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# Differential Transform Method for Differential Equations

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# Differential Transform Method for Differential Equations

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M.Sc. Thesis

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The undersigned hereby certify that they have read and recommend to the Deanship of Graduate Studies and Scientific Research at Palestine Polytechnic University the acceptance of a thesis entitled "Differential Transform Method for Differential Equations" submitted by Alaa Khatib in partial fulfilment of the requirements for the degree of Master in Mathematics.

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# Dedications

*This work is dedicated to my father Bassam Khatib, my mother Gada Khatib, my husband Ibrahim Abu-Eisha and my daughter Leen.*

*Alaa*

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# Abstract

The differential transform method was firstly introduced by Zhou before thirty years ago. This method is a semi-analytical numerical method for solving differential equations. Indeed, the differential transform method is based on Taylor series expansion, in a different manner, in which the differential equation is converted into a recurrence relation to get a series solution in terms of polynomials.

This thesis is mainly concerned with the differential transform method for both ordinary and partial differential equations. Firstly, we use the one dimensional differential transform method to solve initial value problems as well as boundary value problems for ordinary differential equations. In addition, we present recent modifications of differential transform method that improve its algorithm.

Secondly, we solve initial and boundary value problems for partial differential equations by using two dimensional differential transform method, reduced differential transform method and Laplace differential transform method.

# Contents

# Chapter 1

# Introduction

## 1.1 Overview

In real world, many physical and natural phenomena are formulated as differential equations. Most of these differential equations are nonlinear. So there are difficulties in finding the exact or analytical solutions caused by the nonlinear part [27, 38]. Many methods have been proposed to solve or approximate nonlinear differential equations [4, 27]. For example, the *Adomian Decomposition Method* (ADM), *Variational Iteration Method* (VIM), *Homotopy Analysis Method* (HAM), and *Homotopy Perturbation Method* (HPM), see [2, 3, 24, 25, 35]. But these methods need calculations with some restrictions, also in some cases to get a good convergence more terms are needed.

There is a need for a method that handel nonlinear terms easily without any restrictions and with less size of computations. Indeed, the so called *Differential Transform Method* (DTM) which gives a series solutions can overcome some of the above difficulties.

The DTM is very effective numerical and analytical method for solving different types of differential equations as well as integral equations. This method converts the differential equations into a recurrence relations, then by Taylor series expansion, with a different approach, we obtain convergent series solutions.

The concept of DTM was first introduced by Zhou in 1986 to solve linear and nonlinear initial value problems in electrical circuit analysis [49]. Later, the DTM has been applied to solve different problems. Some of these problems are enumerated in the following discussion:

1. **Eigenvalue problems:** Chen and Ho (1996) in [10] used the DTM to solve eigenvalue problems. They took the differential transformation of Sturm-Lioville problem and made some calculations to get eigenvalues and eigenfunctions. Hassan (2002) solved the eigenvalue problems by DTM in [17]. In 2007, he applied the DTM on the one-dimensional planar Bratu problem which is a nonlinear eigenvalue problem, see [18].

2. **Initial value problems:** Jang and Chen (2000) in [29] used the DTM to approximate the solutions of linear and nonlinear initial value problems. Ïbis in [27] used DTM to get approximate analytical solution of nonlinear Emden-Fowler equation which is singular initial value problem. Ïbis showed that this method is reliable to solve this kind of equations. Also, Hassan (2004) in [19] solved higher order initial value problems, and in (2008) (see[22]) he made a comparison between DTM and ADM for solving initial value problems of partial differential equations and gave some examples to emphasize that the solution obtained from DTM coincides with the approximate solution of ADM and analytical solution. Recently, several authors have considered the initial value problems by the DTM, see [13, 16, 33, 38, 44].

3. **Boundary value problems:** Hassan (2009) in [21] solved linear and nonlinear boundary value problems by choosing boundary value problems with different orders to show that the DTM has high accuracy solution comparing with the exact one. See [4, 23, 26, 28].

4. **Partial differential equations:** Chen and Ho (1999) in [9] used the so called *two-dimensional differential transform method* (TDDTM) to get a series solution of partial differential equations. Ayaz (2003) in [6] applied the TDDTM to solve initial value problems for partial differential equations and compare the result with decomposition method. Also, the *reduced differential transform method* (RDTM) was used by Haghbin and Hesam (2012) in [16] to solve Sawada-Katera equations as an effective and convenient alternative method. Alquran et al. (2010) in [5] use the Laplace DTM to solve non-homogeneous linear partial differential equations. Also, in (2015) Kumari use the Laplace DTM to solve wave equations and wave-like equations, see [30].

5. **Integral and integro-differential equations:** Volterra integral equations were solved by Tari et al. (2009) in [45]. Kajani and Shehni (2011) in [46] solved nonlinear Volterra integro-differential equations. Two-dimensional nonlinear Volterra integro-differential equations were considered by Darania et al. (2011) in [12].

6. **System of differential equations:** Ayaz (2004) in [7] and Hassan (2008) in [22] used the DTM to solve system of differential equations. Also see [36, 41].

7. **Delay differential equations:** Delay differential equations were solved by DTM by Karakoç and Bereketoglu (2009) in [31].

This thesis is concerned with the DTM for several types of differential equations. Firstly, we present the definition of DTM with some basic theorems. Then we apply this method for ordinary differential equations. Indeed, we solve linear and nonlinear initial value problems as well as boundary value problems. Moreover, we use the DTM to solve second order boundary value problems with the linear shooting method to get the same or more close solution to the exact one.

Next, we present some modifications of the DTM to simplify the solution, or to increase the efficiency of the method. These modifications include the modification of the DTM by using the Adomain polynomials, in which we calculate the Adomain polynomials instead of differential transform of nonlinear functions. Also, the modification of the DTM by using Laplace transform and Padé approximation to handle the periodic behavior of the solution.

Then we introduce the main procedures of the DTM for solving partial differential equations. Namely, we present the TDDTM, the RDTM and the Laplace DTM.

## 1.2 Outline

This thesis contains four chapters. In these chapters we present the basics of DTM and we solve different differential equations by this method.

Chapter 1 was the introduction chapter. It contains the history of DTM and a short literature review.

Then Chapter 2 considers the basic definition and main theorems of differential transform. In addition we solve several examples of initial and boundary value problems using DTM. At the end of this chapter a result for solving second order boundary value problems by differential transform with linear shooting method is given.

In Chapter 3 we give some modifications of the DTM to solve some problems easily. For instance, we use the Adomain polynomials instead of the differential transform of the nonlinear functions. Also, we modify the DTM by using Laplace transform and Padé approximation to handel the periodic behavior of the solution.

Chapter 4 is devoted to present the TDDTM for solving partial differential equations. Also we discuss the RDTM and Laplace DTM.

# Chapter 2

# DTM for ordinary differential equations

In this chapter, the definition and theorems of the DTM are introduced. Then we present a reliable and efficient procedure to calculate the differential transform for some nonlinear functions. After that, we apply the differential transform method to obtain approximate solutions of linear and nonlinear initial value problems as well as boundary value problems. In addition, results concerning differential transform with linear shooting method for solving second order boundary value problems are given. Furthermore, some physical models equations are also presented at the end of this chapter.

As mentioned before, the concept of differential transform was first introduced by Zhou in 1986 [49]. It was applied to solve linear and nonlinear initial value problems in electric circuit analysis. Later, several researches have been conducted in applying differential transform method to different types of equations. These researches confirm the fact that this method is reliable, efficient as well as having a wider applicability, see [4, 11, 23]. Results in this chapter can be found in [4, 11, 23, 26, 27, 38, 42].

## 2.1 Basics of DTM

In this section, we introduce the definition of one-dimensional differential transform or simply the DTM. Then some basic theorems are given and proved by this definition.

**Definition 2.1.** The *differential transform* of a function $y(x)$ is defined as follows

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=0}, \tag{2.1}$$

where $y(x)$ is the original function and $Y(k)$ is the transformed function. Differential inverse transform of $Y(k)$ is defined as

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k \approx y_N(x) = \sum_{k=0}^{N} Y(k)x^k. \tag{2.2}$$

By substituting equation(2.1) in (2.2) we get

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\frac{d^k y(x)}{dx^k}\bigg|_{x=0}, \tag{2.3}$$

which implies that the concept of differential transform is derived from Taylor series expansion.

In the previous definition we consider the case when $x = 0$, but it is true for any fixed real number $x = x_0$.

Through this thesis, we use small letter to denote the original function and capital letter to denote the transformed function.

The next theorems are the main theorems that can be derived from equations (2.1) and (2.2).

**Theorem 2.1.** *If $f(x) = \alpha g(x) + \beta h(x)$, then $F(k) = \alpha G(k) + \beta H(k)$, where $\alpha$ and $\beta$ are constants.*

*Proof.* Let $f(x)$ be the original function, then the differential transform of $f(x)$ is given by

$$\begin{aligned}
F(k) &= \frac{1}{k!}\frac{d^k f(x)}{dx^k}\bigg|_{x=0} \\
&= \frac{1}{k!}\frac{d^k(\alpha g(x)) + (\beta h(x))}{dx^k}\bigg|_{x=0} \\
&= \frac{1}{k!}\left[\frac{d^k(\alpha g(x))}{dx^k} + \frac{d^k(\beta h(x))}{dx^k}\right]_{x=0} \\
&= \frac{1}{k!}\alpha\frac{d^k g(x)}{dx^k}\bigg|_{x=0} + \frac{1}{k!}\beta\frac{d^k h(x)}{dx^k}\bigg|_{x=0} \\
&= \alpha G(k) + \beta H(k).
\end{aligned}$$

□

**Theorem 2.2.** *If $f(x) = g(x)h(x)$, then $F(k) = \sum_{r=0}^{k} G(r)H(k-r)$.*

*Proof.* Let $f(x) = g(x)h(x)$ be the original function, then from Leibnitz formula for the *nth* derivative of a product we have

$$\frac{d^n(g(x)h(x))}{dx^n} = \sum_{r=0}^{n}\binom{n}{r}\frac{d^r g(x)}{dx^r}\frac{d^{n-r}h(x)}{dx^{n-r}}. \tag{2.4}$$

Now, by using (2.4) the differential transform of $f(x)$ is given by

$$\begin{aligned}
F(k) &= \frac{1}{k!}\frac{d^k f(x)}{dx^k}\bigg|_{x=0} \\
&= \frac{1}{k!}\left[\sum_{r=0}^{k}\binom{k}{r}\frac{d^r g(x)}{dx^r}\frac{d^{k-r}h(x)}{dx^{k-r}}\right]_{x=0} \\
&= \frac{1}{k!}\left[\sum_{r=0}^{k}\frac{k!}{r!(k-r)!}\frac{d^r g(x)}{dx^r}\frac{d^{k-r}h(x)}{dx^{k-r}}\right]_{x=0} \\
&= \sum_{r=0}^{k} G(r)H(k-r).
\end{aligned}$$

□

The next theorem is a generalization of the previous one.

**Theorem 2.3.** *If $f(x) = g_1(x)g_2(x)...g_n(x)$, then*

$$F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} ... \sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2 - k_1)...G_n(k - k_{n-1}).$$

*Proof.* By using mathematical induction, the statement is true for $n = 2$ by Theorem 2.2. Assume that the statement is true for $n = m$ . Now, for $n = m + 1$, we have

$$f(x) = (g_1(x)g_2(x)...g_m(x))g_{m+1}(x).$$

Let $g(x) = g_1(x)g_2(x)...g_m(x)$, by Theorem 2.2 we get

$$F(k) = \sum_{k_m=0}^{k} G(k_m)G_{m+1}(k - k_m),$$

where $G(k_m)$ is the transformed function of $g(x)$. Then

$$F(k) = \sum_{k_m=0}^{k} \left( \sum_{k_{m-1}=0}^{k_m} \sum_{k_{m-2}=0}^{k_{m-1}} ... \sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2 - k_1)...G_m(k_m - k_{m-1}) \right) G_{m+1}(k-k_m).$$

The statement is true for $n = m + 1$, so it is true for $n \geq 2$. $\square$

**Theorem 2.4.** *If $f(x) = x^n$, then*

$$F(k) = \delta(k - n) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

*Proof.* Let $f(x)$ be the original function, then the differential transform of $f(x)$ is given by

$$\begin{aligned} F(k) &= \frac{1}{k!} \frac{d^k f(x)}{dx^k}\bigg|_{x=0} \\ &= \frac{1}{k!} \frac{d^k x^n}{dx^k}\bigg|_{x=0}. \end{aligned}$$

From differentiation rule we have

$$\frac{d^k x^n}{dx^k} = n(n-1)(n-2)...(n-k+1)x^{n-k}.$$

So,

$$F(k) = \frac{n(n-1)(n-2)...(n-k+1)}{k!}x^{n-k}.$$

If $n = k$,

$$F(k) = \frac{k(k-1)(k-2)...1}{k!},$$

so,

$$F(k) = 1.$$

If $n \neq k$, and $x = 0$ we get

$$F(k) = 0.$$

Then

$$F(k) = \delta(k-n) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n. \end{cases}$$

□

**Theorem 2.5.** *If* $f(x) = \dfrac{d^n g(x)}{dx^n}$, *then* $F(k) = (k+1)(k+2)...(k+n)G(k+n)$.

*Proof.* Let $f(x)$ be the original function, then the differential transform of $f(x)$ is given by

$$\begin{aligned} F(k) &= \frac{1}{k!}\frac{d^k f(x)}{dx^k}\bigg|_{x=0} \\ &= \frac{1}{k!}\frac{d^k \frac{d^n g(x)}{dx^n}}{dx^k}\bigg|_{x=0} \\ &= \frac{1}{k!}\frac{d^{k+n} g(x)}{dx^{k+n}}\bigg|_{x=0} \\ &= \frac{(k+n)!}{k!}\left[\frac{1}{(k+n)!}\frac{d^{k+n} g(x)}{dx^{k+n}}\right]_{x=0} \\ &= (k+1)(k+2)...(k+n)G(k+n). \end{aligned}$$

□

The following theorems give the differential transform of some functions.

**Theorem 2.6.** *If* $f(x) = e^x$*, then* $F(k) = \dfrac{1}{k!}$.

*Proof.* Let $f(x)$ be the original function, then the differential transform of $f(x)$ is given by

$$\begin{aligned} F(k) &= \frac{1}{k!}\frac{d^k f(x)}{dx^k}\bigg|_{x=0} \\ &= \frac{1}{k!}\frac{d^k e^x}{dx^k}\bigg|_{x=0} \\ &= \frac{1}{k!}e^x\bigg|_{x=0} \\ &= \frac{1}{k!}. \end{aligned}$$

$\square$

**Theorem 2.7.** *If* $f(x) = e^{\lambda x}$*, then* $F(k) = \dfrac{\lambda^k}{k!}$*, where* $\lambda$ *is constant.*

*Proof.* Let $f(x)$ be the original function, then the differential transform of $f(x)$ is given by

$$\begin{aligned} F(k) &= \frac{1}{k!}\frac{d^k f(x)}{dx^k}\bigg|_{x=0} \\ &= \frac{1}{k!}\frac{d^k e^{\lambda x}}{dx^k}\bigg|_{x=0} \\ &= \frac{1}{k!}\lambda^k e^{\lambda x}\bigg|_{x=0} \\ &= \frac{\lambda^k}{k!}. \end{aligned}$$

$\square$

**Theorem 2.8.** *If* $f(x) = \sin(wx + \alpha)$*, then* $F(k) = \dfrac{w^k}{k!}\sin\left(\dfrac{k\pi}{2} + \alpha\right)$*, where* $w$ *and* $\alpha$ *are constants.*

*Proof.* Take the differential transform of $f(x) = \sin(wx + \alpha)$

$$\begin{aligned} F(k) &= \frac{1}{k!} \frac{d^k \sin(wx+\alpha)}{dx^k}\bigg|_{x=0} \\ &= \frac{w^k}{k!} \sin\left(\frac{k\pi}{2} + \alpha\right). \end{aligned}$$

□

**Theorem 2.9.** *If $f(x) = \cos(wx + \alpha)$, then $F(k) = \frac{w^k}{k!} \cos\left(\frac{k\pi}{2} + \alpha\right)$, where $w$ and $\alpha$ are constants.*

*Proof.* Take the differential transform of $f(x) = \cos(wx + \alpha)$

$$\begin{aligned} F(k) &= \frac{1}{k!} \frac{d^k \cos(wx+\alpha)}{dx^k}\bigg|_{x=0} \\ &= \frac{w^k}{k!} \cos\left(\frac{k\pi}{2} + \alpha\right). \end{aligned}$$

□

**Theorem 2.10.** *If $f(x) = (1 + x)^b$, then $F(k) = \frac{b(b-1)...(b-k+1)}{k!}$, where $b$ is constant.*

*Proof.* Take the differential transform of $f(x) = (1 + x)^b$

$$\begin{aligned} F(k) &= \frac{1}{k!} \frac{d^k (1+x)^b}{dx^k}\bigg|_{x=0} \\ &= \frac{b(b-1)...(b-k+1)(1+x)^{b-k}}{k!}\bigg|_{x=0} \\ &= \frac{b(b-1)...(b-k+1)}{k!}. \end{aligned}$$

□

**Theorem 2.11.** *If* $f(x) = \int_0^x g(t)\ dt$*, then* $F(k) = \dfrac{G(k-1)}{k}$*, for* $k \geq 1$*.*

*Proof.* Take the differential transform of $f(x)$

$$\begin{aligned} F(k) &= \frac{1}{k!}\frac{d^k \int_0^x g(t)dt}{dx^k}\bigg|_{x=0} \\ &= \frac{1}{k!}\frac{d^{k-1}g(x)}{dx^{k-1}}\bigg|_{x=0} \\ &= \frac{1}{k(k-1)!}\frac{d^{k-1}g(x)}{dx^{k-1}}\bigg|_{x=0} \\ &= \frac{G(k-1)}{k}. \end{aligned}$$

□

## 2.2 The differential transform for some nonlinear functions

In this section, we calculate the differential transform for some nonlinear functions. These calculations based on the differentiation and Theorems 2.1 - 2.5. Here, we refer to [27, 38].

**Theorem 2.12.** *If* $f(y) = y^m$*, then*

$$F(k) = \begin{cases} Y^m(0) & \text{if } k = 0, \\ \\ \dfrac{1}{Y(0)} \displaystyle\sum_{r=1}^{k} \frac{(m+1)r-k}{k} Y(r)F(k-r) & \text{if } k \geq 1. \end{cases}$$

*Proof.* If $k = 0$, by using Definition 2.1 we have,

$$F(0) = y^m(0) = Y^m(0). \tag{2.5}$$

If $k \geq 1$, differentiate $f(y) = y^m$ with respect to $x$, we get

$$\frac{df(y)}{dx} = my^{m-1}\frac{dy(x)}{dx}. \tag{2.6}$$

Multiply both sides of Equation (2.6) by $y(x)$, we get

$$y(x)\frac{df(y)}{dx} = mf(y(x))\frac{dy(x)}{dx}. \tag{2.7}$$

Apply the DTM on Equation (2.7)

$$\sum_{r=0}^{k} Y(r)(k-r+1)F(k-r+1) = m\sum_{r=0}^{k}(r+1)Y(r+1)F(k-r),$$

then

$$(k+1)Y(0)F(k+1) = m\sum_{r=0}^{k}(r+1)Y(r+1)F(k-r) - \sum_{r=1}^{k} Y(r)(k-r+1)F(k-r+1),$$

or

$$\begin{aligned}(k+1)Y(0)F(k+1) &= m\sum_{r=1}^{k+1} rY(r)F(k-r+1) - \sum_{r=1}^{k} Y(r)(k-r+1)F(k-r+1) \\ &= \sum_{r=1}^{k+1}((m+1)r-k-1)Y(r)F(k-r+1),\end{aligned}$$

putting $k$ instead of $k+1$ gives

$$kY(0)F(k) = \sum_{r=1}^{k}((m+1)r-k)Y(r)F(k-r),$$

from this, we have

$$F(k) = \frac{1}{Y(0)}\sum_{r=1}^{k}\left[\left(\frac{(m+1)r-k}{k}\right)Y(r)F(k-r)\right]. \tag{2.8}$$

From (2.5) and (2.8) we get,

$$F(k) = \begin{cases} Y^m(0) & \text{if } k = 0, \\ \\ \dfrac{1}{Y(0)}\displaystyle\sum_{r=1}^{k}\frac{(m+1)r-k}{k}Y(r)F(k-r) & \text{if } k \geq 1. \end{cases}$$

□

**Theorem 2.13.** *If $f(y) = e^{ay}$, then*

$$F(k) = \begin{cases} e^{aY(0)} & \text{if } k = 0, \\ a\displaystyle\sum_{r=0}^{k-1} \frac{r+1}{k} Y(r+1)F(k-r-1) & \text{if } k \geq 1. \end{cases}$$

*Proof.* If $k = 0$, by using Definition 2.1 we have,

$$F(0) = f(y)\Big|_{x=0} = e^{ay(0)} = e^{aY(0)}. \tag{2.9}$$

Now, differentiate $f(y)$ with respect to $x$

$$\frac{df(y)}{dx} = ae^{ay}\frac{dy(x)}{dx} = af(y)\frac{dy(x)}{dx}. \tag{2.10}$$

By taking differential transform to both sides of Equation (2.10), we get

$$(k+1)F(k+1) = a\sum_{r=0}^{k}(r+1)Y(r+1)F(k-r),$$

putting $k$ instead of $k+1$ gives

$$F(k) = a\sum_{r=0}^{k-1}\frac{r+1}{k}Y(r+1)F(k-r-1), \ k \geq 1. \tag{2.11}$$

From (2.9) and (2.11) we get

$$F(k) = \begin{cases} e^{aY(0)} & \text{if } k = 0, \\ a\displaystyle\sum_{r=0}^{k-1} \frac{r+1}{k} Y(r+1)F(k-r-1) & \text{if } k \geq 1. \end{cases}$$

□

**Theorem 2.14.** *If* $f(y) = \ln(a + by), \quad a + by > 0$, *then*

$$F(k) = \begin{cases} \ln(a + bY(0)) & \text{if } k = 0, \\ \dfrac{b}{a + bY(0)} Y(1) & \text{if } k = 1, \\ \dfrac{b}{a + bY(0)} \left[ Y(k) - \sum_{r=0}^{k-2} \dfrac{r+1}{k} F(r+1) Y(k-r-1) \right] & \text{if } k \geq 2. \end{cases}$$

*Proof.* If $k = 0$, by using Definition 2.1 we have

$$F(0) = f(y)\Big|_{x=0} = \ln(a + by(0)) = \ln(a + bY(0)). \tag{2.12}$$

Now, by differentiating $f(y)$ with respect to $x$ we get

$$\frac{df(y)}{dx} = \frac{b}{a + by} \frac{dy(x)}{dx},$$

or

$$a \frac{df(y)}{dx} = b \left[ \frac{dy(x)}{dx} - y \frac{df(y)}{dx} \right]. \tag{2.13}$$

Taking the differential transform of Equation (2.13) we obtain

$$aF(k+1) = b \left[ Y(k+1) - \sum_{r=0}^{k} \frac{r+1}{k+1} F(r+1) Y(k-r) \right],$$

putting $k$ instead of $k + 1$ gives

$$aF(k) = b \left[ Y(k) - \sum_{r=0}^{k-1} \frac{r+1}{k} F(r+1) Y(k-r-1) \right], k \geq 1. \tag{2.14}$$

Substituting $k = 1$ in Equation (2.14), we get

$$\begin{aligned} F(1) &= \frac{b}{a}[Y(1) - F(1)Y(0)] \qquad (2.15) \\ &= \frac{b}{a + bY(0)} Y(1). \end{aligned}$$

We can rewrite Equation (2.14) as

$$F(k) = \frac{b}{a+bY(0)}\left[Y(k) - \sum_{r=0}^{k-2}\frac{r+1}{k}F(r+1)Y(k-r-1)\right], \quad k \geq 2. \tag{2.16}$$

From (2.12), (2.15) and (2.16) we have

$$F(k) = \begin{cases} \ln(a+bY(0)) & \text{if } k=0, \\ \\ \dfrac{b}{a+bY(0)}Y(1) & \text{if } k=1, \\ \\ \dfrac{b}{a+bY(0)}\left[Y(k) - \displaystyle\sum_{r=0}^{k-2}\frac{r+1}{k}F(r+1)Y(k-r-1)\right] & \text{if } k \geq 2. \end{cases}$$

□

**Theorem 2.15.** *If $f(y) = \sin(ay)$ and $g(y) = \cos(ay)$, then*

$$F(k) = \begin{cases} \sin(aY(0)) & \text{if } k=0, \\ \\ a\displaystyle\sum_{r=0}^{k-1}\frac{k-r}{k}G(r)Y(k-r) & \text{if } k \geq 1 \end{cases}$$

*and*

$$G(k) = \begin{cases} \cos(aY(0)) & \text{if } k=0, \\ \\ -a\displaystyle\sum_{r=0}^{k-1}\frac{k-r}{k}F(r)Y(k-r) & \text{if } k \geq 1. \end{cases}$$

*Proof.* If $k=0$, by using Definition 2.1 we have

$$F(0) = f(y)\Big|_{x=0} = \sin(ay(0)) = \sin(aY(0)). \tag{2.17}$$

Now, for $k \geq 1$ differentiate $f(y)$ with respect to $x$ to get

$$\begin{aligned} \frac{df(y)}{dx} &= a\cos(ay)\frac{dy(x)}{dx} \\ &= ag(y)\frac{dy(x)}{dx}. \end{aligned} \tag{2.18}$$

Applying the differential transform to both sides of Equation (2.18) gives

$$(k+1)F(k+1) = a\sum_{r=0}^{k}(k-r+1)G(r)Y(k-r+1),$$

replacing $k+1$ by $k$ yields:

$$kF(k) = a\sum_{r=0}^{k-1}(k-r)G(r)Y(k-r),$$

or

$$F(k) = a\sum_{r=0}^{k-1}\frac{k-r}{k}G(r)Y(k-r). \tag{2.19}$$

From (2.17) and (2.19) we obtain,

$$F(k) = \begin{cases} \sin(aY(0)) & \text{if } k=0, \\ \\ a\displaystyle\sum_{r=0}^{k-1}\frac{k-r}{k}G(r)Y(k-r) & \text{if } k \geq 1. \end{cases}$$

The proof is similar for $g(y) = \cos(ay)$. □

**Theorem 2.16.** *If $f(y) = \sinh(ay)$ and $g(y) = \cosh(ay)$, then*

$$F(k) = \begin{cases} \sinh(aY(0)) & \text{if } k=0, \\ \\ a\displaystyle\sum_{r=0}^{k-1}\frac{k-r}{k}G(r)Y(k-r) & \text{if } k \geq 1 \end{cases}$$

*and*

$$G(k) = \begin{cases} \cosh(aY(0)) & \text{if } k = 0, \\ \\ a\displaystyle\sum_{r=0}^{k-1} F(r)Y(k-r) & \text{if } k \geq 1. \end{cases}$$

*Proof.* If $k = 0$, by using Definition 2.1 we have

$$F(0) = f(y)\Big|_{x=0} = \sinh(ay(0)) = \sinh(aY(0)). \tag{2.20}$$

Now, for $k \geq 1$ differentiating $f(y)$ with respect to $x$, we obtain

$$\begin{aligned} \frac{df(y)}{dx} &= a\cosh(ay)\frac{dy(x)}{dx} \\ &= ag(y)\frac{dy(x)}{dx}. \end{aligned} \tag{2.21}$$

Applying the differential transform to both sides of Equation (2.21) gives

$$(k+1)F(k+1) = a\sum_{r=0}^{k}(k-r+1)G(r)Y(k-r+1),$$

replacing $k+1$ by $k$ yields

$$kF(k) = a\sum_{r=0}^{k-1}(k-r)G(r)Y(k-r)$$

or

$$F(k) = a\sum_{r=0}^{k-1}\frac{k-r}{k}G(r)Y(k-r). \tag{2.22}$$

From (2.20) and (2.22) we obtain,

$$F(k) = \begin{cases} \sinh(aY(0)) & \text{if } k = 0, \\ \\ a\displaystyle\sum_{r=0}^{k-1} G(r)Y(k-r) & \text{if } k \geq 1. \end{cases}$$

For $g(y) = \cosh(ay)$, the proof is similar. $\square$

## 2.3 Linear initial value problems

In this section, we give solutions for linear initial value problems by using differential transform method. The technique is explained through examples.

**Example 2.1.** Consider the first order linear initial value problem

$$1 - \frac{dy}{dx} + y = 0, \tag{2.23}$$

with an initial condition

$$y(0) = a. \tag{2.24}$$

**Solution.** By applying the DTM on Equation (2.23) in view of Section 2.2 we have

$$\delta(k) - (k+1)Y(k+1) + Y(k) = 0.$$

This leads to the following recurrence relation

$$Y(k+1) = \frac{1}{(k+1)}[Y(k) + \delta(k)], \tag{2.25}$$

and from initial condition (2.24) we get

$$Y(0) = a. \tag{2.26}$$

Using recurrence relation (2.25) and (2.26) for $k = 0, 1, ..., 4$ we get the following

$$Y(0) = a, \quad Y(1) = a + 1, \quad Y(2) = \frac{a+1}{2}, \quad Y(3) = \frac{a+1}{6}, \quad Y(4) = \frac{a+1}{24}.$$

We can write the solution as

$$\begin{aligned} y(x) &\approx \sum_{k=0}^{4} Y(k)x^k \\ &\approx a + (a+1)x + \frac{a+1}{2}x^2 + \frac{a+1}{6}x^3 + \frac{a+1}{24}x^4. \end{aligned}$$

Where the exact solution of this example is

$$y(x) = -1 + (a+1)e^x.$$

Note that the first terms of the Taylor series expansion of the exact solution equal first terms of the solution obtained by the DTM. To obtain extra terms we should consider extra values for $k \geq 4$.

**Example 2.2.** Consider the second order linear initial value problem

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 24 \tag{2.27}$$

with initial conditions

$$y(0) = 10 \text{ and } y'(0) = 0. \tag{2.28}$$

**Solution.** By applying the DTM on Equation (2.27)

$$(k+1)(k+2)Y(k+2) + 3(k+1)Y(k+1) + 2Y(k) = 24\delta(k). \tag{2.29}$$

We get the following recurrence relation

$$Y(k+2) = \frac{1}{(k+1)(k+2)}(-3(k+1)Y(k+1) - 2Y(k) + 24\delta(k)). \tag{2.30}$$

The initial conditions become

$$Y(0) = 10 \text{ and } Y(1) = 0. \tag{2.31}$$

Using recurrence relation (2.30) and conditions (2.31) for $k = 0, 1, 2, ...$ we get the following

$$Y(0) = 10, Y(1) = 0, Y(2) = 2, Y(3) = -2, Y(4) = \frac{7}{6}, ...$$

We can write the solution as

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k \tag{2.32}$$

$$= 10 + 2x^2 - 2x^3 + \frac{7}{6}x^4 + ... \tag{2.33}$$

Where the exact solution of this example is

$$y(x) = 12 - 4e^{-x} + 2e^{-2x} \tag{2.34}$$

Note that the first terms of the Taylor series expansion of the exact solution equal the first terms of the solution obtained by the DTM.

For more examples see [33, 42].

## 2.4 Nonlinear initial value problems

In this section, we apply the DTM to solve nonlinear initial value problem. By using theorems in Section 2.2 we decompose the nonlinear terms. Therefore the solution can be obtained by iteration procedure. Next, we consider several examples with different forms of nonlinearity.

**Example 2.3.** Consider the nonlinear initial value problem

$$y''(x) + 2(y'(x))^2 + 8y(x) = 0, \quad 0 \leq x < \infty \tag{2.35}$$

with initial conditions

$$y(0) = 0 \text{ and } y'(0) = 1. \tag{2.36}$$

**Solution.** Apply the DTM on Equation (2.35), we have

$$(k+1)(k+2)Y(k+2)+2\sum_{r=0}^{k}(r+1)(k-r+1)Y(r+1)Y(k-r+1)+8Y(k)=0.$$

Thus, we get the following recurrence relation

$$Y(k+2)=\frac{-1}{(k+1)(k+2)}\left[2\sum_{r=0}^{k}(r+1)(k-r+1)Y(r+1)Y(k-r+1)+8Y(k)\right], \tag{2.37}$$

from the initial condition (2.36) we get

$$Y(0)=0 \text{ and } Y(1)=1. \tag{2.38}$$

Using recurrence relation (2.37) and (2.38) for $k=0,1,...,4$ we get the following

$$Y(0)=0, Y(1)=1, Y(2)=-1, Y(3)=0, Y(4)=0.$$

For $k\geq 3$, $Y(k)=0$, so we can write the solution as

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} Y(k)x^k \\ &= x-x^2. \end{aligned}$$

Which is the exact solution.

**Example 2.4.** Consider the nonlinear initial value problem

$$xy''(x)+8y'(x)+18axy=-4axy\ln y, \quad x>0 \tag{2.39}$$

with initial conditions

$$y(0)=1 \quad \text{and} \quad y'(0)=0. \tag{2.40}$$

**Solution.** Apply the DTM on Equation (2.39)

$$\begin{aligned}\sum_{r=0}^{k}\delta(r-1)(k-r+1)(k-r+2)Y(k-r+2)&\\+8(k+1)Y(k+1)+18a\sum_{r=0}^{k}\delta(r-1)Y(k-r)&\\=-4a\sum_{r=0}^{k}\sum_{k_1=0}^{r}\delta(k_1-1)Y(r-k_1)F(k-r),&\end{aligned}\tag{2.41}$$

where $F(k)$is the transformed function of $f(y)=\ln y$.

Now, we simplify (2.41) to get

$$\begin{aligned}k(k+1)Y(k+1)+8(k+1)Y(k+1)+18a\sum_{r=0}^{k}\delta(r-1)Y(k-r)&\\=-4a\sum_{r=0}^{k}\sum_{k_1=0}^{r}\delta(k_1-1)Y(r-k_1)F(k-r),&\end{aligned}$$

Then we get the following recurrence relation

$$\begin{aligned}Y(k+1)\quad=\quad&\frac{-18a}{(k+1)(k+8)}\sum_{r=0}^{k}\delta(r-1)Y(k-r)\\&-\frac{4a}{(k+1)(k+8)}\sum_{r=0}^{k}\sum_{k_1=0}^{r}\delta(k_1-1)Y(r-k_1)F(k-r),\end{aligned}\tag{2.42}$$

from (2.40) we get

$$Y(0)=1 \text{ and } Y(1)=0.\tag{2.43}$$

Using recurrence relation (2.42) and (2.43) for $k=0,1,2,...$ we get the following series solution

$$\begin{aligned}y(x)\quad&=\quad\sum_{k=0}^{\infty}Y(k)x^k\\&=\quad1-ax^2+\frac{1}{2}a^2x^4-\frac{1}{6}a^3x^6+\frac{1}{24}a^4x^8-\frac{1}{120}a^5x^{10}+\frac{1}{720}a^6x^{12}+...\end{aligned}$$

which is the Taylor series expansion of the exact solution $e^{-ax^2}$.

Note: The result is computed by MAPLE program.

## 2.5 Boundary value problems

In this section, we solve boundary value problems by the DTM. Like the previous sections, we get a recurrence relation after applying the DTM on the given ordinary differential equation, but there is a difference in finding the transformed boundary conditions. We need to solve a system of linear equations by using the original boundary conditions to get a series solution for the given boundary value problem from the recurrence relation. This is illustrated in the next examples.

**Example 2.5.** Consider the fifth-order linear boundary value problem

$$y^{(5)}(x) = y(x) - 10e^x - 5xe^x + x^2e^x, \tag{2.44}$$

with boundary conditions

$$\begin{gathered} y(0) = 0, \quad y'(0) = \frac{4}{5}, \quad y''(0) = \frac{-1}{5}, \\ y'(1) = -2.265234857, \quad y''(1) = -8.245454881. \end{gathered} \tag{2.45}$$

**Solution.** Applying the DTM on Equation (2.44) gives the following recurrence relation

$$(k+1)(k+2)(k+3)(k+4)(k+5)Y(k+5) = Y(k) - \frac{10}{k!} - 5\sum_{r=0}^{k}\frac{\delta(r-1)}{(k-r)!} + \sum_{r=0}^{k}\frac{\delta(r-2)}{(k-r)!},$$

or

$$Y(k+5) = \frac{k!}{(k+5)!}\left[Y(k) - \frac{10}{k!} - 5\sum_{r=0}^{k}\frac{\delta(r-1)}{(k-r)!} + \sum_{r=0}^{k}\frac{\delta(r-2)}{(k-r)!}\right]. \tag{2.46}$$

Applying the DTM on boundary conditions at $x = 0$ gives the following transformed boundary conditions

$$\begin{aligned} Y(0) &= 0, \;\; Y(1) = \frac{4}{5}, \;\; Y(2) = \frac{y''(0)}{2!} = \frac{-1}{10}, \\ Y(3) &= \frac{y'''(0)}{3!} = a, \;\; Y(4) = \frac{y^{(4)}(0)}{4!} = b. \end{aligned}$$

Using Definition 2.1 for $k = 0, 1, ..., 15$, at $x = 1$ and transformed boundary conditions, we get the following linear system of two equations

$$\frac{15443560997}{435891456000} + \frac{239595841}{79833600}a + \frac{148284463}{37065600}b = -2.265234857, \tag{2.47}$$

$$\frac{-20554253119}{7783776000} + \frac{39972241}{6652800}a + \frac{239595841}{19958400}b = -8.245454881. \tag{2.48}$$

Solving (2.47) and (2.48) for $a$ and $b$ give

$$a = -0.4333333329 \text{ and } b = -0.2500000003.$$

For $k = 1, 2, ..., 15$ we get the following series solution

$$\begin{aligned} y(x) &= 0.8x - 0.1x^2 - 0.4333333329x^3 - 0.2500000003x^4 - 0.08333333333x^5 \\ &\quad - 0.01972222222x^6 - 0.003611111111x^7 - 0.0005357142856x^8 \\ &\quad - 0.00006613756615x^9 - 0.000006889329806x^{10} \\ &\quad - 6.062610229 \times 10^{-7}x^{11} - 4.425872481 \times 10^{-8}x^{12} \\ &\quad - 2.505210838 \times 10^{-9}x^{13} - 8.029521927 \times 10^{-11}x^{14} \\ &\quad + 3.823581866 \times 10^{-12}x^{15}. \end{aligned}$$

The calculation in this example made by MAPLE program.

Figure 2-1 shows that DTM give high accuracy solution. The exact solution values is taken from [26].

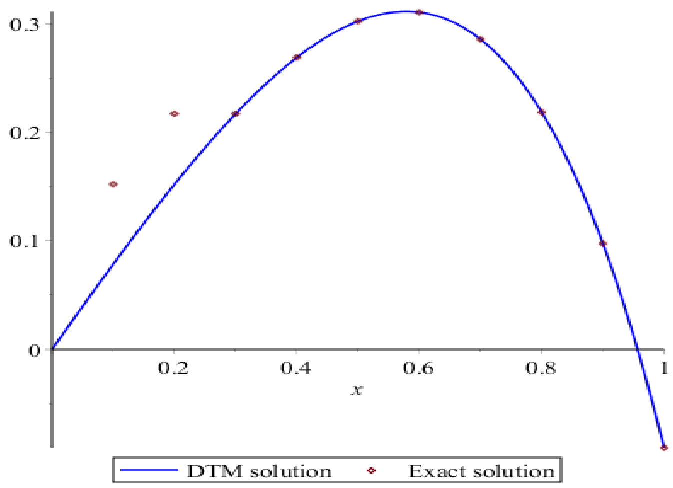


Figure 2-1: The exact solution together with the solution obtained from DTM.

**Example 2.6.** Consider the sixth-order linear boundary value problem

$$y^{(6)}(x) = y(x) + 15e^x + 10xe^x + x^3e^x, \tag{2.49}$$

with boundary conditions

$$y(0) = 10, \quad y'(0) = -\frac{25}{8}, \quad y''(0) = -\frac{5}{3},$$

$$y(1) = -3.397852287, \quad y'(1) = -2.831543570, \quad y''(1) = 3.397852297. \tag{2.50}$$

**Solution.** Applying the DTM on Equation (2.49) gives the following recurrence relation

$$\begin{aligned}(k+1)(k+2)(k+3)(k+4)(k+5)(k+6)Y(k+6) = \\ Y(k) + \frac{15}{k!} + 10\sum_{r=0}^{k}\frac{\delta(r-1)}{(k-r)!} + \sum_{r=0}^{k}\frac{\delta(r-3)}{(k-r)!},\end{aligned}$$

or

$$Y(k+6) = \frac{k!}{(k+6)!}\left[Y(k) + \frac{15}{k!} + 10\sum_{r=0}^{k}\frac{\delta(r-1)}{(k-r)!} + \sum_{r=0}^{k}\frac{\delta(r-3)}{(k-r)!}\right]. \qquad (2.51)$$

Applying the DTM on boundary conditions at $x = 0$ gives the following transformed boundary conditions

$$\begin{aligned} Y(0) &= 10, \quad Y(1) = -\frac{25}{8}, \quad Y(2) = \frac{y''(0)}{2!} = -\frac{5}{6}, \\ Y(3) &= \frac{y'''(0)}{3!} = a, Y(4) = \frac{y^{(4)}(0)}{4!} = b, \quad Y(5) = \frac{y^{(5)}(0)}{5!} = c. \end{aligned}$$

Using Definition 2.1 for $k = 0, 1, ..., 15$, at $x = 1$ and boundary conditions, we get the following linear system of three equations

$$\begin{aligned} &- \frac{4113595263593}{1046139494400} + \frac{217949331601}{217945728000}a + \frac{151201}{151200}b + \frac{332641}{332640}c = -3.397852287 \\ &- \frac{4841688336097}{1046139494400} + \frac{43591307761}{14529715200}a + \frac{60481}{15120}b + \frac{151201}{30240}c = -2.83154357 \\ &- \frac{29911705387}{37362124800} + \frac{889750903}{148262400}a + \frac{20161}{1680}b + \frac{60481}{3024}c = 3.397852297. \end{aligned}$$

Solving the above system of equations for $a$, $b$ and $c$ gives

$$a = 0.3124999935, \ b = 0.2500000111 \text{ and } c = 0.08506943967.$$

For $k = 1, 2, ..., 15$ we get the following series solution

$$\begin{aligned} y(x) &= -3.125x - 0.8333333333x^2 + 0.35124999935x^3 + 0.2500000111x^4 \\ &- 0.08506943967x^5 - 0.02083333333x^6 + 0.004340277778x^7 \\ &+ 0.0008267195767x^8 + 0.0001457093252x^9 \\ &+ 0.00002342372142x^{10} + 0.000003387253807x^{11} \\ &+ 4.384118967 \times 10^{-7}x^{12} + 5.088709516 \times 10^{-8}x^{13} \\ &+ 5.326249539 \times 10^{-9}x^{14} + 5.061466495 \times 10^{-10}x^{15}. \end{aligned}$$

Figure 2-2 shows that DTM give high accuracy solution. The exact solution values is taken from [26].

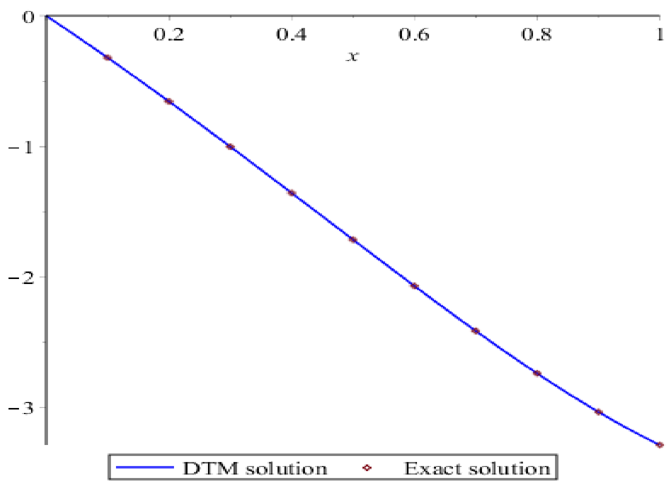


Figure 2-2: The exact solution together with the solution obtained from DTM.

For more examples refer to [4, 23, 26, 28].

## 2.6 DTM with linear shooting

In solving initial value problems the initial conditions are transformed directly to the values $Y(k)$, for $k = 0$ or $k \geq 1$, i.e we obtain the values $Y(0)$ and may $Y(1), Y(2), ...$ directly. But the situation is different for boundary value problems. In boundary value problems usually we require to solve linear system of equations that use the boundary conditions to obtain the values of $Y(0)$ or $Y(1), Y(2), ...$ as in the previous section.

In this section, we apply the linear shooting method to solve the second order boundary value problems. In this method we can solve the second order boundary value problems by combination of two initial value problems.

We see that solving initial value problems by the DTM give a series solution that converges fast to the exact solution, this gives us a motivation to use this solution in solving second order linear boundary value problems by linear shooting method.

Consider the second order linear boundary value problem

$$y'' = p(x)y' + q(x)y + r(x), \tag{2.52}$$

for $a \leq x \leq b$, subject to

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta.$$

Suppose that (2.52) has a unique solution, we can use linear shooting method to find this approximate unique solution by dividing this problem into two initial value problems, see[8]. So we have

$$u'' = p(x)u' + q(x)u + r(x), \tag{2.53}$$

subject to

$$u(a) = \alpha \quad \text{and} \quad u'(a) = 0$$

and

$$v'' = p(x)v' + q(x)v, \tag{2.54}$$

subject to

$$v(a) = 0 \quad and \quad v'(a) = 1.$$

Let $u(x)$ and $v(x)$ be solutions of the initial value problems (2.53) and (2.54) respectively. These solutions are obtained by applying the DTM on (2.53) and (2.54).

Define

$$y(x) = u(x) + \frac{\beta - u(b)}{v(b)} v(x), \tag{2.55}$$

where $v(b) \neq 0$. It is clearly that $y(x)$ is a solution of the boundary value problem (2.52).

The next examples will illustrate the idea.

**Example 2.7.** Consider the second-order linear boundary value problem

$$y''(x) - y(x) - 2 - x^2 = 0, \tag{2.56}$$

where $0 \leq x \leq 2$, with boundary conditions

$$y(0) = 1, \quad y(2) = e^2 + 4. \tag{2.57}$$

**Solution.** Divide (2.56) into two initial value problems

$$\begin{cases} u''(x) - u(x) - 2 - x^2 = 0, \\ u(0) = 1, u'(0) = 0 \end{cases}$$

and

$$\begin{cases} v''(x) - v(x) = 0, \\ v(0) = 0, v'(0) = 1. \end{cases}$$

After applying DTM on these initial value problems we get the following solutions

$$\begin{aligned} u(x) &\approx 1 + \frac{3}{2}x^2 + \frac{5}{24}x^4 + \frac{1}{144}x^6 + \frac{1}{8064}x^8 + \frac{1}{725760}x^{10}. \\ v(x) &\approx x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \frac{1}{362880}x^9 + \frac{1}{39916800}x^{11}. \end{aligned}$$

By shooting the solution is

$$\begin{aligned} y(x) &= 1 + \frac{3}{2}x^2 + \frac{5}{24}x^4 + \frac{1}{144}x^6 + \frac{1}{8064}x^8 + \frac{1}{725760}x^{10} + 0.1594000096x \\ &\quad + 0.02656666827x^3 + 0.001328333413x^5 + 0.0000316269603x^7 + 4.392636949 \times 10^{-7}x^9 \\ &\quad + 3.993306317 \times 10^{-9}x^{11}. \end{aligned} \tag{2.58}$$

Now, By applying the DTM on Equation (2.56) for $k = 1, 2, ..., 11$ we get the following series solution

$$\begin{aligned} y(x) &= 1 + 0.1594000065x + \frac{3}{2}x^2 + 0.2656666775x^3 + \frac{5}{24}x^4 + 0.1328333388 \times 10^{-2}x^5 \\ &\quad + \frac{1}{144}x^6 + 0.3162698542 \times 10^{-4}x^7 + \frac{1}{8064}x^8 + 4.392636863 \times 10^{-7}x^9 \\ &\quad + \frac{1}{725760}x^{10} + 3.993306239 \times 10^{-9}x^{11}. \end{aligned} \tag{2.59}$$

By comparing these two solutions with $y(x) = e^x + x^2$ which is the exact solution we see that the solution obtained from the DTM with shooting is almost the same as the solution obtained by DTM without shooting. by zooming this figure solution obtained from the DTM with shooting is closer to the exact one. See Figure 2-3.

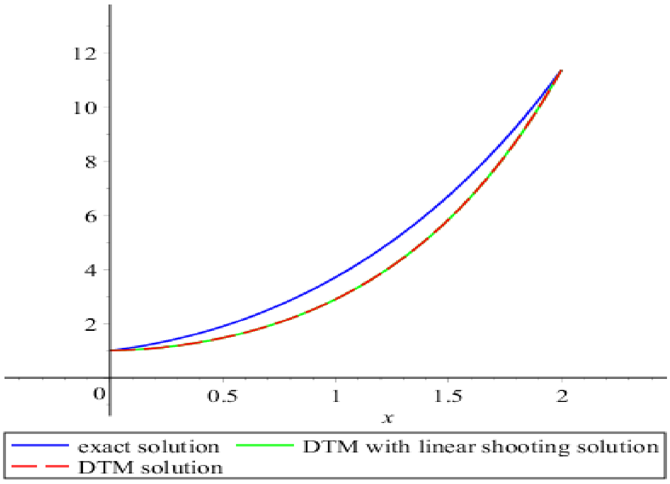


Figure 2-3: The solution obtained by DTM with shooting , the solution obtained by DTM without shooting and the exact one in Example 2.7.

**Example 2.8.** Consider the second-order linear boundary value problem

$$y''(x) + \frac{1}{2x}y'(x) - 3 = 0, \tag{2.60}$$

where $0 \leq x \leq 1$, with boundary conditions

$$y(0) = 1, \quad y(1) = 3. \tag{2.61}$$

**Solution.** Divide (2.60) into two initial value problems

$$\begin{cases} u''(x) + \frac{1}{2x}u'(x) - 3 = 0, \\ u(0) = 1, u'(0) = 0 \end{cases}$$

and

$$\begin{cases} v''(x) + \frac{1}{2x}v'(x) = 0, \\ v(0) = 0, v'(0) = 1. \end{cases}$$

After applying DTM on those two initial value problems we get the following solutions

$$\begin{aligned} u(x) &= 1+x^2. \\ v(x) &= x. \end{aligned}$$

By shooting the solution is

$$y(x) = 1 + x + x^2. \tag{2.62}$$

Now, By applying the DTM on Equation (2.60) for $k = 1, 2, ..., 11$ we get the following series solution

$$y(x) = 1 + x + x^2 \tag{2.63}$$

By comparing these two solutions with $y(x) = 1+\sqrt{x}+x^2$ which is the exact solution we see that the solution obtained from DTM with shooting is almost the same as the solution obtained by DTM without shooting, see Figure 2-4.

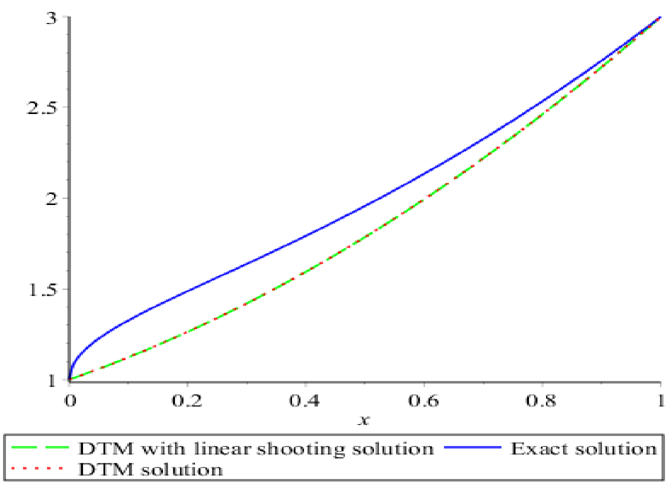


Figure 2-4: The solution obtained by DTM with shooting is the same as solution obtained DTM without shooting in Example 2.8.

## 2.7 Solution of physical models by the DTM

Many problems in mathematical physics are described by initial value problems. So the aim of this section is to solve special forms of differential equations by using DTM, see [13, 27].

**The Emden-Fowler equation:**

The Emden-Fowler equation is a second order ordinary differential equations with initial conditions ( singular initial value problem) which have been used to model several phenomena in mathematical physics and astrophysics. The following form is the Emden-Fowler equation

$$y''(x) + \frac{a}{x}y'(x) + bf(x)g(y(x)) = 0,$$

subject to

$$y(0) = c, \quad y'(0) = 0.$$

Where $a, b, c$ are constants.

The following example is taken from [27].

**Example 2.9.** Consider the nonlinear second order initial value problem

$$y''(x) + \frac{5}{x}y'(x) + 8a(e^y + 2e^{\frac{y}{2}}) = 0, \quad 0 < x \tag{2.64}$$

with initial conditions

$$y(0) = 0 \text{ and } y'(0) = 0. \tag{2.65}$$

**Solution.** Multiply both sides of Equation (2.64) by $x$ gives

$$xy''(x) + 5y'(x) + 8ax(e^y + 2e^{\frac{y}{2}}) = 0. \tag{2.66}$$

Apply the differential transform method on Equation (2.66)

$$\sum_{r=0}^{k} \delta(r-1)(k-r+1)(k-r+2)Y(k-r+2) + 5(k+1)Y(k+1)$$
$$+ 8a \left[ \sum_{r=0}^{k} \delta(r-1)(E_1(k-r) + 2E_2(k-r)) \right] = 0.$$

where $E_i(k) = b\sum_{r=0}^{k-1} \frac{r+1}{k} Y(r+1)E_i(k-r-1)$, i=1,2 is the differential transform of $e^{by}$.

We get the following recurrence relation

$$Y(k+1) = \frac{-8a}{(k+1)(k+5)} \sum_{r=0}^{k} \delta(r-1)(E_1(k-r) + 2E_2(k-r)), \tag{2.67}$$

the initial conditions become

$$Y(0) = 0 \text{ and } Y(1) = 0. \tag{2.68}$$

Using recurrence relation (2.67) and conditions (2.68) for $k = 0, 1, 2, ...$ we get the following

$$\begin{aligned} Y(0) &= 0, \;\; Y(1) = 0, \;\; Y(2) = -2a, \;\; Y(3) = 0, \\ Y(4) &= a^2, Y(5) = 0, \;\; Y(6) = -\frac{2}{3}a^3, \;\; Y(7) = 0, \\ Y(8) &= \frac{1}{2}a^4, \;\; Y(9) = 0, \;\; Y(10) = -\frac{2}{5}a^5, ... \end{aligned}$$

We can write the solution as

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} Y(k)x^k \\ &= -2ax^2 + a^2x^4 - \frac{2}{3}a^3x^6 + \frac{1}{2}a^4x^8 - \frac{2}{5}a^5x^{10} + ... \end{aligned}$$

which equal the Taylor series of the exact solution $-2\ln(1 + ax^2)$.

**The standard Lane-Emden equation:**

The standard Lane-Emden equation is a second order ordinary differential equation arising in the study of stellar interiors, it is also called the polytropic differential equations.

The following example is taken from [27].

**Example 2.10.** Consider the nonlinear second order initial value problem

$$y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0, \quad 0 < x, m \geq 0 \tag{2.69}$$

with initial conditions

$$y(0) = 1 \text{ and } y'(0) = 0. \tag{2.70}$$

**Solution.** By multiplying both sides of Equation (2.69) by $x$, we get

$$xy''(x) + 2y'(x) + xy^m(x) = 0. \tag{2.71}$$

Apply the differential transform on Equation (2.71), we obtain

$$\sum_{r=0}^{k} \delta(r-1)(k-r+1)(k-r+2)Y(k-r+2) + 2(k+1)Y(k+1) + \sum_{r=0}^{k} \delta(r-1)G(k-r) = 0, \tag{2.72}$$

where $G(k)$ is the differential transform of $g(y) = y^m$.

Hence, we get the following recurrence relation

$$Y(k+1) = \frac{-1}{(k+1)(k+2)}G(k-1), \tag{2.73}$$

the initial conditions become

$$Y(0) = 1 \text{ and } Y(1) = 0. \tag{2.74}$$

Using recurrence relation (2.73) and conditions (2.74) for $k = 0, 1, 2, ...$ we get the following

$$\begin{aligned} Y(0) &= 1, \quad Y(1) = 0, \quad Y(2) = \frac{-1}{6}, \quad Y(3) = 0, \\ Y(4) &= \frac{m}{120}, \quad Y(5) = 0, \quad Y(6) = \frac{-m(8m-5)}{3.7!}, ... \end{aligned}$$

For $m = 1$ we can write the solution as

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} Y(k)x^k \\ &= 1 - \frac{1}{6}x^2 - \frac{1}{120}x^4 - \frac{1}{5040}x^6 + \frac{1}{362880}x^8 - \frac{1}{39916800}x^{10} + ... \quad (2.75) \end{aligned}$$

which equal the Taylor series exact solution $\dfrac{\sin x}{x}$.

## 2.8 On the convergence analysis

In this section, we present two theorems about the convergence of the DTM for a class of singular boundary value problems, these theorems are due to Lin et al. [47].

Consider the singular boundary value problem of the form

$$y''(x) + \frac{f(x)}{x(x-1)}y'(x) + \frac{g(x)}{x(x-1)}N(y(x)) = \frac{h(x)}{x(x-1)}, \; 0 < x < 1,$$

subject to

$$y(0) = p, \quad y(1) = q.$$

where $f(x)$, $g(x)$ and $h(x)$ are known and continuous functions on $(0, 1)$, also $N(y(x))$ is a nonlinear function of $y$.

**Theorem 2.17.** *Consider the following two singularly linear boundary value problems*

$$x(1-x)y''(x) + (1-x)y'(x) + g(x)y(x) = f(x), \ 0 < x < 1, \tag{2.76}$$

*where* $f(x) = f_0 + f_1x + f_2x^2 + \ldots$ *and* $g(x) = g_0 + g_1x + g_2x^2 + \ldots$.
*If there exist a fixed* $n$ *such that* $n \geq m$, $\mid f_k \mid \leq Mr^k$ *for some fixed* $M$, $0 < r < 1$, *all* $k \geq n$, *and all* $Y(n) \leq Mr^n$, *then the numerical solution using the DTM absolutely converges.*

*Proof.* Let $g(x) = g_0$ for simplicity. By applying the DTM on (2.76) we have

$$\begin{aligned}
&\sum_{i=0}^{k} \delta(i-1)(k-i+1)(k-r+2)Y(k-i+2) \\
&\quad - \sum_{i=0}^{k} \delta(i-2)(k-i+1)(k-i+2)Y(k-i+2) + (k+1)Y(k+1) \\
&\quad\quad - \sum_{i=0}^{k} \delta(i-1)(k-i+1)Y(k-i+1) + g_0Y(k) = F(k),
\end{aligned} \tag{2.77}$$

by simplifying (2.77) we obtain the following recurrence relation

$$Y(k+1) = \frac{1}{(k+1)^2}[F(k) + (k^2 + g_0)Y(k)]. \tag{2.78}$$

Now,

$$\begin{aligned}
|Y(k+1)| &\leq \frac{1}{(k+1)^2}[Mr^k + |k^2 + g_0|Mr^k] \\
&= Mr^k \left[\frac{1 + k^2 + |g_0|}{(k+1)^2}\right], \ k \geq n \geq m
\end{aligned} \tag{2.79}$$

Let $r = \left[\frac{1 + k^2 + |g_0|}{(k+1)^2}\right]$, so for $k$ is large enough $r < 1$. By induction the hypothesis is true. Then we have

$$|y(x)| \leq \sum_{k=0}^{\infty} |Y(k)|x^k \leq \sum_{k=0}^{\infty} |Y(k)| \leq M \sum_{k=0}^{\infty} r^k.$$

Hence,

$$|y(x)| \leq \frac{M}{1-r}.$$

□

**Theorem 2.18.** *Consider the following two singularly linear boundary value problems*

$$x(1-x)y''(x)+(1-x)y'(x)+g(x)y(x)+y^2(x)=f(x),\ 0<x<1, \tag{2.80}$$

*where* $f(x)=f_0+f_1x+f_2x^2+\ldots$ *and* $g(x)=g_0+g_1x+g_2x^2+\ldots$.
*If there exist a fixed* $n$ *such that* $n\geq m$, $\mid f_k\mid\leq r^k$, $0<r<1$, *all* $k\geq n$, *and all* $Y(n)\leq r^n$, *then the numerical solution using the DTM absolutely converges.*

*Proof.* Let $g(x)=g_0$ for simplicity. By applying the DTM on (2.80) we have

$$\begin{aligned}
&\sum_{i=0}^{k}\delta(i-1)(k-i+1)(k-i+2)Y(k-i+2)\\
&\quad-\sum_{i=0}^{k}\delta(i-2)(k-i+1)(k-i+2)Y(k-i+2)+(k+1)Y(k+1)\\
&\quad\quad-\sum_{i=0}^{k}\delta(i-1)(k-i+1)Y(k-i+1)+g_0Y(k)\\
&\quad\quad\quad+\sum_{i=0}^{k}Y(i)Y(k-i)=F(k),
\end{aligned}$$

by simplifying the above relation we obtain the following recurrence relation

$$Y(k+1)=\frac{1}{(k+1)^2}[F(k)+(k^2+g_0)Y(k)-B(k)], \tag{2.81}$$

where $B(k)=\sum_{i=0}^{k}Y(i)Y(k-i)$.

Also we have $|B(k)|\leq(k+1)r^k$ since $|Y(i)Y(k-i)|\leq r^ir^{k-i}=r^k$.

Now,

$$\begin{aligned}
|Y(k+1)| &\leq \frac{1}{(k+1)^2}[r^k+|k^2+g_0|r^k+(k+1)r^k]\\
&= r^k\left[\frac{1+k^2+|g_0|+k+1}{(k+1)^2}\right],\ k\geq n\geq m
\end{aligned} \tag{2.82}$$

Let $r=\left[\dfrac{1+k^2+|g_0|+k+1}{(k+1)^2}\right]$, so for $k$ is large enough $r<1$. By induction the hypothesis is true. Then we have

$$|y(x)|\leq\sum_{k=0}^{\infty}|Y(k)|x^k\leq\sum_{k=0}^{\infty}|Y(k)|\leq\sum_{k=0}^{\infty}r^k.$$

Hence,

$$|y(x)| \leq \frac{1}{1-r}.$$

$\square$

# Chapter 3

# Modifications of DTM for ordinary differential equations

The DTM is a powerful technique for solving linear and nonlinear ordinary differential equations as mentioned before. But sometimes when we solve nonlinear ordinary differential equations we need hard calculations due to nonlinear terms. Therefore the DTM is modified by using Adomain polynomials instead of the differential transform of nonlinear terms. On the other hand, the DTM is modified by using Laplace transform and Padé approximation to handel the periodic behavior of the solution.

## 3.1 DTM with Adomain polynomials

In this section, we give a brief summary of the *Adomain Decomposition Method* (ADM) needed for the rest of the section. After that, we present a modification of the DTM to solve a class of nonlinear singular boundary value problems. Next, some examples are given to illustrate the idea.

**ADM for ordinary differential equations:**

Many methods were proposed to solve nonlinear ordinary differential equations. One of these methods is the Adomain decomposition method (ADM). The ADM is a technique for solving algebraic equations, ordinary differential equations, partial differential equations, integral and integro differential equations. This method was developed by Adomain from 1970s to 1990s, see [2, 3].

Consider the nonlinear ordinary differential equation

$$\mathcal{F}y(x) = g(x), \tag{3.1}$$

where $\mathcal{F}$ is a nonlinear ordinary differential operator which may be contain linear and nonlinear terms.

Now, we decompose Equation (3.1) as follows:

- **Decomposition of the linear term:** We decompose the linear term into two parts. The first part denoted by $Ly$ is the linear differential operator which is the highest order derivative. The second part denoted by $Ry$ which is the remainder of the linear part. So Equation (3.1) becomes

  $$Ly(x) + Ry(x) + Ny(x) = g(x), \tag{3.2}$$

  where $Ny(x)$ is the nonlinear terms. Solve (3.2) for $Ly(x)$ and take the inverse of the operator to get

  $$L^{-1}Ly(x) = L^{-1}g(x) - L^{-1}Ry(x) - L^{-1}Ny(x). \tag{3.3}$$

  Here $L^{-1}$ is the integration $n$ times, where $n$ is the highest order of the derivative, i.e.

  $$L^{-1} = \int_0^x \int_0^x ... \int_0^x \cdot \ dx\ dx\ ...\ dx.$$

Thus,

$$L^{-1}Ly(x) = y(x) - y(0) - xy'(0) - \frac{x^2}{2!}y''(0) - ... - \frac{x^{n-1}}{(n-1)!}y^{(n-1)}(0). \quad (3.4)$$

By substituting (3.4) in (3.3), we have

$$y(x) - y(0) - xy'(0) - \frac{x^2}{2!}y''(0) - ... - \frac{x^{n-1}}{(n-1)!}y^{(n-1)}(0) = L^{-1}g(x) - L^{-1}Ry(x) - L^{-1}Ny(x).$$

- **Decomposition of the unknown function $y(x)$:** We decompose this function as an infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x).$$

- **Decomposition of the nonlinear terms:** We decompose the nonlinear terms $Ny(x)$ as a sum of polynomials called the Adomain polynomials and denoted by $A_n$. i.e

$$Ny(x) = \sum_{n=0}^{\infty} A_n,$$

To compute $A_n$, take $Ny = f(y)$ to be a nonlinear function in $y$, where $y = y(x)$, then the Taylor series expansion of $f(y)$ around $y_0$ is given as

$$f(y) = f(y_0) + f'(y_0)(y - y_0) + \frac{1}{2!}f''(y_0)(y - y_0)^2 + \frac{1}{3!}f'''(y_0)(y - y_0)^3 + ...,$$

but $y = y_0 + y_1 + y_2 + y_3 + ...$, then

$$\begin{aligned} f(y) &= f(y_0) + f'(y_0)(y_1 + y_2 + y_3 + ...) + \frac{1}{2!}f''(y_0)(y_1 + y_2 + y_3 + ...)^2 \\ &+ \frac{1}{3!}f'''(y_0)(y_1 + y_2 + y_3 + ...)^3 + .... \end{aligned}$$

By expanding all terms we get

$$\begin{aligned} Ny = f(y) &= f(y_0) + f'(y_0)(y_1) + f'(y_0)y_2 + f'(y_0)y_3 + ... + \frac{1}{2!}f''(y_0){y_1}^2 \\ &\quad + \frac{1}{2!}f''(y_0)y_1y_2 + \frac{1}{2!}f''(y_0)y_1y_3 + ... + \frac{1}{3!}f'''(y_0){y_1}^3 \\ &\quad + \frac{1}{3!}f'''(y_0){y_1}^2y_2 + \frac{1}{3!}f'''(y_0){y_1}^2y_3 + ... \end{aligned}$$

Now, let $(l)(i)$ be the order of ${y_l}^i$ and $(l)(i) + (m)(j)$ be the order of ${y_l}^i{y_m}^j$. Then $A_n$ consists of all terms of order $n$. So, we have

$$\begin{aligned} A_0 &= f(y_0) \\ A_1 &= f'(y_0)y_1, \\ A_2 &= f'(y_0)y_2 + \frac{1}{2!}f''(y_0)y_1^2, \\ A_3 &= f'(y_0)y_3 + \frac{2}{2!}f''(y_0)y_1y_2 + \frac{1}{3!}f'''(y_0)y_1^3, \\ A_4 &= f'(y_0)y_4 + \frac{1}{2!}f''(y_0)(2y_1y_3 + y_2^2) + \frac{3}{3!}f'''(y_0)y_1^2y_2 + \frac{1}{4!}f^{(4)}(y_0)y_1^4, \\ &\vdots \end{aligned}$$

or

$$\begin{aligned} A_0 &= N(y_0) \\ A_1 &= N'(y_0)y_1 = \frac{d}{d\lambda}N(y_0 + \lambda y_1)\bigg|_{\lambda=0}, \\ A_2 &= N'(y_0)y_2 + \frac{1}{2!}N''(y_0)y_1^2 = \frac{1}{2!}\frac{d^2}{d\lambda^2}N(y_0 + \lambda y_1 + \lambda^2 y_2)\bigg|_{\lambda=0}, \\ A_3 &= N'(y_0)y_3 + \frac{2}{2!}N''(y_0)y_1y_2 + \frac{1}{3!}N'''(y_0)y_1^3, \\ &= \frac{1}{3!}\frac{d^3}{d\lambda^3}N(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3)\bigg|_{\lambda=0} \\ A_4 &= N'(y_0)y_4 + \frac{1}{2!}N''(y_0)(2y_1y_3 + y_2^2) + \frac{3}{3!}N'''(y_0)y_1^2y_2 + \frac{1}{4!}N^{(4)}(y_0)y_1^4, \\ &= \frac{1}{4!}\frac{d^4}{d\lambda^4}N(y_0 + \lambda y_1 + \lambda^2 y_2 + \lambda^3 y_3 + \lambda^4 y_4)\bigg|_{\lambda=0}. \\ &\vdots \end{aligned}$$

Hence,

$$A_n = A_n(y_0, y_1, ..., y_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n} N\left[\sum_{m=0}^{\infty} \lambda^m y_m\right]_{\lambda=0}.$$

Now, substitute the above decompositions in (3.3), then solve it for $y$ to get

$$y = \phi_0 + L^{-1}g(x) - L^{-1}\sum_{n=0}^{\infty} Ry_n - L^{-1}\sum_{n=0}^{\infty} A_n(y_0, y_1, ..., y_n)$$

or

$$\sum_{n=0}^{\infty} y_n = \phi_0 + L^{-1}g(x) - L^{-1}\sum_{n=0}^{\infty} Ry_n - L^{-1}\sum_{n=0}^{\infty} A_n,$$

where

$$\phi_0 = \begin{cases} y(0) & \text{if } L = \dfrac{d}{dx}, \\ \\ y(0) + xy'(0) & \text{if } L = \dfrac{d^2}{dx^2} \\ \vdots & \\ y(0) + xy'(0) + \dfrac{x^2}{2!}y''(0) + ... + \dfrac{x^n}{n!}y^{(n)}(0) & \text{if } L = \dfrac{d^{n+1}}{dx^{n+1}}. \end{cases}$$

Therefore

$$\begin{aligned} y_0 &= \phi_0 + L^{-1}g(x), \\ y_{n+1} &= -L^{-1}Ry_n - L^{-1}A_n. \end{aligned}$$

To illustrate the ADM, we consider two examples, one for an initial value problem and the other for a boundary value problem.

**Example 3.1.** Consider the initial value problem

$$y'(x) - e^{-y(x)} = 0, \tag{3.5}$$

subject to

$$y(0) = 0. \tag{3.6}$$

**Solution.** We can rewrite the Equation (3.5) as

$$y'(x) = e^{-y(x)}.$$

Let $L = \dfrac{d}{dx}$, then

$$Ly = e^{-y(x)}. \tag{3.7}$$

The Admain polynomials are

$$\begin{aligned}
A_0 &= e^{-y_0},\\
A_1 &= -y_1 e^{-y_0},\\
A_2 &= \frac{e^{-y_0}}{2}(-2y_2 + y_1^2),\\
A_3 &= \frac{e^{-y_0}}{6}(-3y_3 + 2y_1y_2 - y_1^3),\\
&\vdots
\end{aligned}$$

Take the

$$L^{-1} = \int_0^x \cdot \ dx$$

of (3.7), we get

$$y(x) = L^{-1}e^{-y(x)},$$

or

$$\sum_{n=0}^{\infty} y_n(x) = L^{-1}\sum_{n=0}^{\infty} A_n.$$

Hence,

$$
\begin{aligned}
y_0 &= \phi_0 + L^{-1}g(x) = 0, \\
y_1 &= \int_0^x A_0 \, dx = x, \\
y_2 &= \int_0^x A_1 \, dx = -\frac{x^2}{2}, \\
y_3 &= \int_0^x A_2 \, dx = \frac{x^3}{3}, \\
y_4 &= \int_0^x A_3 \, dx = -\frac{x^4}{8}, \\
&\vdots
\end{aligned}
$$

Then the solution is

$$y(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{8} + \ldots$$

By comparing $y(x)$ up to $x^4$ with $\ln(x+1)$ which is the exact solution, we see that $y(x)$ is very close to the exact solution.

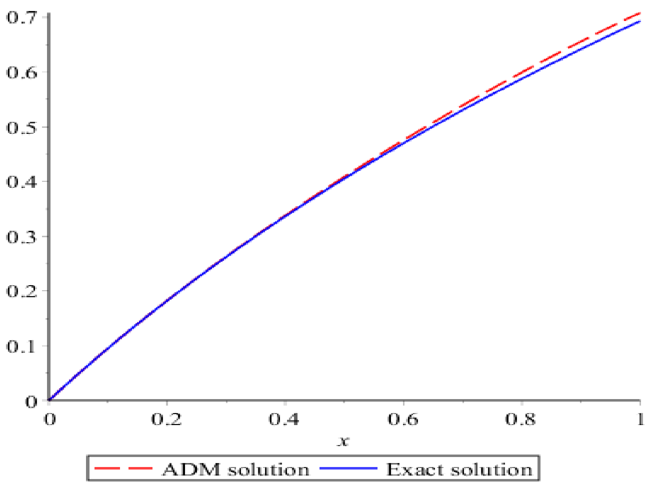


Figure 3-1: Comparison between the solution obtained from ADM and the exact solution.

**Example 3.2.** Consider the nonlinear boundary value problem

$$y^{(6)}(x) = \frac{1}{64}e^{-\frac{x}{2}}\sqrt{y(x)}, \quad 0 < x < 1, \tag{3.8}$$

subject to

$$y(0) = 1, \quad y''(0) = \frac{1}{4}, \quad y^{(4)}(0) = \frac{1}{16}, \tag{3.9}$$

$$y(1) = \sqrt{e}, \quad y''(1) = \frac{\sqrt{e}}{4}, \quad y^{(4)}(1) = \frac{\sqrt{e}}{16}. \tag{3.10}$$

**Solution.** Let $Ly(x) = \dfrac{d^6 y(x)}{dx^6} = y^{(6)}(x)$, so (3.8) becomes

$$Ly(x) = \frac{1}{64}e^{-\frac{x}{2}}\sqrt{y(x)}, \tag{3.11}$$

Now, take

$$L^{-1} = \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x \int_0^x \cdot \; dx\; dx\; dx\; dx\; dx\; dx$$

to both sides of (3.11) and use (3.9), to get

$$y(x) - 1 - y'(0)x - \frac{1}{8}x^2 - \frac{1}{6}x^3 y'''(0) - \frac{1}{384}x^4 - \frac{1}{120}x^5 y^{(5)}(0) = \frac{1}{64}L^{-1}(e^{-\frac{x}{2}}\sqrt{y(x)}),$$

thus,

$$y(x) = 1 + y'(0)x + \frac{1}{8}x^2 + \frac{1}{6}x^3 y'''(0) + \frac{1}{384}x^4 + \frac{1}{120}x^5 y^{(5)}(0) + \frac{1}{64}L^{-1}(e^{-\frac{x}{2}}\sqrt{y(x)}),$$

or

$$\sum_{n=0}^{\infty} y_n(x) = 1+y'(0)x+\frac{1}{2}x^2+\frac{1}{6}x^3 y'''(0)+\frac{1}{24}x^4+\frac{1}{120}x^5 y^{(5)}(0)+\frac{1}{64}L^{-1}\left(e^{-\frac{x}{2}}\sum_{n=0}^{\infty} A_n(x)\right).$$

The Adomain polynomials are

$$
\begin{aligned}
A_0 &= 1, \\
A_1 &= \frac{y_1}{2\sqrt{y_1}}, \\
A_2 &= \frac{1}{2\sqrt{y_0}}\left(y_2 - \frac{{y_1}^2}{4y_0},\right), \\
&\vdots
\end{aligned}
$$

Now, Let

$$
\begin{aligned}
y_0(x) &= 1, \\
y_1(x) &= y'(0)x + \frac{1}{2}x^2 + \frac{1}{6}x^3 y'''(0) + \frac{1}{24}x^4 + \frac{1}{120}x^5 y^{(5)}(0) + \frac{1}{64}L^{-1}(e^{-\frac{x}{2}}A_0), \\
&= -1 + \left(y'(0) + \frac{1}{2}\right)x + \left(\frac{y'''(0)}{6} + \frac{1}{48}\right)x^3 + \left(\frac{y^{(5)}(0)}{120} + \frac{1}{3840}\right)x^5 + e^{-\frac{x}{2}}, \\
y_2(x) &= \frac{1}{64}L^{-1}(e^{-\frac{x}{2}}A_1), \\
&\vdots
\end{aligned}
$$

So,

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} y_n(x) \\
&= \left(y'(0) + \frac{1}{2}\right)x + \left(\frac{y'''(0)}{6} + \frac{1}{48}\right)x^3 + \left(\frac{y^{(5)}(0)}{120} + \frac{1}{3840}\right)x^5 + e^{-\frac{x}{2}} \quad (3.12)
\end{aligned}
$$

But by Taylor series

$$e^{-\frac{x}{2}} = 1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} - \frac{x^5}{3840} + ...,$$

so (3.12) becomes

$$y(x) = 1 + y'(0)x + \frac{x^2}{8} + \frac{y'''(0)}{6}x^3 + \frac{x^4}{384} + \frac{y^{(5)}(0)}{120}x^5 + ... \quad (3.13)$$

By using (3.10) and (3.16) for $n = 1$ we have the following system of equations

$$\begin{aligned}
\frac{433}{384} + y'(0) + \frac{y'''(0)}{6} + \frac{y^{(5)}(0)}{120} &= \sqrt{e}, \\
\frac{9}{32} + y'''(0) + \frac{y^{(5)}(0)}{6} &= \frac{\sqrt{e}}{4}, \\
\frac{1}{16} + y^{(5)}(0) &= \frac{\sqrt{e}}{16}.
\end{aligned}$$

After solving the previous system, we get

$$y'(0) = 0.4098060,$$

$$y'''(0) = 0.1241728,$$

$$y^{(5)}(0) = 0.0405451.$$

Hence,

$$y(x) = 1 + 0.4098060x + \frac{1}{8}x^2 + 0.0206955x^3 + \frac{1}{384}x^4 + 0.0003379x^5 + ...$$

**DTM with Adomain polynomials:**

Since there is no systematic methodology for calculating the differential transform of the nonlinear functions the original DTM is modified via using Adomain polynomials. In this modification which is called the DTM with Adomain polynomials we calculate the Adomain polynomials instead of calculating the differential transform of nonlinear functions to get a series solution easily, see Elsaid [14] and Xie et al. [48].

Consider the nonlinear singular boundary value problem which appears frequently in applied science and engineering of the form

$$y''(x) + \frac{\alpha}{x}y'(x) = f(x, y), \quad 0 < x \leq 1, \quad \alpha \geq 1 \tag{3.14}$$

subject to the boundary conditions

$$y'(0) = 0 \text{ and } ay(1) + by'(1) = c, \tag{3.15}$$

where $a, b$ and $c$ are real constants, see[48].

Note : $f(x,y)$ is continuous and $\dfrac{\partial f}{\partial y} \geq 0$ exist and continuous to set a unique solution.

Now, after applying the DTM on (3.14) we have the following recurrence relation

$$Y(k+1) = \frac{F(k-1)}{(k+1)(k+\alpha)}, \quad k = 1, 2, ...N-1$$

and from (3.15) we get the transformed conditions

$$\text{Let } Y(0) = \beta \text{ and } Y(1) = 0.$$

Where $F(k)$ is the differential transform of the nonlinear function $f(x,y)$.

Let

$$y(x) \approx y_N(x) = \sum_{k=0}^{N} Y(k)x^k$$

be the solution of this problem, then

$$\begin{aligned}
y_N(x) &= \sum_{k=0}^{N} Y(k)x^k \\
&= \beta + \sum_{k=1}^{N} Y(k)x^k \\
&= \beta + \sum_{k=0}^{N-1} Y(k+1)x^{k+1} \\
&= \beta + \sum_{k=1}^{N-1} Y(k+1)x^{k+1} \\
&= \beta + \sum_{k=1}^{N-1} \frac{F(k-1)}{(k+1)(k+\alpha)}x^{k+1}.
\end{aligned}$$

Now, the differential transform of the nonlinear function $f(y)$ as follows

$$\begin{aligned}
F(0) &= f(Y(0)), \\
F(1) &= Y(1)f'(Y(0)), \\
F(2) &= Y(2)f'(Y(0)) + \frac{1}{2!}(Y(1))^2 f''(Y(0)), \\
F(3) &= Y(3)f'(Y(0)) + Y(1)Y(2)f''(Y(0)) + \frac{1}{3!}(Y(1))^3 f'''(Y(0)), \\
&\vdots
\end{aligned}$$

Also, the Adomain polynomials of this function are given as

$$\begin{aligned}
A_0 &= f(y_0) \\
A_1 &= f(y_0)y_1, \\
A_2 &= f'(y_0)y_2 + \frac{1}{2!}f''(y_0)y_1^2, \\
A_3 &= f'(y_0)y_3 + \frac{2}{2!}f''(y_0)y_1y_2 + \frac{1}{3!}f'''(y_0)y_1^3, \\
&\vdots
\end{aligned}$$

By comparing the differential transform and the Adomain polynomials of the nonlinear function $f$ we see that when we replace $y_k$ by $Y(k)$ they have the same structure. Then the solution becomes

$$y_N(x) = \beta + \sum_{k=1}^{N-1} \frac{A_{k-1}}{(k+1)(k+\alpha)} x^{k+1}, \tag{3.16}$$

where $A_k$, $k = 0, 1, ..., N$ are the Adomain polynomials given by the following formula

$$A_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} f\left(\sum_{m=0}^{\infty} Y(m)\lambda^m\right)_{\lambda=0}.$$

By using (3.16), the boundary condition $ay(1) + by'(1) = c$ becomes

$$a\left[\beta + \sum_{k=1}^{N-1} \frac{A_{k-1}}{(k+1)(k+\alpha)}\right] + b\frac{d}{dx}\left[\beta + \sum_{k=1}^{N-1} \frac{A_{k-1}}{(k+1)(k+\alpha)} x^{k+1}\right]_{x=1} = c, \tag{3.17}$$

by solving (3.17) for $\beta$ and substituting the value in (3.16) we obtain the solution.

Note that the DTM with Aomain polynomials called the improved DTM or simply IDTM.

The next two examples are taken from [48].

**Example 3.3.** Consider the nonlinear singular boundary value problem

$$y''(x) + \frac{2}{x}y'(x) = -y^5(x), \tag{3.18}$$

subject to the boundary conditions

$$y'(0) = 0, \quad y(1) = \frac{\sqrt{3}}{2}. \tag{3.19}$$

**Solution.** After applying the IDTM on (3.18) we get

$$\begin{aligned} Y(k+1) &= \frac{F(k-1)}{(k+1)(k+2)} \\ &= \frac{A_{k-1}}{(k+1)(k+2)}, \end{aligned} \tag{3.20}$$

where $F$ is the differential transform of the nonlinear function $-y^5(x)$ and $A_k$ is the Adomain polynomials.

Let

$$Y(0) = \beta,$$

and from (3.19)

$$Y(1) = 0.$$

The Aadomain polynomials of the function $-y^5(x)$ are

$$\begin{aligned} A_0 &= -Y^5(0), \\ A_1 &= -5Y^4(0)Y(1), \\ A_2 &= -10Y^3(0)Y^2(1) - 5Y(0)^4Y(2), \\ A_3 &= -10Y^2(0)Y^3(1) - 20Y(0)^3Y(1)Y(2) - 5Y^4(0)Y(3), \\ &\vdots \end{aligned}$$

From (3.20) we have

$$
\begin{aligned}
Y(2) &= -\frac{1}{6}\beta^5, \\
Y(3) &= 0, \\
Y(4) &= \frac{1}{24}\beta^9, \\
Y(5) &= 0, \\
Y(6) &= -\frac{5}{432}\beta^{13}, \\
Y(7) &= 0, \\
Y(8) &= \frac{35}{10368}\beta^{17}, \\
Y(9) &= 0, \\
Y(10) &= -\frac{7}{6912}\beta^{21}, \\
&\vdots
\end{aligned}
$$

For $N = 10$, we use (3.20) to obtain the following truncated series solution

$$
y_{10}(x) = \beta - \frac{1}{6}\beta^5 x^2 + \frac{1}{24}\beta^9 x^4 - \frac{5}{432}\beta^{13} x^6 + \frac{35}{10368}\beta^{17} x^8 - \frac{7}{6912}\beta^{21} x^{10}. \quad (3.21)
$$

Now, solve (3.21) for $\beta$ using boundary condition at $x = 1$,

$$
\beta = 1.000553890.
$$

Substituting $\beta$ value in (3.21) to get

$$
\begin{aligned}
y_{10}(x) &= 1.000553890 - 0.1671287533x^2 + 0.04187483621x^4 - 0.01165769154x^6 \\
&\quad + 0.003407699551x^8 - 0.001024576736x^{10}. \quad (3.22)
\end{aligned}
$$

By comparing solution in (3.22) by the solution obtained by DTM we see that this solution is almost the same as the exact one $\sqrt{\dfrac{3}{3+x^2}}$, see Figure 3-2.

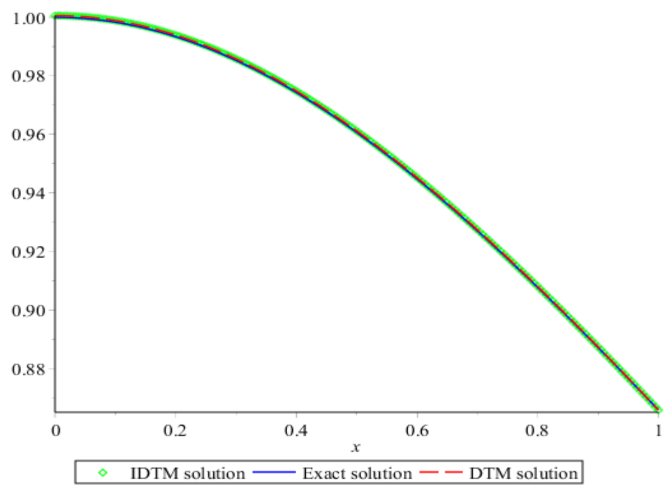


Figure 3-2: Comparison between the solution obtained by DTM the solution obtained by IDTM.

**Example 3.4.** Consider the nonlinear singular boundary value problem

$$y''(x) + \frac{1}{x}y'(x) = -e^{y(x)}, \tag{3.23}$$

subject to the boundary conditions

$$y'(0) = 0, \quad y(1) = 0. \tag{3.24}$$

**Solution.** After applying the IDTM on (3.23) we get

$$\begin{aligned} Y(k+1) &= \frac{F(k-1)}{(k+1)^2} \\ &= \frac{A_{k-1}}{(k+1)^2}, \end{aligned} \tag{3.25}$$

where $F(k)$ is the differential transform of the nonlinear function $-e^{y(x)}$ and $A_k$ is the Adomain polynomials.

Let

$$Y(0) = \beta,$$

and from (3.24)

$$Y(1) = 0.$$

The Aadomain polynomials of the function $-e^{y(x)}$ are

$$\begin{aligned}
A_0 &= -e^{Y(0)}, \\
A_1 &= -Y(1)e^{Y(0)}, \\
A_2 &= -Y(2)e^{Y(0)} - \frac{1}{2}Y^2(1)e^{Y(0)}, \\
A_3 &= -Y(3)e^{Y(0)} - Y(1)Y(2)e^{Y(0)} - \frac{1}{6}Y^3(1)e^{Y(0)}, \\
A_4 &= e^{Y(0)}(-Y(4) - Y(3)Y(1) - \frac{1}{2}Y(2)^2 - \frac{1}{2}Y(2)Y(1)^2 - \frac{1}{24}Y(1)^4),
\end{aligned}$$

$$\begin{aligned}
A_5 &= e^{Y(0)}(-Y(5) - Y(4)Y(1) - Y(3)Y(2) - \frac{1}{2}Y(3)Y(1)^2 - \frac{1}{2}Y(2)^2Y(1) \\
&\quad - \frac{1}{6}Y(2)Y(1)^3 - \frac{1}{120}Y(1)^5),
\end{aligned}$$

$$\begin{aligned}
A_6 &= e^{Y(0)}(-Y(5)Y(1) - Y(4)Y(2) - \frac{1}{2}Y(4)Y(1)^2 - \frac{1}{2}Y(3)^2 - Y(3)Y(2)Y(1) \\
&\quad - \frac{1}{6}Y(3)Y(1)^3 - \frac{1}{6}Y(2)^3 - \frac{1}{4}Y(2)^2Y(1)^2 - \frac{1}{24}Y(2)Y(1)^4 - \frac{1}{720}Y(1)^6),
\end{aligned}$$

$$\begin{aligned}
A_7 &= e^{Y(0)}(-\frac{1}{2}Y(5)Y(1)^2 - Y(4)Y(2)Y(1) - \frac{1}{6}Y(4)Y(1)^3 \\
&\quad - \frac{1}{2}Y(3)Y(2)^2 - \frac{1}{24}Y(3)Y(1)^4 - \frac{1}{12}Y(2)^2Y(1)^3 \\
&\quad - \frac{1}{120}Y(2)Y(1)^5 - Y(5)Y(2) - Y(4)Y(3) - \frac{1}{2}Y(3)^2Y(1) \\
&\quad - \frac{1}{6}Y(2)^3Y(1) - \frac{1}{5040}Y(1)^7 - \frac{1}{2}Y(3)Y(2)Y(1)^2),
\end{aligned}$$

$$\begin{aligned}
A_8 &= e^{Y(0)}(-\frac{1}{40320}Y(1)^8 - Y(5)Y(3) - \frac{1}{2}Y(4)^2 - \frac{1}{2}Y(3)^2Y(2) \\
&\quad - \frac{1}{24}Y(2)^4 - \frac{1}{720}Y(1)^6Y(2) - \frac{1}{6}Y(3)Y(2)Y(1)^3 - \frac{1}{2}Y(4)Y(2)Y(1)^2 \\
&\quad - \frac{1}{6}Y(5)Y(1)^3 - \frac{1}{2}Y(4)Y(2)^2 - \frac{1}{24}Y(4)Y(1)^4 - \frac{1}{120}Y(3)Y(1)^5 \\
&\quad - \frac{1}{48}Y(2)^2Y(1)^4 - Y(5)Y(2)Y(1) - Y(4)Y(3)Y(1) - \frac{1}{4}Y(3)^2Y(1)^2 \\
&\quad - \frac{1}{2}Y(2)^2Y(1)Y(3) - \frac{1}{12}Y(2)^3Y(1)^2), \\
&\vdots
\end{aligned}$$

Then we have

$$\begin{aligned}
Y(2) &= -\frac{e^{\beta}}{4}, \\
Y(3) &= 0, \\
Y(4) &= \frac{e^{2\beta}}{64}, \\
Y(5) &= 0, \\
Y(6) &= -\frac{3e^{3\beta}}{2304}, \\
Y(7) &= 0, \\
Y(8) &= \frac{10e^{4\beta}}{98304}, \\
Y(9) &= 0, \\
Y(10) &= -\frac{76e^{5\beta}}{9830400}.
\end{aligned}$$

For $N = 10$ the series solution is

$$y_{10}(x) = \beta - \frac{e^{\beta}}{4}x^2 + \frac{e^{2\beta}}{64}x^4 - \frac{3e^{3\beta}}{2304}x^6 + \frac{10e^{4\beta}}{98304}x^8 - \frac{76e^{5\beta}}{9830400}x^{10}. \tag{3.26}$$

Now, solve (3.26) for $\beta$ using boundary condition at $x = 1$,

$$\beta = 0.31519599235,$$

substituting $\beta$ value in (3.26) to get

$$\begin{aligned} y_{10}(x) &= 0.31519599235 - \frac{1}{4}e^{0.31519599235}x^2 + \frac{1}{64}e^{0.6303918470}x^4 - \frac{1}{1152}e^{0.9455877705}x^6 \\ &\quad + \frac{5}{49152}e^{1.260783694}x^8 - \frac{19}{2457600}e^{1.575979618}x^{10}, \\ &= 0.3151959235 - 0.3426319508x^2 + 0.02934916344x^4 - 0.002234658027x^6 \\ &\quad + 0.0003589055809x^8 - 0.00003738364589x^10. \end{aligned}$$

Figure 3-3 present the previous solution and the exact solution $2\ln\dfrac{4-2\sqrt{2}}{(3-2\sqrt{2})x^2+1}$.

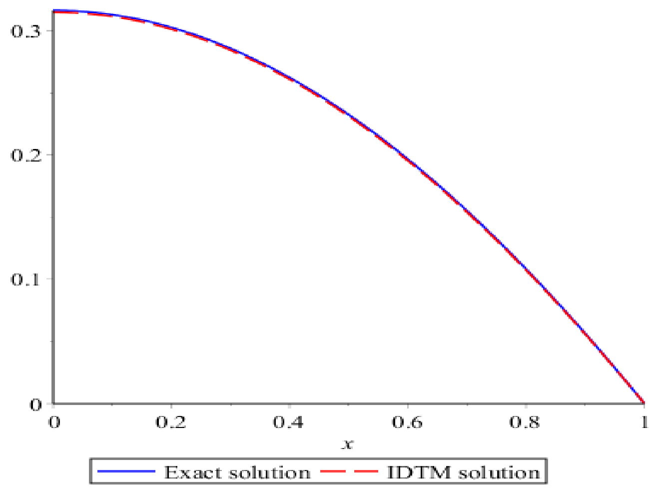


Figure 3-3: Comparison between the solution obtained by DTM and the solution obtained by IDTM.

## 3.2 DTM with Laplace transform and Padé approximation

In this section, we make a quick review of Laplace transform . Also we present the Padé approximation definition and proceeder briefly in order to introduce another modification of the DTM which is DTM with Laplace transform and Padé approximation.

**Laplace transform:**

**Definition 3.1.** Let $f(x)$ be a function defined for $x \geq 0$, then the *Laplace transform* of $f(x)$ denoted by $\mathcal{F}(s) = \mathcal{L}\{f(x)\}$ is defined as follows

$$\begin{aligned} \mathcal{F}(s) = \mathcal{L}\{f(x)\} &= \int_0^\infty e^{-sx} f(x)dx, \\ &= \lim_{a\to\infty} \int_0^a e^{-sx} f(x)dx. \end{aligned} \tag{3.27}$$

Also, the inverse Laplace transform denoted by $L^{-1}\{\mathcal{F}(s)\}$ is equal $f(x)$ if and only if $\mathcal{F}(s) = \mathcal{L}\{f(x)\}$.

The next table gives the Laplace transform of some functions calculated with the help of Definition 3.1.

| $f(x)$ | $\mathcal{F}(s)$ |
|---|---|
| 1 | $\frac{1}{s}$ |
| $x^n, \ n = 1, 2, ...$ | $\frac{n!}{s^{n+1}}$ |
| $e^{bx}$ | $\frac{1}{s-b}, \ s > a$ |
| $\sin(bx)$ | $\frac{b}{s^2+b^2}$ |
| $\cos(bx)$ | $\frac{s}{s^2+b^2}$ |
| $\sinh(bx)$ | $\frac{b}{s^2-b^2}$ |
| $\cosh(bx)$ | $\frac{s}{s^2-b^2}$ |
| $\mathcal{L}\{f'(x)\}$ | $s\mathcal{L}\{f(x)\} - f(0)$ |
| $\mathcal{L}\{f^{(n)}(x)\}$ | $s^n\mathcal{L}\{f(x)\} - s^{n-1}f'(0) - s^{n-2}f''(0) - ... - sf^{(n-1)}(0) - f^{(n)}(0)$ |

Table 3.1: Laplace transform of some functions

**Padé approximation:**

**Definition 3.2.** *Padé approximation* is a ratio of two polynomials come from the

Taylor series expansion of a function $y(x)$ and defined as

$$P_m^l = \frac{\sum_{n=0}^{l} a_n x^n}{\sum_{n=0}^{m} b_n x^n},$$

where $b_0 = 1$ by normalization.

Now, we can write the function $y(x)$ as

$$y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Also,

$$y(x) - P_m^l = O(x^{l+m+1}).$$

Thus,

$$\sum_{n=0}^{\infty} c_n x^n = \frac{\sum_{n=0}^{l} a_n x^n}{\sum_{n=0}^{m} b_n x^n},$$

or

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ... = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...}{1 + b_1 x + b_x^2 + b_3 x^3 + ...}. \tag{3.28}$$

From (3.28) we obtain the following system of equations

$$\begin{aligned}
a_0 &= c_0, \\
a_1 &= c_1 + c_0 b_1, \\
a_2 &= c_2 + c_1 b_1 + c_0 b_2, \\
a_3 &= c_3 + c_2 b_1 + c_1 b_2 + c_0 b_3, \\
a_4 &= c_4 + c_3 b_1 + c_2 b_2 + c_1 b_3 + c_0 b_4, \\
&\vdots
\end{aligned}$$

where $c_n$ is given.

In order to solve the above system for $a_n$ and $b_n$ we take the numerator degree to be $l$ and the denominator degree to be $m$, also we take the Taylor series expansion of $y(x)$ up to $x^{l+m}$. i.e. we want to solve the following system

$$\begin{aligned}
a_0 &= c_0, \\
a_1 &= c_1 + c_0 b_1, \\
a_2 &= c_2 + c_1 b_1 + c_0 b_2, \\
a_3 &= c_3 + c_2 b_1 + c_1 b_2 + c_0 b_3, \\
a_4 &= c_4 + c_3 b_1 + c_2 b_2 + c_1 b_3 + c_0 b_4, \\
&\vdots \\
a_l &= c_l + c_{l-1} b_1 + c_{l-2} b_2 + ... + c_0 b_l, \\
0 &= c_{l+1} + c_l b_1 + c_{l-1} b_2 + ... + c_{l-m+1} b_m, \\
0 &= c_{l+2} + c_{l+1} b_1 + c_l b_2 + ... + c_{l-m+2} b_m, \\
&\vdots \\
0 &= c_{l+m} + c_{l+m-1} b_1 + c_{l+m-2} b_2 + ... + c_l b_m.
\end{aligned}$$

The next example illustrates this approximation.

**Example 3.5.** Consider the exponential function $e^x$ .
The Maclaurin series of $e^x$ is $f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + ....$

**Solution.** To find $P_3^2$ of $f(x)$ we must solve the following system of equations

$$
\begin{aligned}
a_0 &= 1,\\
a_1 &= 1 + b_1,\\
a_2 &= \frac{1}{2} + b_1 + b_2,\\
0 &= \frac{1}{6} + \frac{1}{2}b_1 + b_2 + b_3,\\
0 &= \frac{1}{24} + \frac{1}{6}b_1 + \frac{1}{2}b_2 + b_3,\\
0 &= \frac{1}{120} + \frac{1}{24}b_1 + \frac{1}{6}b_2 + \frac{1}{2}b_3.
\end{aligned}
$$

We obtain the following

$$
\begin{aligned}
b_1 &= -\frac{3}{5},\\
b_2 &= \frac{3}{20},\\
b_3 &= -\frac{1}{60},\\
a_0 &= 1,\\
a_1 &= \frac{2}{5},\\
a_2 &= \frac{1}{20}.
\end{aligned}
$$

Then so,

$$
1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 = \frac{1 + \frac{2}{5}x + \frac{1}{20}x^2}{1 - \frac{3}{5}x + \frac{3}{20}x^2 - \frac{1}{60}x^3}.
$$

Figure 3-4 shows both the Padé approximation and Maclaurin series expansion of $e^x$ . Also, by zooming Figure 3-5 confirm the accuracy of Padé approximation.

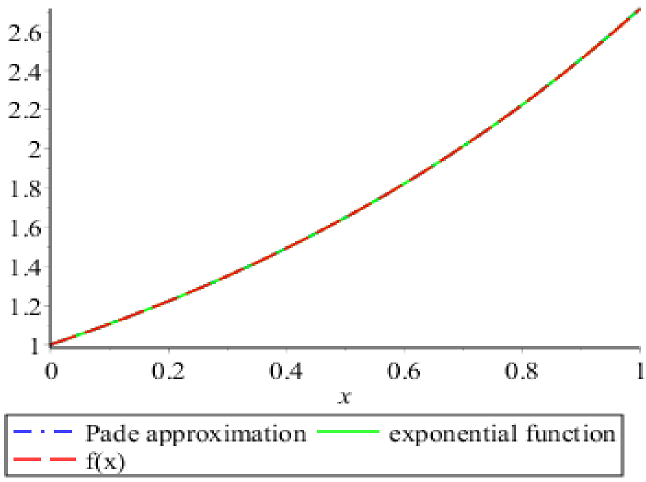


Figure 3-4: The Padé approximation of $f(x)$.

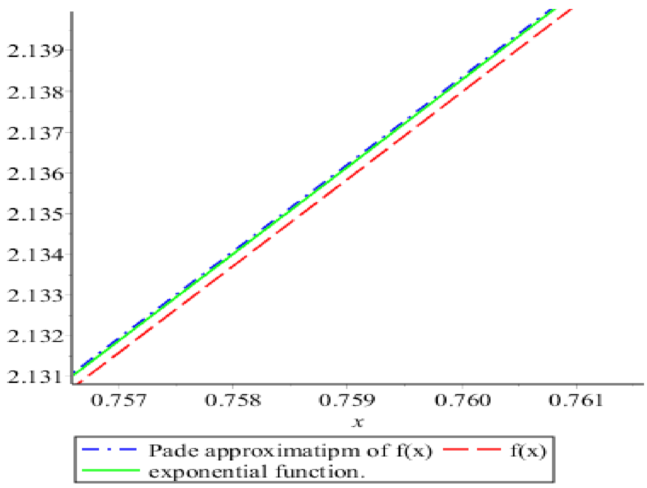


Figure 3-5: The Padé approximation of $f(x)$.

**DTM with Laplace transform and Padé approximation:**

This modification uses the series solution obtained by using DTM and take the Laplace transform of this solution. After that we apply the Padé approximation. To get the final solution by this modification we take the inverse Laplace transform of the Padé approximation. this method can keep the periodic behavior of the solution. The next examples illustrate the idea. The results are calculated by using MAPLE. See [37, 39].

**Example 3.6.** Consider the following initial value problem

$$y''(x) + 2y^2(x) + 4y(x) = 1 + \cos(4x), \tag{3.29}$$

subject to

$$y(0) = 1 \text{ and } y'(0) = 0. \tag{3.30}$$

**Solution.** Firstly, we apply the DTM on (3.29) to obtain the following recurrence relation

$$Y(k+2) = \frac{1}{(k+1)(k+2)}\left(-2\sum_{r=0}^{k} Y(r)Y(k-r) - 4Y(k) + \delta(k) + \frac{4^k}{k!}\cos\left(\frac{k\pi}{2}\right)\right)$$

Also from (3.30) we get

$$Y(0) = 1 \text{ and } Y(1) = 0. \tag{3.31}$$

So, by using (3.31) and (3.31) for $k = 0, ..., 3$ we have the following values

$$\begin{aligned} Y(2) &= -2, \\ Y(3) &= 0, \\ Y(4) &= \frac{2}{3}, \\ Y(5) &= 0, \end{aligned}$$

Thus, the series solution is

$$y(x) \approx 1 - 2x^2 + \frac{2}{3}x^4. \tag{3.32}$$

Now, by taking Laplace transform of (3.32) we get

$$\mathcal{L}\{y(x)\} = \frac{1}{s} - \frac{4}{s^3} + \frac{16}{s^5}. \tag{3.33}$$

To simplify $\mathcal{L}\{y(x)\}$ let $s = \dfrac{1}{x}$ then (3.33) becomes

$$\mathcal{L}\{y(x)\} = x - 4x^3 + 16x^5. \tag{3.34}$$

The $P_3^3$ of (3.34) equal

$$P_3^3 = \frac{x}{1 + 4x^2}, \tag{3.35}$$

replacing $x$ by $\dfrac{1}{s}$ in (3.35)

$$P_3^3 = \frac{1}{s(\frac{4}{s^2} + 1)}. \tag{3.36}$$

After that, we take the inverse Laplace transform of (3.36) to get the following solution

$$y(x) = \cos(2x).$$

By this method we obtain the solution in terms of cosine function.

Figure 3-6 shows that the solution obtained by DTM with Laplace transform and Padê approximation, the DTM solution and the exact solution.

After zooming, Figure 4.24 show that the solution obtained by DTM with Laplace transform and Padê approximation is closer than the DTM solution to the exact

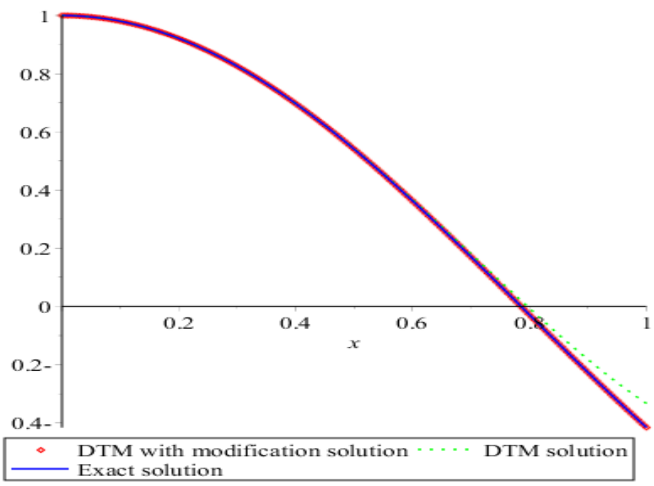


Figure 3-6: The solution obtained by DTM with Laplace transform and Padê approximation and DTM solution together with the exact solution.

one.

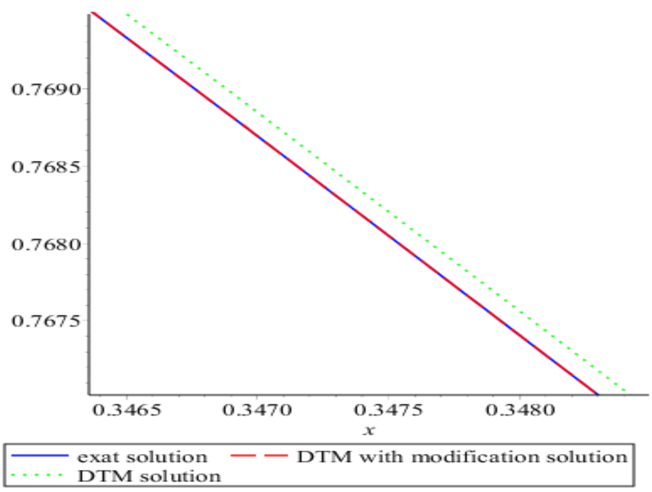


Figure 3-7: Comparison between the solution obtained by DTM with Laplace transform and Padê approximation and DTM solution.

**Example 3.7.** Consider the following initial value problem

$$y''(x) - 0.04(1 - y^2(x))y'(x) + y(x) = 0.04\sin(1.4x), \tag{3.37}$$

subject to

$$y(0) = 1 \text{ and } y'(0) = 0. \tag{3.38}$$

**Solution.** Firstly, we apply the DTM on (3.37) to obtain the following recurrence relation

$$\begin{aligned} Y(k+2) &= \frac{1}{(k+1)(k+2)}\left(0.04(k+1)Y(k+1) - y(k) + 0.04\frac{(1.4)^k}{k!}\sin\left(\frac{k\pi}{2}\right)\right) \\ &\quad - \frac{0.04}{(k+1)(k+2)}\left(\sum_{r=0}^{k}\sum_{m=0}^{r}(m+1)Y(m+1)Y(r-m)Y(k-r)\right). \end{aligned} \tag{3.39}$$

Also from (3.38) we get

$$Y(0) = 1 \text{ and } Y(1) = 0. \tag{3.40}$$

So, by using (3.39) and (3.40) for $k = 0, ..., 6$ we have the following values

$$\begin{aligned} Y(2) &= -0.5000000000, \\ Y(3) &= 0.009333330373, \\ Y(4) &= 0.04167186932, \\ Y(5) &= -0.003381330574, \\ Y(6) &= -0.001327519922, \\ Y(7) &= 0.0005989144050, \\ Y(8) &= -0.000008721286018, \end{aligned}$$

Thus, the series solution is

$$\begin{aligned} y(x) \quad &\approx \quad 1 - 0.5000000000x^2 + 0.009333330373x^3 + 0.04167186932x^4 - 0.003381330574x^5 \\ &- 0.001327519922x^6 + 0.0005989144050x^7 - 0.000008721286018x^8. \end{aligned} \tag{3.41}$$

Now, by taking Laplace transform of (3.41) we get

$$\begin{aligned} \mathcal{L}\{y(x)\} &= \frac{1}{s} - \frac{1}{s^3} - \frac{0.05599998224}{s^4} + \frac{1.000124864}{s^5} - \frac{0.4057596689}{s^6} - \frac{0.9558143438}{s^7} \\ &+ \frac{3.018528601}{s^8} - \frac{0.3516422530}{s^9}. \end{aligned} \tag{3.42}$$

Let $s = \dfrac{1}{x}$ for simplicity then (3.42) becomes

$$\begin{aligned} \mathcal{L}\{y(x)\} &= x - x^3 + 0.05599998224x^4 + 1.000124864x^5 - 0.4057596689x^6 - 0.9558143438x^7 \\ &+ 3.018528601x^8 - 0.3516422530x^9. \end{aligned} \tag{3.43}$$

The $P_4^4$ of (3.43) equal

$$P_4^4 = \frac{x + 0.1703725200x^2 + 7.727570742x^3 + 0.1390053934x^4}{1 + 0.1703725200x + 8.727570742x^2 + .2533779311x^3 + 7.717905020x^4}, \tag{3.44}$$

replacing $x$ by $\dfrac{1}{s}$ in (3.44)

$$P_4^4 = \frac{s^3 + 0.1703725200s^2 + 7.727570742s + 0.1390053934}{s^4 + 0.1703725200s^3 + 8.727570742s^2 + .2533779311s + 7.717905020}. \tag{3.45}$$

After that, we take the inverse Laplace transform of (3.45) to get the following solution

$$\begin{aligned} y(x) \quad &= \quad 0.0003101444994\cos(2.778582680x)e^{-0.07900300730x} \\ &- 0.002314542034\sin(2.778582680x)e^{-0.07900300730x} \\ &+ 0.9996898554\cos(0.9994074949x)e^{-0.006183252709x} \\ &+ 0.01264447568\sin(0.9994074949x)e^{-0.006183252709x}. \end{aligned}$$

For more examples refer to [1, 37, 39].

# Chapter 4

# DTM for partial differential equations

Partial differential equations are used to formulate several phenomena in real world. For example, wave , heat and fluid flows. There are many methods to solve partial differential equations, one of these methods is the DTM. In this chapter, we use the so called *Two Dimensional Differential Transform Method* (TDDTM) to solve partial differential equations. Then, we introduce the *Reduced Differential Transform Method* (RDTM) and *Laplace Differential Transform Method* (LDTM) which used also to solve this type of equations.

## 4.1 The TDDTM

In this section, we intoduce the TDDTM definition. Also, some theorems are given to help us in solving partial differential equations. In addition, some examples are presented to illustrate this method. For the material of this section we refer to [15, 40, 43].

**Definition 4.1.** Let $y(x,t)$ be a function of two variables which is analytic and continuously differentiable on the nonnegative integers. Then the *two dimensional differential transform* of the function $y(x,t)$ defined as follows

$$Y(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h} y(x,t)}{\partial x^k \partial t^h}\right]_{(x,t)=(0,0)}, \tag{4.1}$$

where $y(x,t)$ is the original function and $Y(k,h)$ is the transformed function. Differential inverse transform of $Y(k,h)$ is defined as

$$y(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} Y(k,h) x^k t^h. \tag{4.2}$$

By substituting equation(4.1) in (4.2) we get

$$y(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{x^k t^h}{k!h!} \frac{\partial^{k+h} y(x,t)}{\partial x^k \partial t^h}\bigg|_{(x,t)=(0,0)}. \tag{4.3}$$

Definition 4.1 is true when $(x,t) = (x_0, t_0)$.

The next theorem indicates the linearity property of the TDDTM.

**Theorem 4.1.** *If $y(x,t) = \alpha u(x,t) + \beta v(x,t)$, then $Y(k,h) = \alpha U(k,h) + \beta V(k,h)$, where $\alpha$ and $\beta$ are constants.*

*Proof.* Let $y(x,t)$ be the original function, then the differential transform of $y(x,t)$ is given by

$$\begin{aligned}
Y(k,h) &= \frac{1}{k!h!} \frac{\partial^{k+h} y(x,t)}{\partial x^k \partial t^h}\bigg|_{(x,t)=(0,0)} \\
&= \frac{1}{k!h!} \frac{\partial^{k+h} (\alpha u(x,t)) + (\beta v(x,t))}{\partial x^k \partial t^h}\bigg|_{(x,t)=(0,0)} \\
&= \frac{1}{k!h!}\left[\frac{\partial^{k+h}(\alpha u(x,t))}{\partial x^k \partial t^h} + \frac{\partial^{k+h}(\beta v(x,t))}{\partial x^k \partial t^h}\right]_{(x,t)=(0,0)} \\
&= \frac{\alpha}{k!h!} \frac{\partial^{k+h} u(x,t)}{\partial x^k \partial t^h}\bigg|_{(x,t)=(0,0)} + \frac{\beta}{k!h!} \frac{\partial^{k+h} v(x,t)}{\partial x^k \partial t^h}\bigg|_{(x,t)=(0,0)} \\
&= \alpha U(k,h) + \beta V(k,h).
\end{aligned}$$

□

The next theorems give the two dimensional differential transform of some functions.

**Theorem 4.2.** *If* $y(x,t) = \dfrac{\partial u(x,t)}{\partial x}$, *then* $Y(k,h) = (k+1)U(k+1,h)$.

*Proof.* Let $y(x,t)$ be the original function, then the differential transform of $y(x,t)$ is given by

$$\begin{aligned}
Y(k,h) &= \frac{1}{k!h!}\frac{\partial^{k+h}y(x,t)}{\partial x^k \partial t^h}|_{(x,t)=(0,0)} \\
&= \frac{1}{k!h!}\frac{\partial^{k+h}(\frac{\partial u(x,t)}{\partial x})}{\partial x^k \partial t^h}\Big|_{(x,t)=(0,0)} \\
&= \frac{1}{k!h!}\left[\frac{\partial^{(k+1)+h}(u(x,t))}{\partial x^{k+1}\partial t^h}\right]_{(x,t)=(0,0)} \\
&= (k+1)U(k+1,h).
\end{aligned}$$

□

Similarly, when $y(x,t) = \dfrac{\partial u(x,t)}{\partial t}$, then $Y(k,h) = (h+1)U(k,h+1)$.

**Theorem 4.3.** *If* $y(x,t) = \dfrac{\partial^{r+s}u(x,t)}{\partial x^r \partial t^s}$, *then*

$$Y(k,h) = \frac{(k+r)!(h+s)!}{k!h!}U(k+r,h+s).$$

*Proof.* Let $y(x,t)$ be the original function, then the differential transform of $y(x,t)$ is given by

$$\begin{aligned}
Y(k,h) &= \frac{1}{k!h!}\frac{\partial^{k+h}y(x,t)}{\partial x^k \partial t^h}\Big|_{(x,t)=(0,0)} \\
&= \frac{1}{k!h!}\frac{\partial^{k+h}(\frac{\partial^{r+s}u(x,t)}{\partial x^r \partial t^s})}{\partial x^k \partial t^h}\Big|_{(x,t)=(0,0)} \\
&= \frac{1}{k!h!}\left[\frac{\partial^{(k+r)+(h+s)}(u(x,t))}{\partial x^{k+r}\partial t^{h+s}}\right]_{(x,t)=(0,0)} \\
&= \frac{(k+r)!(h+s)!}{k!h!}U(k+r,h+s)
\end{aligned}$$

□

**Theorem 4.4.** *If $y(x,t) = u(x,t)v(x,t)$, then*

$$Y(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)V(k-r,s).$$

*Proof.* See [6]. □

**Theorem 4.5.** *If $y(x,t) = \dfrac{\partial u(x,t)}{\partial x}\dfrac{\partial v(x,t)}{\partial x}$, then*

$$Y(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h}(r+1)(k-r+1)U(r+1,h-s)V(k-r+1,s).$$

*Proof.* Let $\dfrac{\partial u(x,t)}{\partial x} = u^*(x,t)$ and $\dfrac{\partial v(x,t)}{\partial x} = v^*(x,t)$. Then by using Theorem 4.2 and Theorem4.4 we get

$$\begin{aligned} Y(k,h) &= \sum_{r=0}^{k}\sum_{s=0}^{h} U^*(r,h-s)V^*(k-r,s) \\ &= \sum_{r=0}^{k}\sum_{s=0}^{h}(r+1)(k-r+1)U(r+1,h-s)V(k-r+1,s). \end{aligned}$$

□

Similarly, if $y(x,t) = \dfrac{\partial u(x,t)}{\partial t}\dfrac{\partial v(x,t)}{\partial t}$, then

$$Y(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h}(s+1)(h-s+1)U(r,h-s+1)V(k-r,s+1).$$

**Theorem 4.6.** *If $y(x,t) = x^m t^n$, then $Y(k,h) = \delta(k-m)\delta(h-n)$, where*

$$\delta(k-m) = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k \neq m. \end{cases}$$

$$\delta(h-n) = \begin{cases} 1 & \text{if } h = n, \\ 0 & \text{if } h \neq n. \end{cases}$$

*Proof.* Let $y(x,t)$ be the original function, then the differential transform of $y(x,t)$ is given by

$$\begin{aligned} Y(k,h) &= \frac{1}{k!h!}\frac{\partial^{k+h}y(x,t)}{\partial x^k \partial t^h}\bigg|_{(x,t)=(0,0)} \\ &= \frac{1}{k!h!}\frac{\partial^{k+h}(x^m t^n)}{\partial x^k \partial t^h}\bigg|_{(x,t)=(0,0)} \\ &= \left[\frac{1}{k!}\frac{\partial^k x^m}{\partial x^k}\right]_{x=0}\left[\frac{1}{h!}\frac{\partial^h t^n}{\partial t^h}\right]_{t=0} \\ &= \left[\frac{1}{k!}\frac{d^k x^m}{dx^k}\right]_{x=0}\left[\frac{1}{h!}\frac{d^h t^n}{dt^h}\right]_{t=0} \\ &= \delta(k-m)\delta(h-n). \end{aligned}$$

□

**Theorem 4.7.** *If $y(x,t) = x^m e^{\lambda t}$, then $Y(k,h) = \delta(k-m)\dfrac{\lambda^h}{h!}$, where $\lambda$ is constant.*

*Proof.* Let $y(x,t)$ be the original function, then the differential transform of $y(x,t)$ is given by

$$\begin{aligned} Y(k,h) &= \frac{1}{k!h!}\frac{\partial^{k+h}y(x,t)}{\partial x^k \partial t^h}\bigg|_{(x,t)=(0,0)} \\ &= \frac{1}{k!h!}\frac{\partial^{k+h}(x^m e^{\lambda t})}{\partial x^k \partial t^h}\bigg|_{(x,t)=(0,0)} \\ &= \left[\frac{1}{k!}\frac{\partial^k x^m}{\partial x^k}\right]_{x=0}\left[\frac{1}{h!}\frac{\partial^h e^{\lambda t}}{\partial t^h}\right]_{t=0} \\ &= \left[\frac{1}{k!}\frac{d^k x^m}{dx^k}\right]_{x=0}\left[\frac{1}{h!}\frac{d^h e^{\lambda t}}{dt^h}\right]_{t=0} \\ &= \frac{\lambda^h}{h!}\delta(k-m). \end{aligned}$$

□

**Theorem 4.8.** *If* $y(x,t) = x^m \sin(\alpha t+\beta)$, *then* $Y(k,h) = \delta(k-m)\dfrac{\alpha^h}{h!}\sin\left(\dfrac{h\pi}{2}+\beta\right)$, *where* $\alpha$ *and* $\beta$ *are constants.*

*Proof.* Let $y(x,t)$ be the original function, then the differential transform of $y(x,t)$ is given by

$$\begin{aligned}
Y(k,h) &= \frac{1}{k!h!}\frac{\partial^{k+h}y(x,t)}{\partial x^k \partial t^h}\bigg|_{(x,t)=(0,0)} \\
&= \frac{1}{k!h!}\frac{\partial^{k+h}(x^m \sin(\alpha t+\beta))}{\partial x^k \partial t^h}\bigg|_{(x,t)=(0,0)} \\
&= \left[\frac{1}{k!}\frac{\partial^k x^m}{\partial x^k}\right]_{x=0}\left[\frac{1}{h!}\frac{\partial^h \sin(\alpha t+\beta)}{\partial t^h}\right]_{t=0} \\
&= \left[\frac{1}{k!}\frac{d^k x^m}{dx^k}\right]_{x=0}\left[\frac{1}{h!}\frac{d^h \sin(\alpha t+\beta)}{dt^h}\right]_{t=0} \\
&= \frac{\alpha^h}{h!}\delta(k-m)\sin\left(\frac{h\pi}{2}+\beta\right).
\end{aligned}$$

□

Similarly, if $y(x,t) = x^m \cos(\alpha t+\beta)$, then $Y(k,h) = \dfrac{\alpha^h}{h!}\delta(k-m)\cos\left(\dfrac{h\pi}{2}+\beta\right)$, where $\alpha$ and $\beta$ are constants.

Now, we explain the TDDTM through the following examples.

The following example is taken from [15].

**Example 4.1.** Consider the following nonlinear partial differential equation

$$y_{tt}(x,t) - y(x,t)y_{xx}(x,t) = 1 - \frac{t^2+x^2}{2}, \quad x \geq 1,\ t \geq 0, \tag{4.4}$$

with initial conditions

$$y(x,0) = \frac{x^2}{2}, \quad y_t(x,0) = 0 \tag{4.5}$$

**Solution.** By applying the TDDTM on Equation (4.4) we have

$$(h+1)(h+2)Y(k,h+2) - \sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)(k-r+2)Y(k,h-s)Y(k-r+2,s)$$
$$= \delta(k)\delta(h) - \frac{1}{2}\delta(k)\delta(h-2) - \frac{1}{2}\delta(k-2)\delta(h),$$

This leads to the following relation

$$\begin{aligned} Y(k,h+2) &= \sum_{r=0}^{k}\sum_{s=0}^{h}\frac{(k-r+1)(k-r+2)}{(h+1)(h+2)}Y(k,h-s)Y(k-r+2,s) \\ &\quad + \frac{\big(\delta(k)\delta(h) - \frac{1}{2}\delta(k)\delta(h-2) - \frac{1}{2}\delta(k-2)\delta(h)\big)}{(h+1)(h+2)}. \end{aligned} \tag{4.6}$$

From the first initial condition in (4.5) we get

$$\sum_{k=0}^{\infty} Y(k,0)x^k = \sum_{k=0}^{\infty}\frac{x^k}{k!}\frac{d^k(\frac{x^2}{2})}{dx^k}|_{x=0},$$

thus,

$$Y(k,0) = \begin{cases} \dfrac{1}{2} & \text{if } k = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Also, from the second initial condition we have

$$Y(k,1) = 0, \quad k = 0, 1, ....$$

Using the last new conditions and relation (4.6) we obtain the following

$$\begin{aligned}
Y(0,2) &= \frac{1}{2}, \\
Y(1,2) &= \frac{1}{2}(6Y(0,0)Y(3,0) + 2Y(1,0)Y(2,0) + 0 = 0, \\
Y(2,2) &= \frac{1}{2}(12Y(0,0)Y(4,0) + 6Y(1,0)Y(3,0) + 2Y(2,0)Y(2,0) - \frac{1}{2} = 0, \\
&\vdots
\end{aligned}$$

Continue in this way, we obtain

| k/h | 0 | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | $\frac{1}{2}$ | 0 | 0 | ... |
| 1 | 0 | 0 | 0 | 0 | 0 | ... |
| 2 | $\frac{1}{2}$ | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | ... |
| 4 | 0 | 0 | 0 | 0 | 0 | ... |
| $\vdots$ | ... | | | | | |

Table 4.1: Valves of $Y(k,h)$.

Hence, we can write the solution as

$$y(x,t) = \frac{x^2 + t^2}{2}$$

which is the exact solution.

**Example 4.2.** Consider the following nonlinear partial differential equation

$$y_{tt}(x,t) = y_{xx}(x,t) + y(x,t) + y^2(x,t) - xt - x^2t^2, \quad 0 < x, \ t > 0, \tag{4.7}$$

with initial conditions

$$y(x,0) = 0, \quad y_t(x,0) = x \tag{4.8}$$

and boundary condition

$$y(0,t) = 0. \tag{4.9}$$

**Solution.** By applying the TDDTM on Equation (4.7) we have

$$\begin{aligned}(h+1)(h+2)Y(k,h+2) &= (k+1)(k+2)Y(k+2,h)+Y(k,h)\\ &\quad+\sum_{r=0}^{k}\sum_{s=0}^{h}Y(r,h-s)Y(k-r,s)\\ &\quad-\delta(k-1)\delta(h-1)-\delta(k-2)\delta(h-2),\end{aligned}$$

This leads to the following relation

$$\begin{aligned}Y(k,h+2) &= \frac{1}{(h+1)(h+2)}[(k+1)(k+2)Y(k+2,h)+Y(k,h)\\ &\quad+\sum_{r=0}^{k}\sum_{s=0}^{h}Y(r,h-s)Y(k-r,s)\\ &\quad-\delta(k-1)\delta(h-1)-\delta(k-2)\delta(h-2)].\end{aligned} \tag{4.10}$$

From the first initial condition in (4.8) we get

$$Y(k,0)=0, \quad k=0,1,....$$

Also, from the second initial condition we have

$$Y(k,1)=\begin{cases}1 & \text{if } k=1,\\ 0 & \text{otherwise.}\end{cases}$$

From the boundary condition in (4.9) we get

$$Y(0,h)=0, \quad h=2,3,...,$$

Using the last new conditions and relation (4.10) we obtain the following

$$
\begin{aligned}
Y(0,2) &= 0, \\
Y(1,2) &= \frac{1}{2}(6Y(3,0)+Y(1,0)+2Y(0,0)Y(1,0)) = 0, \\
Y(2,2) &= \frac{1}{2}(12Y(4,0)+Y(2,0)+2Y(0,0)Y(2,0)+Y(1,0)Y(1,0)) = 0, \\
&\vdots
\end{aligned}
$$

By continue in this way, we obtain

| k/h | 0 | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | ... |
| 1 | 0 | 1 | 0 | 0 | 0 | ... |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 0 | 0 | 0 | ... |
| 4 | 0 | 0 | 0 | 0 | 0 | ... |
| $\vdots$ | ... | | | | | |

Table 4.2: Valves of $Y(k,h)$.

Hence, we can write the solution as

$$y(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} Y(k,h)x^k t^h = xt.$$

which is the exact solution.

For more examples refer to [15, 40, 43].

## 4.2 The RDTM

When we use the TDDTM to solve partial differential equations sometimes we have complex calculations. Therefore, in this section we introduce the concept of RDTM. In addition, we give some basic theorems with proofs. See [16, 30, 32, 44].

**Definition 4.2.** Let $y(x,t)$ be analytic and continuously differentiable function. Then the *reduced differential transform* of the function $y(x,t)$ defined as follows

$$Y_k(x) = \frac{1}{k!}\left[\frac{\partial^k y(x,t)}{\partial t^k}\right]_{t=0}, \tag{4.11}$$

where $y(x,t)$ is the original function and $Y_k(x)$ is the transformed function. Differential inverse transform of $Y_k(x)$ is defined as

$$y(x,t) = \sum_{k=0}^{\infty} Y_k(x)t^k. \tag{4.12}$$

By substituting equation(4.11) in (4.12) we get

$$y(x,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\frac{\partial^k y(x,t)}{\partial t^k}\bigg|_{t=0}. \tag{4.13}$$

The next theorem indicates the linearity property of the RDTM.

**Theorem 4.9.** *If $y(x,t) = \alpha u(x,t) + \beta v(x,t)$, then $Y_k(x) = \alpha U_k(x) + \beta V_k(x)$, where $\alpha$ and $\beta$ are constants.*

*Proof.* Let $y(x,t)$ be the original function, then the differential transform of $y(x,t)$ is given by

$$\begin{aligned}
Y_k(x) &= \frac{1}{k!}\frac{\partial^k y(x,t)}{\partial t^k}\bigg|_{t=0} \\
&= \frac{1}{k!}\frac{\partial^k(\alpha u(x,t)) + (\beta v(x,t))}{\partial t^k}\bigg|_{t=0} \\
&= \frac{1}{k!}\left[\frac{\partial^k(\alpha u(x,t))}{\partial t^k} + \frac{\partial^k(\beta v(x,t))}{\partial t^k}\right]_{t=0} \\
&= \frac{\alpha}{k!}\frac{\partial^k u(x,t)}{\partial t^k}\bigg|_{t=0} + \frac{\beta}{k!}\frac{\partial^k v(x,t)}{\partial t^k}\bigg|_{t=0} \\
&= \alpha U_k(x) + \beta V_k(x).
\end{aligned}$$

□

The following theorems give the reduced differential transform of some functions.

**Theorem 4.10.** *If* $y(x,t) = \dfrac{\partial^r u(x,t)}{\partial t^r}$, *then* $Y_k(x) = \dfrac{(k+r)!}{k!} U_{k+r}(x)$.

*Proof.* Let $y(x,t)$ be the original function, then the differential transform of $y(x,t)$ is given by

$$\begin{aligned}
Y_k(x) &= \frac{1}{k!} \frac{\partial^k y(x,t)}{\partial t^k}\bigg|_{t=0} \\
&= \frac{1}{k!} \frac{\partial^k (\frac{\partial^r u(x,t)}{\partial t^r})}{\partial t^k}\bigg|_{t=0} \\
&= \frac{(k+r)!}{k!(k+r)!} \left[\frac{\partial^{k+r} u(x,t)}{\partial t^{k+r}}\right]_{t=0} \\
&= \frac{(k+r)!}{k!} U_{k+r}(x).
\end{aligned}$$

□

**Theorem 4.11.** *If* $y(x,t) = \dfrac{\partial u(x,t)}{\partial x}$, *then*

$$Y_k(x) = \frac{\partial}{\partial x} U_k(x).$$

*Proof.* Let $y(x,t)$ be the original function, then the differential transform of $y(x,t)$ is given by

$$\begin{aligned}
Y_k(x) &= \frac{1}{k!} \frac{\partial^k y(x,t)}{\partial t^k}\bigg|_{t=0} \\
&= \frac{1}{k!} \frac{\partial^k (\dfrac{\partial u(x,t)}{\partial x})}{\partial t^k}\Bigg|_{t=0} \\
&= \frac{\partial}{\partial x} U_k(x).
\end{aligned}$$

□

**Theorem 4.12.** *If* $y(x,t) = u(x,t)v(x,t)$, *then*

$$Y_k(x) = \sum_{r=0}^{k} U_r(x) V_{k-r}(x).$$

*Proof.* Let $y(x,t)$ be the original function, then the differential transform of $y(x,t)$

is given by

$$\begin{aligned} Y_k(x) &= \frac{1}{k!}\frac{\partial^k y(x,t)}{\partial t^k}\bigg|_{t=0} \\ &= \frac{1}{k!}\frac{\partial^k (u(x,t)v(x,t))}{\partial t^k}\bigg|_{t=0} \end{aligned}$$

Now, the Leibnitz rule for partial derivatives of function of several variables

$$\frac{\partial^k}{\partial t^k}(u(x,t)v(x,t)) = \sum_{r=0}^{k}\binom{k}{r}\frac{\partial^r}{\partial t^r}u(x,t)\frac{\partial^{k-r}}{\partial t^{k-r}}v(x,t). \tag{4.14}$$

By using (4.14) we get

$$\begin{aligned} Y_k(x) &= \frac{1}{k!}\left[\sum_{r=0}^{k}\binom{k}{r}\frac{\partial^r u(x,t)}{\partial t^r}\bigg|_{t=0}\frac{\partial^{k-r} v(x,t)}{\partial t^{k-r}}\bigg|_{t=0}\right], \\ &= \frac{1}{r!(k-r)!}\left[\sum_{r=0}^{k}\frac{\partial^r u(x,t)}{\partial t^r}\bigg|_{t=0}\frac{\partial^{k-r} v(x,t)}{\partial t^{k-r}}\bigg|_{t=0}\right], \\ &= \sum_{r=0}^{k} U_r(x)V_{k-r}(x). \end{aligned}$$

□

**Theorem 4.13.** *If $y(x,t) = x^m t^n$ then $Y_k(x) = x^m\delta(k-n)$, where*

$$\delta(k-n) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases}$$

*Proof.* Let $y(x,t)$ be the original function, then the differential transform of $y(x,t)$ is given by

$$\begin{aligned} Y_k(x) &= \frac{1}{k!}\frac{\partial^k y(x,t)}{\partial t^k}\bigg|_{t=0} \\ &= \frac{1}{k!}\frac{\partial^k (x^m t^n)}{\partial t^n}\bigg|_{t=0} \\ &= \frac{x^m}{k!}\frac{\partial^k t^n}{\partial t^k}\bigg|_{t=0} \\ &= x^m\delta(k-n). \end{aligned}$$

□

**Theorem 4.14.** *If $y(x,t) = x^m t^n u(x,t)$, then $Y_k(x) = x^m U_{k-n}(x)$.*

*Proof.* Let $y(x,t)$ be the original function, then the differential transform of $y(x,t)$ is given by

$$\begin{aligned} Y_k(x) &= \frac{1}{k!}\frac{\partial^k y(x,t)}{\partial t^k}\bigg|_{t=0} \\ &= \frac{1}{k!}\frac{\partial^k (x^m t^n u(x,t))}{\partial t^k}\bigg|_{t=0} \\ &= \frac{x^m}{k!}\frac{\partial^k t^n u(x,t)}{\partial t^k}\bigg|_{t=0} \\ &= x^m \sum_{r=0}^{k} \delta(r-n) U_{k-r}(x) \\ &= x^m U_{k-n}(x). \end{aligned}$$

□

Now, we apply the RDTM on the next examples.

**Example 4.3.** Consider the following nonlinear partial differential equation

$$y_t(x,t) - y(x,t)y_{xx}(x,t) - {y^2}_x(x,t) - y(x,t) = 0, \tag{4.15}$$

with initial condition

$$y(x,0) = \sqrt{x}. \tag{4.16}$$

**Solution.** By applying the RDTM on Equation (4.15) we have

$$(k+1)Y_{k+1}(x) - \sum_{r=0}^{k} Y_r(x)\frac{\partial^2}{\partial x^2}Y_{k-r}(x) - \sum_{r=0}^{k}\frac{\partial}{\partial x}Y_r(x)\frac{\partial}{\partial x}Y_{k-r}(x) - Y_k(x) = 0,$$

This leads to the following relation

$$Y_{k+1}(x) = \frac{1}{k+1}\left[\sum_{r=0}^{k} Y_r(x)\frac{\partial^2}{\partial x^2}Y_{k-r}(x) + \sum_{r=0}^{k}\frac{\partial}{\partial x}Y_r(x)\frac{\partial}{\partial x}Y_{k-r}(x) + Y_k(x)\right]. \tag{4.17}$$

From the initial condition (4.16) we get $Y_0(x) = \sqrt{x}$.

Now, by using $Y_0(x) = \sqrt{x}$ and (4.17)we obtain

$$\begin{aligned}
Y_1(x) &= Y_0(x)\frac{\partial^2}{\partial x^2}Y_0(x) + (\frac{\partial}{\partial x}Y_0(x))^2 + Y_0(x) = \sqrt{x}, \\
Y_2(x) &= \frac{1}{2}\left(Y_0(x)\frac{\partial^2}{\partial x^2}Y_1(x) + Y_1(x)\frac{\partial^2}{\partial x^2}Y_0(x) + 2\frac{\partial}{\partial x}Y_0(x)\frac{\partial}{\partial x}Y_1(x) + Y_1(x)\right) = \frac{\sqrt{x}}{2!}, \\
Y_3(x) &= \frac{1}{3}\left(Y_0(x)\frac{\partial^2}{\partial x^2}Y_2(x) + Y_1(x)\frac{\partial^2}{\partial x^2}Y_1(x) + Y_2(x)\frac{\partial^2}{\partial x^2}Y_0(x) + 2\frac{\partial}{\partial x}Y_0(x)\frac{\partial}{\partial x}Y_2(x)\right) \\
&\quad + \frac{1}{3}Y_2(x) = \frac{\sqrt{x}}{3!}, \\
&\vdots
\end{aligned}$$

Hence, we can write the solution as

$$\begin{aligned}
y(x,t) &= \sum_{k=0}^{\infty} Y_k(x)t^k, \\
&= \sqrt{x} + \sqrt{x}t + \frac{\sqrt{x}}{2!}t^2 + \frac{\sqrt{x}}{3!}t^3 + ... \\
&= \sqrt{x}e^t.
\end{aligned}$$

Which is the exact solution.

**Example 4.4.** Consider the following nonlinear partial differential equation

$$y_{tt}(x,t) = y_{xx}(x,t) + y(x,t) + y^2(x,t) - 2xt - 4x^2t^2, \quad 0 < x \leq \pi, \ t > 0, \qquad (4.18)$$

with initial conditions

$$y(x,0) = 0, \quad y_t(x,0) = 2x \qquad (4.19)$$

**Solution.** By applying the RDTM on Equation (4.18) we have

$$\begin{aligned}(k+1)(k+2)Y_{k+2}(x) &= \frac{\partial^2}{\partial x^2}Y_k(x) + Y_k(x) + \sum_{r=0}^{k} Y_r(x)Y_{k-r}(x) \\ &\quad - 2x\delta(k-1) - 4x^2\delta(k-2),\end{aligned}$$

This leads to the following relation

$$\begin{aligned}Y_{k+2}(x) &= \frac{1}{(k+1)(k+2)}\left(\frac{\partial^2}{\partial x^2}Y_k(x) + Y_k(x) + \sum_{r=0}^{k} Y_r(x)Y_{k-r}(x)\right) \\ &\quad + \frac{1}{(k+1)(k+2)}\left(-2x\delta(k-1) - 4x^2\delta(k-2)\right). \qquad (4.20)\end{aligned}$$

From the initial condition (4.19) we get $Y_0(x) = 0$.

Now, by using $Y_0(x) = 0$ and (4.20) we obtain

$$\begin{aligned}Y_1(x) &= 2x, \\ Y_2(x) &= 0, \\ Y_3(x) &= 0, \\ &\vdots\end{aligned}$$

Hence, we can write the solution as

$$\begin{aligned}y(x,t) &= \sum_{k=0}^{\infty} Y_k(x)t^k, \\ &= 2xt.\end{aligned}$$

Which is the exact solution.

For more examples refer to [16, 30, 32, 44].

## 4.3 The LDTM

In this section, we present the LDTM for solving partial differential equations. This modification joint the Laplace transform with DTM to handel the deficiency come from the unsatisfied boundary conditions in using DTM. An example is given to illustrate this modification. For more examples see [5, 34].

Consider the partial differential equation of the form

$$y_{tt}(x,t) + a_0(x)y(x,t) + a_1(x)y_x(x,t) + a_2(x)y_{xx}(x,t) = f(x,t), \tag{4.21}$$

with initial conditions

$$y(x,0) = g_1(x), \quad y_t(x,0) = g_2(x), \tag{4.22}$$

and boundary conditions

$$y(0,t) = h_1(t), \quad y_x(0,t) = h_2(t). \tag{4.23}$$

To solve this problem by using LDTM we follow the next steps:

- Apply the Laplace transform with respect to $t$ on (4.21) we have

  $$\mathcal{L}\{y_{tt}(x,t) + a_0(x)y(x,t) + a_1(x)y_x(x,t) + a_2(x)y_{xx}(x,t)\} = \mathcal{L}\{f(x,t)\},$$

  which equals

  $$\begin{aligned}\int_0^\infty e^{-st}(y_{tt}(x,t) + a_0(x)y(x,t) + a_1(x)y_x(x,t) + a + 2(x)y_{xx}(x,t))dt \\ = \int_0^\infty e^{-st}f(x,t)dt,\end{aligned}$$

after integrating we have

$$s^2y^*(x,s) - sy(x,0) - y_t(x,0) + a_0(x)y^*(x,s) + a_1(x)\frac{d}{dx}y^*(x,s)$$
$$+ a_2(x)\frac{d^2}{dx^2}y^*(x,s) = f^*(x,s),$$

where $y^*(x,s) = \mathcal{L}\{y(x,t)\}$.

Now, by using (4.22) we obtain

$$s^2y^*(x,s) - sg_1(x) - g_2(x) + a_0(x)y^*(x,s) + a_1(x)\frac{d}{dx}y^*(x,s)$$
$$+ a_2(x)\frac{d^2}{dx^2}y^*(x,s) = f^*(x,s). \tag{4.24}$$

Also, we Apply the Laplace transform with respect to $t$ on (4.23), then the boundary conditions become

$$y^*(0,s) = h_1^*(s), \quad y_x^*(0,s) = h_2^*(s). \tag{4.25}$$

Hence, we get an initial value problem for ordinary differential equation.

- Apply the DTM on (4.24) and (4.25) to get the following solution

$$y^*(x,s) = \sum_{k=0}^{\infty} Y(k)x^k. \tag{4.26}$$

- Take the inverse Laplace transform of (4.26) to get $y(x,t)$.

Now, we apply this modification on an example.

**Example 4.5.** Consider the following partial differential equation

$$y_{tt}(x,t) - y_{xx}(x,t) = \alpha t e^x, \quad x,t > 0 \tag{4.27}$$

subject to

$$y(x,0) = b\cos x, \quad y_t(x,0) = 0, \tag{4.28}$$

$$y(0,t) = \alpha(\sinh t - t) + b\cos t, \quad y_x(0,t) = \alpha(\sinh t - t). \tag{4.29}$$

**Solution.** By applying the Laplace transform on (4.27) and by using (4.28) we get

$$s^2 y^*(x,s) - sy(x,0) - y_t(x,0) - \frac{d^2}{dx^2} y^*(x,s) = \frac{\alpha e^x}{s^2},$$

or

$$s^2 y^*(x,s) - sb\cos x - \frac{d^2}{dx^2} y^*(x,s) = \frac{\alpha e^x}{s^2}, \tag{4.30}$$

also, by applying the Laplace transform on (4.29) we get

$$y^*(0,s) = \alpha\left(\frac{1}{s^2-1} - \frac{1}{s^2}\right) + b\frac{s}{s^2+1}, \quad y_x^*(0,s) = \alpha\left(\frac{1}{s^2-1} - \frac{1}{s^2}\right). \tag{4.31}$$

Now, apply the DTM on (4.30) to obtain the following relation

$$s^2 Y(k) - bs\frac{d^k \cos x}{dx^k}|_{x=0} - (k+1)(k+2)Y(k+2) = \frac{\alpha}{k! s^2},$$

or

$$Y(k+2) = \frac{-\dfrac{\alpha}{k! s^2} + s^2 Y(k) - bs\frac{d^k \cos x}{dx^k}|_{x=0}}{(k+1)(k+2)}, \tag{4.32}$$

also, on (4.31) to get

$$Y(0) = \alpha\left(\frac{1}{s^2-1} - \frac{1}{s^2}\right) + b\frac{s}{s^2+1}, \quad Y(1) = \alpha\left(\frac{1}{s^2-1} - \frac{1}{s^2}\right). \tag{4.33}$$

By using (4.32) and (4.33) we obtain the following

$$\begin{aligned}
Y(2) &= \frac{\alpha}{2}\frac{1}{s^2-1} - \frac{b}{2}\frac{s}{s^2+1} - \frac{\alpha}{2s^2}, \\
Y(3) &= \frac{\alpha}{6}\frac{1}{s^2-1} - \frac{\alpha}{6s^2}, \\
Y(4) &= \frac{\alpha}{24}\frac{1}{s^2-1} + \frac{b}{24}\frac{s}{s^2+1} - \frac{\alpha}{24s^2}. \\
&\vdots
\end{aligned}$$

Thus,

$$
\begin{aligned}
y^*(x,s) &= \sum_{k=0}^{\infty} Y(k)x^k, \\
&= \alpha\left(\frac{1}{s^2-1}-\frac{1}{s^2}\right)+b\frac{s}{s^2+1}+\alpha\left(\frac{1}{s^2-1}-\frac{1}{s^2}\right)x \\
&\quad +\left(\frac{\alpha}{2}\frac{1}{s^2-1}-\frac{b}{2}\frac{s}{s^2+1}-\frac{\alpha}{2s^2}\right)x^2 \\
&\quad +\left(\frac{\alpha}{6}\frac{1}{s^2-1}-\frac{\alpha}{6s^2}\right)x^3 \\
&\quad +\left(\frac{\alpha}{24}\frac{1}{s^2-1}+\frac{b}{24}\frac{s}{s^2+1}-\frac{\alpha}{24s^2}\right)x^4 \\
&\quad + ...
\end{aligned}
$$

Hence,

$$
\begin{aligned}
y(x,t) &= \mathcal{L}^{-1}\{y^*(x,s)\}, \\
&= \alpha(\sinh t - t) + b\cos t + (\alpha\sinh t - \alpha t)x + (\frac{\alpha}{2}\sinh t - \frac{b}{2}\cos t - \frac{\alpha}{2}t)x^2 \\
&\quad + (\frac{\alpha}{6}\sinh t - \frac{\alpha}{6}t)x^3 + (\frac{\alpha}{24}\sinh t + \frac{b}{24}\cos t - \frac{\alpha}{24}t)x^4 + ...,
\end{aligned}
$$

or

$$
\begin{aligned}
y(x,t) &= \alpha\sinh t(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ...) \\
&\quad + b\cos t(1 - \frac{x^2}{2} + \frac{x^4}{24} + ...) \\
&\quad - \alpha t(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ...),
\end{aligned}
$$

so

$$y(x,t) = \alpha e^x \sinh t + b\cos t\cos x - \alpha t e^x.$$

Which is the exact solution.

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