Linear Algebra and its Applications 436 (2012) 2494–2503

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Bounds and majorization relations for the critical points of polynomials

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ARTICLE INFO

Article history:
Received 18 May 2011
Accepted 25 October 2011
Available online 25 November 2011

Submitted by S. Kirkland

AMS classification:
15A18
15A42
15A60
26C10
30C15

Keywords:
D-companion matrix
Polynomial
Zero
Critical point
Eigenvalue
Bound
Majorization
Inequality

ABSTRACT

We apply several matrix inequalities to the derivative companion matrices of complex polynomials to establish new bounds and majorization relations for the critical points of these polynomials in terms of their zeros.

1. Introduction

Matrix analysis methods have been successfully used by several mathematicians to obtain new proofs of classical bounds for the zeros of polynomials and to derive new bounds and geometric relations between the zeros and critical points of polynomials. These methods include eigenvalue locations, matrix norms computations, eigenvalue-singular value majorization relations, numerical

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0024-3795/$ - see front matter
doi:10.1016/j.laa.2011.10.025

radius and spectral radius compression inequalities, differentiators, and D-companion matrices. See, e.g., [2–9,12–14], and the references therein.

Let f be a polynomial of degree $n \geqslant 3$, with complex coefficients, and let $z_1, z_2, \dots, z_n$ be the zeros of f.

Let $D = \begin{bmatrix} z_1 & 0 & \dots & 0 \\ 0 & z_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & z_{n-1} \end{bmatrix}$, I, and J be the identity matrix of order $(n-1)$ and the $(n-1) \times (n-1)$ matrix with all entries equal to 1, respectively. Then the $(n-1) \times (n-1)$ derivative companion matrix of f is given by

$$C(f') = D\left(I - \frac{1}{n}J\right) + \frac{z_n}{n}J, \tag{1}$$

which is called a D-companion matrix of f. See [2].

It has been shown in [2] that the critical points of f, i.e., the zeros of f', are exactly the eigenvalues of $C(f')$.

In this paper, we apply several matrix inequalities to $C(f')$ to obtain bounds for the critical points of f in terms of its zeros. In particular, we apply eigenvalue inequalities to the real part of $C(f')$ to establish new bounds and majorization relations for the real parts of the critical points of f.

2. Preliminary results

Let $w_1, w_2, \dots, w_{n-1}$ be the critical points of f (or the eigenvalues of $C(f')$), and let $z_1, z_2, \dots, z_n$ be the zeros of f. To obtain our new bounds and majorization relations for $\operatorname{Re} w_1, \operatorname{Re} w_2, \dots, \operatorname{Re} w_{n-1}$, we need several lemmas involving inequalities and majorization relations for eigenvalues, together with basic facts about the Schatten p-norms of matrices.

Let $M_n(\mathbf{C})$ denote the algebra of all $n \times n$ complex matrices. For $A \in M_n(\mathbf{C})$, the eigenvalues of A are denoted by $\lambda_1(A), \lambda_2(A), \dots, \lambda_n(A)$. If A is Hermitian, then the eigenvalues of A are arranged in such a way that $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \dots \geqslant \lambda_n(A)$.

For two sequences of real numbers arranged in decreasing order,

$$x = (x_1, x_2, \dots, x_n) \quad \text{and} \quad y = (y_1, y_2, \dots, y_n),$$

we say that x is weakly majorized by y if

$$\sum_{i=1}^{k} x_i \leqslant \sum_{i=1}^{k} y_i, \quad \text{for } k = 1, 2, \dots, n-1.$$

If, in addition,

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,$$

then x is said to be majorized by y. For the theory of majorization, we refer to [1,10,15].

Lemma 1. *Let $A \in M_n(\mathbf{C})$ with real part $\operatorname{Re} A = \frac{A+A^*}{2}$. Then*

$$\lambda_n(\operatorname{Re} A) \leqslant \operatorname{Re} \lambda_j(A) \leqslant \lambda_1(\operatorname{Re} A) \quad \text{for } j = 1, 2, \dots, n. \tag{2}$$

Lemma 2. *Let $A_1, A_2, \dots, A_m \in M_n(\mathbf{C})$ be Hermitian. Then*

$$\lambda_j(A_1) + \sum_{i=2}^{m} \lambda_n(A_i) \leqslant \lambda_j\left(\sum_{i=1}^{m} A_i\right) \leqslant \lambda_j(A_1) + \sum_{i=2}^{m} \lambda_1(A_i) \quad \text{for } j = 1, 2, \dots, n. \tag{3}$$

In particular,

$$\lambda_1\left(\sum_{i=1}^{m} A_i\right) \leqslant \sum_{i=1}^{m} \lambda_1(A_i) \tag{4}$$

and

$$\sum_{i=1}^{m} \lambda_n(A_i) \leqslant \lambda_n\left(\sum_{i=1}^{m} A_i\right). \tag{5}$$

Lemma 3. *Let* $A \in M_n(\mathbf{C})$ *with eigenvalues arranged in such a way that* $\operatorname{Re}\lambda_1(A) \geqslant \operatorname{Re}\lambda_2(A) \geqslant \cdots \geqslant \operatorname{Re}\lambda_n(A)$. *Then*

$$\sum_{j=1}^{k} \operatorname{Re}\lambda_j(A) \leqslant \sum_{j=1}^{k} \lambda_j(\operatorname{Re}A), \quad \textit{for } k = 1, 2, \ldots, n-1 \tag{6}$$

and

$$\sum_{j=1}^{n} \operatorname{Re}\lambda_j(A) = \sum_{j=1}^{n} \lambda_j(\operatorname{Re}A). \tag{7}$$

Lemma 4. *Let* $A_1, A_2, \ldots, A_m \in M_n(\mathbf{C})$ *be Hermitian. Then*

$$\sum_{j=1}^{k} \lambda_j\left(\sum_{i=1}^{m} A_i\right) \leqslant \sum_{j=1}^{k}\left(\sum_{i=1}^{m} \lambda_j(A_i)\right) \quad \textit{for } k = 1, 2, \ldots, n-1 \tag{8}$$

and

$$\sum_{j=1}^{n} \lambda_j\left(\sum_{i=1}^{m} A_i\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{m} \lambda_j(A_i)\right). \tag{9}$$

For $A \in M_n(\mathbf{C})$, let $s_1(A), s_2(A), \ldots, s_n(A)$ be the singular values of A, i.e., the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$. Then for $p \geqslant 1$, the Schatten p-norm of A is defined by

$$\|A\|_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{\frac{1}{p}}. \tag{10}$$

One of the basic facts about the Schatten p-norms that will be used in this paper is a submultiplicativity property, which says that if $A, B, C \in M_n(\mathbf{C})$, then

$$\|ABC\|_p \leqslant \|A\|\,\|B\|_p\,\|C\|. \tag{11}$$

See, e.g., [1, p. 94].

Lemma 5. *Let* $A = [a_{ij}] \in M_n(\mathbf{C})$. *If* $0 < p \leqslant 2$, *then*

$$\sum_{j=1}^{n} s_j^p(A) \leqslant \sum_{i,j=1}^{n} |a_{ij}|^p. \tag{12}$$

Lemma 6. *Let* $A = [a_{ij}] \in M_n(\mathbf{C})$. *Then for* $p > 0$,

$$\sum_{j=1}^{n} |\lambda_j(A)|^p \leqslant \sum_{j=1}^{n} s_j^p(A). \tag{13}$$

Lemma 7. *Let* $A = [a_{ij}] \in M_n(\mathbf{C})$. *Then*

$$\sum_{j=1}^{n} |\lambda_j(A)|^2 \leqslant \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} |a_{ji}|^2\right)^{\frac{1}{2}}. \tag{14}$$

The inequality (14) represents an improvement of the classical Schur's inequality

$$\sum_{j=1}^{n} |\lambda_j(A)|^2 \leqslant \sum_{i,j=1}^{n} |a_{ij}|^2. \tag{15}$$

Except for Lemma 7, which can be found in [11], Lemmas 1–6 can be found in [1,15].

3. Main results

Our new bounds for the real parts of the critical points of f can be stated as follows.

Theorem 1. *Let* $z_1, z_2, \ldots, z_n$ *be the zeros of a polynomial* f *of degree* $n \geqslant 3$ *and* $w_1, w_2, \ldots, w_{n-1}$ *be the critical points of* f. *Then for* $j = 1, 2, \ldots, n-1$, *we have*

$$\begin{aligned}&\frac{n-1}{n} \min_{1\leqslant j\leqslant n-1} \{\operatorname{Re} z_j\} - \frac{\sqrt{n-2}}{2n} \sum_{i=1}^{n-1} |z_i| \leqslant \operatorname{Re} w_j \\ &\leqslant \frac{n-1}{n} \max_{1\leqslant j\leqslant n-1} \{\operatorname{Re} z_j\} + \frac{\sqrt{n-2}}{2n} \sum_{i=1}^{n-1} |z_i| + \frac{n-1}{n} \operatorname{Re} z_n, \quad \textit{if } \operatorname{Re} z_n \geqslant 0\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\frac{n-1}{n} \min_{1\leqslant j\leqslant n-1} \{\operatorname{Re} z_j\} - \frac{\sqrt{n-2}}{2n} \sum_{i=1}^{n-1} |z_i| + \frac{n-1}{n} \operatorname{Re} z_n \\ &\leqslant \operatorname{Re} w_j \leqslant \frac{n-1}{n} \max_{1\leqslant j\leqslant n-1} \{\operatorname{Re} z_j\} + \frac{\sqrt{n-2}}{2n} \sum_{i=1}^{n-1} |z_i|, \quad \textit{if } \operatorname{Re} z_n < 0.\end{aligned} \tag{17}$$

Proof. It follows from (1) that

$$\operatorname{Re} C(f') = \begin{bmatrix} \operatorname{Re} z_1 & 0 & \ldots & 0 \\ 0 & \operatorname{Re} z_2 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & \operatorname{Re} z_{n-1} \end{bmatrix} - \frac{1}{2n} \begin{bmatrix} 2\operatorname{Re} z_1 & z_1 + \overline{z_2} & \ldots & z_1 + \overline{z_{n-1}} \\ z_2 + \overline{z_1} & 2\operatorname{Re} z_2 & \ldots & z_2 + \overline{z_{n-1}} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ z_{n-1} + \overline{z_1} & z_{n-1} + \overline{z_2} & \ldots & 2\operatorname{Re} z_{n-1} \end{bmatrix}$$

$$+ \frac{\operatorname{Re} z_n}{n} \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \ldots & 1 \end{bmatrix}.$$

Thus, $\operatorname{Re} C(f') = A_1 + A_2 + A_3$, where

$$A_1 = \frac{n-1}{n}\begin{bmatrix} \operatorname{Re} z_1 & 0 & \dots & 0 \\ 0 & \operatorname{Re} z_2 & \dots & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & \dots & \operatorname{Re} z_{n-1} \end{bmatrix},$$

$$A_2 = \frac{-1}{2n}\begin{bmatrix} 0 & z_1+\overline{z_2} & \dots & z_1+\overline{z_{n-1}} \\ z_2+\overline{z_1} & 0 & \dots & z_2+\overline{z_{n-1}} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ z_{n-1}+\overline{z_1} & z_{n-1}+\overline{z_2} & \dots & 0 \end{bmatrix},$$

$$A_3 = \frac{\operatorname{Re} z_n}{n} J,$$

and these matrices are Hermitian.

Now, A_2 can be written as $A_2 = B_1 + B_2 + \cdots + B_{n-1}$, where

$$B_1 = \begin{bmatrix} 0 & \frac{-z_1}{2n} & \frac{-z_1}{2n} & \dots & \frac{-z_1}{2n} \\ \frac{-\overline{z_1}}{2n} & 0 & 0 & \dots & 0 \\ \frac{-\overline{z_1}}{2n} & 0 & 0 & \dots & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{-\overline{z_1}}{2n} & 0 & 0 & \dots & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & \frac{-\overline{z_2}}{2n} & 0 & \dots & 0 \\ \frac{-z_2}{2n} & 0 & \frac{-z_2}{2n} & \dots & \frac{-z_2}{2n} \\ 0 & \frac{-\overline{z_2}}{2n} & 0 & \dots & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & \frac{-\overline{z_2}}{2n} & 0 & \dots & 0 \end{bmatrix}, \dots,$$

$$B_{n-1} = \begin{bmatrix} 0 & 0 & 0 & \dots & \frac{-\overline{z_{n-1}}}{2n} \\ 0 & 0 & 0 & \dots & \frac{-\overline{z_{n-1}}}{2n} \\ 0 & 0 & 0 & \dots & \frac{-\overline{z_{n-1}}}{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{-z_{n-1}}{2n} & \frac{-z_{n-1}}{2n} & \frac{-z_{n-1}}{2n} & \dots & 0 \end{bmatrix},$$

and all of these matrices are Hermitian.

After some simple computations, we have

$$\sigma(A_1) = \left\{ \frac{n-1}{n}\operatorname{Re} z_1, \frac{n-1}{n}\operatorname{Re} z_2, \dots, \frac{n-1}{n}\operatorname{Re} z_{n-1} \right\},$$

$$\sigma(B_j) = \left\{ \frac{\sqrt{n-2}}{2n}\left|z_j\right|, 0, -\frac{\sqrt{n-2}}{2n}\left|z_j\right| \right\} \quad \text{for } j = 1, 2, \dots, n-1,$$

where 0 is of multiplicity $n-3$, and

$$\sigma(A_3) = \left\{ \frac{n-1}{n} \mathrm{Re}\, z_n, 0 \right\},$$

where 0 is of multiplicity $n-2$. Here $\sigma(A)$ denotes the spectrum (or the set of all eigenvalues) of A.

Applying Lemmas 1 and 2 to $C(f')$and to the Hermitian matrices $A_1, B_1, B_2, \ldots, B_{n-1}, A_3$, we obtain

$$\begin{aligned} &\lambda_n(A_1) + \lambda_n(B_1) + \lambda_n(B_2) + \cdots + \lambda_n(B_{n-1}) + \lambda_n(A_3) \\ &\quad \leqslant \lambda_n(\mathrm{Re}\, C(p')) \leqslant \mathrm{Re}\, w_j \leqslant \lambda_1(\mathrm{Re}\, C(p')) \\ &\quad \leqslant \lambda_1(A_1) + \lambda_1(B_1) + \lambda_1(B_2) + \cdots + \lambda_1(B_{n-1}) + \lambda_1(A_3) \end{aligned}$$

for $j = 1, 2, \ldots, n-1$. Hence our desired result follows. □

In the following theorem, we give a majorization relation for the critical points of a polynomial.

Theorem 2. *Let $z_1, z_2, \ldots, z_n$ be the zeros of a polynomial f of degree $n \geqslant 3$ and $w_1, w_2, \ldots, w_{n-1}$ be the critical points of f arranged in such a way that $\mathrm{Re}\, z_1 \geqslant \mathrm{Re}\, z_2 \geqslant \cdots \geqslant \mathrm{Re}\, z_n$ and $\mathrm{Re}\, w_1 \geqslant \mathrm{Re}\, w_2 \geqslant \cdots \geqslant \mathrm{Re}\, w_{n-1}$. Then for $k = 1, 2, \ldots, n-2$, we have*

$$\sum_{j=1}^{k} \mathrm{Re}\, w_j \leqslant \frac{n-1}{n} \sum_{j=1}^{k} \mathrm{Re}\, z_j + \frac{\sqrt{n-2}}{2n} \sum_{j=1}^{n-1} |z_j| + \frac{n-1}{n} \mathrm{Re}\, z_n, \quad \textit{if}\ \mathrm{Re}\, z_n \geqslant 0 \tag{18}$$

and

$$\sum_{j=1}^{k} \mathrm{Re}\, w_j \leqslant \frac{n-1}{n} \sum_{j=1}^{k} \mathrm{Re}\, z_j + \frac{\sqrt{n-2}}{2n} \sum_{j=1}^{n-1} |z_j|, \quad \textit{if}\ \mathrm{Re}\, z_n < 0. \tag{19}$$

Proof. Applying Lemmas 3 and 4 to $C(f')$ and the Hermitian matrices $A_1, B_1, B_2, \ldots, B_{n-1}, A_3$, we obtain, in view of our analysis in the proof of Theorem 1, that

$$\begin{aligned} \sum_{j=1}^{k} \mathrm{Re}\, w_j &= \sum_{j=1}^{k} \mathrm{Re}\, \lambda_j(C(f')) \leqslant \sum_{j=1}^{k} \lambda_j(\mathrm{Re}\, C(f')) \\ &\leqslant \sum_{j=1}^{k} \lambda_j(A_1) + \sum_{j=1}^{k} \lambda_j(B_1) + \cdots + \sum_{j=1}^{k} \lambda_j(B_{n-1}) + \sum_{j=1}^{k} \lambda_j(A_3) \\ &= \begin{cases} \frac{n-1}{n} \sum_{j=1}^{k} \mathrm{Re}\, z_j + \frac{\sqrt{n-2}}{2n} \sum_{j=1}^{n-1} |z_j| + \frac{n-1}{n} \mathrm{Re}\, z_n, & \text{if } \mathrm{Re}\, z_n \geqslant 0 \\ \frac{n-1}{n} \sum_{j=1}^{k} \mathrm{Re}\, z_j + \frac{\sqrt{n-2}}{2n} \sum_{j=1}^{n-1} |z_j|, & \text{if } \mathrm{Re}\, z_n < 0 \end{cases} \end{aligned}$$

for $k = 1, 2, \ldots, n-2$. This completes the proof of the theorem.

It should be mentioned here that it follows from (1), in view of the fact that the trace of a matrix is the sum of its eigenvalues, that

$$\sum_{j=1}^{n-1} w_j = \frac{n-1}{n} \sum_{j=1}^{n} z_j. \tag{20}$$

Using the theory of majorization, many well-known geometric problems for polynomials have been recently solved. In fact, among other majorization relations, it has been shown by Schmeisser [14] (see also [2,12,13]) that if $z_1, z_2, \ldots, z_n$ and $w_1, w_2, \ldots, w_{n-1}$ are the zeros and critical points of a polynomial f arranged in such a way that $|z_1| \geqslant |z_2| \geqslant \cdots \geqslant |z_n|$ and $|w_1| \geqslant |w_2| \geqslant \cdots \geqslant |w_{n-1}|$, then $(|w_1|, |w_2|, \ldots, |w_{n-1}|, 0)$ is weakly majorized by $(|z_1|, |z_2|, \ldots, |z_n|)$. This majorization relation is stronger than the classical Gauss-Lucas theorem, which says that all the critical points of a polynomial f lie inside the closed convex hull of the zeros of f.

Matrix analysis methods, together with majorization tools, have been employed in [2,8,12,13] to bound $\sum_{i=1}^{n-1}|w_i|^2$ by a suitable combination of the terms $\left|\sum_{i=1}^{n} z_i\right|^2$ *and* $\sum_{i=1}^{n}|z_i|^2$, and to bound $\sum_{i=1}^{n-1}|w_i|^4$ by a suitable combination of the terms $\sum_{i=1}^{n}|z_i|^4$*and* $\left(\sum_{i=1}^{n}|z_i|^2\right)^2$, under the condition that $\sum_{i=1}^{n} z_i = 0$. These bounds lead to proofs of the Schoenberg conjecture and the de Bruijn–Sharma conjecture. In what follows, we obtain related bounds for $\sum_{i=1}^{n-1}|w_i|^p$ for general $p > 0$.

Estimating the Schatten p-norm of $C(f')$, we have the following inequalities relating the zeros and critical points of f. $\square$

Theorem 3. *Let $z_1, z_2, \ldots, z_n$ be the zeros of a polynomial f of degree $n \geqslant 3$ and $w_1, w_2, \ldots, w_{n-1}$ be the critical points of f. Then for $p \geqslant 1$, we have*

$$\left(\sum_{i=1}^{n-1}|w_i|^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n-1}|z_i|^p\right)^{\frac{1}{p}} + \left(\frac{n-1}{n}\right)|z_n|, \tag{21}$$

$$\left(\sum_{i=1}^{n-1}|w_i|^p\right)^{\frac{1}{p}} \leqslant \left((n-2)+\frac{1}{n^p}\right)^{\frac{1}{p}} \max_{1\leqslant i\leqslant n-1}\{|z_i|\} + \left(\frac{n-1}{n}\right)|z_n|, \tag{22}$$

and

$$\left(\sum_{i=1}^{n-1}|w_i|^p\right)^{\frac{1}{p}} \leqslant \left(\sum_{i=1}^{n-1}|z_i|^p\right)^{\frac{1}{p}} + \sqrt{\frac{n-1}{n^2}\sum_{i=1}^{n-1}|z_n-z_i|^2}. \tag{23}$$

Proof. It follows from (1), together with basic properties of the Schatten p-norms, that

$$\begin{aligned}\left\|C(f')\right\|_p &= \left\|D\left(I-\frac{1}{n}J\right)+\frac{z_n}{n}J\right\|_p \leqslant \left\|D\left(I-\frac{1}{n}J\right)\right\|_p + \left\|\frac{z_n}{n}J\right\|_p \leqslant \|D\|_p\left\|I-\frac{1}{n}J\right\| + \left\|\frac{z_n}{n}J\right\|_p \\ &= \left(\sum_{i=1}^{n-1}|z_i|^p\right)^{\frac{1}{p}} + \left(\frac{n-1}{n}\right)|z_n|.\end{aligned}$$

This proves (21).

For (22), we have

$$\begin{aligned}\left\|C(f')\right\|_p &= \left\|D\left(I-\frac{1}{n}J\right)+\frac{z_n}{n}J\right\|_p \leqslant \left\|D\left(I-\frac{1}{n}J\right)\right\|_p + \left\|\frac{z_n}{n}J\right\|_p \leqslant \left\|I-\frac{1}{n}J\right\|_p\|D\| + \left\|\frac{z_n}{n}J\right\|_p \\ &= \left((n-2)+\frac{1}{n^p}\right)^{\frac{1}{p}} \max_{1\leqslant i\leqslant n-1}\{|z_i|\} + \left(\frac{n-1}{n}\right)|z_n|.\end{aligned}$$

This proves (22).

Now for (23), we have

$$C(f') = D + \frac{1}{n}(z_nJ - DJ) = D + \frac{1}{n}\begin{bmatrix} z_n-z_1 & z_n-z_1 & \ldots & z_n-z_1 \\ z_n-z_2 & z_n-z_2 & \ldots & z_n-z_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ z_n-z_{n-1} & z_n-z_{n-1} & \ldots & z_n-z_{n-1} \end{bmatrix} = D + \frac{1}{n}E,$$

where

$$E = \begin{bmatrix} z_n - z_1 & z_n - z_1 & \dots & z_n - z_1 \\ z_n - z_2 & z_n - z_2 & \dots & z_n - z_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ z_n - z_{n-1} & z_n - z_{n-1} & \dots & z_n - z_{n-1} \end{bmatrix}.$$

Note that $\operatorname{rank}(E) \leqslant 1$ and

$$E^*E = \begin{bmatrix} \overline{z_n - z_1} & \overline{z_n - z_2} & \dots & \overline{z_n - z_{n-1}} \\ \overline{z_n - z_1} & \overline{z_n - z_2} & \dots & \overline{z_n - z_{n-1}} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \overline{z_n - z_1} & \overline{z_n - z_2} & \dots & \overline{z_n - z_{n-1}} \end{bmatrix} \begin{bmatrix} z_n - z_1 & z_n - z_1 & \dots & z_n - z_1 \\ z_n - z_2 & z_n - z_2 & \dots & z_n - z_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ z_n - z_{n-1} & z_n - z_{n-1} & \dots & z_n - z_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{n-1} |z_n - z_i|^2 & \sum_{i=1}^{n-1} |z_n - z_i|^2 & \dots & \sum_{i=1}^{n-1} |z_n - z_i|^2 \\ \sum_{i=1}^{n-1} |z_n - z_i|^2 & \sum_{i=1}^{n-1} |z_n - z_i|^2 & \dots & \sum_{i=1}^{n-1} |z_n - z_i|^2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \sum_{i=1}^{n-1} |z_n - z_i|^2 & \sum_{i=1}^{n-1} |z_n - z_i|^2 & \dots & \sum_{i=1}^{n-1} |z_n - z_i|^2 \end{bmatrix} = \left(\sum_{i=1}^{n-1} |z_n - z_i|^2 \right) J.$$

Hence, $\sigma(E^*E) = \left\{ (n-1) \sum_{i=1}^{n-1} |z_n - z_i|^2, 0 \right\}$, where 0 is of multiplicity $n-2$.

So, $s_1\left(\frac{1}{n}E\right) = \sqrt{\frac{n-1}{n^2} \sum_{i=1}^{n-1} |z_n - z_i|^2}$, $s_j(\frac{1}{n}E) = 0$ for $j = 2, 3, \dots, n-2$.

Now,

$$\left(\sum_{i=1}^{n-1} |w_i|^p \right)^{\frac{1}{p}} \leqslant \left\| C(f') \right\|_p = \left\| D + \frac{1}{n}E \right\|_p \leqslant \|D\|_p + \left\| \frac{1}{n}E \right\|_p$$
$$= \left(\sum_{i=1}^{n-1} |z_j|^p \right)^{\frac{1}{p}} + \sqrt{\frac{n-1}{n^2} \sum_{i=1}^{n-1} |z_n - z_i|^2}.$$

This proves (23) and competes the proof of theorem. □

Related to the inequalities (21)–(23), it follows from the verified de Bruijn-Springer conjecture that, for $p \geqslant 1$,

$$\left(\sum_{i=1}^{n-1} |w_i|^p \right)^{\frac{1}{p}} \leqslant \left(\frac{n-1}{n} \sum_{i=1}^{n} |z_i|^p \right)^{\frac{1}{p}}. \tag{24}$$

See, e.g., [8,12]. We remark here that for $p = 1$, the inequality (24) is better than the inequalities (21)-(23). However, for $p > 1$, non of these inequalities is uniformly better than the others.

For p = 2, another inequality can be obtained by using the improvement of Schur's inequality given in Lemma 7.

Theorem 4. *Let $z_1, z_2, \ldots, z_n$ be the zeros of a polynomial f of degree $n \geqslant 3$ and $w_1, w_2, \ldots, w_{n-1}$ be the critical points of f. Then*

$$\sum_{i=1}^{n-1} |w_i|^2 \leqslant \frac{1}{n^2} \sum_{i=1}^{n-1} \left(\sqrt{(n-2)\,|z_n - z_i|^2 + |(n-1)z_i + z_n|^2} \times \sqrt{\sum_{\substack{j=1 \\ j \neq i}}^{n-1} |z_n - z_j|^2 + |(n-1)z_i + z_n|^2} \right) \tag{25}$$

Proof. It follows from (1) that

$$C(f') = D\left(I - \frac{1}{n}J\right) + \frac{z_n}{n}J = D - \frac{1}{n}DJ + \frac{z_n}{n}J,$$

and so

$$C(f') = D + \frac{1}{n}(z_nJ - DJ) = \begin{bmatrix} \frac{(n-1)z_1+z_n}{n} & \frac{z_n-z_1}{n} & \cdots & \frac{z_n-z_1}{n} \\ \frac{z_n-z_2}{n} & \frac{(n-1)z_2+z_n}{n} & \cdots & \frac{z_n-z_2}{n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{z_n-z_{n-1}}{n} & \frac{z_n-z_{n-1}}{n} & \cdots & \frac{(n-1)z_{n-1}+z_n}{n} \end{bmatrix}.$$

Then by the improvement of Schur's inequality given in Lemma 7, we have

$$\sum_{i=1}^{n-1} |w_i|^2 \leqslant \frac{1}{n^2} \sum_{i=1}^{n-1} \left(\sqrt{(n-2)\,|z_n - z_i|^2 + |(n-1)z_i + z_n|^2} \times \sqrt{\sum_{\substack{j=1 \\ j \neq i}}^{n-1} |z_n - z_j|^2 + |(n-1)z_i + z_n|^2} \right).$$

This completes the proof of the theorem. □

We conclude this section with the following inequality, which is based on Lemmas 5 and 6.

Theorem 5. *Let $z_1, z_2, \ldots, z_n$ be the zeros of a polynomial f of degree $n \geqslant 3$ and $w_1, w_2, \ldots, w_{n-1}$ be the critical points of f. Then for $0 < p \leqslant 2$, we have*

$$\sum_{i=1}^{n-1} |w_i|^p \leqslant \frac{1}{n^p} \left[\sum_{i=1}^{n-1} \left(|(n-1)z_i + z_n|^p + (n-2)\,|z_n - z_i|^p \right) \right]. \tag{26}$$

Proof. It follows from (1) that

$$C(f') = D\left(I - \frac{1}{n}J\right) + \frac{z_n}{n}J = D - \frac{1}{n}DJ + \frac{z_n}{n}J,$$

and so

$$C(f') = D + \frac{1}{n}(z_nJ - DJ) = \begin{bmatrix} \frac{(n-1)z_1+z_n}{n} & \frac{z_n-z_1}{n} & \cdots & \frac{z_n-z_1}{n} \\ \frac{z_n-z_2}{n} & \frac{(n-1)z_2+z_n}{n} & \cdots & \frac{z_n-z_2}{n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{z_n-z_{n-1}}{n} & \frac{z_n-z_{n-1}}{n} & \cdots & \frac{(n-1)z_{n-1}+z_n}{n} \end{bmatrix}.$$

Now, using Lemma 5 and 6, we have

$$\sum_{i=1}^{n-1} |w_i|^p \leqslant \sum_{i,j=1}^{n-1} \left|C(f')_{i,j}\right|^p = \frac{1}{n^p}\left[\sum_{i=1}^{n-1}\left(|(n-1)z_i + z_n|^p + (n-2)\,|z_n - z_i|^p\right)\right].$$

This completes the proof of the theorem. □

4. Acknowledgment

The first author would like to thank the German Academic Exchange Service (DAAD) for supporting his M.Sc. Study at the University of Jordan.

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