4.14 with associated Dirichlet and Neumann boundary conditions is unique. The Neumann boundary condition is considered necessary for the solution of a harmonic function to exist, namely,

$$\oint_{\partial\Omega} \frac{\partial\xi}{\partial n} = 0. \tag{4.17}$$

This argument is verified using Gauss's Theorem,

$$\oint_{\partial\Omega} \frac{\partial \xi}{\partial n} ds = \oint_{\partial\Omega} \nabla\xi \cdot \mathbf{n} ds = \iint_{\Omega} \nabla \cdot \nabla\xi dA = 0.$$

It should be noted that the mathematical condition attributed the normal derivative $\frac{\partial \xi}{\partial n} = 0$ has a physical meaning. For example, the total rate at which heat flows into the object is complete (otherwise it will never reach the state of equilibrium), namely the steady state modeled by Laplace equation. One also can think of this condition as *non-existence* condition. For instance, if $\partial\Omega$ is the unit circle, and the normal derivative is prescribed to be $\frac{\partial \xi}{\partial n} = 1$ on $\partial\Omega$. Then no harmonic function ξ can exist satisfying this condition, since the integral in 4.17 has the value 2π, not 0.

In the case at hand, we can also prove the solution ξ of the boundary-value problem 4.14, if it exists, it is unique. This argument uses Green's Theorem which gives $|\nabla\xi|^2 = 0$ everywhere in Ω,

$$\iint_{\Omega} |\nabla\xi|^2 dA = \oint_{\partial\Omega} \xi \frac{\partial \xi}{\partial n} ds = 0.$$

Later in the section, we prove that Neumann boundary conditions are the necessary conditions for the the solution of the boundary-value problem 4.14 to coincide with the quasiconformal, or conformal in the case $\mu = 1$. One observation worth noting, from 4.4 and equation 4.10, the Neumann boundary conditions identified by *normal derivative*

$$\frac{\partial \xi}{\partial n} = 0 \quad \text{and} \quad \frac{\partial \eta}{\partial n} = 0,$$

are equivalent to coordinate *boundary orthogonality* condition

$$\nabla\xi \cdot \nabla\eta = 0, \tag{4.18}$$

whereas, the boundary *tangential derivative* conditions

$$\frac{\partial \xi}{\partial s} = -\mu\eta_n \quad \text{and} \quad \frac{\partial \eta}{\partial s} = \frac{1}{\mu}\xi_n,$$

are equivalent to *boundary spacing* condition

$$|\nabla\xi| = \mu|\nabla\eta|. \tag{4.19}$$

Next, we present an important mathematical tool in the study of elliptic differential equations called *maximum principle.* The usefulness of maximum principle is restricted to second-order equations since the second derivatives of the function provide information on function's extrema. Also the effect of the maximum principle can be seen in its use to prove the uniqueness of the solution to many elliptic equations. A comprehensive reference on the this topic by Protter and Weinberger [22] which also include the proof of this next result.

Theorem 4.6. *Maximum Principle - Let Ω be a closed and bounded region in $\mathbb{R}^2$ and the real-valued function $\varphi : \Omega \to \mathbb{R}$ be continuous on $\Omega \cup \partial\Omega$. Suppose that φ has the following two properties:*

(1) φ_x and φ_y are C^1-differentiable at every point $(x, y) \in \Omega$;

(2) φ_{xx} and φ_{yy} exist at every point $(x, y) \in \Omega$.

If φ is a non-constant solution of the Laplace equation $\nabla^2\varphi = 0$ in Ω, then φ assumes its maximum (and minimum) values on $\partial\Omega$.

In other words, harmonic functions satisfy the maximum principle cannot attain local maxima or minima in Ω, other than the exceptional case where φ is constant. As a result of this theorem, we conclude that the solution of the Laplace equation in 4.14, if it exists, then it is unique, For further details on the use of maximum principle in partial differential equations the book of Myint and Debnath [21] is indispensable reference.

Now we turn our attention to the retrieval process of quasicoformal map $\zeta = \xi + i\eta$ as the solution to the elliptic equations system of partial equations given by 4.14. To put thing in prospective, let Ω be a simply connected region in $\mathbb{R}^2$ bounded by a smooth closed curve $\partial\Omega$. Again, we decompose $\partial\Omega$ into four arcs Γ_1, Γ_2, Γ_3, and Γ_4 having only endpoints in common. The ordering and orientation of the arcs are in a counterclockwise fashion around $\partial\Omega$ and each mapped to a side of the square Ω_s. Let $c_1, c_2 \in \mathbb{R}$ with $c_1 < c_2$.

Theorem 4.7. *Let Ω be a simply connected region and $\xi : \Omega \to \mathbb{R}$ be a harmonic function such that $\xi = c_1$ on Γ_4, $\xi = c_2$ on Γ_2, ξ is increasing on Γ_1, and ξ is decreasing on Γ_3, then gradient $\nabla\xi \neq 0$ on Ω.*

Proof. Suppose there is $z_0 = (x_0, y_0) \in \Omega$ such that the gradient $\nabla\xi(z_0) = 0$. Since Ω is simply connected region and ξ is harmonic on Ω, then there exists a harmonic conjugate ξ^* of ξ such that $\phi(z) = \xi(z) + i\xi^*(z)$ is holomorphic on Ω. It is easy to see that $\phi'(z_0) = 0$. Define $\psi = \phi - \phi(z_0)$ on Ω, then the function ψ has a zero of multiplicity at least two at $z = z_0$. By the argument principle, the change in the argument of ψ around $\partial\Omega$ (or a curve in Ω arbitrarily close to $\partial\Omega$ in case ψ has a zero on $\partial\Omega$) must be at least 4π. This contradicts the boundary values of ξ since $\xi(z) - \xi(z_0)$ assumes the value zero only twice on $\partial\Omega$. $\square$

Suppose Ω is mapped to a region Ω_s by the function $\zeta = \xi + i\eta$, where ξ and η are harmonic functions. Moreover, if ζ maps the boundary components Γ_1, Γ_2, Γ_3, and Γ_4 to the edges of Ω_s in a one-one manner, then both $\nabla\xi$ and $\nabla\eta$ will be nonzero on Ω by the previous theorem.

Theorem 4.8. *Let Ω be a simply connected region mapped to a square region Ω_s by the function $\zeta = \xi + i\eta$. Suppose that ξ and η are harmonic functions. (i) If the Jacobian of transformation $J = \xi_x\eta_y - \xi_y\eta_x > 0$ on $\partial\Omega$, then $J > 0$ in the interior of Ω. (ii) If the mappings ξ and η are orthogonal,* i.e., *$\nabla\xi \cdot \nabla\eta = 0$, on $\partial\Omega$ then they are orthogonal in Ω. (iii) Assume (i) and (ii) hold and if conformal module μ of the mapping ζ satisfies $|\nabla\xi| = \mu|\nabla\eta|$ on $\partial\Omega$ then it is satisfied in Ω.*

Proof. (i) and (ii)— Let ψ be the mapping defined by

$$\psi(z) = \frac{\eta_x - i\eta_y}{\xi_x - i\xi_y}$$

We see that ξ_x, ξ_y, η_x, and η_y are harmonic functions, since ξ and η are harmonic. It follows that $\eta_x - i\eta_y$ and $\xi_x - i\xi_y$ are holomorphic functions, *i.e.*, the real and the imaginary parts of both functions satisfy Cauchy-Riemann equations. By the virtue of Theorem 4.7, we have $\nabla\xi \neq 0$ on Ω. Therefore, the mapping $\psi(z)$ is holomorphic. We rewrite the function ψ as

$$\psi(z) = \frac{(\xi_x\eta_x + \xi_y\eta_y) - i(\xi_x\eta_y - \xi_y\eta_x)}{\xi_x^2 + \xi_y^2}$$

since ψ is holomorphic, we have

$$\begin{aligned} \frac{J}{|\nabla\xi|^2} &= \frac{\xi_x\eta_y - \xi_y\eta_x}{\xi_x^2 + \xi_y^2} \\ \frac{\nabla\xi \cdot \nabla\eta}{|\nabla\xi|^2} &= \frac{\xi_x\eta_x + \xi_y\eta_y}{\xi_x^2 + \xi_y^2} \end{aligned} \tag{4.20}$$

are harmonic functions. Applying the maximum principle to the above equations in 4.20, we conclude that the metric tensors $J > 0$, and $\nabla\xi \cdot \nabla\eta = 0$ in the interior of Ω.

(iii)— Let ψ be the mapping defined by

$$\psi(z) = \frac{(\xi_x + \mu^2\eta_x) - i(\xi_y + \mu^2\eta_y)}{(\xi_x - \eta_x) - i(\xi_y - \eta_y)}$$

By the same argument above we see the components of the function ψ are harmonic functions, since and that $(\xi_x + \mu^2\eta_x) - i(\xi_y + \mu^2\eta_y)$ and $(\xi_x - \eta_x) - i(\xi_y - \eta_y)$ are holomorphic functions. Moreover, from Theorem 4.7 we have $\nabla\xi \neq 0$ and $\nabla\eta \neq 0$ on Ω and the mapping $\psi(z)$ is holomorphic. We rewrite the function ψ as

$$\psi(z) = \frac{|\nabla\xi|^2 - \mu^2|\nabla\eta|^2}{|\nabla\xi|^2 + |\nabla\eta|^2} - i\frac{2J}{|\nabla\xi|^2 + |\nabla\eta|^2}$$

since ψ is holomorphic, therefore the maximum principle yields

$$|\nabla\xi| = \mu|\nabla\eta|$$

in the interior of Ω. □

At this end, We are ready to synchronize the overall discussion presented in this section on harmonic maps and boundary-value problems in relation to the quasiconformal mapping equivalence Theorem 4.5, we now state our main result:

Theorem 4.9. *Let the mapping $\zeta : \Omega \to \Omega_s$ be a homeomorphism with non-vanishing Jacobian. Then $\zeta = \xi + i\eta$ is quasiconformal map if and only if the real-valued functions $\xi, \eta \in C^1(\Omega)$ and solution to the Laplace equation*

$$\nabla^2\zeta = 0$$

with boundary conditions

$$\nabla\xi \cdot \nabla\eta = 0 \quad \text{and} \quad |\nabla\xi| = \mu|\nabla\eta| \quad \text{on} \quad \partial\Omega$$

4.3 Curvilinear Coordinate Transformations

The motivation for the use of quasiconformal mapping in computational geometry is the ease of expressing orthogonal curvilinear coordinate systems on simply connected two-dimensional regions. The primary concern of this section is to establish conformal and quasiconformal maps as an enabling mathematical tools in solving practical problems in geometry. In real-world applications in geometric modeling and analysis, it is however, the map from the computational domain Ω_s to the physical domain Ω is foremost important. For the purpose of ease, the process of theoretical analysis is performed in the forward direction $\Omega \mapsto \Omega_s$. On the other hand, it is the inverse mapping process from $\Omega_s \mapsto \Omega$ what is sought after in practice.

We now begin by setting the stage for the inverse coordinate transformation from the computational region Ω_s to the physical region Ω. In most of the two-dimensional problems which have been solved using a numerical transformation method, it is not the transformation from the physical region Ω to the square region Ω_s that is constructed, but rather the transformation from the region Ω_s to the region Ω. Our work proceeds in the same direction. The first task is to invert the elliptic 4.14; that is, to develop an equivalent system so that the computational variables (ξ, η) are independent variables and the physical variables (x, y) become the dependent variables. Define the inverse map $\zeta^{-1} : \Omega_s \to \Omega$ by the function $\chi(\xi,\eta) = x(\xi,\eta) + iy(\xi,\eta)$. The function $\chi := \zeta^{-1}$ describes a quasiconformal mapping from a square region to a simply connected domain:

$$\chi : \Omega_s \to \Omega,$$

whose gradient

$$\mathbf{F} := \nabla_\zeta \chi, \tag{4.21}$$

has nonzero determinant for every point ζ of Ω_s, that is, the Jacobian

$$\mathbf{J} = \det \mathbf{F} \neq 0, \quad \forall \zeta \in \Omega_s. \tag{4.22}$$

We now proceed with the coordinate inverse transformation by assuming that the physical variables x and y are C^1–differentiable and the Jacobian of the inverse transformation $\mathbf{J} = x_\xi y_\eta - x_\eta y_\xi$ is nonvanishing in the region under consideration. The metrics ("$\xi_x, \xi_y, \eta_x, \eta_y$") appear in previous differential equations in this chapter can be determined uniquely in the following manner. Using the chain rule of partial differentiation, the partial derivatives become

$$\frac{\partial}{\partial \xi} = x_\xi \frac{\partial}{\partial x} + y_\xi \frac{\partial}{\partial y} \quad \text{and}$$
$$\frac{\partial}{\partial \eta} = x_\eta \frac{\partial}{\partial x} + y_\eta \frac{\partial}{\partial y}.$$

We first write the differential expressions

$$du = x_\xi d\xi + x_\eta d\eta \quad \text{and}$$
$$dv = y_\xi d\xi + y_\eta d\eta,$$

which in matrix form become

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix} \begin{pmatrix} d\xi \\ d\eta \end{pmatrix}.$$

In a like manner we can write

$$\begin{pmatrix} d\xi \\ d\eta \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Therefore,

$$
\begin{aligned}
\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} &= \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix}^{-1} \\
&= \frac{1}{\mathbf{J}} \begin{pmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{pmatrix}.
\end{aligned}
$$

Thus, the metrics are

$$
\begin{aligned}
\xi_x &= \frac{1}{\mathbf{J}} y_\eta, \quad \xi_y = -\frac{1}{\mathbf{J}} x_\eta, \quad \text{and} \\
\eta_x &= -\frac{1}{\mathbf{J}} y_\xi, \quad \eta_y = \frac{1}{\mathbf{J}} x_\xi.
\end{aligned} \tag{4.23}
$$

Note that the Jacobian of the transformation ζ and its inverse χ are interchangeable, *i.e.*, $\mathbf{J} = \frac{1}{J}$, where $J = \xi_x \eta_y - \xi_y \eta_x$.

By combining the transformational relations above and from the definition of quasiconformal mapping, we observe that the function χ maps Ω_s homeomorphically onto a region Ω, and χ is a solution to Beltrami equation

$$
\chi_{\bar{\zeta}} = \mu \chi_{\zeta}, \tag{4.24}
$$

where $\bar{\zeta} = \xi - i\eta$ and $\zeta = \xi + i\eta$. By definition, the real and imaginary parts of χ are solutions to the expanded Beltrami equations

$$
\begin{aligned}
\mu x_\xi &= y_\eta, \\
x_\eta &= -\mu y_\xi.
\end{aligned} \tag{4.25}
$$

We now establish the mapping $\zeta = \xi + i\eta$ in which ξ and η are solutions to the linear elliptic system

$$
\begin{aligned}
\xi_{xx} + \xi_{yy} &= 0, \\
\eta_{xx} + \eta_{yy} &= 0,
\end{aligned} \tag{4.26}
$$

is equivalent to argument that $\chi = x + iy$ in which x and y are solutions to the quasi-linear elliptic system

$$
\begin{aligned}
\mu^2 x_{\xi\xi} + x_{\eta\eta} &= 0, \\
\mu^2 y_{\xi\xi} + y_{\eta\eta} &= 0.
\end{aligned} \tag{4.27}
$$

To complete the mathematical specification of the boundary value problem represented by system 4.27, it is necessary to address the boundary conditions. The solution existence and uniqueness of the elliptic system depends on boundary conditions expressed in terms of orthogonality and spacing constraints. Again, under coordinate transformation, the original boundary conditions for the system 4.26

$$\nabla\xi \cdot \nabla\eta = 0 \quad and \quad |\nabla\xi| = \mu|\nabla\eta| \quad on \quad \partial\Omega \tag{4.28}$$

lead to the boundary conditions for the system 4.27

$$\chi_\xi \cdot \chi_\eta = 0 \quad and \quad |\chi_\eta| = \mu|\chi_\xi| \quad on \quad \partial\Omega_s. \tag{4.29}$$

It is the boundary-value problem governed by the elliptic system 4.27 modulo boundary conditions 4.29 which forms the basis of our computational geometry and grid generation system. This method based on quasiconformal mapping has emerged in an attempt to unify and improve several existing geometric discretization methods. Geometric elliptic methods make use of harmonic functions that satisfy combinations of some intuitive geometric requirements. The physical boundary conditions in 4.29 have an important value from numerical implementation view point. Looking back at Theorem 4.8, we make the case that knowing $\mathbf{J} > 0$ on $\partial\Omega$ implies $\mathbf{J} > 0$ in Ω since $\mathbf{J} = \frac{1}{J}$. In addition, the orthogonality conditions on Ω and Ω_s are related by $\nabla\xi \cdot \nabla\eta = -\frac{1}{\mathbf{J}^2}\chi_\xi \cdot \chi_\eta$, where $\chi_\xi \cdot \chi_\eta = x_\xi x_\eta + y_\xi y_\eta$ is the orthogonality condition of the grid on Ω. Therefore, imposed orthogonality on the boundaries results in orthogonal network of coordinate (or grid) lines throughout the domain. In the course of generating grids, as a result of the maximum principle, the mapping defined as a solution to the elliptic system 4.27 with corresponding boundary conditions satisfies the following fundamental geometric property: All points of Ω_s map into Ω, that is, there is no grid folding or spill-over. For the computational treatment of the geometric boundary-value problem 4.27, we refer the reader to Khamayseh *et al.* [29] and Khamayseh and Hansen [34].

We conclude this chapter with the demonstration of geometric comparison between conformal and quasiconformal mappings. The numerical result exhibited in Figure 4.3 show the analogy and contrast between the two mapping. Fundamentally, the mappings produced consistent orthogonal/quasi-orthogonal grids on a curvilinear geometry as expected. However, the most appealing feature of the quasiconformal grid (right) is the preservation of the edge spacing throughout the domain, while the conformal grid (left) exhibited lack of control over the grid spacing. This undesired conformal geometric property is due to absence of the domain module μ in the conformal map, which exerts strong control over the spacing between grid lines. It is a known fact that conformal maps are weak near convex boundaries and strong near concave boundaries, and as result we see grid lines repulsed from the convex boundary segments while attracted in the vicinity of the concave boundary segments. The quantity μ in the quasiconformal mapping acts as a spatial mobility function which can be computed in advance over the whole domain or user-specified.

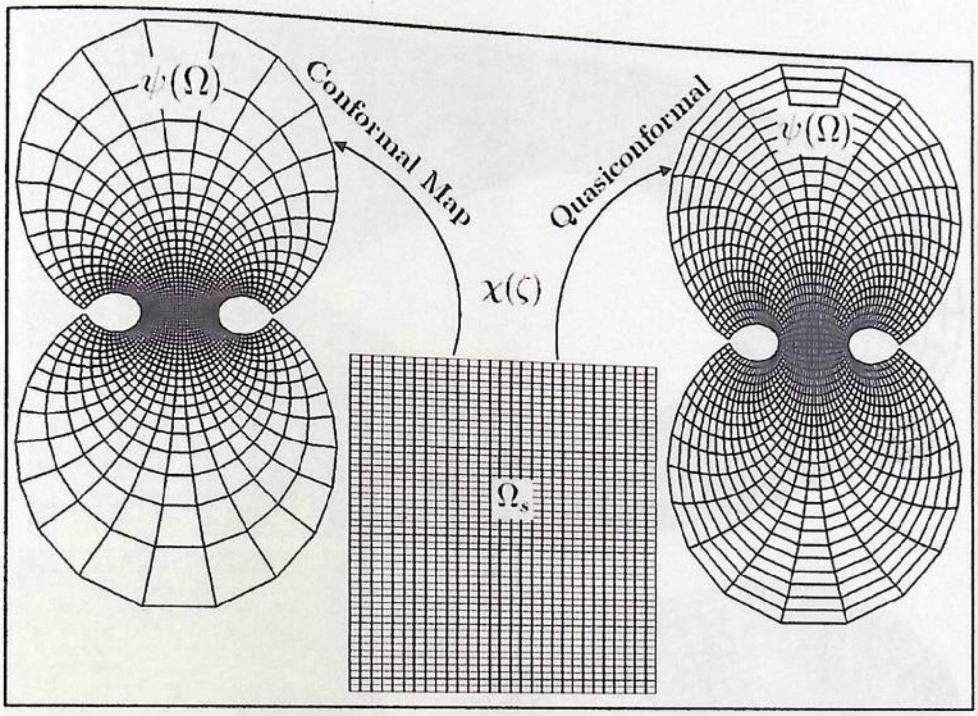

Figure 4.3: Comparison between conformal and quasiconformal mappings.

The elliptic approach to computational geometry is widely used because of its novelty in producing smooth-orthogonal coordinate lines suitable for scientific computations. The method can be incorporate consistently into the computational frame work using most of the standard methods, *e.g.*, finite-difference methods, finite-element methods, or finite-volume methods. Furthermore, what the method appears to produce satisfactory results in a wide range of complicated geometric designs. The multiple grid geometries presented in Figure 4.3 were constructed using finite-difference methods to solve the associated elliptic system 4.27. Boundary conditions 4.29 were imposed on the internal and external boundaries of curvilinear geometries. The end result is high quality spatial discretization of challenging geometries fully controlled by rigorous mathematical methods. Last but not least, robust methods of solution are often required, however, the method still has much to be developed and profit from as applied to other areas of mathematics and computational sciences.

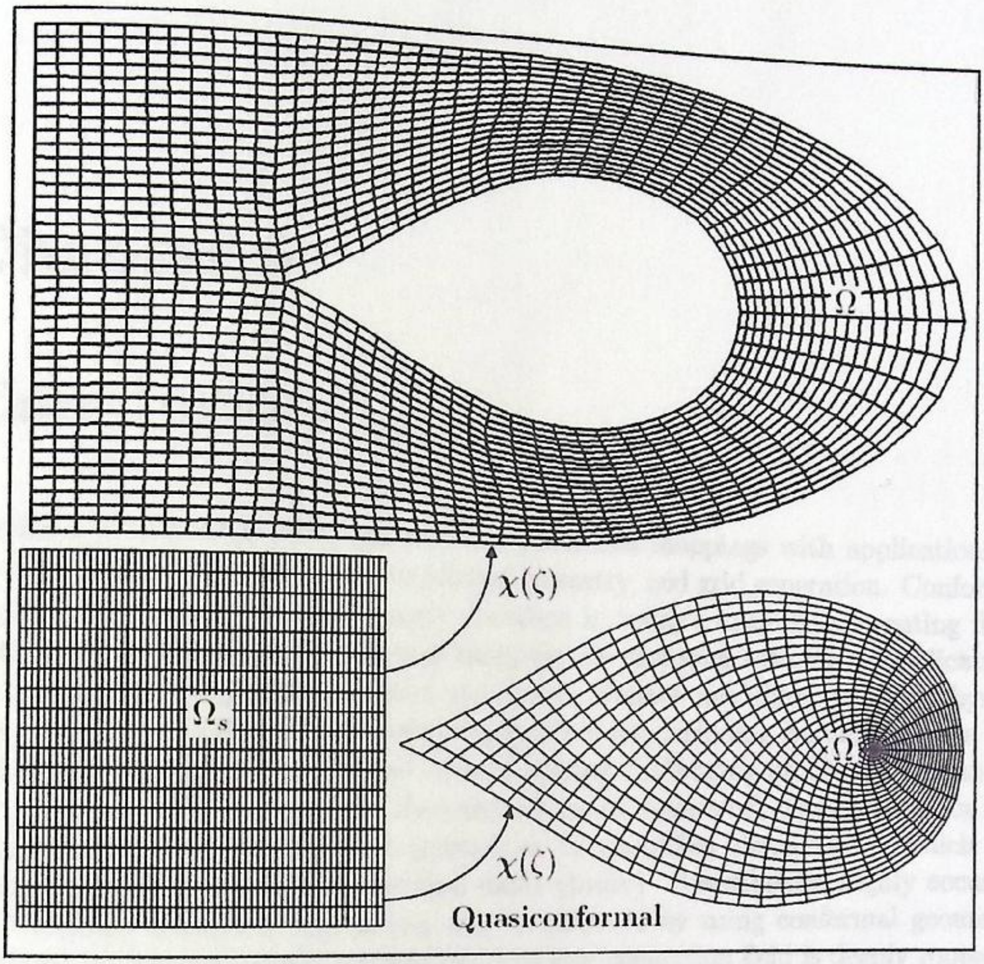

Figure 4.4: Quasiconformal mappings on curvilinear geometries.

It must be noted here, even though the theory developed and presented in this chapter is focused on the study of quasiconformal mapping, harmonic maps, and boundary-value problem on simply connected domains, it has also an extension to multi-connected domains. Needless to say, a simple way to overcome the complexity of boundary-value problems on multi-connected domains is to decomposed the multi-connected domain into multi-simply connected sub-domains, and however the theory and analysis presented here hold.

Chapter 5

Conclusion

We have presented a thorough study of conformal mappings with applications to boundary value problems, computational geometry, and grid generation. Conformal mappings have become increasingly prevalent in today's scientific computing field. Fundamental methods of conformal mapping are emerging with many applications ranging from computational fluid dynamics, climate modeling, to astrophysics. We analyzed a family of conformal geometric maps including harmonic maps and quasiconformal maps with an eye toward solving challenging problem in geometry. As a result, we developed a theoretically novel framework that allows for the transformation of complicated geometries to simplified domains, in which the resulting conformal mapping method easily studied. Nevertheless, highly accurate and efficient numerical algorithms can be achieved by using conformal geometric maps in the discipline grid generation. The grid generation field is deeply rooted in the theory of conformal mapping and differential equations.

One prominent feature of this study is the combining of several theoretical and applied mathematical ideas pertaining to conformal geometry in a unified and well-integrated presentation. Another distinctive aspect of this work is aimed at rigorously addressing basic conformal geometric properties that otherwise have been viewed as facts and definitions. The extension of conformal mapping to quasiconformal mapping is not to the level of fully developed theory, however, it is adequate for the formulation of the elliptic boundary-value problem for the purpose of geometric modeling. We have established quasiconformal mapping as indispensable geometrical and mathematical tool in solving a multitude of problems, most important, the invariance of the Laplace equation under conformal transformation. Finally, this presentation has stopped short of addressing the numerical methodology and algorithmic issues in solving the developed elliptic boundary-value problem. However, initial computational experiments performed consistent with the merits of the theoretical premise of the conformal mapping theory, as shown in the numerical results.

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