UNIVERSITY NAME (IN BLOCK CAPITALS)

Congruence Relations On Almost Distributive Lattices

by

Jihan

A thesis submitted in partial fulfillment for the degree of Doctor of Philosophy

in the

Faculty Name
Department or School Name

March 26, 2019

- –

Abstract

Abstract

The Program of Graduated Studies

Department of Mathematics

Deanship of Graduate Studies and Scientific Research

Congruence Relations On Almost Distributive Lattices

Prepared by: Jihan AL-Zughayyar

Supervisor

Dr. Nureddin Rabie

Master thesis submitted and accepted, Date: ——————.

The name and signature of the examining committee members:

Dr. Nureddin Rabie Head of committee signature ————

Dr. Ali Altawaiha External Examiner signature ————

Dr. Iyad Alhribat Internal Examiner signature ————

Dean of Graduate Studies and Scientific Research, signature ————

Palestine Polytechnic University, Palestine.

2019

Declaration

I declare that the master thesis entitled ”Congruence Relations On Almost Distributive Lattices” is my own work, and hereby certify that unless stated, all work con- tained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgment is made in the text.

Jihan AL-Zughayyar

Signature: ______________________________

Date: ______________________________

Dedication

To my parents,
To Dr. Nureddin Rabie, my thesis advisor,
To myself and
To my brothers and sisters.

Jihan AL-Zughayyar

Acknowledgements

I am very grateful to my supervisor Dr. Nureddin Rabie for all his help and encouragement.

I am very grateful to my internal referee Dr. Iyad Alhribat for his valuable suggestions on this thesis.

I thank my external referee Dr. Ali Altawaiha for his useful comments and advice.

Also, my thanks to the members of the department of mathematics at the Palestine Polytechnic University.

Abstract

This thesis aims to develop a better understanding of Almost Distributive Lattices and it's congruence relations. We present the definition of Almost Distributive Lattice, and state important properties of an ADL that will be used in developing further theory, we also introduce the concept of congruence relations, and discuss special congruence relations in an ADL. Furthermore we discuss the notion of derivation in an ADL and it's properties, then two kinds of congruence are proposed in an ADL. Also, we introduced the concept of Regular ring and study spacial congruence on it. Finally, we introduce the concept of $\theta-$ filters and $\theta-$ ideals in an ADL and characterized them in terms of ADL congruence.

Contents

Introduction

In 1854, George Boole introduced an important class of algebraic structures ”Boolean Algebra”, Pierce and Schroder continued to investigate the Boolean Algebra at the end of the nineteenth century and introduced the lattice as a generalization of a Boolean algebra. Independently, Richard Dedekind’s research on ideals of algebraic numbers led to the same discovery. Although some of the early results of these mathematicians were very elegant and far from trivial, they did not attract the attention of the mathematical community.

In the 19th century, important results due to Minkowshi, motivated the use of lattice theory in the theory and geometry of numbers. The evolution of computer science, in the 20th century led to lattice applications in various theoretical areas such as information theory, information access controls and cryptanalysis.

The notion of lattice ideals played an important role in lattice theoretical researches. On the other hand, the study of congruence relations on lattices had became a special interest to many authors.

In 1981, U.M. Swamy and G.C. Rao, introduced another type of generalization of a Boolean algebra called Almost Distributive Lattice(ADL) which is neither complemented nor distributive. For that matter, which is not even a lattice, the class of distributive lattices has occupied a major part of the present lattice theory since lattices were abstracted from Boolean algebras though the class of distributive lattices and the class of distributive lattices has many interesting properties which lattices, in general, do not have. On the other hand, as observed by M.H. Ston, a Boolean algebra also has a ring structure (called a Boolean ring). Several mathematicians have worked on the ring theoretic generalization of a Boolean algebra. Prominent among them are P-rings, regular rings and biregular rings. In order to obtain common abstraction to almost all the existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other hand, in 1981, Swamy and Rao, introduced the concept of an Almost Distributive Lattice as an algebra $(R, \wedge, \vee, 0)$.

An ADL satisfies all the properties of a distributive lattice with 0 except possibly the commutativity of $\vee$, the commutativity of $\wedge$ and the right distributivity of $\vee$ over $\wedge$ and every non- empty set can be regarded as an ADL with any arbitrary chosen element as 0 (called discrete ADL). Many important fundamental concepts like ideals, filters, prime ideals etc. Were extended to the class of ADLs. It was observed that the set of all principal ideals of an ADL forms a distributive lattice. This provided a path to extend many existing concepts of lattice theory to the class of ADLs.

Recently, Sambasiva rao introduced the concepts of congruence with the help of derivation in distributive lattice and studied their properties. The study of congruence is important both from theoretical stand point and for its applications in the field of logic based approaches to uncertainty.

In 2012, and based on the concept of multiplicatively closed subsets of an ADL, special congruence relations ψ^s and ϕ^s are introduced on an ADL by Y. Pawar and I. Shaikh. Later in 2013, N. Rafi, G. Rao and R. Banbaru introduced the concept of $\theta-$ Filter in ADL, and then characterized it in terms of ADL congruence. Also they introduce the concept of $\theta-$ prime filters and established a set of equivalent conditions for every $\theta-$ filter which becomes a $\theta-$ prime filter. In 2014, since the usual lattice theoretic duality principle doesn't hold in ADL, N. Rafi, introduced the concept of $\theta-$ ideal, and then characterized it in terms of ADL congruence. In 2015, N. Rafi, R. Bandaru and G. Rao, introduced the concept of a derivation in ADL, and discussed two kinds of congruence in ADL.

The thesis is mainly divided into 3 chapters. We give briefly the summary of the main results proved in this theses.

In Chapter 1, we collect the necessary definitions and results (from different sources) which are required in the subsequent chapters. It consists of three sections. Section 1.1 contains definitions and results related to partially ordered sets. In section 1.2, we give the definitions of lattice, homomorphism, ideal and filter. In section 1.3, we introduce the concept of congruence in some algebra and then on lattices.

Chapter 2 is divided into four sections. In section 2.1, we give the definitions of an ADL and state important properties of an ADL that will be used in developing the further theory, also we collect definitions and some preliminary results related to ideals, filters, homomorphism, congruence ect. In section 2.2, special congruence relations ψ^s and ϕ^s are introduced on an ADL. In section 2.3, we introduce the concept of a derivation in an ADL, then two kinds of congruences are proposed in ADL. In section 2.4, we present the definition of regular ring, some of it's properties and theorems are studied, we find a relation between ADLs and regular rings, we also discuss some congruence relations in regular rings.

Chapter 3 is divided into two sections. In section 3.1, we introduce the concept of $\theta-$ filters in an ADL, and then characterized it in terms of ADL congruence's. The lattice theoretic duality principle dose not hold good in case of an ADL for simple reason that an ADL satisfies the right distributivity of $\wedge$ over $\vee$, but dose not satisfy the right distributivity of $\vee$ over $\wedge$. For this reason, we also introduce the concepts of $\theta-$ ideals in an ADL, and then characterized it in terms of ADL congruence.

Chapter 1

Preliminaries

This chapter summarizes some basic definitions and results that will be used in succeeding chapters. It isn't intended to be an exhaustive study of any topic, but it's intended to be a collection of those results which will play important roles in next chapters.

1.1 Partially Ordered Sets

In this section, we give the basic theory of partially ordered sets. We introduce the definition of a partially ordered set and totally ordered set, then we learn how to represent any finite partially ordered set graphically. Also we discuss some results and examples. Duality which is a very important concept will be introduced.

Definition 1.1.1. *[13] A binary relation $\leq$ defined on a set P, which satisfies for all $x, y, z \in P$ the following conditions:*

$P1.$	*For all x, $x \leq x$.*	(*Reflexivity*)
$P2.$	*If $x \leq y$ and $y \leq x$, then $x = y$.*	(*Antisymmetry*)
$P3.$	*If $x \leq y$ and $y \leq z$, then $x \leq z$.*	(*Transitivity*)

is called a partial order.

A set P together with a partial order $\leq$ is called a partially ordered set or a poset for short and is denoted by $(P, \leq)$

Sometimes we shall say that P (rather than $(P, \leq)$) is a poset, when the partial ordering is understood. For $x \leq y$ we say that x is less than or equal y, and if $x \leq y$ and $x \neq y$, one writes $x < y$, and says that x is less than y.

The notation of ordered sets plays an important role not only throughout mathematics but also in adjacent disciplines such as computer science, so now we introduce some examples of partially ordered sets.

Example 1.1.1. *Let R be the set of real numbers, and let $x \leq y$ have its usual meaning for real numbers, then $(R, \leq)$ is a poset.*

Example 1.1.2. *Let N be the set of natural numbers, and let $x|y$ mean that x divides y, then $(N, |)$ is a poset.*

Example 1.1.3. *The set $P(X)$ of all subset of a non - empty set X with the relation $\subseteq$ of set inclusion $(P(X), \subseteq)$ is a poset.*

Example 1.1.4. *If $(E_1, \leq_1)$, . . .,$(E_n, \leq_n)$ are Partially ordered sets then the Cartesian product set $\prod_{i=1}^{n} E_i$ can be given the Cartesian order $\leq$ defined by:*

$$(x_1, ..., x_n) \leq (y_1, ..., y_n) \Leftrightarrow x_i \leq y_i.$$

Definition 1.1.2. *[13] If $(A, \leq)$ is a poset, $a, b \in A$, then a and b are comparable if $a \leq b$ or $b \leq a$. Otherwise, a and b are incomparable.*

The word "partial" in the names " partial order " or "partially ordered set" is used as an indication that not every pair of elements need to be comparable. That is, there may be pairs of elements for which neither element precedes the other in the poset.

Definition 1.1.3. *If $(P, \leq)$ is a poset and every two elements of P are comparable, then P is called a totally ordered set or a chain. In this case the relation $\leq$ is said to be a total order.*

In other words, for any two distinct elements in a chain, one is less and the other greater than. Clearly, the poset of example 1.1.1 is a chain and the posets of example 1.1.2 and 1.1.3 are not chains.

Theorem 1.1.1. *Any subset S of a poset P is itself a poset under the same inclusion relation(restricted to S). Any subset of a chain is a chain*

Proof. This is true since if S is any subset of a poset P, then the elements of S satisfy $P1$ - $P3$ as elements of P. Also if P is a chain, then so is S, since every two elements of S must be comparable as elements of P. □

Thus, the set N of natural numbers ($N \subseteq R$) is a chain under the relation $\leq$ given in example 1.1.1, though a poset which is not a chain under the partial ordering is given in example 1.1.2.

Definition 1.1.4. *[4] The converse of a relation $\leq$ is, by definition, the relation $\geq$ such that $x \geq y$ if and only if $y \leq x$.*

In other words, we can say that the converse relation of a binary relation is the relation that occurs when the order of the elements is switched in the relation. The converse relation is also called the dual of the original relation. Now we have the following theorem.

Theorem 1.1.2. *[4] The converse of any partial ordering is itself a partial ordering on the same set.*

Proof. Let P be any set. Then since $x \leq x$ for every $x \in P$, it follows that $x \geq x$ for every $x \in P$, so $\geq$ is reflexive.

The antisymmetry of $\geq$ follows from the antisymmetry of $\leq$, since if $x \geq y$ and $y \geq x$, then $y \leq x$ and $x \leq y \Rightarrow x = y$, thus $\geq$ is antisymmetry.

To prove the transitivity of $\geq$, assume that $x \geq y$ and $y \geq z$ this means that $y \leq x$ and $z \leq y$, and because of the transitivity of $\leq$, we obtain $z \leq x$, so $x \geq z$, which proves that $\geq$ is transitive. □

Definition 1.1.5. *[4] The dual of a poset X is that poset X defined by the converse partial ordering relation on the same elements.*

Now, if Φ is a "statement" a bout posets, and if in Φ we replace occurrences of $\leq$ by $\geq$, we get the dual of Φ. The importance of this simple definition stems from the fact that every definition and theorem of order theory can readily be transferred to the dual order. Formally, this is captured by the Duality principle for ordered sets.

Duality principle[13]: If a statement Φ is true in all posets, then it's dual is also true in all posets.
This is true simply because Φ holds for $(P, \leq)$ if and only if the dual of Φ holds for $(P, \geq)$ which also ranges over all posets.

Definition 1.1.6. *Let $(P, \leq)$ be a poset, then:*

1. *An element g in P is **a greatest element** if for every element a in P, $a \leq g$.*
2. *An element m in P is **a least element** if for every element a in P, $a \geq m$.*
3. *An element g in P is **a maximal element** if there is no element a in P such that $a > g$.*
4. *An element m in P is **a minimal element** if there is no element a in P such that $a < m$.*

Remark 1.1.

1. A poset can only have one greatest or least element (if it exist), and is denoted by 1 and 0 respectively.
2. If a poset has a greatest element, it must be the unique maximal element, but otherwise there can be more than one maximal element and minimal element.
3. A greatest element must be maximal and a least element must be minimal, but the converse is not true.

Definition 1.1.7. *Let $H \subseteq P$. Then*

1. *$a \in P$ is an **upper bound** of H if and only if $h \leq a$, for all $h \in H$. An upper bound a of H is the **least upper bound** of H "lub" or **supremum** of H if and only if, for any upper bound b of H, we have $a \leq b$. We shall write a = sup H or $a = \bigvee H$.*

2. *$a \in P$ is a **lower bound** of H if and only if $h \geq a$, for all $h \in H$. A lower bound a of H is the **greatest lower bound** of H "glb" or **infimum** of H if and only if, for any lower bound b of H, we have $a \geq b$. We shall write a = inf H or $a = \bigwedge H$.*

Definition 1.1.8. *[8] A partially ordered set P is complete if for every subset H of P both sup H and inf H exist (in P).*

Definition 1.1.9. *A poset $(P, \leq)$ with 0 and 1 is called a bounded poset.*

Remark 1.2.

1. a need not be in H to be an upper bound of H. For example; let $P = \{1, 2, 3, 6, 12\}$ under divisibility and $H = \{1, 2, 3\}$, then 6 is an upper bound of H since $x|6$ $\forall x \in H$, but $6 \notin H$.

2. A greatest element of P is an upper bound of P it self, and a least element is a lower bound of P.

3. By $P2$, sup H is unique if it exists. To show the uniqueness of the supremum, let a_0 and a_1 be both suprema of H, then $a_0 \leq a_1$, since a_1 is an upper bound and a_0 is a supremum. Similarly, $a_1 \leq a_0$, thus $a_0 = a_1$, and similarly for the inf.

4. The dual statement of " if sup H exists, then it is unique " is the statement " if inf H exists, then it is unique ", and the dual of "$(P, \leq)$ has a 0 "is "$(P, \geq)$ has a 1".

Example 1.1.5. *Consider the natural numbers, ordered by divisibility $(N, |)$, then:*

1 is a least element, as it divides all other elements, on the other hand this poset dose not have a greatest element(although if one would include 0 in the poset, which is a multiple of any integer, that would be a greatest element). This partially ordered set dose not even have any maximal elements.

if the number 1 is excluded, then the resulting poset $(N\backslash\{1\}, |)$ doesn't have a least element, but any prime number is a minimal element for it.

In a poset there may be no maximal element, or there may be more than one. But in a finite poset there is always at least one maximal element. Dually, a finite poset must contain minimal elements.

Infinite posets(such as N), as we remarked, need not contain maximal elements. Zorns lemma gives a sufficient condition for maximal elements to exist.

Lemma 1.3 (Zorns lemma)**.** *[7] Let $(P, \leq)$ be a poset in which every chain has an upper bound. Then P contains a maximal element.*

Hasse diagrams

As with relations and functions, there is a convenient graphical representation for partial orders_ Hasse Diagrams.

Definition 1.1.10. *[8] In a poset $(P, \leq)$, we define the interval $[x, y]$ to be the set*

$$[x, y] = \{z \in P : x \leq z \leq y\}$$

.

Definition 1.1.11. *[8] Let x and y be distinct elements of a poset $(P, \leq)$. We say that y covers x if*

$$[x, y] = \{x, y\}$$

that is, $x < y$ but no element z satisfies $x < z < y$.

In general, there may be no pairs x and y such that y covers x (this is the case in the rational numbers by it's natural order, for example).

Using the covering relation, one can obtain a graphical representation of any finite poset P as follows.

Definition 1.1.12. *[8] The Hasse diagram of a poset $(P, \leq)$ is the directed graph whose vertex set is P and whose arcs are the covering pairs (x, y) in the poset.*

we usually draw the Hasse diagram of a finite poset in the plane in such a way that, if y covers x, then the point representing y is higher than the point representing x. Then no arrows are required in the drawing, since the direction of the arrows are implicit.

Example 1.1.6. *The Hasse diagram of the poset of subsets of* $\{1,2,3\}$ *is shown in figure 1.1*

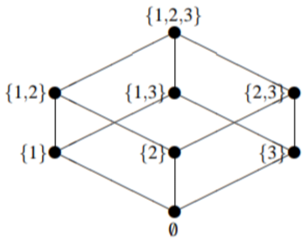

FIGURE 1.1: A Hasse diagram

Example 1.1.7. *The Hasse diagram for the partial ordering* $\{a \leq b : a|b\}$ *on* $\{1,2,3,4,5,6,10,12,15,20,30,60\}$ *is*

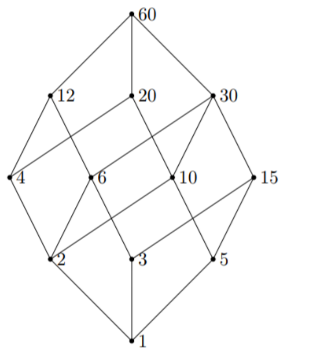

FIGURE 1.2

Example 1.1.8. *Depending on the following Hasse diagram, the lower/upper bounds and glb/lub of the sets* $\{d,e,f\}$*,* $\{a,c\}$ *and* $\{b,d\}$ *are as follows:*

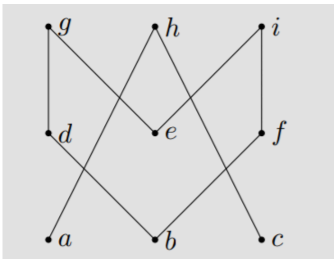

FIGURE 1.3

- *for* $\{d,e,f\}$
  Lower Bounds: ϕ*, thus no glb either.*
  Upper Bounds: ϕ*, thus no lub either.*

- *for* $\{a, c\}$
  Lower Bounds: ϕ*, thus no glb either.*
  Upper Bounds: $\{h\}$*, since its unique, lub is also* h*.*

- *Finally, for* $\{b, d\}$
  Lower Bounds: $\{b\}$ *and so also glb is* b*.*
  Upper Bounds: $\{g, d\}$ *and since* $d \leq g$*, the lub is* d*.*

1.2 Lattices

In this section we introduce the concept of lattice and sublattice in two ways. Examples are presented to illustrate these concepts, also we introduce the concept of distributive lattice which is very important type of lattices. Furthermore we discuss lattice homomorphism and it's properties. Finally we talk about ideals and filters in lattices.

1.2.1 Definition and Example

Many important properties of a poset P are expressed in terms of the existence of certain upper bounds or lower bounds of subsets of P. The most important class of posets defined in this way is lattices.

lattices can be defined in two ways: one based on the existence of an order relation satisfying certain properties and the other based on the existence of binary operations satisfying certain algebraic properties.

In partially ordered sets, the least upper bound of two elements may fail to exist for different reasons: one because they may have no common upper bound, and the other because they have no least upper bound, similar statements can be made for greatest lower bound. A special structure arises when every pair of elements in a poset has a least upper bound and a greatest lower bound.

Definition 1.2.1. *[4] A lattice is a poset* P *where any two of whose elements have a glb and a lub.*

We shall use the notations

$$a \wedge b = \inf\{a, b\}$$
$$a \vee b = \sup\{a, b\}$$

and call $\wedge$ the meet and $\vee$ the join. In lattices, they are both binary operations, which means that they can be applied to a pair of elements a, b of L to yield a gain an element of L. Thus $\wedge$ is a map of $L \times L$ into L and so is $\vee$.

Remark 1.4.

1. If $(P, \leq)$ is a lattice, then so is it's dual $(P, \geq)$.
2. A lattice need not have a 1 or a 0, such as the real numbers(in their natural order).
3. The lattice of all subsets of a given set X (Example1.1.3) has the empty set ϕ it's 0, and X itself it's 1.
4. A singleton set $L = \{a\}$ is a lattice under the only possible order on L. This is a trivial lattice. Any lattice with more than one element is a nontrivial lattice.
5. Any totally ordered set is a lattice.

Definition 1.2.2. *[4] A sublattice of a lattice L is a subset X of L such that $a \in X$, $b \in X$ imply $a \wedge b \in X$ and $a \vee b \in X$.*

Note.

- A sublattice X is a lattice under the same join and meet operations of L.
- To show that a partially ordered set is not a lattice, it suffices to find a pair that doesn't have a lub or glb. For a pair not to have a lub or glb, they must first be incomparable.

Example 1.2.1.

1. *The empty subset is a sublattice; so is any one - element subset.*
2. *Any interval of a lattice is a sublattice, and so is any intersection of intervals. For example, let $L = P(X)$, where $X = \{a, b, c\}$, then the intervals $L_1 = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}\}$ and $L_2 = \{\phi, \{b\}, \{a, b\}, \{a, b, c\}\}$ are a sublattices and so is $L_1 \cap L_2 = \{\phi, \{a, b\}, \{a, b, c\}\}$.*

3. *let* $L = \{1, 2, 3, 6, 12\}$ *under divisibility, where* $a \wedge b = gcd(a, b)$ *and* $a \vee b = lcm(a, b)$. *The subset* $S = \{1, 2, 3, 12\}$ *is a lattice under divisibility but not a sublattice of* L *since* $2 \vee 3 = lcm\{2, 3\} = 6 \notin S$. *The subset* $T = \{1, 2, 3, 6\}$ *is a sublattic of* L.

Example 1.2.2.

1. *Let* $S(G)$ *be the set of the subgroups of any group* G*, and let* $\leq$ *mean set-inclusion. Then* $S(G)$ *is a lattice, with* $H \wedge K = H \cap K$ *(set-intersection) and* $H \vee K$ *the least subgroup in* $S(G)$ *containing* H *and* K *(which is not their set-theoretical union).*

2. *The normal subgroups of any group is a sublattice of* $S(G)$.

Example 1.2.3. *The set of all natural number* $N = \{1, 2, 3, ...\}$ *with the usual order of* $\leq$ *is a poset. By defining sup* $\{a, b\}$ *as the bigger of the two elements and inf* $\{a, b\}$ *as the smaller of the two elements, it forms a lattice.*

Example 1.2.4. *[14] For a positive integer* n*, let* L_n *be the set of all positive divisors of* n*. Let us define a relation* $\leq$ *on* L_n *as:*

$$a \leq b \Leftrightarrow a|b$$

Define sup $\{a, b\}$ *as Lcm* (a, b)*, and inf* $\{a, b\}$ *as gcd (a,b). Then* L_n *becomes a lattice. The lattice* L_6 *is the following.*

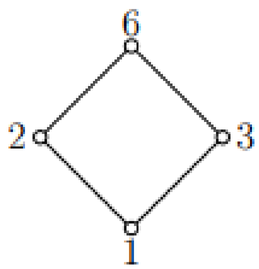

FIGURE 1.4: lattice L6

Example 1.2.5. *Not every poset is a lattice: consider* $S = \{2, 3, ...\}$*, the set of natural numbers deleted* 1*. Let us define the partial order as in Example 1.2.4 above, then* S *is not a lattice as, for example, the gcd of* 2 *and* 3 *dose not belong to* S.

Definition 1.2.3. *A subset K of a lattice L is called convex if and only if*

$$a, b \in L \text{ and } a \leq c \leq b \text{ imply that } c \in K.$$

Example 1.2.6. *For $a, b \in L$, $a \leq b$, the interval*

$$[a, b] = \{x : a \leq x \leq b\}$$

is an important example of a convex sublattice.

Lattices as Algebraic Structures

It is obvious that $\leq$ can be characterized by $\wedge$ and $\vee$. So we can characterize $(L, \leq)$ as $(L, \wedge, \vee)$, which is an algebra (that is, a set equipped with operation, in this case, we have two binary operations)[15].

Note that $\leq$ is a subset of L x L, where $\wedge$ and $\vee$ are maps from L x L into L. We want such a characterization because if we can treat lattices as algebras, then all the concepts and methods of universal algebra will become applicable.

Theorem 1.2.1. *[13] The following properties hold in a lattice L:*

$L1)$ ***idempotency***

$$a \vee a = a$$
$$a \wedge a = a$$

$L2)$ ***commutativity***

$$a \vee b = b \vee a$$
$$a \wedge b = b \wedge a$$

$L3)$ ***associativity***

$$(a \vee b) \vee c = a \vee (b \vee c)$$
$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$

$L4)$ ***absorption***

$$a \vee (a \wedge b) = a$$
$$a \wedge (a \vee b) = a$$

Conversely, if L *is a nonempty set with two binary operations* $\wedge$ *and* $\vee$ *satisfying* $L1)$ *-* $L4)$*, then* L *is a lattice where meet is* $\wedge$*, join is* $\vee$ *and the order relation is given by*

$$a \leq b \text{ if } a \vee b = b$$

or equivalently,

$$a \leq b \text{ if } a \wedge b = a$$

Moreover, since the set of axioms $L1)$ *-* $L4)$ *is self - dual, it follows that if a statement holds in every lattice, then any dual statement holds in every lattice.*

Proof.
$(\Rightarrow)$ Let L be a lattice. Then the idempotency and commutativity are evident. The associativity is also evident since $(a \vee b) \vee c$ and $a \vee (b \vee c)$ are both equal to the least upper bound of $\{a, b, c\}$.

For $(L4)$, since $a \vee b$ is the least upper bound for $\{a, b\}$, so $a \leq a \vee b$. Therefore, $a \wedge (a \vee b) = a$. By duality, $a \vee (a \wedge b) = a$.

$(\Leftarrow)$ We set $a \leq b$ to mean that $a \wedge b = a$. Now $\leq$ is reflexive since $\wedge$ is idempotent; $\leq$ is antisymmetric since $a \leq b$ and $b \leq a$ mean that $a \wedge b = a$ and $b \wedge a = b$, which, by the commutativity of $\wedge$, imply that $a = a \wedge b = b \wedge a = b$; $\leq$ is transitive, since if $a \leq b$ and $b \leq c$, and so

$$a = a \wedge b = a \wedge (b \wedge c)$$

($\wedge$ is associative)

$$= (a \wedge b) \wedge c = (a \wedge c),$$

that is, $a \leq c$. Thus $(L, \leq)$ is a poset.

To prove that $(L, \leq)$ is a lattice, we shall verify that $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$(these are not definitions). Indeed, $a \wedge b \leq a$, since

$$(a \wedge b) \wedge a = a \wedge (b \wedge a) = a \wedge (a \wedge b) = (a \wedge a) \wedge b = a \wedge b,$$

using the associativity, commutativity, and idempotency of $\wedge$; similarly, $a \wedge b \leq b$. Now if $c \leq a$ and $c \leq b$, that is, $c \wedge a = c$ and $c \wedge b = c$, then

$$c \wedge (a \wedge b) = (c \wedge a) \wedge b = c \wedge b = c$$

thus $a \wedge b = \inf\{a, b\}$.

Finally, $a \leq a \vee b$ and $b \leq a \vee b$, because $a = a \wedge (a \vee b)$ and $b = b \wedge (a \vee b)$ by the first absorption identity; if $a \leq c$ and $b \leq c$, that is, $a = a \wedge c$ and $b = b \wedge c$, then $a \vee c = (a \wedge c) \vee c = c$ and $b \vee c = c$ (by the second absorption identity). Thus

$$(a \vee b) \wedge c = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge (a \vee (b \vee c))$$

$$= (a \vee b) \wedge ((a \vee b) \vee c) = a \vee b,$$

that is, $a \vee b \leq c$, completing the proof of $a \vee b = \sup\{a, b\}$. □

Definition 1.2.4. *[13] An algebra $(L, \wedge, \vee)$ is called a lattice if and only if L is a nonempty set, $\wedge$ and $\vee$ are binary operations on L, both $\wedge$ and $\vee$ are idempotent, commutative, and associative, and they satisfy the two absorption identities.*

The following theorem states that a lattice as an algebra and a lattice as a poset are ”equivalent” concepts.

Theorem 1.2.2. *[13]*

1. *Let the poset $\sum = (L, \leq)$ be a lattice. Set*

$$a \wedge b = \mathit{inf}\{a, b\},$$
$$a \vee b = \mathit{sup}\{a, b\},$$

   Then the algebra $\sum^a = (L, \wedge, \vee)$ is a lattice.

2. *Let the algebra $\sum = (L, \wedge, \vee)$ be a lattice. Set*

$$a \leq b \Leftrightarrow a \wedge b = a$$

Then $\sum^p = (L, \leq)$ *is a poset, and the poset* $\sum^p$ *is a lattice.*

3. *Let the poset* $\sum = (L, \leq)$ *be a lattice. Then* $(\sum^a)^p = \sum$.

4. *Let the algebra* $\sum = (L, \wedge, \vee)$ *be a lattice. Then* $(\sum^p)^a = \sum$.

Remark, (1) and (2) describe how we pass from a poset to an algebra and back, whereas (3) and (4) state that going there and back takes us back to where we started.

Example 1.2.7.

1. *Any chain is a lattice, in which* $x \wedge y$ *is simply the smaller and* $x \vee y$ *is the larger of* x *and* y.

2. *The set* N *of natural numbers is a lattice where*

$$a \wedge b = gcd(a, b) \text{ and } a \vee b = lcm(a, b).$$

Lemma 1.5. *[4] If a poset* P *has a* 0, *then*

$$0 \wedge x = 0 \text{ and } 0 \vee x = x \text{ for all } x \in P$$

Dually, if P *has a universal upper bound* 1, *then*

$$x \wedge 1 = x \text{ and } x \vee 1 = 1 \text{ for all } x \in P$$

Lemma 1.6. *[4] In any lattice, the operations of join and meet are isotone, that is*

$$\text{if } y \leq z, \text{ then } x \wedge y \leq x \wedge z \text{ and } x \vee y \leq x \vee z \tag{1.1}$$

Proof. By $L1$ - $L4$ and consistency, $y \leq z$ implies

$$x \wedge y = (x \wedge x) \wedge (y \wedge z) = (x \wedge y) \wedge (x \wedge z)$$

whence $x \wedge y \leq x \wedge z$ by consistency. The second inequality of 1.1 can proved dually. □

Lemma 1.7. *[4] In any lattice, we have the distributive inequalities*

(L5) $$x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z),$$

(L5́) $$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$$

Proof. Clearly $x \wedge y \leq x$, and $x \wedge y \leq y \leq y \vee z$, hence $x \wedge y \leq x \wedge (y \vee z)$. Also $x \wedge z \leq x$, $x \wedge z \leq z \leq y \vee z$; hence $x \wedge z \leq x \wedge (y \vee z)$. That is, $x \wedge (y \vee z)$ is an upper bound of $x \wedge y$ and $x \wedge z$, from which (5) follows. The distributive inequality ($L\acute{5}$) follows from ($L5$) by duality.

□

Lemma 1.8. *[4] The elements of any lattice satisfy the modular inequality:*

(L6) $$x \leq z \textit{ implies } x \vee (y \wedge z) \leq (x \vee y) \wedge z$$

Proof. $x \leq x \vee y$ and $x \leq z$. Hence $x \leq (x \vee y) \wedge z$. Also $y \wedge z \leq y \leq x \vee y$ and $y \wedge z \leq z$. Therefore $y \wedge z \leq (x \vee y) \wedge z$, whence $x \vee (y \wedge z) \leq (x \vee y) \wedge z$ □

A finite lattice can always be described by a meet - table and a join - table. For example, the following two tables describe a lattice on the set $L = \{0, a, b, 1\}$:

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

TABLE 1.1

Since both operations are commutative, the tables are symmetric with respect to the diagonal.

Now we introduce one of the most important types of lattices, namely distributive lattices. Since lattices came with two binary operations $\wedge$ and $\vee$, it's natural to ask whether one of them distributes on the other.

Theorem 1.2.3. *[4] In any lattice, the following identities are equivalent:*

$(L6')$ $\quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad$ *for all* x, y, z

$(L6'')$ $\quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad$ *for all* x, y, z .

Proof. We shall prove that $L6'$ implies $L6''$. The converse implication $L6'' \Rightarrow L6'$ will then follow by duality. We have, for any x, y and z,

$$\begin{aligned} (x \vee y) \wedge (x \vee z) &= [(x \vee y) \wedge x] \vee [(x \vee y) \wedge z] && \text{by } L6' \\ &= x \vee [z \wedge (x \vee y)] && \text{by } L4, L2 \\ &= x \vee [(z \wedge x) \vee (z \wedge y)] && \text{by } L6' \\ &= [x \vee (z \wedge x)] \vee (z \wedge y) && \text{by } L3 \\ &= x \vee (y \wedge z) && \text{by } L4 \end{aligned}$$

□

Remark 1.9. In any lattice L, we have the following property:

$$a \wedge b = b \iff a \vee b = a$$

Proof. If $a \vee b = a$; Then

$$a \wedge b = (a \vee b) \wedge b = b.$$

□

Remark 1.10. In any lattice $(L, \wedge, \vee)$, the following are equivalent:

1. $c \le a \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee c$.
2. $c \le a \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
3. $a \wedge (b \vee (a \wedge c)) = (a \wedge b) \vee (a \wedge c) \qquad \forall\, a, b, c \in L$.

Definition 1.2.5. *[4] A lattice is distributive if and only if the identity* $(L6')$ *(and hence* $(L6'')$*) holds in it.*

Example 1.2.8. *The following are examples of a distributive lattice:*

1. *Any chain is a distributive lattice. In fact,* $x \wedge y$ *is the smaller of* x *and* y*,* $x \vee y$ *is the greater of* x *and* y*,* $x \wedge (y \vee z)$ *and* $(x \wedge y) \vee (y \wedge z)$ *are both equal to* x *if* x *is smaller than* y *or* z*; and both equal to* $y \vee z$ *in the alternative case that* x *is bigger than* y *and* z*, thus , the real numbers(example1.1.1) form a distributive lattice.*

2. *The dual of any distributive lattice is distributive, and any sublattice of a distributive lattice is distributive.*

3. *The lattice* $(S(G), \subseteq)$ *of subgroups of a group* G *is distributive if and only if* G *is locally cyclic (That is, any finite nonempty subset of* G *generates a cyclic subgroup). For instance, the lattice of subgroups of the group* $(Z, +)$ *is distributive. Thus, if* G *is finite, then* $(S(G))$ *is distributive if and only if* G *is cyclic.*

Theorem 1.2.4. *[4] In a distributive lattice, if* $c \wedge x = c \wedge y$ *and* $c \vee x = c \vee y$*, then* $x = y$*.*

Proof. Using repeatedly the hypotheses, $L4$, $L2$, and $L6$, we have

$$\begin{aligned} x &= x \wedge (c \vee x) \\ &= x \wedge (c \vee y) \\ &= (x \wedge c) \vee (x \wedge y) \\ &= (c \wedge y) \vee (x \wedge y) \\ &= (c \vee x) \wedge y \\ &= (c \vee y) \wedge y \\ &= y \end{aligned}$$

This completes the proof. □

Definition 1.2.6. *A bounded lattice is an algebraic structure of the form* $(L, \vee, \wedge, 0, 1)$ *such that* $(L, \vee, \wedge)$ *is a lattice,* 0 *is (the lattices bottom) the identity element of the join operation* $\vee$ *and* 1 *(the lattices top) is the identity element of meet operation* $\wedge$*.*

Example 1.2.9. *Let* $X = \{a, b, c\}$ *and let* $L = (X, \vee, \wedge, 0, 1)$ *be the power set of* X*, then* L *is a bounded lattice with* $0 = \phi$ *and* $1 = X$*. Where the join and meet here are the union and intersection respectively.*

Definition 1.2.7. *[21] Let L be a bounded lattice. A **complement** of $a \in L$ is an element $b \in L$ for which*

$$a \wedge b = 0 \text{ and } a \vee b = 1$$

*In this case, we say that a and b are **complementary**.*

Complements need not exist and they need not be unique when they do exist. For instance, in a bounded chain, no elements other than 0 and 1 have complements.

Definition 1.2.8. *A bounded lattice L for which each element has a complement is called a complemented lattice.*

Definition 1.2.9. *[21] Let $u \leq v$ be elements of a lattice L and let $a \in [u, v]$. A **relative complement** of a with respect to $[u, v]$ is a complement of a in the sublattice $[u, v]$, that is, an element $x \in [u, v]$ for which*

$$x \wedge a = u \text{ and } x \vee a = v$$

the set of all relative complements of a with respect to $[u, v]$ is denoted by $a_{[u,v]}$.

Definition 1.2.10. *A lattice L is **relatively complemented** if every closed interval $[u, v]$ in L is complemented.*

Example 1.2.10. *The lattice of subsets of a set A is complemented lattice, for we identify the whole set A as 1 and the empty set as 0, then define the complement of any subset of A as the collection of all elements of A which are not in the subset.*

1.2.2 Homomorphisms, Ideals and Filters in Lattice

A map $f : L \to M$ between lattices need not, in general, preserve meets and joins. For example, consider the lattices of integers

$$L = \{1, 2, 3, 12\}, M = \{1, 2, 3, 6, 12\}$$

Where the order is divisibility.

The inclusion map $f : L \to M$ defined by $f(n) = n$ dose not preserve joins. Since

$$f(2 \vee 3) = f(12) = 12$$

and,

$$f(2) \vee f(3) = 2 \vee 3 = 6$$

Definition 1.2.11. *[21] Let L and M be lattices. A function $f : L \to M$ that preserves finite meets and joins, that is, for which*

$$f(a \wedge b) = f(a) \wedge f(b)$$

$$f(a \vee b) = f(a) \vee f(b)$$

is called a lattice homomorphism.

Some kinds of homomorphisms:

1. A lattice monomorphism is an injective lattice homomorphism.
2. A lattice epimorphism is a surjective lattice homomorphism.
3. A lattice endomorphism of L is a lattice homomorphism from L to itself.
4. A lattice isomorphism is a bijective lattice homomorphism.

Note. If $f : L \to M$ is a lattice homomorphism, then $f(L)$ is a sublattice of M.

In addition, if L is a bounded lattice with top 1 and bottom 0, then

$$f(a) = f(1 \wedge a) = f(1) \wedge f(a) \text{ and } f(a) = f(0 \vee a) = f(0) \vee f(a) \text{ for all } a \in L.$$

Thus L is mapped onto a bounded sublattice $f(L)$ of M, with top $f(1)$ and bottom $f(0)$.

If both L and M are bounded with lattice homomorphism

$$f : L \to M \text{ for which } f(0) = 0 \text{ and } f(1) = 1$$

then f is called a $\{0,1\}$ - homomorpism.

If $f(1)$ and $f(0)$ are the top and bottom of M. In other words,

$$f(1_L) = 1_M \text{ and } f(0_L) = 0_M$$

Where 1_L, 1_M, 0_L, 0_M are the top and bottom elements of L and M respectively.

We see that the supremum and infimum of any finite set in a lattice are preserved under a lattice homomorphism. This is, however, not true for infinite sets in general. For instance, let $P = [0,1) \cup 2$ and $Q = [0,1] \cup 2$, the map $f : P \to Q$ defined by $f(x) = x$, is a lattice homomorphism from $(P, \leq)$ into $(Q, \leq)$, but $f(\vee[0,1)) = f(2) = 2$ while $\vee f([0,1)) = \vee[0,1) = 1$.

Example 1.2.11. *Let L be the lattice $(P(\{a,b\}), \cap, \cup)$ and let S be the lattice $(\{0,1\}, \wedge, \vee)$ where $m \wedge n$ = min$\{a,b\}$ and $m \vee n$ = max$\{a,b\}$.*
Define $g : P(\{a,b\}) \to \{0,1\}$ as follows:

$$g(X) = \begin{cases} 1 & if \quad a \in X, \\ 0 & if \quad a \notin X. \end{cases}$$

It is easy to check that g is a lattice homomorphism by checking that

$$g(X \cap Y) = g(X) \wedge g(Y) \quad \text{and} \quad g(X \cup Y) = g(X) \vee g(Y)$$

by taking four cases :

$$g(X \cap Y) = 0,\ g(X \cap Y) = 1,\ g(X \cup Y) = 0 \text{ and } g(X \cup Y) = 1.$$

Note that since g is not a bijection, it certainly is not a lattice isomorphism.

Definition 1.2.12. *[21] If a lattice M has a smallest element (zero element), then the **kernel** of a lattice homomorphism $f : L \to M$ is the set*

$$ker(f) = f^{-1}(0) = \{a \in L : f(a) = 0\}$$

Definition 1.2.13. *[21] Tow lattices L and M are isomorphic, if there is an isomorpism between them.*

$$f : L \to M$$

In this case we write $L \approxeq M$.

Example 1.2.12. *Let $(L, \leq)$ and $(S, \leq)$ be two latices, where $L = \{1, 3, 9\}$ and $S = \{2, 4, 8\}$ under the relation "divisibility". Define $f : L \to S$ as follows:*

$$f(1) = 2,\ f(3) = 4,\ f(9) = 8.$$

Then we can easily check that $L \approxeq S$.

Now we will introduce the concepts of ideals and filters in lattices, and discuss some of their properties.

Definition 1.2.14. *[21] Let L be a lattice*

1. *A non - empty subset I of L is said to be an* ***ideal*** *of L if it satisfies the following:*
   - $a, b \in I \Rightarrow a \vee b \in I$.
   - $a \in I,\ x \in L \Rightarrow a \wedge x \in I$.

   In this case, we write $I \trianglelefteq L$. A proper ideal, that is, an ideal $I \neq L$. The set of all ideals of L is denoted by ***Id***(L).

2. *Dually, a non - empty subset F of L is said to be a* ***filter*** *of L if it satisfies the following:*
   - $a, b \in F \Rightarrow a \wedge b \in F$.
   - $a \in F,\ x \in L \Rightarrow a \vee x \in F$.

   In this case, we write $F \trianglerighteq L$. A proper filter, that is, a filter $F \neq L$. The set of all filters of L is denoted by ***F***(L).

Every ideal I of a lattice L is a sublattice, since $a \wedge b \leq a$, $\forall a, b \in L$, and hence, $a \wedge b \in I$, every lattice have two trivial ideals itself and $\{0_L\}$; and every intersection of ideals of L is an ideal.

Note. The union of ideals of a Lattice need not be an ideal. For example, let L be the ADL whose hasse diagram is given in Figure 1.5.Then $I_1 = \{1, 2, 4\}$ and $I_2 = \{1, 2, 5\}$ are ideals of L, and $I_1 \cup I_2 = \{1, 2, 4, 5\}$ isn't an ideal since $5 \vee 4 = 10$.

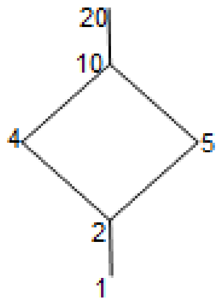

FIGURE 1.5

Definition 1.2.15. *[21] Let H be a non - empty subset of a lattice L*

1. *The **ideal generated** by H, denoted by $(H]$, is the smallest ideal of L containing H, and is obtained by intersecting all ideals containing H.*
2. *The **filter generated** by H, denoted by $[H)$, is the smallest filter of L containing H, and is obtained by intersecting all filters containing H.*

*If $H = \{a\}$, we write $(a]$ for $(\{a\}]$; we shall call $(a]$ **a principal** ideal and it's given by*

$$(a] = \{x : x \leq a\} = \{x \wedge a : x \in L\}$$

*and, we write $[a)$ for $[\{a\})$; we shall call $[a)$ **a principal** filter and it's given by*

$$[a) = \{x : a \leq x\} = \{x \vee a : x \in L\}$$

Example 1.2.13. *Consider the lattice $L = \{1, 2, 4, 5, 10, 20\}$ with Hasse diagram given in Figure1.6. The ideal generated by $\{2, 4\}$ is $(\{2, 4\}] = \{1, 2, 4\} = (4]$, and the ideal* *generated by* $\{4,5\} = \{1,2,4,5,10\} = (10]$. *Of course,* $(20] = L$, *since* 20 *is the top element in* L.

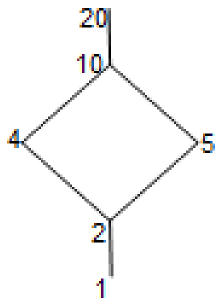

FIGURE 1.6

Theorem 1.2.5. *[4] The set of all ideals of any lattice* L*, ordered by set inclusion, is a lattice Id*(L)*. The set of all principal ideals in* L *forms a sublattice of this lattice, which is isomorphic with* L*.*

Note. If J and K are principal ideals of L with generators a and b. Then $J \vee K$ and $J \wedge K$ are the principal ideals generated by $a \vee b$ and $a \wedge b$ respectively.

Remark 1.11. We can also define the lattice Id(L) by $(Id(L), \wedge, \vee)$ where

$$J \wedge K = J \cap K,\, J \vee K = \{x \in L : (\exists j \in J)(\exists k \in K), x \leq j \vee k\}.$$

By dualizing, the set of all filters, ordered by set inclusion, is it self a Lattice $F(L)$. The following special types of ideals and filters play a key role in lattice theory.

Definition 1.2.16. *[21] Let* L *be a lattice*

1. *A proper ideal* I *of* L *is* ***maximal*** *if for any ideal* J*,*

$$I \subseteq J \subseteq L \Rightarrow J = I \text{ or } J = L$$

A proper filter F *of* L *is* ***maximal*** *if for any filter* X*,*

$$F \subseteq X \subseteq L \Rightarrow X = F \text{ or } X = L$$

A maximal filter is also called an ultrafilter.

2. *A proper ideal I is **prime** if*

$$a \wedge b \in I \Rightarrow a \in I \text{ or } b \in I$$

*The set of prime ideals of L is called the **spectrum** of L and is denoted by spec(L). Dually, a proper filter F is **prime** if*

$$a \vee b \in F \Rightarrow a \in F \text{ or } b \in F.$$

Note that in F(L) the largest element is L; if L has 0 and 1, then L = $[0)$ is the largest and $\{1\}$ = $[1)$ is the smallest element of F(L).

Example 1.2.14. *Let L be the lattice given in Example 1.2.13. The ideal $(10]$ = $\{1, 2, 4, 5, 10\}$ is prime ideal. The ideal I = $\{1, 2\}$ is not a prime ideal since $4 \wedge 5$ = $2 \in I$ but neither $4 \notin I$ nor $5 \notin I$. The ideal $(4]$= $\{1, 2, 4\}$ is maximal.*

Example 1.2.15. *The kernel of a lattice homomorphism $f : L \to M$ is an ideal of L.*

Example 1.2.16. *Let L and M be bounded lattices and $f : L \to M$ a $\{0, 1\}$ - homomorphism, then $f^{-1}(0)$ is an ideal and $f^{-1}(1)$ is a filter in L.*

It is easy to show that an ideal I of a lattice with 1 is proper if and only if $1 \notin I$, and dually, a filter F of a lattice with 0 is proper if and only if $0 \notin F$.

1.3 Congruence Relations On Lattices

Congruence relations play a central role in lattice theory. This section develops the rudiments of a theory which goes way beyond the scope of an introductory text such as this. In this section we introduce the congruence relations on groups and rings. Then we discuss the concept of congruence relation on lattices. Some examples and properties are given to illustrate these concepts.

1.3.1 Congruence

We know that, If S is an algebraic structure, then an equivalence relation $\equiv$ is a binary relation that is at the same time reflexive, symmetric and transitive relation. We write $a \equiv b$ or $a\theta b$ to indicate that a and b are related under the relation θ. The relation "is equal to" on the set of real numbers is a primary example of an equivalence relation. For example, $\frac{1}{2}$ is equal to $\frac{4}{8}$.

Note.

1. Any equivalence relation, as a consequence of the reflexive, symmetric, and transitive properties, provides a partition of a set into equivalence classes or blocks of θ. A typical block is of the form $[a]_\theta = \{x \in A : x \equiv a\}$.

2. Equality is both an equivalence relation and a partial order. Equality is also the only relation on a set that is reflexive, symmetric and antisymmetric.

Definition 1.3.1. *[8] If S is an algebraic structure, then an equivalence relation on S that also preserves the algebraic operations of S is called **a congruence relation** on S.*

For example if G is a group with operation $*$, a congruence relation on G is an equivalence relation $\equiv$ on the elements of G satisfying

$$g_1 \equiv g_2 \text{ and } h_1 \equiv h_2 \Rightarrow g_1 * h_1 \equiv g_2 * h_2$$

for all $g_1, g_2, h_1, h_2 \in G$.

Example 1.3.1. *The prototypical example of a congruence relation is congruence modulo n on the set of integers. For a given positive integer n, two integers a and b are called congruent modulo n, written*

$$a \equiv b$$

if $a - b$ is divisible by n (or equivalently if a and b have the same remainder when divided by n).

Now in elementary algebra, one teaches that there is a correspondence between certain special types of substructures and quotient structures. In the case of groups, for example, a more complete story for groups given by the next theorem.

Theorem 1.3.1. *[25] Every normal subgroup has corresponding congruence relation and vice versa.*

Proof. ($\Rightarrow$) let G be a group where $H \lhd G$ is a normal subgroup in G, define a relation $\rho \subseteq G$ x G as follows:

$$g_1 \sim g_2 \text{ under } \rho \text{ if and only if } g_1 g_2^{-1} \in H$$

1. $\sim$ is an equivalence relation
   - **Reflexivity**: it is easy to see that $g \sim g$ since $gg^{-1} = e \in H$.
   - **symmetry**: suppose $g_1 \sim g_2$, so by definition we have $g_1 g_2^{-1} \in H$, so $g_1 g_2^{-1}$ = h for some $h \in H$, and then
   $$g_1 g_2^{-1} = h \Rightarrow g_2 g_1^{-1} = h^{-1} \in H \Rightarrow g_2 \sim g_1.$$
   - **Transitivity**: suppose $g_1 \sim g_2$ and $g_2 \sim g_3$, then:
   $$g_1 g_2^{-1} = h \text{ and } g_2 g_3^{-1} = k, \text{ where } h, k \in H$$
   now from the last identity we have $g_2^{-1} = g_3^{-1} k^{-1}$, and then
   $$g_1 g_3^{-1} k^{-1} = h \Rightarrow g_1 g_3^{-1} = hk \Rightarrow g_1 \sim g_3$$
2. This relation preserves the group structure, since if :
   $$g_1 \sim g_2 \text{ and } g_3 \sim g_4,$$
   then
   $$g_1 \sim g_2 \Rightarrow g_1 g_2^{-1} = h_1 \in H$$
   $$g_3 \sim g_4 \Rightarrow g_3 g_4^{-1} = h_2 \in H$$
   $$\Rightarrow g_1 g_3 (g_2 g_4)^{-1} = g_1 g_3 g_4^{-1} g_2^{-1} = g_1 h_2 g_2^{-1} = g_1 g_2^{-1} h' = h_1 h' \in H$$
   Because normality implies that for all $x \in G$, and all $h \in H$ there exist $h' \in H$ with $xh = h'x$.

   ($\Leftarrow$) Now let $\sim$ be a congruence relation on a group G. Define the set $H = \{g \in G : g \sim e\}$. Firstly, we prove H is a subgroup

   (a) Of course, e $\sim$ e, thus e $\in H$.

(b) Suppose $h_1, h_2 \in H$, then $h_1 \sim e$ and $h_2 \sim e$. Since $\sim$ is a congruence relation, we have

$$h_1h_2 \sim ee \Rightarrow h_1h_2 \sim e \Rightarrow h_1h_2 \in H$$

(c) Suppose $h \in H$, then $h \sim e$ since $\sim$ is an equivalence, we have $h^{-1} \sim h^{-1}$, and since it's also a congruence relation, we get

$$hh^{-1} \sim eh^{-1} \Rightarrow e \sim h^{-1} \Rightarrow h^{-1} \in H$$

We want to show that H is normal, that is, $\forall g \in G$ and $\forall h \in H$, we have $ghg^{-1} \in H$

Since $\sim$ is an equivalence relation, we have $g \sim g$ and $g^{-1} \sim g^{-1}$. Furthermore, as $\sim$ is congruence, and $h \in H \Rightarrow h \sim e$ and so $ghg^{-1} \sim geg^{-1} = e$ for any $g \in G, h \in H$.

□

So every congruence relation has a corresponding quotient structure, whose elements are the equivalence classes (or, congruence classes) for the relation.

Example 1.3.2. *for the group (S_3, o) we have three congruence relations, since we have three normal subgroups of S_3 which are A_3, ρ_0 and S_3, the first congruence relation defined by: $a \equiv b$, if a and b have the same parity, and the second congruence relation defined by $a \equiv b$, if $a = b$, and the third congruence relation defined by $a \equiv b$, for all a and b.*

Remark 1.12. For a congruence on a group, the equivalence class containing the identity element is always a normal subgroup, and the other equivalence classes are the cosets of this subgroup. Together, these equivalence classes are the elements of a quotient group.

When an algebraic structure includes more than one operation, congruence relations are required. For example, a ring possesses both addition and multiplication, and a congruence relation on a ring must satisfy

$$r_1 + s_1 \equiv r_2 + s_2 \text{ and } r_1s_1 \equiv r_2s_2$$

whenever

$$r_1 \equiv r_2 \text{ and } s_1 \equiv s_2.$$

For a congruence on a ring, the equivalence class containing 0 is always a two - sided ideal, and the two operations on the set of equivalence classes define the corresponding quotient ring.

For example, congruence modulo n (for fixed n) is compatible with both addition and multiplication on the integers. That is if

$$a_1 \equiv a_2 \quad \text{and} \quad b_1 \equiv b_2$$

Then,

$$a_1 + a_2 \equiv a_2 + b_2$$

and

$$a_1 b_1 \equiv a_2 b_2$$

The corresponding addition and multiplication of equivalence classes is known as modular arithmetic. From the point of view of abstract algebra, congruence modulo n is a congruence relation on the ring of integers.

1.3.2 Congruence relations on lattices

We begin with the definition of a congruence relation on a lattice.

Definition 1.3.2. *[21] An equivalence relation θ on a lattice L is a **congruence relation** on L, if for all $a, b, x, y \in L$*

$$a\theta x \text{ and } b\theta y \Rightarrow (a \wedge b)\theta(x \wedge y) \text{ and } (a \vee b)\theta(x \vee y)$$

We will use the notations

$$a\theta b\ ,\ a \equiv_\theta b,\ a \equiv b(\theta) \text{ and } a \stackrel{\theta}{\equiv} b$$

and write $a \equiv b$ when the specific congruence is understood. Also, the notation

$$a \stackrel{\theta}{\equiv} b \stackrel{\theta}{\equiv} c$$

is shorthand for $a \stackrel{\theta}{\equiv} b$ and $b \stackrel{\theta}{\equiv} c$. The equivalence classes under a congruence relation θ are called **congruence classes** or blocks. The congruence class containing $a \in L$ is denoted by $[a]$ or $[a]_\theta$, that is $[a] = \{x : x\theta a\}$. The set of all congruence relations on L is denoted by Con(L).

Example 1.3.3. *In any lattice there are always two trivial congruence relations, the congruence relation θ_1 where each element is it's own equivalence class (block), this is called the smallest congruence relation, and the congruence relation θ_2 with a single block i.e.*

$$x \overset{\theta_1}{\equiv} y \text{ if and only if } x = y$$

$$x \overset{\theta_2}{\equiv} y \text{ for all } x, y \in L$$

Example 1.3.4. *Let L be a lattice with Hasse diagram in Figure 1.7*

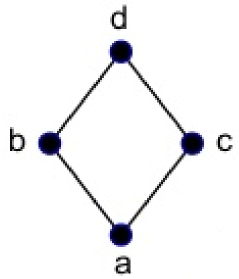

FIGURE 1.7

The following are all congruence of L :
$\theta_1 = \{\{a\}, \{b\}, \{c\}, \{d\}\}$, $\theta_2 = \{a, b, c, d\}$, $\{\{a, b\}\{c, d\}\}$, $\{\{a, c\}\{b, d\}\}$, $\{\{a\}\{c\}\{b, d\}\}$, $\{\{a\}\{b\}\{c, d\}\}$.

Example 1.3.5. *In a finite chain C, a congruence relation is any decomposition of C into disjoint closed intervals as in the figure.*

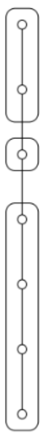

FIGURE 1.8: A congruence of a finite chain C

Example 1.3.6. *A congruence relation of a lattice is shown in figure 1.9*

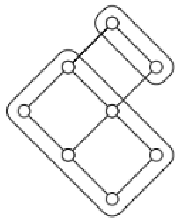

FIGURE 1.9: Congruence of a lattice

Remark 1.13. If $u \leq v$ and $u \equiv v$, then all elements in the interval $[u, v]$ are congruent, for if $x \in [u, v]$, then

$$x = v \wedge x \equiv u \wedge x = u$$

and so every element of $[u, v]$ is congruent to u. Thus, congruence classes are convex subsets of L. Moreover, if $a \equiv b$, then

$$a \wedge b \equiv b \wedge b = b \text{ and } a \vee b \equiv b \vee b = b$$

and so all elements of the interval $[a \wedge b, a \vee b]$ are congruent.

Theorem 1.3.2. *[21] Let θ and σ be congruence relations on a lattice L.*

1. *For all $a, b \in L$*

$$a\theta b \Leftrightarrow (a \wedge b)\theta(a \vee b)$$

in which case every element of the interval $[a \wedge b, a \vee b]$ is congruent to a.

2. *$\theta = \sigma$ if and only if*

$$a\theta b \Leftrightarrow a\sigma b$$

for all $a < b$ in L.

3. *If μ is a binary relation on L satisfying*

$$a\mu b \Leftrightarrow (a \wedge b)\mu(a \vee b)$$

for all $a, b \in L$, then $\theta = \mu$ if and only if

$$a\theta b \Leftrightarrow a\mu b$$

for all $a \leq b$ in L.

Proof. For part 3 ($\Rightarrow$) suppose $\theta = \mu$

$$\begin{aligned} a\theta b &\Leftrightarrow (a \wedge b)\theta(a \vee b) \\ &\Leftrightarrow (a \wedge b)\mu(a \vee b) \\ &\Leftrightarrow a\mu b. \end{aligned}$$

□

The following simple result is very useful.

Theorem 1.3.3. *[21] An equivalence relation $\equiv$ on a lattice L is a congruence relation if and only if for all $a, b, x \in L$,*

$$a \equiv b \Rightarrow a \wedge x \equiv b \wedge x \text{ and } a \vee x \equiv b \vee x$$

Proof. ($\Rightarrow$) Assume that $\equiv$ is a congruence on a lattice L. If $a \stackrel{\theta}{\equiv} b$, then since $x \stackrel{\theta}{\equiv} x$, we have

$$a \vee x \equiv b \vee x \text{ and } a \wedge x \equiv b \wedge x$$

($\Leftarrow$)If the stated property holds, then

$$\begin{aligned} a \equiv b, x \equiv y &\Rightarrow a \wedge x \equiv b \wedge x, b \wedge x \equiv b \wedge y \\ &\Rightarrow a \wedge x \equiv b \wedge y \end{aligned}$$

and similarly $a \vee x \equiv b \vee y$

□

Example 1.3.7. *[21] Let L be a distributive lattice and let $t \in L$. Then the binary relations defined by*

$$a\delta b \text{ if } a \vee t = b \vee t$$

and

$$a\mu b \text{ if } a \wedge t = b \wedge t$$

are both congruence relations on L. It is easy to see that those relations are equivalence relations. For δ, it is clear that

$$a\delta b \Rightarrow (a \vee x)\delta(b \vee x)$$

and for meet, the distributivity of L gives

$$\begin{aligned} a\delta b &\Rightarrow a \vee t = b \vee t \\ &\Rightarrow (a \vee t) \wedge (x \vee t) = (b \vee t) \wedge (x \vee t) \\ &\Rightarrow (a \wedge x) \vee t = (b \wedge x) \vee t \\ &\Rightarrow (a \wedge x)\delta(b \wedge x) \end{aligned}$$

A similar argument can be made for μ.

Quotient Lattices and Kernels

Homomorpisms and congruence relations express two sides of the same phenomenon. To establish this fact, we first define quotient lattices (also called factor lattices).
Let L be a lattice and let θ be a congruence relation on L. Let L/θ denote the collection of all congruence classes induced by the congruence θ, that is,

$$L/\theta = \{[a]_\theta : a \in L\}$$

set

$$[a]_\theta \wedge [b]_\theta = [a \wedge b]_\theta$$
$$[a]_\theta \vee [b]_\theta = [a \vee b]_\theta$$

This defines $\wedge$ and $\vee$ on L/θ. Indeed, if $[a]_\theta = [a_1]_\theta$ and $[b]_\theta = [b_1]_\theta$, then $a \overset{\theta}{\equiv} a_1$ and $b \overset{\theta}{\equiv} b_1$
therefore, $a \wedge b \equiv a_1 \wedge b_1$, that is $[a \wedge b]_\theta = [a_1 \wedge b_1]_\theta$. Thus $\wedge$ and dually $\vee$, are well defined on L/θ. The lattice axioms are easily verified. The lattice L/θ is the quotient lattice of L modulo θ.

Lemma 1.14. *[13] The map*

$$\pi_\theta \colon x \mapsto [x]_\theta, \text{ for } x \in L$$

is a homomorphism of L onto L/θ

Remark 1.15. The lattice K is a homomorphic image of the lattice L if and only if there is a homomorphisme of L onto K. Lemma 1.14 states that any quotient lattice is a homomorphic image.

Theorem 1.3.4. *[21]*

1. *Every lattice homomorphism $f : L \to M$ defines a congruence relation θ_f given by*

$$a \overset{\theta_f}{\equiv} b \Leftrightarrow f(a) = f(b).$$

and is called the congruence kernel of f. The congruence classes of θ_f are the sets

$$L/\theta_f = \{f^{-1}(x) : x \in im(f)\}$$

Thus, f is injective if and only if θ_f is equality.

2. *Every congruence relation θ on a lattice L defines a lattice epimorphism π_θ : $L \to L/\theta$ given by*

$$\pi_\theta(a) = [a]$$

and called the natural projection or canonical projection of L modulo θ. The congruence kernel of π_θ is θ, that is,

$$\theta_{\pi\theta} = \theta$$

Proof. For part (1), if $a \overset{\theta_f}{\equiv} b$ and $x \overset{\theta_f}{\equiv} y$, then $f(a) = f(b)$ and $f(x) = f(y)$ and so

$$f(a \vee x) = f(a) \vee f(x) = f(b) \vee f(y) = f(b \vee y)$$

which implies that $(a \vee x) \overset{\theta_f}{\equiv} (b \vee y)$. A similar argument can be made for meets. For part $2)$, it is clear that π_θ is surjective. Also,

$$\pi_\theta(a \wedge b) = [a \wedge b] = [a] \wedge [b] = \pi_\theta(a) \wedge \pi_\theta(b)$$

and similarly for join. Hence, π_θ is a lattice epimorphism. Finally, the congruence kernel of π_θ is θ, since

$$a \overset{\theta}{\equiv} b \Leftrightarrow [a]_\theta = [b]_\theta \Leftrightarrow \pi_\theta(a) = \pi_\theta(b).$$

□

Theorem 1.3.5. *[13][The Homomorphism Theorem]*
Let L be a lattice. Any homomorphic image of L is isomorphic to a suitable quotient lattice of L. In fact, if $\phi : L \to L_1$ is a homomorphisme of L onto L_1 and if θ is the congruence relation of L defined by $x \stackrel{\theta}{\equiv} y$ if and only if $\phi(x) = \phi(y)$, then $L/\theta \cong L_1$; an isomorphism is given by

$$\psi : [x]_\theta \to \phi(x), \quad x \in L$$

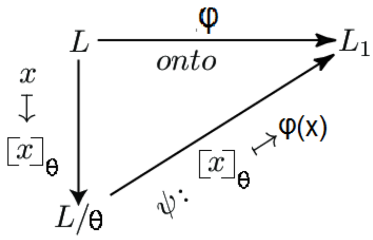

FIGURE 1.10

Proof. It is easy to check that ϕ is a congruence relation. To prove that ψ is an isomorphism we have to check that ψ is well define, is one_to_one, onto, and preserves the operations.

1. Let $[x]_\theta = [y]_\theta$. Then $x \stackrel{\theta}{\equiv} y$, thus $\phi(x) = \phi(y)$, that is, $\psi([x]_\theta) = \psi([y]_\theta)$.
2. Let $\psi([x]_\theta) = \psi([y]_\theta)$, that is $\phi(x) = \phi(y)$. Then $x \stackrel{\theta}{\equiv} y$; and so $[x]_\theta = [y]_\theta$.
3. Let $a \in L_1$. Since ϕ is onto, there is an $x \in L$ with $\phi(x) = a$. Thus $\psi([x]_\theta) = a$.
4. $\psi([x]_\theta \wedge [y]_\theta) = \psi([x \wedge y]_\theta) = \phi(x \wedge y) = \phi(x) \wedge \phi(y) = \psi([x]_\theta) \wedge \psi([y]_\theta)$.The computation for $\vee$ is identical.

□

Definition 1.3.3. *[21]Let $f : L \to M$ be a lattice homomorphism.*

1. *The congruence kernel of f is the congruence relation θ_f.*
2. *If M has a smallest element 0 and if $f^{-1}(0)$ is nonempty, then the set*

$$ker(f) = f^{-1}(0)$$

is called the ideal kernel of f

The ideal kernel of a homomorphism is easily seen to be an ideal. Also, the ideal kernel of the congruence kernel θ_f is the ideal kernel of f, as illustrated in the figure below.

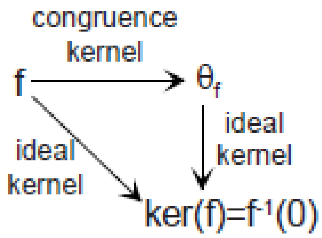

FIGURE 1.11

Chapter 2

Almost Distributive Lattices and Congruence Relations

In this chapter we introduce the concept of Almost Distributive Lattices(ADL), and state important properties of an ADL, also we collect definitions and some preliminary results related to ideals, filters, homomorphism and congruence. Then four special congruence relations are introduced on an ADL, two of them based on the concept of multiplicative closed subset in ADL, and the other based on the concept of a derivation on ADL. In addition this chapter discuss the relation between ADL and regular rings, also a special congrounce relation is introduced in regular rings.

2.1 Almost Distributive Lattice

The concept of an Almost Distributive Lattice (ADL) was introduced by swamy U.M and Rao G.C. They observed that the class of ADL's include most of the existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattice on the other, and they proved that the set of all principal ideals (filters) of an ADL forms a distributive lattice through which many concepts were extended from the class of distributive lattices to the class of ADL's.
In this section we give the definition of an ADL, and we give some basic results and elementary properties of ADL's.

Definition 2.1.1. *[24] An algebraic structure* $(R, \wedge, \vee, 0)$ *is called an Almost Distributive Lattice, abbreviated as ADL with* 0 *if it satisfies the following axioms:*

$(L1)$ $a \vee 0 = a$,
$(L2)$ $0 \wedge a = 0$,
$(L3)$ $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$,
$(L4)$ $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
$(L5)$ $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
$(L6)$ $(a \vee b) \wedge b = b$,
for all $a, b, c \in R$.

It can be verified, by means of non - trivial examples, that every distributive lattice is an ADL. The following example shows that every non - empty set X can be regarded as an ADL with any arbitrarily preassigned element as its zero as follows.

Example 2.1.1. *Let* X *be a non - empty set. Fix* $x_0 \in X$. *For any* $x, y \in X$, *define:*

$$x \wedge y = \begin{cases} x_0 & if \quad x = x_0 \\ y & if \quad x \neq x_0 \end{cases} \qquad x \vee y = \begin{cases} y & if \quad x = x_0 \\ x & if \quad x \neq x_0 \end{cases}$$

Then $(X, \wedge, \vee, x_0)$ *is an ADL with* x_0 *as zero element, and is called a discrete ADL.*

One can directly note that this example isn't a distributive lattice since the commutativity doesn't hold.
From now onwards by R we mean an ADL $(R, \wedge, \vee, 0)$ unless otherwise mentioned.

Example 2.1.2. *Let* $R = \{0, a, b, c\}$ *and define* $\wedge$ *and* $\vee$ *as follows:*

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c

TABLE 2.1

Clearly $(R, \wedge, \vee, 0)$ *is a discrete ADL.*

An ADL $(R, \wedge, \vee, 0)$ satisfies many properties satisfied by a distributive lattice with 0. Now we give certain elementary properties of an ADL.

Lemma 2.1. *[24] For any $a \in R$, we have*

1. $a \wedge 0 = 0$,
2. $a \wedge a = a$,
3. $a \vee a = a$,
4. $0 \vee a = a$.

Proof.

$$
\begin{array}{llll}
1) & 0 & = (a \vee 0) \wedge 0 & \text{by } (L6) \\
 & & = a \wedge 0 & \text{by } (L1)
\end{array}
$$

$$
\begin{array}{llll}
2) & a & = a \vee 0 & \text{by } (L1) \\
 & & = a \vee (0 \wedge 0) & \text{by } (L2) \\
 & & = (a \vee 0) \wedge (a \vee 0) & \text{by } (L5) \\
 & & = a \wedge a & \text{by } (L2)
\end{array}
$$

$$
\begin{array}{llll}
3) & a & = (a \vee a) \wedge a & \text{by } (L6) \\
 & & = (a \wedge a) \vee (a \wedge a) & \text{by } (L3) \\
 & & = a \vee a & \text{by } 2
\end{array}
$$

$$
\begin{array}{llll}
4) & a & = (0 \vee a) \wedge a & \text{by } (L6) \\
 & & = (0 \wedge a) \vee (a \wedge a) & \text{by } (L3) \\
 & & = 0 \vee a & \text{by } (L2), 2
\end{array}
$$

□

Here we have the absorption laws that are valid in ADL's in general.

Theorem 2.1.1. *[24] For any $a, b \in R$, we have:*

1. $(a \wedge b) \vee b = b$,
2. $a \vee (a \wedge b) = a = a \wedge (a \vee b)$,

3. $a \vee (b \wedge a) = a = (a \vee b) \wedge a$.

Proof.

$$\begin{aligned} 1) \quad b &= (a \vee b) \wedge b \\ &= (a \wedge b) \vee (b \wedge b) \\ &= (a \wedge b) \vee b \end{aligned}$$

$$\begin{aligned} 2) \quad a &= a \vee 0 \\ &= a \vee (0 \wedge b) \\ &= (a \vee 0) \wedge (a \vee b) \\ &= a \wedge (a \vee b) \end{aligned}$$

also

$$\begin{aligned} a \vee (a \wedge b) &= (a \vee a) \wedge (a \vee b) \\ &= a \wedge (a \vee b) \end{aligned}$$

$$\begin{aligned} 3) \quad a &= a \vee 0 \\ &= a \vee (b \wedge 0) \\ &= (a \vee b) \wedge (a \vee 0) \\ &= (a \vee b) \wedge a \end{aligned}$$

also

$$\begin{aligned} a \vee (b \wedge a) &= (a \vee b) \wedge (a \vee a) \\ &= (a \vee b) \wedge a \end{aligned}$$

□

Corollary 2.2. *[24] For any $a, b \in R$*

1. $a \vee b = a \Leftrightarrow a \wedge b = b$,

2. $a \vee b = b \Leftrightarrow a \wedge b = a$.

Proof. 1) $(\Rightarrow)$ Let $a \vee b = a$, then from $(L6)$ $b = (a \vee b) \wedge b = a \wedge b$.

$(\Leftarrow)$ Let $a \wedge b = b$, then

$$\begin{aligned} a &= a \vee (0 \wedge b) && \text{by } (L1), (L2) \\ &= (a \vee 0) \wedge (a \vee b) && \text{by } (L5) \\ &= a \wedge (a \vee b) && \text{by } (L1) \\ &= (a \wedge a) \vee (a \wedge b) && \text{by } (L4) \\ &= a \vee b. \end{aligned}$$

□

In view of the above corollary, and if $(R, \wedge, \vee, 0)$ is an ADL , and for any $a, b \in R$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on R.

In theorem 2.1.1, we mentioned the absorption laws that are valid in ADLs in general. Regarding the remaining absorption laws we have the following.

Theorem 2.1.2. *[24] For any $a, b \in R$, the following are equivalent.*

1 $(a \wedge b) \vee a = a.$

2 $a \wedge (b \vee a) = a.$

3 $(b \wedge a) \vee b = b.$

4 $b \wedge (a \vee b) = b.$

5 $a \wedge b = b \wedge a.$

6 $a \vee b = b \vee a.$

7 *The supremum of a and b exists in R and equals $a \vee b$.*

8 *There exists $x \in R$ such that $a \leq x$ and $b \leq x$.*

9 *The infimum of a and b exists in R and equals $a \wedge b$.*

Proof. Firstly, we'll show the equivalence of (1) through (6), then $(6) \Rightarrow (7) \Rightarrow (8) \Rightarrow (1)$, and finally $(5) \Leftrightarrow (9)$.

The equivalence of (1) and (2) as well as that of (3) and (4) follow from $(L4)$. $(5) \Rightarrow (1)$ and $(6) \Rightarrow (2)$ are clear by using theorem 2.1.2. We'll prove $(1) \Rightarrow (5)$ and $(2) \Rightarrow (6)$. Assume (1)

$$\begin{aligned} b \wedge a &= b \wedge \{(a \wedge b) \vee a\} = \{b \wedge (a \wedge b)\} \vee (b \wedge a) \\ &= (a \wedge b) \vee \{a \wedge (b \wedge a)\} \text{ [since } \inf\{b \wedge (a \wedge b)\} = a \wedge b] \\ &= \{(a \wedge b) \vee a\} \wedge \{(a \wedge b) \vee (a \wedge b)\} \\ &= a \wedge (a \wedge b) = a \wedge b \end{aligned}$$

Now assume (2). Then

$$\begin{aligned} a \vee b &= \{a \wedge (b \vee a)\} \vee \{b \wedge (b \vee a)\} = (a \vee b) \wedge (b \vee a) \\ &= \{(a \vee b) \wedge b\} \vee \{(a \vee b) \wedge a\} = b \vee a. \end{aligned}$$

Thus $(2) \Rightarrow (6)$. By interchanging the roles of a and b, we get the equivalence of (1) through (6).

Assume (6). Since for any $a, b \in R$, $a \leq a \vee b$, by (6), we have $a \vee b$ is an upper bound of a and b, to show that $a \vee b$ is the least upper bound, let c be an upper bound of a and b. Then

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) = a \vee b$$

and hence, $a \vee b$ is the supremum of a and b. $(7) \Rightarrow (8)$ is clear. Now we prove $(8) \Rightarrow (1)$. Assume (8), then

$$(a \wedge b) \vee a = (a \wedge b) \vee (a \wedge x) = a \wedge (b \vee x) = a \wedge x = a.$$

The equivalence of (5) and (9) follows, dually, from that of (6) and (7). □

Lemma 2.3. *[24] For any $a, b \in R$, $(a \vee b) \vee a = a \vee b = a \vee (b \vee a)$.*

Proof.

$$\begin{aligned} a \vee b &= a \vee (b \wedge (b \vee a)) = (a \vee b) \wedge (a \vee (b \vee a)) \\ &= ((a \vee b) \wedge a) \vee [((a \vee b) \wedge b) \vee ((a \vee b) \wedge a)] \\ &= a \vee (b \vee a). \end{aligned}$$

□

For any $a, b, c \in R$, we have $a \wedge c \leq c$ and hence the following are a consequences of theorem 2.1.2

Lemma 2.4. *[24] For any $a, b, c \in R$, $(a \vee b) \wedge c = (b \vee a) \wedge c$.*

Lemma 2.5. *[24] $\wedge$ is associative in R.*

Proof.

$$\begin{aligned}
(a \wedge b) \wedge c &= (a \wedge b) \wedge [c \vee \{a \wedge (b \wedge c)\}] \\
&= \{(a \wedge b) \wedge c\} \vee [(a \wedge b) \wedge \{a \wedge (b \wedge c)\}] \\
&= \{(a \wedge b) \wedge c\} \vee \{a \wedge (b \wedge c)\} [\text{by corollary 2.2, and since} \\
&\qquad (a \wedge b) \vee \{a \wedge (b \wedge c)\} = a \wedge \{b \vee (b \wedge c)\} = a \wedge b] \\
&= [\{(a \wedge b) \wedge c\} \vee a] \wedge [\{(a \wedge b) \wedge c\} \vee (b \wedge c)] \\
&= a \wedge (b \wedge c).
\end{aligned}$$

□

from lemmas 2.4 and 2.5, we have the following.

Lemma 2.6. *[24] For any $a, b, c \in R$, $a \wedge b \wedge c = b \wedge a \wedge c$.*

More generally, and by mathematical induction, we have

Lemma 2.7. *[24] If $a_1, a_2, ..., a_n \in R$ and $(i_1, i_2, ..., i_n)$ is any permutation of $(1, 2, ..., n)$, then*

$$a_{i_1} \wedge a_{i_2} \wedge ... \wedge a_{i_n} \wedge b = a_1 \wedge a_2 \wedge ... \wedge a_n \wedge b.$$

Theorem 2.1.3. *[19] If $(R, \wedge, \vee, 0)$ is an ADL, for any $a, b, c \in R$, we have the following*

$$a \wedge b = 0 \Leftrightarrow b \wedge a = 0$$

Proof. let $a \wedge b = 0$, then

$$b \wedge a = (b \wedge a) \wedge b = (a \wedge b) \wedge b = 0 \wedge b = 0$$

□

Theorem 2.1.4. *[19] If $a \leq c$, $b \leq c$ then $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$.*

Proof.

$$\begin{aligned} a \wedge b &= a \wedge (b \wedge c) &= (a \wedge b) \wedge c \\ &= (b \wedge a) \wedge c &= b \wedge (a \wedge c) \\ &= b \wedge a. \end{aligned}$$

□

It can be observed that an ADL R satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. It was also observed that any one of these three properties converts an ADL into a distributive lattice.

Theorem 2.1.5. *[19] Let $(R, \wedge, \vee, 0)$ be an ADL With 0. Then the following are equivalent*

1. $(R, \wedge, \vee, 0)$ *is a distributive lattice with smallest element* 0.
2. $a \vee b = b \vee a$, *for all* $a, b \in R$.
3. $a \wedge b = b \wedge a$, *for all* $a, b \in R$.
4. $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, *for all* $a, b, c \in R$.

Proof. $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (4) \Rightarrow (1)$. All are trivial except $(3) \Rightarrow (2)$

$$\begin{aligned} a \vee b &= \{(b \vee a) \wedge a\} \vee \{b \wedge (b \vee a)\} \\ &= \{a \wedge (b \vee a)\} \vee \{b \wedge (b \vee a)\} \\ &= (a \vee b) \wedge (b \vee a) \\ &= \{(a \vee b) \wedge b\} \vee \{(a \vee b) \wedge a\} \\ &= b \vee a. \end{aligned}$$

□

As in distributive lattices, we have the following definition.

Definition 2.1.2. *[19]*

1. *A non - empty subset I of an ADL R is called an ideal of R if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in R$.*

2. *A non - empty subset F of an ADL R is called a filter of R if $a \wedge b \in F$ and $x \vee a \in F$ for any $a, b \in F$ and $x \in R$.*

Remark 2.8. The set $Id(R)$ of all ideals of R is a bounded distributive lattice with least element $\{0\}$ and greatest element R under set inclusion in which, for any $I, J \in I(R)$, $I \cap J$ is the infimum of I and J, while the supremum is given by $I \cup J := \{a \vee b : a \in I, b \in J\}$.

Here we have some special types of ideals in ADLs

Definition 2.1.3. *[19]*

1. *A proper ideal P of R is called a prime ideal if, for any $x, y \in R$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$.*

2. *A proper filter P of R is called a prime filter if, for any $x, y \in R$, $x \vee y \in P \Rightarrow x \in P$ or $y \in P$.*

3. *A proper ideal (filter) M of R is said to be maximal if it is not properly contained in any proper ideal (filter) of R.*

Definition 2.1.4. *[19] An element $m \in R$ is called maximal if it is a maximal element in the partially ordered set $(R, \leq)$. That is, for any $a \in R$, $m \leq a \Rightarrow m = a$.*

Theorem 2.1.6. *[19] Let R be an ADL and $m \in R$. Then the following are equivalent:*

1. *m is maximal with respect to $\leq$.*

2. *$m \vee a = m$, for all $a \in R$.*

3. *$m \wedge a = a$, for all $a \in R$.*

4. *$a \vee m$ is maximal, for all $a \in R$.*

Remark 2.9.

1. Every maximal ideal of R is a prime ideal.

2. Every proper ideal of R is contained in a maximal ideal.

3. For any subset S of R, the smallest ideal containing S is given by

$$(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x : s_i \in S, x \in R \text{ and } n \in N\}.$$

If $S = \{s\}$, we write $(s]$ insted of $(S]$. Similarly, for any $S \in R$,

$$[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) : s_i \in S, x \in R \text{ and } n \in N\}.$$

If $S = \{s\}$, we write $[s)$ insted of $[S)$.

Theorem 2.1.7. *[19]For any x, y in R the following are equivalent:*

1. $(x] \subseteq (y]$.

2. $y \wedge x = x$.

3. $y \vee x = y$.

4. $[y) \subseteq [x)$.

For any $x, y \in R$, it can be verified that $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$. Hence the set $PI(R)$ of all principal ideals of R is a sublattice of the distributive lattice $I(R)$ of ideals of R.

Theorem 2.1.8. *[19] Let I be an ideal and F a filter of R such that $I \cap F = \phi$. Then there exist a prime ideal P such that $I \subseteq P$ and $P \cap F = \phi$*

Definition 2.1.5. *[16] Let R and R' be any two ADLs. A mapping $f : R \to R'$ is called a homomorphism if it satisfies the following:*

1. $f(a \vee b) = f(a) \vee f(b)$.

2. $f(a \wedge b) = f(a) \wedge f(b)$.

3. $f(0) = 0'$, *for all* $a, b \in R$.

Definition 2.1.6. *[16] An equivalence relation* θ *on* R *is called a congruence relation if for all* $a, b, c, d \in R$,

$$a \overset{\theta}{\equiv} b,\ c \overset{\theta}{\equiv} d \Rightarrow a \wedge c \overset{\theta}{\equiv} b \wedge d,\ a \vee c \overset{\theta}{\equiv} b \vee d$$

Definition 2.1.7. *[16] For any congruence relation* θ *on an ADL* R *and* $a \in R$, *we define*

$$[a]_\theta = \{b \in R : a \overset{\theta}{\equiv} b\}$$

and it is called the congruence class containing a, *and the set of all congruence classes of* R *is denoted by* R/θ.

Now we prove the following results.

Theorem 2.1.9. *[19] Let* θ *be a congruence relation on an ADL R and* $R/\theta = \{[x]_\theta : x \in L\}$. *Define binary operations* $\wedge, \vee$ *on* R/θ *by*

$$[x]_\theta \wedge [y]_\theta = [x \wedge y]_\theta \text{ and } [x]_\theta \vee [y]_\theta = [x \vee y]_\theta,$$

then $(R/\theta, \wedge, \vee)$ *is an ADL.*

Proof. Let θ be a congruence relation on an ADL R. For any $x \in R$, $[x]_\theta = \{y \in R : x \overset{\theta}{\equiv} y\}$. Write $R/\theta = \{[x]_\theta : x \in R\}$. Define binary operations $\wedge, \vee$ on R/θ by

$$[x]_\theta \wedge [y]_\theta = [x \wedge y]_\theta \text{ and } [x]_\theta \vee [y]_\theta = [x \vee y]_\theta.$$

1. $\wedge, \vee$ **are well defined**: Let $[x]_\theta = [x_1]_\theta$, $[y]_\theta = [y_1]_\theta \Rightarrow (x, x_1), (y, y_1) \in \theta \Rightarrow (x \vee y, x_1 \vee y_1) \in \theta \Rightarrow [x \vee y]_\theta = [x_1 \vee y_1]_\theta \Rightarrow [x]_\theta \vee [y]_\theta = [x_1]_\theta \vee [y_1]_\theta$. Similarly, we can prove $[x]_\theta \wedge [y]_\theta = [x_1]_\theta \wedge [y_1]_\theta$. Therefore θ is well defined.

2. **(L1)**: $[x]_\theta \vee 0 = [x]_\theta \vee [0]_\theta = [x \vee 0]_\theta = [x]_\theta$.

3. **(L2)**: $0 \wedge [x]_\theta = [0]_\theta \wedge [x]_\theta = [0 \wedge x]_\theta = [0]_\theta$.

4. **(L3)**: Let $[x]_\theta, [y]_\theta, [z]_\theta \in R/\theta$. Consider $([x]_\theta \vee [y]_\theta) \wedge [z]_\theta = [x \vee y]_\theta \wedge [z]_\theta = [(x \vee y) \wedge z]_\theta = [(x \wedge z) \vee (y \wedge z)]_\theta = [x \wedge z]_\theta \vee [y \wedge z]_\theta = ([x_\theta] \wedge [z]_\theta) \vee ([y]_\theta \wedge [z]_\theta)$.

5. (**L4**): Consider $[x]_\theta \wedge ([y]_\theta \vee [z]_\theta) = [x]_\theta \wedge [y \vee z]_\theta = [(x) \wedge (y \vee z)]_\theta = [(x \wedge y) \vee (x \wedge z)]_\theta$ $= [x \wedge y]_\theta \vee [x \wedge z]_\theta = ([x_\theta] \wedge [y]_\theta) \vee ([x]_\theta \wedge [z]_\theta)$.

6. (**L5**): Consider $[x]_\theta \vee ([y]_\theta \wedge [z]_\theta) = [x]_\theta \vee [y \wedge z]_\theta = [(x) \vee (y \wedge z)]_\theta = [(x \vee y) \wedge (x \vee z)]_\theta$ $= [x \vee y]_\theta \wedge [x \vee z]_\theta = ([x_\theta] \vee [y]_\theta) \wedge ([x]_\theta \vee [z]_\theta)$.

7. (**L6**): Consider $([x]_\theta \vee [y]_\theta) \wedge [y]_\theta = [x \vee y]_\theta \wedge [y]_\theta = [(x \vee y) \wedge y]_\theta = [y]_\theta$.

□

Theorem 2.1.10. *[17] An equivalence relation θ on an ADL R is a **congruence relation** if and only if for any $a \overset{\theta}{\equiv} b$, $x \in R$, $a \vee x \overset{\theta}{\equiv} b \vee x$, $a \wedge x \overset{\theta}{\equiv} b \wedge x$.*

Definition 2.1.8. *[16] A subset S of R is said to be multiplicatively closed subset of R if $S \neq \phi$ and for any $a, b \in S \Rightarrow a \wedge b \in S$.*

Note. Any filter F of R is a multiplicatively closed subset.

Corollary 2.10. *[16] Let I be an ideal and S be a multiplicatively closed subset of R such that $I \cap S = \phi$. Then there is a prime ideal M of R such that $I \subseteq M$ and $M \cap S = \phi$.*

2.2 Special Congruence Relations on Almost Distributive Lattices

In this section, and based on the concept of multiplicatively closed subset S of an ADL R special congruence relations ψ^S and ϕ^S are introduced on an ADL R. Also some Properties of ψ^S and ϕ^S are discussed. Further we show for any prime ideal P and a filter F of an ADL R, there exist an order preserving onto map between the set of all prime ideals of R/ψ^F and the set of prime ideals of R disjoint with F.

Definition 2.2.1. *[16] Let S be a multiplicatively closed subset of an ADL R. We define the relations ψ^S and ϕ^S on R as follows:*
for all $a, b \in R$,

$$a \overset{\psi^S}{\equiv} b \Leftrightarrow a \wedge s = b \wedge s \text{ for some } s \in S.$$
$$a \overset{\phi^S}{\equiv} b \Leftrightarrow s \wedge a = s \wedge b \text{ for some } s \in S.$$

In the following theorem we show that ψ^S is a congruence relation.

Theorem 2.2.1. *[16] The relation ψ^S is a congruence relation on the ADL R and S is contained in a single congruence class.*

Proof. 1. Firstly, we prove ψ^S is a congruence relation on R.

Obviously, ψ^S is reflexive and symmetric. Assume $x \overset{\psi^S}{\equiv} y$ and $y \overset{\psi^S}{\equiv} z$ for $x, y, z \in S$. Then

$$x \wedge s = y \wedge s \text{ for some } s \in S$$

and

$$y \wedge t = z \wedge t \text{ for some } t \in S.$$

Also, $s \wedge t \in S$; as S is a multiplicatively closed subset of R. Further $x \wedge s \wedge t = y \wedge s \wedge t = s \wedge y \wedge t = s \wedge z \wedge t = z \wedge s \wedge t$. This shows that $x \overset{\psi^S}{\equiv} z$. Hence ψ^S is transitive.

Let $a \overset{\psi^S}{\equiv} b$ and $c \overset{\psi^S}{\equiv} d$ for $a, b, c, d \in R$. Then $a \wedge s = b \wedge s$ and $c \wedge t = d \wedge t$ for some $s, t \in S$. As S is multiplicatively closed subset of R, $s \wedge t \in S$. Further $a \wedge c \wedge s \wedge t = a \wedge s \wedge c \wedge t = b \wedge s \wedge c \wedge t = b \wedge s \wedge d \wedge t = b \wedge d \wedge s \wedge t$, this show that $a \wedge c \equiv b \wedge d$.

Again

$$\begin{aligned}
(a \vee c) \wedge s \wedge t &= (a \wedge s \wedge t) \vee (c \wedge s \wedge t) \\
&= (b \wedge s \wedge t) \vee (s \wedge c \wedge t) \\
&= (b \wedge s \wedge t) \vee (s \wedge d \wedge t) \\
&= (b \wedge s \wedge t) \vee (d \wedge s \wedge t) \\
&= (b \vee d) \wedge (s \wedge t)
\end{aligned}$$

Hence $a \vee c \overset{\psi^S}{\equiv} b \vee d$ as $s \wedge t \in S$. Therefore ψ^S is a congruence relation on R.

2. We prove that S is contained in one congruence class under ψ^S.

Let $s,\ t \in S$, then $s \wedge t \in S$ and $s \wedge (s \wedge t) = s \wedge t = s \wedge t \wedge t = t \wedge (s \wedge t)$. Then we get $s \overset{\psi^S}{\equiv} t$. This shows that S is contained in one congruence class of the congruence relation ψ^S.

□

Example 2.2.1. *Let R be an ADL whose Hasse diagram is given in figure 2.1.*

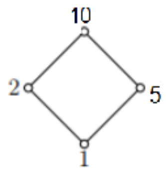

FIGURE 2.1

then $S = \{1, 2, 10\}$ is a multiplicatively closed subset of R, and

$$2 \overset{\psi^S}{\equiv} 5$$

since $1 \in S$ and

$$2 \wedge 1 = 5 \wedge 1$$

Also, $2 \overset{\psi^S}{\equiv} 10$ and $2 \overset{\psi^S}{\equiv} 1$ since S is contained in one congruence class. hence

$$[1]_{\psi^S} = \{1, 2, 5, 10\}$$

We know that if the operation $\wedge$ in ADL R is commutative then R is a distributive lattice.

Theorem 2.2.2. *[16] R/ψ^S is a distributive lattice.*

Proof. Let $x, y \in R$. Since $S \neq \phi$, we can choose $a \in S$. But then $x \wedge y \wedge a = y \wedge x \wedge a$ implies $x \wedge y \overset{\psi^S}{\equiv} y \wedge x$. Hence $[x]_{\psi^S} \wedge [y]_{\psi^S} = [x \wedge y]_{\psi^S} = [y \wedge x]_{\psi^S} = [y]_{\psi^S} \wedge [x]_{\psi^S}$. Thus the operation $\wedge$ is commutative on R/ψ^S and hence R/ψ^S is a distributive lattice. □

The following theorem show in details that ϕ^S is a congruence relation.

Theorem 2.2.3. *[16] The relation ϕ^S is a congruence relation on an ADL R.*

Proof. Obviously, ϕ^S is reflexive and symmetric. Let $a \overset{\phi^S}{\equiv} b$, $c \overset{\phi^S}{\equiv} d$. Then $x \wedge a = x \wedge b$ and $y \wedge c = y \wedge d$, for some $x, y \in S$. As S is multiplicatively closed subset of R, we have $x \wedge y \in S$. Now,

$$x \wedge y \wedge a = y \wedge x \wedge a = y \wedge x \wedge b = x \wedge y \wedge b = x \wedge y \wedge c.$$

Therefore $a \overset{\phi^S}{\equiv} c$. Hence ϕ^S is an equivalence relation on R.
Let $a \overset{\phi^S}{\equiv} b$, $c \overset{\phi^S}{\equiv} d$. Then $x \wedge a = x \wedge b$ and $y \wedge c = y \wedge d$, for some $x, y \in S$. Since S is multiplicatively closed subset of R, we have $x \wedge y \in S$ and $x \wedge y \wedge a \wedge c = x \wedge a \wedge y \wedge c$ $= x \wedge b \wedge y \wedge d = x \wedge y \wedge b \wedge d$. Also

$$\begin{aligned}(x \wedge y) \wedge (a \vee c) &= (x \wedge y \wedge a) \vee (x \wedge y \wedge c) \\ &= (y \wedge x \wedge a) \vee (x \wedge y \wedge d) \qquad (\text{since } y \wedge c = y \wedge d) \\ &= (y \wedge x \wedge b) \vee (x \wedge y \wedge d) \qquad (\text{since } a \wedge a = x \wedge b) \\ &= ((x \wedge y) \wedge b) \vee ((x \wedge y) \wedge d) \\ &= (x \wedge y) \wedge (b \vee d)\end{aligned}$$

Therefore $(a \wedge c) \overset{\phi^S}{\equiv} (b \wedge d)$, $(a \vee c) \overset{\phi^S}{\equiv} (b \vee d)$. Thus ϕ^S is congruence relation on R.

□

From the definitions of ϕ^S and ψ^S we get the following remarks.

Remark 2.11. [16] If R is lattice, then $\phi^S = \psi^S$.

Remark 2.12. [16] R/ϕ^S is an ADL under $\wedge$ and $\vee$ defined by $[x]_{\phi^S} \vee [y]_{\phi^S} = [x \vee y]_{\phi^S}$ and $[x]_{\phi^S} \wedge [y]_{\phi^S} = [x \wedge y]_{\phi^S}$. But R/ϕ^S need not be a lattice.
For this consider the following example which shows that D/ϕ^S(where D is a discreet ADL) is isomorphic to a discreet ADL.

Example 2.2.2. *Let D be a discrete ADL as given in the following tables. Let $S = D\backslash\{0\}$, then S is a multiplicatively closed subset of D*

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	b	1
b	0	a	b	1
1	0	a	b	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	a	a
b	b	b	b	b
1	1	1	1	1

TABLE 2.2

Now since $\phi^S = \{(x, y) \in D \times D : s \wedge x = s \wedge y$ for some $s \neq 0\}$. Then we have

$$[0]_{\phi^S} = \{0\},\ [a]_{\phi^S} = \{a\},\ [b]_{\phi^S} = \{b\},\ [1]_{\phi^S} = \{1\}.$$

Hence $D/\phi^S \cong D$

Remark 2.13. If D is a discrete ADL. Let $S = D\backslash\{0\}$, then $\phi^S = \{(x, y) \in D$ x $D : s \wedge x = s \wedge y$ for some $s \neq 0\}$. Therefore $D/\phi^S \cong D$ which isn't a lattice, unless $|D| \leq 2$.

Remark 2.14. For any multiplicatively closed subset S of R, $\phi^S \subseteq \psi^S$.

Theorem 2.2.4. *[16] Let S and T be two multiplicatively closed subsets of the ADLs R_1 and R_2 respectively. Then for any homomorphism $\Phi : R_1 \to R_2$ such that $\Phi(S) \subseteq T$, there exists a homomorphism $f : R_1/\psi^S \to R_2/\psi^T$ such that $fo\ h = ko\ \Phi$, where $h : R_1 \to R_1/\psi^S$ and $k : R_2 \to R_2/\psi^T$ denote the canonical epimorphisms. Further*

1. *If Φ is a monomorphism and if $\Phi(S) = T$, then f is a monomorphism.*
2. *If Φ is an epimorphism, then f is an epimorphism.*

Proof. Define $f : R_1/\psi^S \to R_2/\psi^T$ by $f([x]_{\psi^S}) = [\Phi(x)]_{\psi^T}$ for each $x \in R_1$. Let $[x]_{\psi^S} = [y]_{\psi^S}$ for some $x, y \in R_1$. Then

$$\begin{aligned}
[x]_{\psi^S} = [y]_{\psi^S} &\Rightarrow x \overset{\psi^S}{\equiv} y \\
&\Rightarrow x \wedge s = y \wedge s \qquad \text{for some } s \in S \\
&\Rightarrow \Phi(x \wedge s) = \Phi(y \wedge s) \\
&\Rightarrow \Phi(x) \wedge \Phi(s) = \Phi(y) \wedge \Phi(s) \\
&\Rightarrow \Phi(x) \overset{\psi^T}{\equiv} \Phi(y) \qquad \text{as } \Phi(s) \in T \\
&\Rightarrow [\Phi(x)]_{\psi^T} = [\Phi(y)]_{\psi^T} \\
&\Rightarrow f([x]_{\psi^S}) = f([y]_{\psi^S})
\end{aligned}$$

This shows that f is well defined.

Let $x, y \in S$.

$$\begin{aligned}
f([x]_{\psi^S} \wedge [y]_{\psi^S}) &= f([x \wedge y]_{\psi^S}) \\
&= [\Phi(x \wedge y)]_{\psi^T} \\
&= [\Phi(x) \wedge \Phi(y)]_{\psi^T} \\
&= [\Phi(x)]_{\psi^T} \wedge [\Phi(y)]_{\psi^T} \\
&= f([x]_{\psi^S}) \wedge f([y]_{\psi^S}).
\end{aligned}$$

Similarly we can prove $f([x]_{\psi^S} \vee [y]_{\psi^S}) = f([x]_{\psi^S}) \vee f([y]_{\psi^S})$ for all $x, y \in R_1$. Hence f is a homomorphism.

Now $fo\, h : R_1 \to R_2/\psi^T$ and for any $x \in R_1$ we have $[fo\, h](x) = f(h(x)) = f([x]_{\psi^S})$ = $[\Phi(x)]_{\psi^T}$. Again $ko\ \Phi : R_1 \to R_2/\psi^T$ and for any $x \in R_1$ we have $[ko\ \Phi](x)$ = $k(\Phi(x)) = [\Phi(x)]_{\psi}$. Hence $[fo\, h](x) = [ko\ \Phi](x), \forall x \in R_1$. This shows that $fo\, h = ko\ \phi$.

1. Let Φ be a monomorphism and let $\Phi(S) = T$. Let $f([x]_{\psi^S}) = f([y]_{\psi^S})$ for some $x, y \in R_1$. Then $[\Phi(x)]_{\psi^T} = [\Phi(y)]_{\psi^T} \Rightarrow \Phi(x) \overset{\psi^T}{\equiv} \Phi(y) \Rightarrow \Phi(x) \wedge t = \Phi(y) \wedge t$, for some $t \in T \Rightarrow \Phi(x) \wedge \Phi(s) = \Phi(y) \wedge \Phi(s)$, for some $s \in S$ (since $\Phi(S) = T$) $\Rightarrow \Phi(x \wedge s) = \Phi(y \wedge s)$ (since Φ is a monomorphism) $\Rightarrow x \wedge s = y \wedge s \Rightarrow x \overset{\psi^S}{\equiv} y \Rightarrow [x]_{\psi^S} = [y]_{\psi^S}$. This shows that f is one-one.
2. Let Φ be an epimorphism. Let $[y]_{\psi^T} \in R_2/\psi^T$. As $\Phi : R_1 \to R_2$ is onto and $y \in R_2$, $\Phi(x) = y$ for some $x \in R_1$. Thus $[x]_{\psi^S} \in R_1/\psi^S$ and $f([x]_{\psi^S}) = [\Phi(x)]_{\psi^T}$ = $[y]_{\psi^T}$. This shows that f is an epimorphism.

□

For any two congruence relations ψ^S and ψ^T induced by two multiplicatively closed subsets S and T of R with $S \subseteq T$ we have the following theorem.

Theorem 2.2.5. *[16] Let R be an ADL and let S, T be any two multiplicatively closed subsets of R with $S \subseteq T$. Then the following are equivalent:*

1. *The mapping $f : R/\psi^S \to R/\psi^T$ defined by $f([x]_{\psi^S}) = [x]_{\psi^T}$ for each $x \in R$, is an isomorphism.*
2. *For each $t \in T$, there exists $s \in S$ such that $t \wedge s \in S$.*
3. *For any prime ideal P of R, $P \cap T \neq \phi \Rightarrow P \cap S \neq \phi$.*

Proof. $i) \Rightarrow ii)$
Obviously f is a well defined map. Let $x, y \in R$. Then

$$x \overset{\psi^T}{\equiv} y \Rightarrow f([x]_{\psi^S}) = f([y]_{\psi^S}) \Rightarrow [x]_{\psi^S} = [y]_{\psi^S}$$

(since f is one-one)

$$\Rightarrow x \overset{\psi^S}{\equiv} y.$$

Therefore

$$x \overset{\psi^T}{\equiv} y \Rightarrow x \overset{\psi^S}{\equiv} y \Rightarrow \psi^T \subseteq \psi^S.$$

As $S \subseteq T$, $\psi^S \subseteq \psi^T$. Hence

$$\psi^S = \psi^T.$$

Hence any $t \in T$ must be congruent to some $s_1 \in S$. Which mean $t \equiv s_1 \in S$. Therefore $t \wedge s = s_1 \wedge s$ for some $s \in S$. As $s_1 \wedge s \in S$, we get $t \wedge s \in S$.

$ii) \Rightarrow iii)$
Let P be a prime ideal in R such that $P \cap T \neq \phi$. Select any $t \in P \cap T$. As $t \in T$ there exists $s \in S$ such that $t \wedge s \in S$. As $t \in P$, $t \wedge s \in P$. Thus $t \wedge s \in P \cap S$. This shows that $P \cap S \neq \phi$.

$iii) \Rightarrow i)$
Claim: $\psi^S = \psi^T$.
As $S \subseteq T \Rightarrow \psi^S \subseteq \psi^T$. To prove that $\psi^T \subseteq \psi^S$. Let $a \overset{\psi^T}{\equiv} b$. Hence $a \wedge t = b \wedge t$ for any $t \in T$. Suppose $S \cap (t] = \phi$. Then there is a prime ideal P such that $(t] \subseteq P$ and $P \cap S = \phi$, (by Corollary 2.10.) which contradicts the assumption (iii) as $t \in P \cap T \Rightarrow P \cap S \neq \phi$. Hence $S \cap (t] \neq \phi$. Therefore $\exists s \in S \cap (t]$. Hence $s = t \wedge x$ for some $x \in R$.
Now $a \wedge s = a \wedge (t \wedge x) = (a \wedge t) \wedge x = (b \wedge t) \wedge x = b \wedge (t \wedge x) = b \wedge s$. But this shows that $a \overset{\psi^S}{\equiv} b$. Thus $\psi^T \subseteq \psi^S$. Combining both the inclusions we get $\psi^T = \psi^S$ and the implication follows. □

Theorem 2.2.6. *[16] Let R be an ADL with maximal elements and F a filter of R and let $h : R \to R/\psi^F$ be the canonical epimorphism. Then we have*

1. *If P' is a prime ideal in R/ψ^F, then $h^{-1}(P')$ is a prime ideal in R disjoint with F.*
2. *Let $\Theta : \mathfrak{P}[R/\psi^F] \to \{Q \in \mathfrak{P} : Q \cap F = \phi\}$ be defined by $\Theta(P') = h^{-1}(P')$.*

Then Θ is an order preserving onto map, where $\mathfrak{P}$ and $\mathfrak{P}[R/\psi^F]$ denote the set of all prime ideals of R and R/ψ^F respectively.

Proof. (I) As $h : R \to R/\psi^F$ is an epimorphism, we get $h^{-1}(P')$ is a prime ideal in R. It's remain only to prove that $h^{-1}(P') \cap F = \phi$.

Let $s \in h^{-1}(P') \cap F$. If m is a maximal element, then m, $s \in F$ and hence $m \stackrel{\psi^F}{\equiv} s$. Therefore $h(m) = h(s) \in P'$. A contradiction since $h(m)$ is a maximal element in R/ψ^F. Hence $h^{-1}(P') \cap F = \phi$. Thus $h^{-1}(P') \in \{Q \in \mathfrak{P} : Q \cap F = \phi\}$.

(II) Let $P', Q' \in \mathfrak{P}[R/\psi^F]$ such that $P' \subseteq Q'$. Let $\Theta(P') = P$ and $\Theta(Q') = Q$. If $P' \subseteq Q'$ then $h^{-1}(P') \subseteq h^{-1}(Q')$ and hence $\Theta(P') \subseteq \Theta(Q')$. Then Θ is order preserving.

Let $P \in \mathfrak{P}$ be such that $P \cap F = \phi$. $P \subseteq h^{-1}(h(P))$ always. To prove that $h^{-1}(h(P)) \subseteq P$. Let $x \in h^{-1}(h(P))$. Then as $h(x) \in h(P)$, $[x]_{\psi^F} = [p]_{\psi^F}$ for some $p \in P$. This means $x \stackrel{\psi^F}{\equiv} p$. Therefore $x \wedge s = p \wedge s$ for some $s \in F$. As $P \cap F = \phi$, $s \notin P$. Again $p \wedge s \in P$, but then $x \wedge s \in P$ implies $x \in P$ as $s \notin P$. This shows that $h^{-1}(h(P)) \subseteq P$. Combining both the inclusions we get $h^{-1}(h(P)) = P$. Hence Θ is onto.

□

Theorem 2.2.7. *[16] Let S denote a multiplicatively closed subset of an ADL R with maximal elements. Let P' be a prime ideal in $[R/\psi^S]$. Define $h^{-1}(P') = P$, where*

$$h : R \to R/\psi^S$$

is the canonical epimorphism. Then the mapping

$$\alpha : R/\psi^T \to (R/\psi^S)/\psi^{T'}$$

defined by $\alpha([x]_{\psi^T}) = [[x]_{\psi^S}]_{\psi^{T'}}$ is an isomorphism, where $T = R \backslash P$ and $T' = [R/\psi^S] \backslash P'$ are the filters in the ADLs R and R/ψ^S respectively.

Proof. Let $[x]_{\psi^T} = [y]_{\psi^T}$. Then $x \wedge t = y \wedge t$ for some $t \in T$ as $x \stackrel{\psi^T}{\equiv} y$. But $t \notin P$ implies $h(t) = [t]_{\psi^S} \notin P'$ and hence $[t]_{\psi^S} \in [R/\psi^S] \backslash P' = T'$. Further

$$\begin{aligned}
[x]_{\psi^S} \wedge [t]_{\psi^S} = [y]_{\psi^S} \wedge [t]_{\psi^S} \quad &\Rightarrow \quad [x]_{\psi^S} \stackrel{\psi^{T'}}{\equiv} [y]_{\psi^S} \\
&\Rightarrow \quad [[x]_{\psi^S}]_{\psi^{T'}} = [[y]_{\psi^S}]_{\psi^{T'}} \\
&\Rightarrow \quad \alpha([x]_{\psi^T}) = \alpha([y]_{\psi^T}).
\end{aligned}$$

This shows that α is well defined.

To prove that α is one-one. Claim : $P \cap S = \phi$.
As P' is a prime ideal in $[R/\psi^S]$, P' is a proper ideal in $[R/\psi^S]$. Hence $[m]_{\psi^S} \notin P'$ for any maximal element in R. But $[m]_{\psi^S} = S$ for all maximal elements m in R. Hence $S \notin P'$. Let $s_1 \in P \cap S$. Then $s_1 \in P \Rightarrow s_1 \in h^{-1}(P') \Rightarrow h(s_1) \in P' \Rightarrow [s_1]_{\psi^S} \in P' \Rightarrow S \in P'$, a contradiction. Hence $P \cap S = \phi$.

$$\begin{aligned}
\text{Let } \alpha([x]_{\psi^T}) = \alpha([y]_{\psi^T}) \quad &\Rightarrow \quad ([x]_{\psi^S}]_{\psi^{T'}}) = [[y]_{\psi^S}]_{\psi^{T'}}) \\
&\Rightarrow \quad [x]_{\psi^S} \stackrel{\psi^{T'}}{\equiv} [y]_{\psi^{T'}} \\
&\Rightarrow \quad [x]_{\psi^S} \wedge [t]_{\psi^S} = [y]_{\psi^S} \wedge [t]_{\psi^S} \quad \text{for some } [t]_{\psi^S} \in T' = [R/\psi^S]\backslash P' \\
&\Rightarrow \quad [x \wedge t]_{\psi^S} = [y \wedge t]_{\psi^S} \quad \text{for some } [t]_{\psi^S} \in T' = [R/\psi^S]\backslash P' \\
&\Rightarrow \quad (x \wedge t \wedge s) = (y \wedge t \wedge s) \quad \text{for some } s \in S.
\end{aligned}$$

As P' is prime ideal in R/ψ^S, by claim $P \cap S = \phi$. Hence $t \in T$ and $s \in T$ imply $t \wedge s \in T$. But then $x \wedge (t \wedge s) = y \wedge (t \wedge s)$ for $t \wedge s \in T \Rightarrow x \stackrel{\psi^T}{\equiv} y \Rightarrow [x]_{\psi^T} = [y]_{\psi^T}$. But this shows that α is one-one .

Now to prove that α is a homomorphism. For any $x, y \in R$, we have

$$\begin{aligned}
\alpha([x]_{\psi^T} \wedge [y]_{\psi^T}) &= [[x \wedge y]_{\psi^S}]_{\psi^{T'}} \\
&= [[x]_{\psi^S} \wedge [y]_{\psi^S}]_{\psi^{T'}} \\
&= [[x]_{\psi^S}]_{\psi^{T'}} \wedge [[y]_{\psi^S}]_{\psi^{T'}} \\
&= \alpha([x]_{\psi^T}) \wedge \alpha([y]_{\psi^T})
\end{aligned}$$

Similarly we can prove that $\alpha([x]_{\psi^T} \vee [y]_{\psi^T}) = \alpha([x]_{\psi^T}) \vee \alpha([y]_{\psi^T})$ for any $x, y \in R$. Hence α is a homomorphism.

Obviously α being an onto map, we get α is an isomorphism and hence the result.

□

2.3 d_Congruence of Almost Distributive Lattices

In this section, we introduce the concept of a derivation in an Almost Distributive Lattice (ADL), then two kinds of congruences are proposed on ADL, the first one is considered in terms of ideals generated by derivations and second one is in terms of images of derivations. An equivalent condition is derived for the corresponding quotient ADL to become a Boolean algebra. An another equivalent condition is also established for the existence of a derivation.

Definition 2.3.1. *[19]Let R be an ADL. A self - mapping $d : R \to R$ is called a derivation of R if it satisfies the following properties:*

1. $d(x \wedge y) = d(x) \wedge y$.
2. $d(x \vee y) = d(x) \vee d(y)$, *for all* $x, y \in R$.

Note. The kernel of a derivation is defined as.

$$\ker(d) = \{x \in R : d(x) = 0\}$$

Example 2.3.1. *The identity map on R is a derivation on R with ker$(d) = \{0\}$. This is called the identity derivation on R.*

Example 2.3.2. *The function d on R defined by $d(x) = 0$ for all $x \in R$ is a derivation on R with ker$(d) = R$.*

Example 2.3.3. *Let R_1 and R_2 be two ADL's, and d_1 and d_2 are derivations on R_1 and R_2 respectively. Then, $d_1 \times d_2$ is a derivation on $R_1 \times R_2$ where $(d_1 \times d_2)(x, y) = (d_1(x), d_2(y))$, for all $x \in R_1$, $y \in R_2$.*

Example 2.3.4. *Let $R = \{0, a, b, 1\}$ be a chain as in Figure2.2*

FIGURE 2.2

Let d_1, d_2 on R be two functions defined on R, as follows

$$d_1(x) = \begin{cases} b & if \quad x = 1 \\ x & Otherwise \end{cases} \qquad d_2(x) = \begin{cases} a & if \quad x = 1 \\ x & Otherwise \end{cases}$$

Then $d_1(x)$ is a derivation on R, but $d_2(x)$ is not a derivation. Since $d_2(b \vee 1) = d_2(1) = a$, and $d_2(b) \vee d_2(1) = b \vee a = b$.

Example 2.3.5. *Consider the distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given in the figure.*

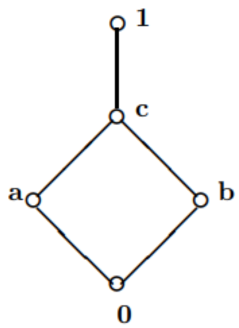

FIGURE 2.3

Define a self- map $d : L \to L$ as follows:

$$d(x) = \begin{cases} a & , \quad x = a, c, 1, \\ 0 & , \quad Otherwise. \end{cases}$$

Then it can be easily verified by trial that d is a derivation of L.

Theorem 2.3.1. *If a function $d : R \to R$ is a derivation on R, then*

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)) \textit{ for all } x, y \in R.$$

Proof. From definition 2.3.1, we get that

$$
\begin{aligned}
(d(x) \wedge y) \vee (x \wedge d(y)) &= d(x \wedge y) \vee (x \wedge d(y)) \\
&= (d(x \wedge y) \vee x) \wedge (d(x \wedge y) \vee d(y)) \\
&= (d(x \wedge y) \vee x) \wedge (d((x \wedge y) \vee y) && \text{by theorem2.1.1} \\
&= d(y) \wedge (d(x \wedge y) \vee x) && \text{since } \wedge \text{ is associative in } R \\
&= (d(y) \wedge d(x \wedge y)) \vee (d(y) \wedge x) \\
&= d(y \wedge x \wedge y) \vee d(y \wedge x) \\
&= d(x \wedge y) \vee d(x \wedge y) = d(x \wedge y)
\end{aligned}
$$

□

Now we give some properties for the derivations of ADL.

Lemma 2.15. *[19] Let R be an ADL and d, any derivation of R. Then we have*

1. $d(0) = 0$
2. $d^2(x) = d(x)$
3. $d(x) \leq x$, *for all* $x \in R$
4. *ker*(d) *is an ideal of* R

Proof. 1. $d(0) = d(0 \wedge 0) = d(0) \wedge 0 = 0$.

2. $d^2(x) = d(d(x)) = d(d(x \wedge x)) = d(d(x) \wedge x) = d(x \wedge d(x) \wedge x) = d(x \wedge d(x)) = d(x) \wedge d(x) = d(x)$.
3. If $x \in R$, then $d(x) = d(x \wedge x) = d(x) \wedge x$. Therefore, $d(x) \leq x$.
4. Let $a, b \in$ker(d) and $x \in R$, then
   - $d(a \vee b) = d(a) \vee d(b) = 0 \vee 0 = 0$. Therefore $a \vee b \in$ ker(d).
   - $d(a \wedge x) = d(a) \wedge x = 0 \wedge x = 0$. Therefore $a \wedge x \in$ ker(d)

   Hence ker(d) is an ideal of R.

□

Lemma 2.16. *[20] Let d be a derivation on R, then the following hold:*

1. $d(x) \wedge d(y) \leq d(x \wedge y)$ *for all* $x, y \in R$.
2. *If I is an ideal of R, then $d(I) \subseteq I$.*

Proof.

1. Let $x, y \in R$. We have $d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$. Therefore, $d(x) \wedge y \leq d(x \wedge y)$. Now by (3) in lemma 2.15, we get that $d(x) \wedge d(y) \leq d(x) \wedge y \leq d(x \wedge y)$.
2. If $a \in I$, then by (3) in lemma 2.15, $d(a) \leq a$ and hence $d(a) \in I$. Thus, $d(I) \subseteq I$.

□

Example 2.3.6. *Consider the distributive lattice $A = \{0, a, b, c, 1\}$ whose Hasse diagram is given below. Let $D = \{0', a'\}$ be the discrete ADL. Then*

$L = D \times A = \{(0', 0), (0', a), (0', b), (0', c), (0', 1), (a', 0), (a', a), (a', b), (a', c), (a', 1)\}$ *is an ADL.*

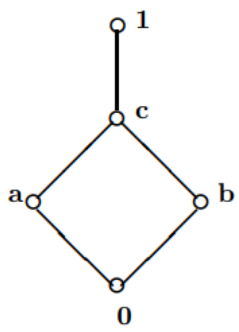

FIGURE 2.4

Define a self map $d : L \to L$ such that
$d((0', 0)) = (0', 0)$
$d((0', a)) = d((0', c)) = (0', a)$
$d((0', b)) = d((0', 1)) = (0', b)$
$d((a', 0)) = (a', 0)$
$d((a', a)) = d((a', c)) = (a', a)$
$d((a', b)) = d((a', 1)) = (a', b)$.

Then clearly by trial d is a derivation on L, and

$$I = \{(0', 0), (0', a), (0', b), (a', 0), (a', a), (a', b)\}.$$

is an ideal of L satisfying $d(I) = I$.

Theorem 2.3.2. *[20] If d is a derivation on a discrete ADL R with 0, then d is either a zero derivation or the identity derivation on R.*

Proof. Suppose $d(a) \neq 0$ for some $a(\neq 0) \in R$. Then, $d(a) = d(a \wedge a) = d(a) \wedge a = a$. Therefore d is either a zero derivation or the identity derivation. $\square$

Definition 2.3.2. *[19] Let d be a derivation of an ADL R. For any $a \in R$, define the set*

$$(a)^d = \{x \in R : x \wedge a \in ker(d)\}.$$

Example 2.3.7. *Consider the ADL R given in Example 2.3.4 with d_1 as a derivation on it, then*

$$ker\ (d_1) = \{0\},\ (0)^{d_1} = R$$

and

$$(a)^{d_1} = (b)^{d_1} = (1)^{d_1} = \{0\}.$$

Example 2.3.8. *Consider the distributive lattice $A = \{0, a, b, c, 1\}$ given in example 2.3.6. Define a self map $d : A \to A$ as follows:*

$$d(x) = \begin{cases} 0 & if \quad x = 0, \\ a & if \quad x = a, c, \\ b & if \quad x = b, 1. \end{cases}$$

It can be easily verified that d is a derivation of A. Note that ker $(d) = \{0\}$, $(0)^d = A$, $(a)^d = \{0, b\}$, $(b)^d = \{0, a\}$, $(c)^d = \{0\}$ and $(1)^d = \{0\}$.

Now, we illustrate some basic properties that hold to any derivation.

Lemma 2.17. *[19] Let d be a derivation of an ADL R. Then for any $a, b, c \in R$, the following conditions hold:*

1. *$ker(d) \subseteq (x)^d$, for all $x \in R$.*

2. *If* $a \in ker(d)$*, then* $(a)^d = R$.
3. *If* $a = 0$*, then* $a \in (a)^d$.
4. $(a)^d$ *is an ideal of* R.
5. *If* $a \leq b$*, then* $(b)^d \subseteq (a)^d$.
6. $(a \vee b)^d = (b \vee a)^d$ *and* $(a \wedge b)^d = (b \wedge a)^d$.
7. $(a \vee b)^d = (a)^d \cap (b)^d$.
8. *If* $(a)^d = (b)^d$*, then* $(a \wedge c)^d = (b \wedge c)^d$ *and* $(a \vee c)^d = (b \vee c)^d$.

Proof.

1. Let $t \in \text{ker}(d)$ and $x \in R$. Then $t \wedge x \in \text{ker}(d)$, since $\text{ker}(d)$ is an ideal of R. That implies $t \in (x)^d$.Therefore $\text{ker}(d) \subseteq (x)^d$, for all $x \in R$.
2. Let $a \in \text{ker}(d)$. Since $\text{ker}(d)$ is an ideal of R, we get $x \wedge a \in \text{ker}(d)$, for all $x \in R$. Therefore $x \in (a)^d$ and hence $(a)^d = R$.
3. Clear.
4. Clearly $0 \in (a)^d$. Therefore $(a)^d \neq \phi$. Let $x, y \in (a)^d$. Then $x \wedge a$, $y \wedge a \in \text{ker}(d)$. Since $\text{ker}(d)$ is an ideal of R, we get $(x \wedge a) \vee (y \wedge a) = (x \vee y) \wedge a \in \text{ker}(d)$ and hence $x \wedge r \wedge a \in \text{ker}(d)$. Therefore $x \wedge r \in (a)^d$. Thus $(a)^d$ is an ideal of R.
5. Let $a \leq b$. We prove that $(b)^d \subseteq (a)^d$. Let $x \in (b)^d$. Then $x \wedge b \in \text{ker}(d)$ and hence $x \wedge b = x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b) \in \text{ker}(d)$. Since $\text{ker}(d)$ is an ideal of R, we get $x \wedge a \in \text{ker}(d)$. Therefore $x \in (a)^d$. Thus $(b)^d \subseteq (a)^d$.
6. It is obtained easily.
7. Clearly we have $(a \vee b)^d \subseteq (a)^d \cap (b)^d$. Let $x \in (a)^d \cap (b)^d$, Then $x \wedge a$, $x \wedge b \in \text{ker}(d)$ and hence $x \wedge (a \vee b) \in \text{ker}(d)$. That implies $x \in (a \vee b)^d$. Therefore $(a)^d \cap (b)^d \subseteq (a \vee b)^d$. Thus $(a \vee b)^d = (a)^d \cap (b)^d$.
8. Assume that $(a)^d = (b)^d$. Now,

$$
\begin{aligned}
x \in (a \wedge c)^d &\Leftrightarrow x \wedge a \wedge c \in \ker(d) \\
&\Leftrightarrow x \wedge c \wedge a \in \ker(d) \\
&\Leftrightarrow x \wedge c \in (a)^d = (b)^d \\
&\Leftrightarrow x \wedge c \wedge b \in \ker(d) \\
&\Leftrightarrow x \wedge b \wedge c \in \ker(d) \\
&\Leftrightarrow x \in (b \wedge c)^d \\
&\Leftrightarrow (a \wedge c)^d = (b \wedge c)^d .
\end{aligned}
$$

Now we prove that $(a \vee c)^d = (b \vee c)^d$.Let

$$
\begin{aligned}
x \in (a \vee c)^d &\Rightarrow x \wedge (a \vee c) \in \ker\,(d) \\
&\Rightarrow (x \wedge a) \vee (x \wedge c) \in \ker(d) \\
&\Rightarrow x \wedge a,\, x \wedge c \in \ker(d) \\
&\Rightarrow x \in (a)^d = (b)^d \\
&\Rightarrow x \wedge b \in \ker(d) \\
&\Rightarrow (x \wedge b) \vee (x \wedge c) \in \ker(d) \\
&\Rightarrow x \wedge (b \vee c) \in \ker(d) \\
&\Rightarrow x \in (b \vee c)^d \\
&\Rightarrow (a \vee c)^d \subseteq (b \vee c)^d.
\end{aligned}
$$

Similarly, we get $(b \vee c)^d \subseteq (a \vee c)^d$. Therefore $(a \vee c)^d = (b \vee c)^d$.

□

Definition 2.3.3. *[19] Let d be a derivation of R. For any $x, y \in R$, define a relation on R with respect to d, as $x \overset{\theta_d}{\equiv} y$ if and only if $(x)^d = (y)^d$.*

It is observed that θ_d is a congruence relation on R.

Definition 2.3.4. *[19] An element x of an ADL R is said to be kernel if $(x)^d = ker(d)$. The set of all kernel elements of R is denoted by K_d.*

Example 2.3.9. *Consider the ADL R given in Example 2.3.4 with d_1 as a derivation on it, then*

$$R/\theta_{d_1} = \{[0]_{d_1}, [1]_{d_1}\} \textit{ where } [0]_{d_1} = \{0\}$$

and

$$[1]_{d_1} = \{a, b, 1\}. \textit{ Also } K_d = \{0\}.$$

Now we have the following lemma which illustrate important properties of K_d.

Lemma 2.18. *[19] Let R be an ADL with maximal elements. Then for any derivation d of R, we have the following:*

1. *K_d is a congruence class with respect to θ_d.*
2. *K_d is closed under $\wedge$ and $\vee$.*
3. *K_d is a filter of R.*

Proof. As 1 and 2 are clear, we only prove 3.
Let m be any maximal element of an ADL R. Clearly, m is a kernel element of R. So that $K_d \neq \phi$. Let $a \in K_d$ and $x \in R$. Then $(a)^d$ = ker(d). Clearly, ker$(d) \subseteq (x \vee a)^d$. Let $t \in (x \vee a)^d$. Then $t \wedge (x \vee a) \in$ ker(d). That implies $t \wedge x, t \wedge a \in$ ker(d). So that $t \in (a)^d$ = ker(d). Therefore $(x \vee a)^d \subseteq$ ker(d) and hence $(x \vee a)^d$ = ker(d). Thus K_d is a filter of R. □

Definition 2.3.5. *[26] A class of elements B together with two binary operations* **+** *and* **.** *is a Boolean algebra if and only if the following postulates hold:*

1. *The operations* **+** *and* **.** *are commutative.*
2. *There exist in B distinct identity elements **0** and **1** relative to the operations* **+** *and* **.**, *respectively.*
3. *Each operation is distributive over the other.*
4. *For every a in B there exists an element a' in B such that $a + a'$ = **1** and $a.a'$ = **0***

Example 2.3.10. *The class B consisting of 0 and 1, together with the operations defined by the following tables, is a Boolean algebra.*

+	0	1
0	0	1
1	1	1

.	0	1
0	0	0
1	0	1

TABLE 2.3

Example 2.3.11. *[9] Let $B_2 = \{0,1\} \times \{0,1\}$ be the Boolean algebra having 4 elements. A part from the zero derivation and the identity derivation, we find the two functions f and g as follows:*

$$f(a,b) = f(a,0), \quad g(a,b) = g(0,b)$$

Moreover, if we denote $D(B_2)$ the set of the derivations of B_2, it can be easily seen that $D(B_2) = \{0, I_{B_2}, f, g\}$

In the following, a necessary and sufficient condition is derived for the quotient algebra R/θ_d to become a Boolean algebra.

Theorem 2.3.3. *[19] Let d be a derivation of R. Then R/θ_d is a Boolean algebra if and only if to each $x \in R$, there exists $y \in R$ such that $x \wedge y \in$ ker(d) and $x \vee y \in K_d$*

Proof. We first prove that ker(d) is the smallest congruence class and K_d is the largest congruence class in R/θ_d. Clearly, ker(d) is a congruence class of R/θ_d. Since ker(d) is an ideal, we get that for any $a \in$ ker(d) and $x \in R$, we have $a \wedge x \in$ ker(d). Hence $[a]_{\theta_d} \wedge [x]_{\theta_d} = [a \wedge x]_{\theta_d} = [a]_{\theta_d} =$ ker(d). This is true for all $x \in R$. Therefore $[a]_{\theta_d}$ = ker(d) is the smsllest congruence class of R/θ_d. Again, clearly K_d is a congruence class of R/θ_d. Let $a \in K_d$ and $x \in R$. Since K_d is a filter, we get that $x \vee a \in K_d$. Therefore $(x \vee a)^d$ = ker(d).

We now prove that K_d is the greatest congruence class of R/θ_d. For any $a \in K_d$ and $x \in R$, we get that $[x]_{\theta_d} \vee [a]_{\theta_d} = [x \vee a]_{\theta_d} = [a]_{\theta_d}$. Therefore K_d is the greatest congruence class of R/θ_d.

We now prove the main part of the Theorem. Assume that R/θ_d is a Boolean algebra. Let $x \in R$ so that $[x]_{\theta_d} \in R/\theta_d$. Since R/θ_d is a Boolean algebra, there exeists $[y]_{\theta_d} \in R/\theta_d$ such that $[x \wedge y]_{\theta_d} = [x]_{\theta_d} \cap [y]_{\theta_d}$ = ker(d) and $[x \vee y]_{\theta_d} = [x]_{\theta_d} \vee [y]_{\theta_d} = K_d$. Hence $x \wedge y \in$ ker(d) and $x \vee y \in K_d$. Converse can be proved in a similar way. □

Theorem 2.3.4. *[19] Let d be a derivation of R. If R/θ_d is a Boolean algebra, then θ_d is the largest congruence relation having congruence class K_d.*

Proof. Clearly, θ_d is a congruence with K_d as a congruence class. Let θ be any congruence with K_d as a congruence class. Let $(x, y) \in \theta$. Then for any $a \in R$, we can have

$$\begin{aligned}(x, y) \in \theta &\Rightarrow (x \vee a, y \vee a) \in \theta \\ &\Rightarrow x \vee a \in K_d \Leftrightarrow y \vee a \in K_d \\ &\Rightarrow (x \vee a)^d = \ker(d) \Leftrightarrow (y \vee a)^d = \ker(d) \\ &\Rightarrow (x)^d \cap (a)^d = \ker(d) \Leftrightarrow (y)^d \cap (a)^d = \ker(d)\end{aligned}$$

Since R/θ_d is a Boolean algebra, there exists $x', a' \in R$ such that $x \wedge x'$, $a \wedge a' \in \ker(d)$ and $(x \vee x')^d = \ker(d)$, $(a \vee a')^d = \ker(d)$. Hence $x' \in (x)^d$ and $a' \in (a)^d$ which implies that $x' \wedge a' \in (x)^d \cap (a)^d = \ker(d)$. Therefore $a' \in (x')^d$.

Similarly, we can get $a' \in (y')^d$ for a suitable $y' \in R$. Then, we get

$$\begin{aligned}a' \in (x')^d \Leftrightarrow a' \in (y')^d &\Rightarrow (x')^d = (y')^d \\ &\Rightarrow (x', y') \in \theta_d \\ &\Rightarrow x' \in K_d \Leftrightarrow y' \in K_d \\ &\Rightarrow (x')^d = \ker(d) \Leftrightarrow (y')^d = \ker(d) \\ &\Rightarrow (x \vee x')^d = (x)^d \Leftrightarrow (y \vee y')^d = (y)^d \\ &\Rightarrow (x)^d = \ker(d) \Leftrightarrow (y)^d = \ker(d) \\ &\Rightarrow (x)^d = (y)^d \\ &\Rightarrow (x, y) \in \theta_d.\end{aligned}$$

□

Now we are ready to define another congruence relation based on the derivation.

Definition 2.3.6. *[19] Let d be a derivation of an ADL R. Then define a relation θ^d with respect to d on R by $(x, y) \in \theta^d$ if and only if $d(x) = d(y)$, for all x, $y \in R$.*

Lemma 2.19. *[19] For any derivation d of an ADL R, we have the following:*

1. *θ^d is a congruence relation on R.*
2. *Ker θ^d = Ker(d).*

Proof.

1. Clearly θ^d is an equivalence relation on R and hence it is easy to prove θ^d is congruence on R.

2. Ker $\theta^d = \{x \in R : (x, 0) \in \theta^d\} = \{x \in R : d(x) = d(0) = 0\} =$ ker (d).

□

Definition 2.3.7. *Let d be a derivation of an ADL R. We define the set*

$$d(R) = \{x \in R : x = d(a) \text{ for some } a \in R\}.$$

Lemma 2.20. *[19] Let d be a derivation of an ADL R. Then we have the following:*

1. *$d(x) = x$, for all $x \in d(R)$.*
2. *If $(x, y) \in \theta^d$ and x, $y \in d(R)$, then $x = y$.*

Proof.

1. Let $x \in d(R)$. Then $x = d(a)$, for some $a \in R$. That implies $x = d(a) = d(d(a))$ $= d(x)$. Therefore $d(x) = x$.
2. Let x, $y \in d(R)$ with $(x, y) \in \theta^d$. Then $d(x) = d(y)$ and $x = d(a)$, $y = d(b)$, for some a, $b \in R$. That implies that $x = d(a) = d(y) = d(b) = y$ and hence $x = y$.

□

Theorem 2.3.5. *[19] Let I be an ideal of an ADL R. Then there exists a derivation d on R such that $d(R) = I$ if and only if there exists a congruence relation θ on R such that $I \cap [x]_\theta$ is a singleton set for all $x \in R$.*

Proof. (⇒) Let d br a derivation of R such that $d(R) = I$. For any $x \in R$, we have $d(x) = d(d(x))$. That implies that $(x, d(x)) \in \theta^d$. Hence $d(x) \in I \cap [x]_{\theta^d}$. Therefore $I \cap [x]_{\theta^d} \neq \phi$. We prove that $I \cap [x]_{\theta^d}$ is a singleton set. Suppose that $a, b \in I \cap [x]_{\theta^d}$. Then $a, b \in I = d(R)$ and $a, b \in [x]_{\theta^d}$. By the above lemma we get $a = b$. Therefore $I \cap [x]_{\theta^d}$ is a singleton set.

(⇐) Assume that there exist a congruence θ on R such that $I \cap [x]_{\theta^d}$ is a singleton set for any $x \in R$. Then choose x_0 is the single element pf $I \cap [x]_{\theta^d}$. Define a map $d : R \to R$ by $d(x) = x_0$, for all $x \in R$. Let $a, b \in R$. Then $d(a \vee b) = x_0 = x_0 \vee x_0$

= $d(a) \vee d(b)$. Now, $d(a \wedge b) = x_0 \in I \cap [x]_{\theta^d}$. Clearly, we have $(d(a), a) \in \theta^d$ and hence $(d(a) \wedge b, a \wedge b) \in \theta^d$. That implies $d(a) \wedge b \in I \cap [a \wedge b]_{\theta^d}$. Therefore $d(a \wedge b)$ = $d(a) \wedge b$ Hence d is a derivation on R. □

Example 2.3.12. *Consider the ADL R given in Example 2.3.4 with derivation d_1, then $I = \{0, a, b\}$ is an ideal. Also $d(R) = I$ and so we have from the last theorem that $[0]_{\theta^d} \cap I$, $[a]_{\theta^d} \cap I$ and $[b]_{\theta^d} \cap I$ are singelton sets.*

2.4 Congruence in Regular Rings related to an Almost Distributive Lattices

As mentioned before the concept of almost distributive lattice was introduced to include almost all existing rings such as, regular rings.

In this section, regular ring is defined, some of it's properties, characterizations and theorems are studied, and several examples are given. Finally, we find a relation between ADLs and regular rings and we discuss the congruence relation in this ADL. In this section by R we mean a commutative regular ring $(R, +, ., 0)$ unless otherwise mentioned.

Definition 2.4.1. *[11] A ring R is said to be regular if for every $r \in R$, there is an $x \in R$ such that $rxr = r$.*

The importance of regular ring emitted from the fact that it contains other types of rings such as Boolean rings and division rings, We start with some examples of regular rings.

Example 2.4.1. *Every division ring is obviously regular becouse if $r = 0$, then $r = rxr$ for all x, and if $r \neq 0$, then $r = rxr$ for $x = r^{-1}$.*

Example 2.4.2. *Every direct product of regular rings is clearly a regular ring.*

Definition 2.4.2. *A Boolean ring R is a ring for which $r^2 = r, \forall r \in R$.*

Example 2.4.3. *Boolean rings are regular, since* $\forall\, r \in R$, $r = rrr$.

Example 2.4.4. *Let* X *be a non - empty set, Consider the ring* $(P(X), \cap, \triangle, 0)$ *where* $\triangle$ *is the symmetric difference (*$A \triangle B = (A \cup B) - (A \cap B)$*), then*

$$A \cap A = A, \forall A \in P(X).$$

So R *is a regular ring since it's Boolean.*

Note. Since $rxrx = rx$, then the element $e = rx$ is idempotent.

Theorem 2.4.1. *[2] Let* R *be a ring. Then* R *is regular if and only if* R *satisfies the condition*

$$\forall a \in R, \exists e^2 = e \in R$$

Such that $Ra = Re$.

Proof. ($\Rightarrow$) Suppose that R is regular, then for any $a \in R$, there exist $x \in R$ such that $a = axa$. Since xa and ax are idempotents in R, taking $xa = e$, we have

$$Ra = Raxa = Rae \subseteq Re$$

and
$$Re = Rxa \subseteq Ra.$$

Hence $Ra = Re$.

($\Leftarrow$) Conversely, assume that R has the given condition $\forall a \in R, \exists e^2 = e \in R$ such that $Ra = Re$, and R has an identity. Then

$$a \in Ra = Re$$

So that ther exists $y \in R$ such that $a = ye$. From the condition, we see that

$$e = ee \in Re = Ra$$

So that there exists $x \in R$ such that $e = xa$. Thus we obtain that

$$a = ye = yee = yexa = axa.$$

Consequently, R is regular. □

Definition 2.4.3. *[10] The order of an element r in a ring R is the smallest positive integer n such that r^n = identity. (In additive notation, this would be $nr = 0$). If no such integer exists, we say r has infinite order.*

Lemma 2.21. *Let $(R, +, ., 0, 1)$ be a commutative regular ring with unity 1, and x_0 is the unique idempotent in R such that $xR = x_0R$. Then :*

1. $(x_0)_0 = x_0$.
2. $(xy)_0 = x_0y_0$.
3. $2x_0 = 0$.
4. $x_0x = x$.

Proof.

1. $x_0x_0x_0 = x_0$, since x_0 is idempotent element. Hence we get $(x_0)_0 = x_0$.
2. $xy(x_0y_0)xy = xx_0xyy_0y = xy$. Hence $(xy)_0 = x_0y_0$.
3. It's true since x_0 is of order 2.
4. $x_0x = x_0xx_0x = xx_0x = x$.

□

Now, we give a theorem that presents a relation between ADL and regular rings under some condition.

Theorem 2.4.2. *Let* $(R, +, ., 0, 1)$ *be a commutative regular ring with unity* 1*, if we define*

$$x \wedge y = x_0 y$$

and

$$x \vee y = x + y + x_0 y \qquad \forall x, y \in R.$$

where x_0 *is the unique idempotent in* R *such that* $xR = x_0 R$*. Then* $(R, \vee, \wedge, 0)$ *is an ADL. where* 0 *is the additive identity in* R*.*

Proof.
(L_1) $x \vee 0 = x + 0 + 0 = x$.
(L_2) $0 \wedge x = x.0 = 0$.
(L_3) $(x \vee y) \wedge z = (x + y + x_0 y) \wedge z = (x + y + x_0 y)_0 z = (x_0 + y_0 + x_0 y_0) z = (x_0 z + y_0 z + x_0 y_0 z)$.
Also $(x \wedge z) \vee (y \wedge z) = x_0 z \vee y_0 z = x_0 z + y_0 z + (x_0 z)_0 y_0 z = x_0 z + y_0 z + x_0 z_0 y_0 z = x_0 z + y_0 z + x_0 y_0 z_0 z = x_0 z + y_0 z + x_0 y_0 z$.
So we get that $(x \vee y) \wedge z = (x \wedge y) \vee (y \wedge z)$.
(L_4) $x \wedge (y \vee z) = x \wedge (y + z + y_0 z) = x_0 (y + z + y_0 z) = x_0 y + x_0 z + x_0 y_0 z$.
Also $(x \wedge y) \vee (x \wedge z) = x_0 y \vee x_0 z = x_0 y + x_0 z + (x_0 y)_0 x_0 z = x_0 y + x_0 z + x_0 y_0 z$.
(L_5) $x \vee (y \wedge z) = x \vee y_0 z = x + y_0 z + x_0 y_0 z$.
Also $(x \vee y) \wedge (x \vee z) = (x + y + x_0 y)_0 (x + z + x_0 z) = (x_0 + y_0 + x_0 y_0)(x + z + x_0 z)$
$= x_0 x + x_0 z + x_0 x_0 z + y_0 x + y_0 z + y_0 x_0 z + x_0 y_0 x + x_0 y_0 z + x_0 y_0 x_0 z + x + y_0 z + x_0 y_0 z + 2x_0 z + 2y_0 x + 2x_0 y_0 z = x + y_0 z + x_0 y_0 z$.
So we get that $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.
(L_6) $(x \vee y) \wedge y = (x + y + x_0 y)_0 y = (x_0 + y_0 + x_0 y_0) y = x_0 y + y_0 y + x_0 y = 2x_0 y + y$
$= y$. □

Note. the set R in previous theorem isn't a ring under the new operations $\wedge, \vee$.

Corollary 2.22. *Let* $R = Z_2 \text{x} Z_2 \text{x} ... \text{x} Z_2$*, then* $(R, \wedge, \vee, 0)$ *is an ADL where* $\wedge, \vee$ *as defined in theorem 2.4.2,* $x_0 = x$*,* $\forall x \in R$ *and* $0 = (0, 0, ..., 0)$.

Example 2.4.5. *Consider* $Z_2 \times Z_2 = \{(0,0), (0,1), (1,0), (1,1)\}$*, then (*$Z_2 \times Z_2$*, +, ., *$(0,0), (1,1)$*) is a commutative regular ring with unity, where + and . are the addition and multiplication mod* 2*.*
By using last theorem, we can define an ADL from this commututive regular ring. Below

are the meet and join tables of this ADL.

x ∧ y	**(0, 0)**	**(0, 1)**	**(1, 0)**	**(1, 1)**
(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
(0, 1)	(0, 0)	(0, 1)	(0, 0)	(0, 1)
(1, 0)	(0, 0)	(0, 0)	(1, 0)	(1, 0)
(1, 1)	(0, 0)	(0, 1)	(1, 0)	(1, 1)

x ∨ y	**(0, 0)**	**(0, 1)**	**(1, 0)**	**(1, 1)**
(0, 0)	(0, 0)	(0, 1)	(1, 0)	(1, 1)
(0, 1)	(0, 1)	(0, 1)	(1, 1)	(1, 1)
(1, 0)	(1, 0)	(1, 1)	(1, 0)	(1, 1)
(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)

TABLE 2.4

The Hasse diagram of this ADL is

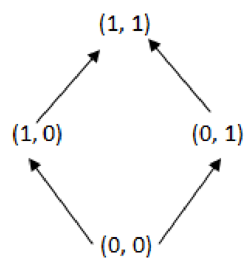

FIGURE 2.5

Now, let $[(0,0)] = \{(0,0)\}$, $[(1,1)] = \{(1,1),(0,1),(1,0)\}$. *Then* θ *is a congruence relation on* R.

Remark 2.23. In the above theorem, since x_0 is an idempotent element in a commutative regular ring, then we have

$$2x_0 = 0 \text{ which mean } x_0 + x_0 = 0.$$

So we can define the join in the above theorem by

$$x \vee y = x + y - x_0 y$$

Example 2.4.6. *Let* $X = \{a, b\}$, *and* $(P(X), \triangle, \cap, 0, 1)$ *be the commutative regular ring with unity. For any* $A, B \in P(X)$ *and by using theorem 2.4.2, define*

$$A \wedge B = A \cap B$$

and

$$A \vee B = (A \triangle B) \cup (A \cap B) = A \cup B.$$

Then $(P(X), \cup, \cap, 0)$ *is an ADL. We can see this easily from the ring's tables and ADL's tables below.*

$\cap$	ϕ	$\{0\}$	$\{a\}$	$\{0,a\}$
ϕ	ϕ	ϕ	ϕ	ϕ
$\{0\}$	ϕ	$\{0\}$	ϕ	$\{0\}$
$\{a\}$	ϕ	ϕ	$\{a\}$	$\{a\}$
$\{0,a\}$	ϕ	$\{0\}$	$\{a\}$	$\{0,a\}$

$\triangle$	ϕ	$\{0\}$	$\{a\}$	$\{0,a\}$
ϕ	ϕ	$\{0\}$	$\{a\}$	$\{0,a\}$
$\{0\}$	$\{0\}$	ϕ	$\{0,a\}$	$\{a\}$
$\{a\}$	$\{a\}$	$\{0,a\}$	ϕ	$\{0\}$
$\{0,a\}$	$\{0,a\}$	$\{a\}$	$\{0\}$	ϕ

TABLE 2.5: Ring $(P(X), \triangle, \cap, 0, 1)$

$\cap$	ϕ	$\{0\}$	$\{a\}$	$\{0,a\}$
ϕ	ϕ	ϕ	ϕ	ϕ
$\{0\}$	ϕ	$\{0\}$	ϕ	$\{0\}$
$\{a\}$	ϕ	ϕ	$\{a\}$	$\{a\}$
$\{0,a\}$	ϕ	$\{0\}$	$\{a\}$	$\{0,a\}$

$\cup$	ϕ	$\{0\}$	$\{a\}$	$\{0,a\}$
ϕ	ϕ	$\{0\}$	$\{a\}$	$\{0,a\}$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0,a\}$	$\{0,a\}$
$\{a\}$	$\{a\}$	$\{0,a\}$	$\{a\}$	$\{0,a\}$
$\{0,a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$	$\{0,a\}$

TABLE 2.6: ADL $(P(X), \cup, \cap, 0)$

Conversely, if we have an ADL $(P(X), \cup, \cap, 0)$*, then we can get a regular ring by defining the operation* **+** *and* **.** *as follows:*

$$A.B = A \cap B.$$

and

$$A + B = (A^c \cap B) \cup (B^c \cap A) = A \triangle B.$$

Definition 2.4.4. *[1] A subset S of a ring R is said to be a multiplicatively closed subset of R if :*

- $1 \in S$.
- *For any $a, b \in S \Rightarrow a.b \in S$.*

Example 2.4.7. *Let $X = \{a, b\}$, then we know that $R = (P(X), \triangle, \cap, 0, 1)$ is a commutative regular ring, and $L = (P(X), \cup, \cap, 0)$ is an ADL.*

$F = \{\{a\}, \{b\}, \{a, b\}\}$ is a multiplicatively closed subset of R and of L.

Note that in R, every element of $P(X)$ which contains X is a multiplicatively closed subset, and in general for any ring R any interval containing X is a multiplicatively closed set.

Definition 2.4.5. *Let R be a commutative regular ring and S be a multiplicatively closed subset of R. We define the relations ψ^S on R as follows:*
for all $a, b \in R$,

$$a \overset{\psi^S}{\equiv} b \Leftrightarrow a.s = b.s \text{ for some } s \in S.$$

Note.

- If $\phi \in S$, then ψ^S is the congruence relation which contains one block.
- If R is a field, then ψ^S is the smallest congruence relation.

Example 2.4.8. *Let $X = \{a, b\}$. Consider $R = (P(X), \triangle, \cap, 0, 1)$ and $S = \{\{a\}, \{a, b\}\}$ be the multiplicatively closed subset of R, then*

$$\phi \overset{\psi^S}{\equiv} \{b\} \text{ and } \{a\} \overset{\psi^S}{\equiv} \{a, b\}$$

In the following theorem we show that ψ^S is a congruence relation on R.

Theorem 2.4.3. *The relation ψ^S is a congruence relation on the commutative regular ring R.*

Proof. Obviously, ψ^S is reflexive and symmetric. Assume $x \overset{\psi^S}{\equiv} y$ and $y \overset{\psi^S}{\equiv} z$ for $x, y, z \in S$. Then

$$x.s = y.s \text{ for some } s \in S$$

and

$$y.t = z.t \text{ for some } t \in S.$$

Also, $s.t \in S$, as S is a multiplicatively closed subset of R. Further $x.s.t = y.s.t = s.y.t = s.z.t = z.s.t$. This shows that $x \overset{\psi^S}{\equiv} z$. Hence ψ^S is transitive.

Let $a \overset{\psi^S}{\equiv} b$ and $c \overset{\psi^S}{\equiv} d$ for $a, b, c, d \in R$. Then $a.s = b.s$ and $c.t = d.t$ for some $s, t \in S$. As S is multiplicatively closed subset of R, $s.t \in S$. Further $a.c.s.t = a.s.c.t = b.s.d.t = b.d.s.t$, this show that $a.c \equiv b.d$.
Again,

$$\begin{aligned} (a+c).st &= (a.st)+(c.st) \\ &= (b.s.t)+(s.c.t) \\ &= (b.st)+(s.d.t) \\ &= (b.st)+(d.st) \\ &= (b+d).st \end{aligned}$$

Hence $a+c \overset{\psi^S}{\equiv} b+d$ as $st \in S$. Therefore ψ^S is a congruence relation on R.

□

Remark 2.24. The class $[0]$ is an ideal of R in the relation ψ^S.

Proof. Let $a, b \in [0]$, then $a.s = 0.s$ and $b.t = 0.t$ for some $s, t \in S$

- $(a+b).st = (a.s.t)+(b.t.s) = 0 + 0 = 0\ .st$. Hence $a+b \overset{\psi^S}{\equiv} 0$ and $a + b \in [0]$.

- $(a.b).st = (a.s).(b.t) = 0 + 0 = 0\ .st$. Hence $a{+}b \overset{\psi^S}{\equiv} 0$ and $a + b \in [0]$.
- Let $r \in R$, then $ra.s = 0$, and hence $ra \in [0]$.

□

Note. R/ψ^S is a ring.

Theorem 2.4.4. *Let S and T be two multiplicatively closed subsets of a commutative regular rings R_1 and R_2 respectively. Then for any homomorphism $\Phi : R_1 \to R_2$ such that $\Phi(S) \subseteq T$, there exists a homomorphism $f : R_1/\psi^S \to R_2/\psi^T$ such that $fo\ h = ko\Phi$, where $h : R_1 \to R_1/\psi^S$ and $k : R_2 \to R_2/\psi^T$ denote the canonical epimorphisms. Further*

1. *If Φ is a monomorphism and if $\Phi(S) = T$, then f is a monomorphism.*
2. *If Φ is an epimorphism, then f is an epimorphism.*

Proof. Define $f : R_1/\psi^S \to R_2/\psi^T$ by $f([x]_{\psi^S}) = [\Phi(x)]_{\psi^T}$ for each $x \in R_1$. Let $[x]_{\psi^S}$ = $[y]_{\psi^S}$ for some $x, y \in R_1$. Then

$$\begin{aligned}
[x]_{\psi^S} = [y]_{\psi^S} &\Rightarrow x \overset{\psi^S}{\equiv} y \\
&\Rightarrow x.s = y.s && \text{for some } s \in S \\
&\Rightarrow \Phi(x.s) = \Phi(y.s) \\
&\Rightarrow \Phi(x).\Phi(s) = \Phi(y).\Phi(s) && \text{as } \Phi \text{ is homomorphism.} \\
&\Rightarrow \Phi(x) \overset{\psi^T}{\equiv} \Phi(y) && \text{as } \Phi(s) \in T \\
&\Rightarrow [\Phi(x)]_{\psi^T} = [\Phi(y)]_{\psi^T} \\
&\Rightarrow f([x]_{\psi^S}) = f([y]_{\psi^S})
\end{aligned}$$

This shows that f is well defined.

Next, let $x, y \in S$, then.

$$\begin{aligned}
f([x]_{\psi^S}.[y]_{\psi^S}) &= f([x.y]_{\psi^S}) \\
&= [\Phi(x.y)]_{\psi^T} \\
&= [\Phi(x).\Phi(y)]_{\psi^T} \\
&= [\Phi(x)]_{\psi^T}.[\Phi(y)]_{\psi^T} \\
&= f([x]_{\psi^S}).f([y]_{\psi^S}).
\end{aligned}$$

Similarly we can prove $f([x]_{\psi^S}+[y]_{\psi^S}) = f([x]_{\psi^S})+f([y]_{\psi^S})$ for all $x, y \in R_1$. Hence f is a homomorphism.

Now $fo\,h : R_1 \to R_2/\psi^T$ and for any $x \in R_1$ we have $[fo\,h](x) = f(h(x)) = f([x]_{\psi^S})$ = $[\Phi(x)]_{\psi^T}$. Again $ko\ \Phi : R_1 \to R_2/\psi^T$ and for any $x \in R_1$ we have $[ko\ \Phi](x) = k(\Phi(x)) = [\Phi(x)]_{\psi}$. Hence $[fo\,h](x) = [ko\ \Phi](x)$, $\forall x \in R_1$. This shows that $fo\,h = ko\ \phi$.

1. Let Φ be a monomorphism and let $\Phi(S) = T$. Let $f([x]_{\psi^S}) = f([y]_{\psi^S})$ for some $x, y \in R_1$. Then $[\Phi(x)]_{\psi^T} = [\Phi(y)]_{\psi^T} \Rightarrow \Phi(x) \overset{\psi^{\mathrm{T}}}{\equiv} \Phi(y) \Rightarrow \Phi(x).t = \Phi(y).t$, for some $t \in T \Rightarrow \Phi(x).\Phi(s) = \Phi(y).\Phi(s)$, for some $s \in S$ (since $\Phi(S) = T$) $\Rightarrow$ $\Phi(x.s) = \Phi(y.s)$ (since Φ is a monomorphism) $\Rightarrow x.s = y.s \Rightarrow x \overset{\psi^{\mathrm{S}}}{\equiv} y \Rightarrow [x]_{\psi^S} = [y]_{\psi^S}$. This shows that f is one-one.

2. Let Φ be an epimorphism. Let $[y]_{\psi^T} \in R_2/\psi^T$. As $\Phi : R_1 \to R_2$ is onto and $y \in R_2$, $\Phi(x) = y$ for some $x \in R_1$. Thus $[x]_{\psi^S} \in R_1/\psi^S$ and $f([x]_{\psi^S}) = [\Phi(x)]_{\psi^T}$ = $[y]_{\psi^T}$. This shows that f is an epimorphism.

□

Example 2.4.9. *Let $R = Z_2$ x Z_2 x Z_2. Consider $S = \{(1,1,0),(1,1,1)\}$,and $T = \{(0,1,0),(0,1,1),(1,1,0)\ ,(1,1,1)\}$ be two multiplicatively closed subsets of R. Define $\phi : R \to R$ by*

$$\phi(a, b, c) = (0, b, c)$$

Then clearly ϕ is a homomorphism and $\phi(S) \subset T$.
For ψ^S we have four classes
$[(0,0,0)] = \{(0,0,0),(0,0,1)\}$
$[(0,1,0)] = \{(0,1,0),(0,1,1)\}$
$[(1,0,0)] = \{(1,0,0),(1,0,1)\}$
$[(1,1,0)] = \{(1,1,0),(1,1,1)\}$

And $R/\psi^S \cong Z_2$ x Z_2

For ψ^T *we have Two classes*
$[(0,0,0)] = \{(0,0,0),(0,0,1),(1,0,0),(1,0,1)\}$
$[(1,1,1)] = \{(0,1,0),(0,1,1),(1,1,0),(1,1,1)\}$

And $R/\psi^T \cong Z_2$.

Now define $f : R_1/\psi^S \to R_2/\psi^T$ *by* $f([x]_{\psi^S}) = [\Phi(x)]_{\psi^T}$, *then by last theorem* f *is a homomorphism.*

Chapter 3

Filters and Ideals of Almost Distributive Lattices with respect to a Congruence

In this chapter, we introduce the concept of $\theta-$ filters in an ADL, and then characterized it in terms of ADL congruence. Since the lattice theoretic duality principle doesn't hold in case of an ADL (for the simple reason that an ADL satisfies the right distributivity of $\wedge$ over $\vee$, but dose not satisfy the right distributivity of $\vee$ over $\wedge$). We also introduce the concepts of $\theta-$ ideals in an ADL, and then characterized it in terms of ADL congruence.

3.1 Filters of Almost Distributive Lattices with respect to a Congruence

In this section, the concept of $\theta-$filters is introduced in an ADL, and then characterized it in terms of ADL congruences. A set of equivalent conditions are derived for every filter of an ADL to become a $\theta-$filter. The concept of $\theta-$prime filters is also introduced and a set of equivalent conditions for every $\theta-$filter which becomes a $\theta-$prime filter is established. Some properties of $\theta-$filters and $\theta-$prime filters are studied. The class of all $\theta-$filters of an ADL can be made into a bounded distributive lattice. Finally, the prime ideal theorem is generalized in the case of $\theta-$prime filter in an ADL.

Definition 3.1.1. *[18] Let θ be a congruence relation on an ADL R. A filter F of R is called a $\theta-$filter of R, if for any $a \in F \Rightarrow [a]_\theta \subseteq F$.*

Example 3.1.1. *Let $D = \{0', a'\}$ be a discrete ADL and $A = \{0, a, b, c, 1\}$ is a distributive lattice as shown in the following figure:*

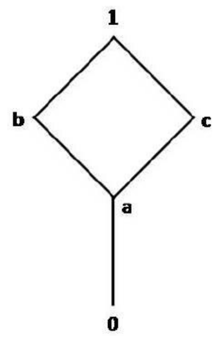

FIGURE 3.1

Then $R = D \times A$ is an ADL. Define $[(0', 1)]_\theta = \{(0', 1)\}$, $[(0', b)]_\theta = \{(0', b)\}$, $[(0', c)]_\theta = \{(0', c)\}$, $[(0', a)]_\theta = [(0', 0)]_\theta = \{(0', a), (0', 0)\}$, $[(a', 1)]_\theta = \{(a', 1)\}$, $[(a', b)]_\theta = \{(a', b)\}$, $[(a', c)]_\theta = \{(a', c)\}$, $[(a', a)]_\theta = [(a', 0)]_\theta = \{(a', a), (a', 0)\}$. Clearly, θ is a congruence relation on R.

Consider a filter $F = \{(a', c), (a', 1)\}$. Clearly F is a $\theta-$filter of R. Now consider the filter $F_1 = \{(a', a), (a', b), (a', c), (a', 1)\}$. Then $(a', 0) \notin F_1$ and $(a', 0) \in [a', a]_\theta$. Therefore $[(a', a)]_\theta \nsubseteq F_1$. Hence F_1 is not a $\theta-$filter of R.

Example 3.1.2. *Let $R = \{0, a, b, 1\}$ be a chain as given in Example 2.3.4. Define $[0]_\theta = \{0\}$, $[1]_\theta = \{a, b, 1\}$. Clearly, θ is a congruence relation on R. Consider a filter $F = \{a, b, 1\}$. Clearly F is a $\theta-$filter of R. Now consider a filter $F_1 = \{b, 1\}$. Then $a \notin F_1$ and $a \in [1]_\theta$. Hence F_1 is not a $\theta-$filter of R.*

Lemma 3.1. *[18] Let R be an ADL with a maximal element m and θ a congruence relation on R. For any filter F of R, the following hold:*

1. *$\{m\}$ is a $\theta-$filter if and only if $[m]_\theta \subseteq \{m\}$.*
2. *If F is a $\theta-$filter, then $[m]_\theta \subseteq F$.*

3. *If F is a proper $\theta-$filter, then $F \cap [0]_\theta = \phi$.*

Proof.

1. It is obvious.

2. Suppose F is a $\theta-$filter of R. We always have $m \in F$. Then $[m]_\theta \subseteq F$.

3. Let F be a proper $\theta-$filter of R. Suppose $F \cap [0]_\theta \neq \phi$. Choose $x \in F \cap [0]_\theta$. Then $x \in F$ and $(x, 0) \in \theta$ and hence $0 \in [x]_\theta \subseteq F$. This implies $0 \in F$, which is a contradiction. Therefore $F \cap [0]_\theta = \phi$.

□

Now, we give the following equivalent condition for the concept $\theta-$filter.

Theorem 3.1.1. *[18] Let θ be a congruence relation on an ADL R. Then for any filter F of R, the following conditions are equivalent:*

1. *F is a $\theta-$filter.*

2. *For any $x, y \in R$, $(x, y) \in \theta$ and $x \in F \Rightarrow y \in F$.*

3. *$F = \bigcup_{x \in F} [x]_\theta$.*

Proof.
$(1) \Rightarrow (2)$: Assume that F is a $\theta-$filter of an ADL. Let $(x, y) \in \theta$ and $x \in F$. Then $y \in [x]_\theta$ and $[x]_\theta \subseteq F$. This implies that $y \in F$.

$(2) \Rightarrow (3)$: Assume (2). Clearly, we have $F \subseteq \bigcup_{x \in F} [x]_\theta$. Let $a \in \bigcup_{x \in F} [x]_\theta$. Then $a \in [y]_\theta$, for some $y \in F$. This implies $(a, y) \in \theta$. By our assumption we get $a \in F$. Hence $F = \bigcup_{x \in F} [x]_\theta$.

$(3) \Rightarrow (1)$: Assume (3). Let $a \in F$. Then $a \in [y]_\theta$, for some $y \in F$. We have to prove that $[a]_\theta \subseteq F$. Let $t \in [a]_\theta$. Then $(a, t) \in \theta$ and hence $(t, y) \in \theta$. That implies $t \in [y]_\theta \subseteq F$. Therefore $[a]_\theta \subseteq F$. Hence F is a $\theta-$filter of an ADL R. □

Theorem 3.1.2. *[18] If θ is the smallest congruence relation on an ADL R, then every filter of R is a $\theta-$filter.*

Now, we can introduce the concept of θ - prime filters in an ADL, which is a special type of θ - filters.

Definition 3.1.2. *[18] Let θ be a congruence relation on an ADL R with any maximal element m. A proper $\theta-$filter P of an ADL R is called a $\theta-$prime filter of R if for any $a, b \in R$ with $a \vee b \in [m]_\theta$ then either $a \in P$ or $b \in P$.*

Example 3.1.3. *Consider the ADL R given in Example 3.1.2. Then F is a $\theta-$prime filter.*

Lemma 3.2. *[18] If θ is the smallest congruence relation on an ADL R with maximal elements, then every prime filter of an ADL R is a $\theta-$prime filter of an ADL R.*

Proof. Let R be an ADL with maximal elements. Let θ be the smallest congruence relation on R. Suppose that P is a prime filter of an ADL R. Then by the above result, we get P is a $\theta-$filter of R. Let $a, b \in R$ with $a \vee b \in [m]_\theta$, where m is any maximal element of R. Then $[a \vee b]_\theta = [m]_\theta$. Since θ is the smallest congruence relation on an ADL R, we get $a \vee b = m$. Since P is a prime filter of R, we have $a \vee b = m \in P$. This implies that either $a \in P$ or $b \in P$. Therefore P is $\theta-$prime filter of an ADL R. □

The following lemma describes the relation between prime $\theta-$filter and $\theta-$prime filter.

Lemma 3.3. *[18] Let θ be a congruence relation on an ADL R with maximal elements. Then every prime $\theta-$filter of R is a $\theta-$prime filter of R.*

Proof. Let P be a prime $\theta-$filter of an ADL R. Let $x, y \in R$ with $x \vee y \in [m]_\theta$, where m is any maximal element of R. Since $m \in P$ and P is a $\theta-$filter of R. We get $[m]_\theta \subseteq P$ and hence $x \vee y \in P$. This implies either $x \in P$ or $y \in P$. Thus P is a $\theta-$prime filter of R.

□

Theorem 3.1.3. *[18] Let θ be a congruence relation on an ADL R with maximal element m and P, a $\theta-$filter of R. If $[a]_\theta = [m]_\theta \Rightarrow (a] \subseteq [m]_\theta$, for all $a \in R$. Then the following conditions are equivalent:*

1. *P is a $\theta-$prime filter of R.*
2. *For any filters I, J of R with $I \cap J \subseteq [m]_\theta \Rightarrow I \subseteq P$ or $J \subseteq P$.*
3. *For any $a, b \in R$, $[a]_\theta \vee [b]_\theta = [m]_\theta \Rightarrow$ either $a \in P$ or $b \in P$.*

Proof.

$(1) \Rightarrow (2)$: Assume that P is a $\theta-$prime filter of R. Let I and J be any filters of R, with $I \cap J \subseteq [m]_\theta$. Let $a \in I$ and $b \in J$. Then $a \vee b \in I \cap J \subseteq [m]_\theta$. This implies that $a \vee b \in [m]_\theta$. By our assumption, we have either $a \in P$ or $b \in P$. Therefore either $I \subseteq P$ or $J \subseteq P$.

$(2) \Rightarrow (3)$: Assume that for any filters I, J of R with $I \cap J \subseteq [m]_\theta \Rightarrow I \subseteq P$ or $J \subseteq P$. Let $a, b \in R$, with $[a]_\theta \vee [b]_\theta = [m]_\theta$. Then $[a \vee b]_\theta = [a]_\theta \vee [b]_\theta = [m]_\theta$. This implies that $(a \vee b] \subseteq [m]_\theta$ and hence $(a] \cap (b] \subseteq [m]_\theta$. By our assumption we get either $(a] \subseteq P$ or $(b] \subseteq P$. Therefore either $a \in P$ or $b \in P$.

$(3) \Rightarrow (1)$: Assume that condition (3). Let $a \vee b \in [m]_\theta$. Then $[a]_\theta \vee [b]_\theta = [a \vee b]_\theta = [m]_\theta$. By our assumption, we have $[a]_\theta \subseteq P$ or $[b]_\theta \subseteq P$. This implies that $a \in P$ or $b \in P$. Hence P is a $\theta-$prime filter of an ADL R. □

Lemma 3.4. *[18] Let θ be a congruence relation on an ADL R. Then every minimal prime filter disjoint from $[0]_\theta$ is a $\theta-$filter of the ADL R.*

Proof. [18] Let M be a minimal prime filter of R such that $M \cap [0]_\theta \neq \phi$. Let $x, y \in R$ with $(x, y) \in \theta$ and $x \in M$. We prove that $y \in M$. Suppose $y \notin M$. Then $M \vee [y) = R$. This implies $a \wedge y = 0$, for some $a \in M$. Since $(x, y) \in \theta$, we get that $(a \wedge x, 0) \in \theta$ and hence $a \wedge x \in [0]_\theta$. So that $a \wedge x \in M$. Therefore $M \cap [0]_\theta \neq \phi$, which is a contradiction. Hence $y \in M$. Thus M is a $\theta-$filter of an ADL R. □

Note. Let θ be a congruence relation on an ADL R. If $[0]_\theta = \{0\}$, then every minimal prime filter of R is a $\theta-$filter of R.

Definition 3.1.3. *[18] Let θ be a congruence relation on an ADL R. For any filter F of R, define the set*

$$F^{\theta} = \{x \in R : (x, a) \in \theta, \text{for some } a \in F\}.$$

Example 3.1.4. *Consider the ADL R given in Example 3.1.2. Then for $F = \{(a', c), (a', 1)\}$ the set $F^{\theta} = \{(a', c), (a', 1)\} = F$, and for $F_1 = \{(a', a), (a', b), (a', c), (a', 1)\}$, the set $F_1^{\theta} = \{(a', 0), (a', a), (a', b), (a', c), (a', 1)\}$.*

Lemma 3.5. *[18] Let θ be a congruence relation on an ADL R. For any filter F of R, the set F^{θ} is a filter of R.*

Proof. Clearly $F^{\theta} \neq \phi$, since $F \neq \theta$. Let $x, y \in F^{\theta}$. Then $(x, a) \in \theta$ and $(y, b) \in \theta$, for some $a, b \in F$. This implies that $(x \wedge y, a \wedge b) \in \theta$ and $a \wedge b \in F$. Therefore $x \wedge y \in F^{\theta}$. Let $x \in F^{\theta}$ and $r \in R$. Then $(x, a) \in \theta$, for some $a \in F$. Then $(r \vee x, r \vee a) \in \theta$ and $r \vee a \in F$. Hence $r \vee x \in F^{\theta}$. Thus F^{θ} is a filter of R. □

Lemma 3.6. *[18] Let θ be a congruence relation on an ADL R. For any two filters I, J of R, we have the following:*

1. *$I \subseteq I^{\theta}$.*
2. *If $I \subseteq J$ then $I^{\theta} \subseteq J^{\theta}$.*
3. *$(I \cap J)^{\theta} = I^{\theta} \cap J^{\theta}$.*
4. *$(I^{\theta})^{\theta} = I^{\theta}$.*

Proof.

1. Let $a \in I$. We have $(a, a) \in \theta$, and hence $a \in I^{\theta}$ and hence $a \in I^{\theta}$. Therefore $I \subseteq I^{\theta}$.
2. Suppose that $I \subseteq J$. Let $x \in I^{\theta}$. Then $(x, a) \in \theta$, for some $a \in I$. Since $I \subseteq J$, we get $(x, a) \in \theta$ and $a \in J$. Therefore $x \in J^{\theta}$. Hence $I^{\theta} \subseteq J^{\theta}$.

3. Clearly $(I \cap J)^\theta \subseteq I^\theta \cap J^\theta$. Conversely, let $x \in (I \cap J)^\theta$. Then $(x, a) \in \theta$, for some $a \in I \cap J$. That implies $(x, a) \in \theta$ and $a \in I$, $a \in J$. Therefore $x \in I^\theta \cap J^\theta$. Hence $(I \cap J)^\theta \subseteq I^\theta \cap J^\theta$.
   Let $x \in I^\theta \cap J^\theta$. This implies $(x, a), (x, b) \in \theta$, for some $a \in I$ and $b \in J$. So that $(x, a \vee b) \in \theta$ and $a \vee b \in I \cap J$. Implies that $x \in (I \cap J)^\theta$. Therefore $I^\theta \cap J^\theta \subseteq (I \cap J)^\theta$. Hence $(I \cap J)^\theta = I^\theta \cap J^\theta$.

4. Let $x \in (I^\theta)^\theta$. Then $(x, a) \in \theta$, for some $a \in I^\theta$. Since $a \in I^\theta$, we have $(a, b) \in \theta$, for some $b \in I$. This implies $(x, b) \in \theta$ and $b \in I$ and hence $x \in I^\theta$. Therefore $(I^\theta)^\theta \subseteq I^\theta$.
   Let $x \in I^\theta$. Then $(x, a) \in \theta$, for some $a \in I$. Since $a \in I$, we have $a \in I^\theta$. That implies $(x, a) \in \theta$, for some $a \in I^\theta$. Therefore $x \in (I^\theta)^\theta$ and hence $I^\theta \subseteq (I^\theta)^\theta$. Thus $(I^\theta)^\theta = I^\theta$.

□

Lemma 3.7. *[18] Let θ be a congruence relation on an ADL R. For any filter F of R, F^θ is the smallest $\theta-$filter of R such that $F \subseteq F^\theta$.*

Proof. Clearly, F^θ is a filter of R and $F \subseteq F^\theta$. Let $x \in F^\theta$. Then $(x, a) \in \theta$, for some $a \in F$. We have prove that $[x]_\theta \subseteq F^\theta$. Let $t \in [x]_\theta$. Then $(t, x) \in \theta$. Since $(x, a) \in \theta$ and $a \in F$, we get $(t, a) \in \theta$ and hence $t \in F^\theta$. Therefore F^θ is a $\theta-$filter of R containing F.

Let K be any $\theta-$filter of R containing F. Now we prove that $F^\theta \subseteq K$. Let $x \in F^\theta$. Then $(x, a) \in \theta$, for some $a \in F$. Since $F \subseteq K$, we have $a \in K$. Since K is a $\theta-$filter of R, we get $x \in K$. Therefore F^θ is the smallest $\theta-$filter of R such that $F \subseteq F^\theta$. □

Now, we give some theorems that help for the change of $\theta-$filter into a $\theta-$prime filter with the help of a set of equivalent condition.

Theorem 3.1.4. *[18] Let θ be a congruence relation on an ADL R with maximal elements. For any proper $\theta-$filter F of R we have $F = \bigcap\{P : P$ is a $\theta-$prime filter and $F \subseteq P\}$.*

Proof. Take $F_0 = \bigcap\{P : P$ is a $\theta-$prime filter and $F \subseteq P\}$. Clearly $F \subseteq F_0$. Let $a \notin F$. Consider $\mathfrak{F} = \{J : J$ is a $\theta-$filter, $F \subseteq J$ and $a \notin F\}$. Clearly $F \in \mathfrak{F}$.

Let $\{J_\alpha\}_{\alpha\in\triangle}$ be a chain of $\theta-$filters in $\mathfrak{F}$. Clearly, $\bigcup_{\alpha\in\triangle} J_\alpha$ is a $\theta-$filter of R such that $F \subseteq \bigcup_{\alpha\in\triangle} J_\alpha$ and $a \notin \bigcup_{\alpha\in\triangle} J_\alpha$. Hence by the Zorn's lemma, $\mathfrak{F}$ has a maximal element M, say. That is M is a $\theta-$filter, $F \subseteq M$ and $a \notin M$.

Let $x, y \in R$ with $x \vee y \in [m]_\theta$, where m is any maximal element of an ADL R. Suppose $x \notin M$ and $y \notin M$. Then $M \subset M \vee [x) \subseteq (M \vee [x))^\theta$ and $M \subset M \vee [y) \subseteq (M \vee [y))^\theta$. By the maximality of M, we get that $a \in (M \vee [x))^\theta \cap (M \vee [y))^\theta = (M \vee [x \vee y))^\theta$. Since $x \vee y \in [m]_\theta$, we get that $a \notin M$, which is a contradiction. Hence M is a $\theta-$prime filter of R. Therefore for any $a \notin F$, there exists a $\theta-$prime filter M of an ADL R such that $F \subseteq M$ and $a \notin M$. Thus $a \notin F_0$. Hence $F_0 \subseteq F$. Therefore $F_0 = F$. □

Corollary 3.8. *[18] Let R be an ADL with maximal element m. Then $[m]_\theta = \bigcap\{P : P$ is a $\theta-$prime filter of $R\}$*

Corollary 3.9. *[18] Let R be an ADL with maximal element m and θ be a congruence relation on R. If $a \notin [m]_\theta$ then there exist a $\theta-$prime filter P of R such that $a \notin P$.*

Theorem 3.1.5. *[18] Let R be an ADL with maximal element m and θ be a congruence on R. Suppose F is a $\theta-$filter and I is an ideal of R such that $F \cap I = \phi$. Then there exist a $\theta-$prime filter P of R such that $F \subseteq P$ and $I \cap P = \phi$.*

Proof. Let F be a $\theta-$filter and I, an ideal of R with $F \cap I = \phi$. Consider $\mathfrak{F} = \{J : J$ is a $\theta-$filter, $F \subseteq J$ and $J \cap I = \phi\}$. Clearly $F \in \mathfrak{F}$. Let $\{J_\alpha\}_{\alpha\in\triangle}$ be a chain of $\theta-$filters in $\mathfrak{F}$. Clearly, $\bigcup_{\alpha\in\triangle} J_\alpha$ is a $\theta-$filter of R such that $F \subseteq \bigcup_{\alpha\in\triangle} J_\alpha$ and $(\bigcup_{\alpha\in\triangle} J_\alpha) \cap I = \phi$. Hence by the Zorn's lemma $\mathfrak{F}$ has a maximal element M, say. Let $x, y \in R$ with $x \vee y \in [m]_\theta$. We prove that $x \in M$ or $y \in M$. Suppose that $x \notin M$ and $y \notin M$. Then $M \subset M \vee [x) \subseteq (M \vee [x))^\theta$ and $M \subset M \vee [y) \subseteq (M \vee [y))^\theta$. By the maximality of M, we get that $(M \vee [x))^\theta \cap F \neq \phi$ and $(M[y))^\theta \cap F \neq \phi$. Choose $a \in (M \vee [x))^\theta \cap F$ and $b \in (M \vee [x))^\theta \cap F$. Then $a \vee b \in (M \vee [x))^\theta \cap (M \vee [y))^\theta = (M \vee [x \vee y))^\theta$ and $a \vee b \in F$. Since $x \vee y \in [m]_\theta$, we get that $x \vee y \in M$. Since $x \vee y \in F$, we have $x \vee y \in M \cap F$, which is a contradiction. Therefore M is a $\theta-$prime filter of an ADL R. □

3.2 Ideals of Almost Distributive Lattices with respect to a Congruence

The concept of $\theta-$filters in an ADL was given in the last section. The usual lattice theoretic duality principle dosen't hold in ADL. So we introduce the concept of $\theta-$ideals in an ADL and study their important properties. The concept of $\theta-$ideals is introduced in an ADL, and then characterized in terms of ADL congruence. Also the concept of $\theta-$prime ideals is introduced and established a set of equivalent conditions for every $\theta-$ideal to becomes a $\theta-$prime ideal. Some properties of $\theta-$ideals and $\theta-$prime ideals are studied. The class of all $\theta-$ideals of an ADL can be made into a bounded distributive lattice. Finally, the prime ideal theorem is generalized in the case of $\theta-$prime ideal in an ADL.

Though many results look similar, the proofs are not similar because of the lack of the properties like commutativity of $\vee$, commutativity of $\wedge$ and the right distributivity of $\vee$ over $\wedge$ in an ADL.

Now we have the following definition of a $\theta-$ ideal in an ADL R.

Definition 3.2.1. *[17] Let θ be a congruence relation on an ADL R. An ideal I of R is called a $\theta-$ideal of R, if for any $a \in I \Rightarrow [a]_\theta \subseteq I$.*

Note. For any congruence θ on an ADL R, it can be easily observed that the zero ideal $\{0\}$ is a $\theta-$ideal if and only if $[0]_\theta = \{0\}$.

Example 3.2.1. *Let $R = \{0, a, b, 1\}$ be a chain as given in Example 2.3.4. Define $[0]_\theta = \{0, a, b\}$, $[1]_\theta = \{1\}$. Clearly, θ is a congruence relation on R. Consider an ideal $I = \{0, a, b\}$. Clearly I is a $\theta-$ideal of R. Now consider the ideal $J = \{0, a\}$. Then $b \notin J$ and $b \in [0]_\theta$. Hence J is not a $\theta-$ideal of R.*

Lemma 3.10. *[17] Let θ be a congruence on R and m be any maximal element of R. For any ideal I of R, the following hold:*

1. *If I is a $\theta-$ideal, then $[0]_\theta \subseteq I$.*

2. *If I is a proper $\theta-$ideal, then $I \cap [m]_\theta = \phi$.*

3. *If θ is the smallest congruence, then every ideal is a $\theta-$ideal.*

Example 3.2.2. *Let $D = \{0', a'\}$ be a discrete ADL and $A = \{0, a, b, c, 1\}$ is a distributive lattice whose Hasse diagram is given in the figure:*

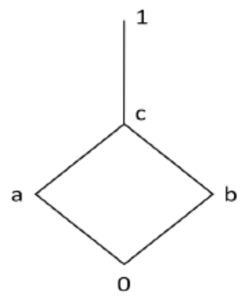

FIGURE 3.2

Then $R = D \times A$ is an ADL under point$-$wise operations. Define $[(0', 0)]_\theta = \{(0', 0)\}$, $[(0', a)]_\theta = \{(0', a)\}$, $[(0', b)]_\theta = \{(0', b)\}$, $[(0', c)]_\theta = [(0', 1)]_\theta = \{(0', c), (0', 1)\}$, $[(a', 0)]_\theta = \{(a', 0)\}$, $[(a', a)]_\theta = \{(a', a)\}$, $[(a', b)]_\theta = \{(a', b)\}$, $[(a', c)]_\theta = \{(a', c)\}$, $[(a', 1)]_\theta = \{(a', 1)\}$. Clearly, θ is a congruence relation on R.

Consider the ideal $I = \{(0', 0), (0', a)\}$. Clearly I is a $\theta-$ideal of R. Now consider the ideal $J = \{(0', 0), (0', a), (0', b), (0', c)\}$. Then $(0', 1) \notin J$ and $(0', 1) \in [0', c]_\theta$. Therefore $[(0', c)]_\theta \nsubseteq J$. Hence J is not a $\theta-$ideal of R.

Example 3.2.3. *Let R be a distributive lattice whose Hasse diagram is given in the Example 3.2.2. For the congruence relation θ whose partition is $\{\{0\}, \{a\}, \{b\}, \{c, 1\}\}$, we can observe that the ideal $I = \{0, a\}$ and $J = \{0, b\}$ are both $\theta-$ideals of the distributive lattice R. But the ideal $I \vee J$ is not a $\theta-$ideal of R.*

Now, we give the following equivalent conditions for $\theta-$ideal.

Theorem 3.2.1. *[17] Let θ be a congruence relation on an ADL R. Then for any ideal I of R, the following conditions are equivalent:*

1. *I is a $\theta-$ideal.*

2. *For any* $x, y \in R$, $(x, y) \in \theta$ *and* $x \in I \Rightarrow y \in I$.

3. $I = \bigcup_{x \in I} [x]_\theta$.

Proof.
$(1) \Rightarrow (2)$: Assume that I is a $\theta-$ideal of R. Let $x, y \in R$ be such that $(x, y) \in \theta$. Suppose $x \in I$. Therefore we get that $y \in [x]_\theta \subseteq I$.

$(2) \Rightarrow (3)$: Assume the condition (2). Let $x \in I$. Since $x \in [x]_\theta$, we get $I \subseteq \bigcup_{x \in I} [x]_\theta$. Conversely, Let $a \in \bigcup_{x \in I} [x]_\theta$. Then $(a, x) \in \theta$, for some $x \in I$. By condition (2), we get that $a \in I$. Therefore $I = \bigcup_{x \in I} [x]_\theta$.

$(3) \Rightarrow (1)$: Assume that the condition (3) holds. Let $a \in I$. Then we get $(x, a) \in \theta$, for some $x \in I$. Let $t \in [a]_\theta$. Then we get $(t, a) \in \theta$ and hence $(x, t) \in \theta$. Thus it yields that $t \in [x]_\theta \subseteq I$. Therefore I is a $\theta-$ideal of R. □

Definition 3.2.2. *[17] Let θ be a congruence relation on an ADL R. A proper $\theta-$ideal P of an ADL R is called a $\theta-$prime ideal of R if for any $a, b \in R$ with $a \wedge b \in [0]_\theta \Rightarrow$ either $a \in P$ or $b \in P$.*

Example 3.2.4. *Consider the ADL R given in Example 3.2.1. Then I is a $\theta-$prime ideal.*

Lemma 3.11. *[17] If θ is the smallest congruence relation on an ADL R, then every prime ideal of R is a $\theta-$prime ideal.*

Proof. Suppose that θ is the smallest congruence relation on R. Let P be a prime ideal of R. Then by the above Lemma 3.10, P is a $\theta-$ideal of R. Let $a, b \in R$ be such that $a \wedge b \in [0]_\theta$. Then we get that $[a \wedge b]_\theta = [0]_\theta$. Since θ is the smallest congruence relation on R, it can be concluded that $a \wedge b = 0 \in P$. Therefore P is $\theta-$prime ideal of R. □

The following lemma gives a relation between prime $\theta-$ideal and $\theta-$prime ideal.

Lemma 3.12. *[17] Let θ be a congruence relation on an ADL R. Then every prime $\theta-$ideal of R is a $\theta-$prime ideal of R.*

Proof. Let P be a prime $\theta-$ideal of an ADL R. Let $x, y \in R$ with $x \wedge y \in [0]_\theta$. Since P and P is a $\theta-$ideal of R. We get that $x \wedge y \in [0]_\theta \subseteq P$. Since P is a prime ideal of R, we get that either $x \in P$ or $y \in P$. Therefore P is a $\theta-$prime ideal of R. □

Theorem 3.2.2. *[17] Let θ be a congruence relation on an ADL R and P, a $\theta-$ideal of R. Then the following conditions are equivalent:*

1. *P is a $\theta-$prime ideal of R.*
2. *For any ideals I, J of R with $I \cap J \subseteq [0]_\theta$ implies that$I \subseteq P$ or $J \subseteq P$.*
3. *For any $a, b \in R$, $[a]_\theta \cap [b]_\theta = [0]_\theta$ implies that either $a \in P$ or $b \in P$.*

Proof. $(1) \Rightarrow (2)$: Assume that P is a $\theta-$prime ideal of R. Let I, J be two ideals of R such that $I \cap J \subseteq [0]_\theta$. Let $a \in I$ and $b \in J$. Then $a \wedge b \in I \cap J \subseteq [0]_\theta$. Since P is $\theta-$prime, we get that either $a \in P$ or $b \in P$.Thus we get that either $I \subseteq P$ or $J \subseteq P$.

$(2) \Rightarrow (3)$: Assume that condition (2). Suppose that $[a]_\theta \cap [b]_\theta = [0]_\theta$ for any $a, b \in R$. Then we get $[a \wedge b]_\theta = [0]_\theta$. Thus it yields that $a \wedge b \in [0]_\theta$ and hence $(a] \cap (b] \subseteq [0]_\theta$. Therefore by the assumed condition (2) we get that either $a \in (a] \subseteq P$ or $b \in (b] \subseteq P$.

$(3) \Rightarrow (1)$: Assume that condition (3) holds. Let $a, b \in R$ be such that $a \wedge b \in [0]_\theta$. Hence we get $[a]_\theta \cap [b]_\theta = [a \wedge b]_\theta = [0]_\theta$. Thus by condition (3), we get that either $a \in P$ or $b \in P$. Therefore P is a $\theta-$prime ideal of R. □

Lemma 3.13. *[17] Let θ be a congruence relation on an ADL R and m be any maximal element of R. Then every maximal ideal disjoint from $[m]_\theta$ is a $\theta-$ideal of R.*

Proof. [17] Let M be a maximal ideal of R and m be any maximal element of R such that $M \cap [m]_\theta = \phi$. Let $x, y \in R$ with $(x, y) \in \theta$ and $x \in M$. Suppose $y \notin M$. Then $M \vee (y] = R$. That implies $a \vee y$ is a maximal element of R for some $a \in M$. Since $(x, y) \in \theta$, we get that $(a \vee x, a \vee y) \in \theta$. Thus we can obtain that $a \vee x \in [a \vee y]_\theta$. Since $a \vee x \in M$. We get that $M \cap [a \vee y]_\theta \neq \phi$, which is a contradiction. Therefore $y \in M$. Which yields that M is a $\theta-$ideal of R. □

Note. [17] Let θ be a congruence relation on an ADL R and m be any maximal element of R. If $[m]_\theta = \{m\}$, then every maximal ideal of R is a $\theta-$ideal of R.

Definition 3.2.3. *[17] Let θ be a congruence relation on an ADL R. For any ideal I of R, define the set I^θ as given by $I^\theta = \{x \in R : (x, a) \in \theta$, for some $a \in I\}$.*

Example 3.2.5. *Consider the ADL R given in Example 3.2.1. Then for $I = \{(0', 0), (0', a)\}$ the set $I^\theta = \{(0', 0), (0', a)\} = I$, and for $J = \{(0', 0), (0', a), (0', b), (0', c)\}$, the set $J^\theta = \{(0', 0), (0', a), (0', b), (0', c), (0', 1)\}$.*

Lemma 3.14. *[17] Let θ be a congruence relation on an ADL R. For any ideal I of R, the set I^θ is an ideal of R.*

Proof. Clearly, $0 \in I^\theta$. Let $x, y \in I^\theta$. Then we get $(x, a) \in \theta$ and $(y, b) \in \theta$, for some $a, b \in I$. Hence we get $(x \vee y, a \vee b) \in \theta$. That implies $x \vee y \in I^\theta$. Again, let $x \in I^\theta$ and $r \in R$. Then $(x, a) \in \theta$, for some $a \in I$. Since θ is a congruence, we get $(x \wedge r, a \wedge r) \in \theta$. Since $a \wedge r \in I$, We get $x \wedge r \in I^\theta$. Therefore I^θ is an ideal of R. □

Lemma 3.15. *[17] Let θ be a congruence relation on an ADL R. For any two ideals I, J of R, we have the following:*

1. *$I \subseteq I^\theta$.*
2. *If $I \subseteq J$ implies $I^\theta \subseteq J^\theta$.*
3. *$(I \cap J)^\theta = I^\theta \cap J^\theta$.*
4. *$(I^\theta)^\theta = I^\theta$.*

Proof.

1. Let $a \in I$. We have $(a, a) \in \theta$, and hence $a \in I^\theta$ and hence $a \in I^\theta$. Therefore $I \subseteq I^\theta$.
2. Suppose that $I \subseteq J$. Let $x \in I^\theta$. Then $(x, a) \in \theta$, for some $a \in I$. Since $I \subseteq J$, we get $(x, a) \in \theta$ and $a \in J$. Therefore $x \in J^\theta$. Hence $I^\theta \subseteq J^\theta$.

3. Clearly $(I \cap J)^\theta \subseteq I^\theta \cap J^\theta$. Conversely, let $x \in (I \cap J)^\theta$. Then $(x, a), (x, b) \in \theta$, for some $a \in I$ and $b \in J$. So that $(x, a \wedge b) \in \theta$ and $a \wedge b \in I \cap J$. Implies that $x \in (I \cap J)^\theta$. Therefore $I^\theta \cap J^\theta \subseteq (I \cap J)^\theta$. Hence $(I \cap J)^\theta = I^\theta \cap J^\theta$.

4. Clearly $(I^\theta)^\theta \subseteq I^\theta$. Again, let $x \in (I^\theta)^\theta$. Then $(x, a) \in \theta$, for some $a \in I^\theta$. Since $a \in I^\theta$, we have $(a, b) \in \theta$, for some $b \in I$. This implies $(x, b) \in \theta$, $b \in I$ and hence $x \in I^\theta$. Therefore $(I^\theta)^\theta \subseteq I^\theta)$. Thus $(I^\theta)^\theta = I^\theta$.

□

Note. [17] Let θ be a congruence relation on an ADL R. For any ideal I of R, I^θ is the smallest $\theta-$ideal of R such that $I \subseteq I^\theta$.

Proof. From Lemma 3.14 and Lemma 3.15(1), we get that I^θ is a $\theta-$ideal of R containing the ideal I. Let K be a $\theta-$ideal of R such that $I \subseteq K$. Let $x \in I^\theta$. Then we get $(x, a) \in \theta$ for some $a \in I \subseteq K$. Hence $x \in [x]_\theta = [a]_\theta \subseteq K$. Therefore $I^\theta \subseteq K$. □

Example 3.2.6. *Let R be a distributive lattice whose Hasse diagram is given in the Example 3.2.2. For any congruence relation θ on a distributive lattice R, one can easily observe that the set Id$_\theta(R)$ of all $\theta-$ideals of R is not a sublattice of the ideal lattice Id(R). For, consider the ideal $I = \{0, a\}$ and $J = \{0, b\}$. Now, for the congruence relation θ whose partition is $\{\{0\}, \{a\}, \{b\}, \{c, 1\}\}$, we can observe that I and J are both the $\theta-$ideals of the distributive lattice R. But the ideal $I \vee J$ is not a $\theta-$ideal of R.*

In view of the operation depicted in the Definition 3.2.3, it can be observed that Id$_\theta(R)$ can be made into a distributive lattice with respect to the following operations: for any $I, J \in$ Id$_\theta(R)$, $I \wedge J = I \cap J$ and $I \vee J = (I \vee J)^\theta$.

Theorem 3.2.3. *[17] Let θ be a congruence relation on an ADL R. For any proper $\theta-$ideal I of R we have $I = \bigcap\{P : P$ is a $\theta-$prime ideal and $I \subseteq P\}$.*

Proof. Take $I_0 = \bigcap\{P : P$ is a $\theta-$prime ideal, $I \subseteq P\}$. Clearly $I \subseteq I_0$. Let $a \notin I$. Consider $\mathfrak{F} = \{J : J$ is a $\theta-$ideal, $I \subseteq J$ and $a \notin F\}$. Clearly $I \in \mathfrak{F}$. Let $\{J_\alpha\}_{\alpha \in \triangle}$ be a chain of $\theta-$ideals in $\mathfrak{F}$. Clearly, $\bigcup_{\alpha \in \triangle} J_\alpha$ is a $\theta-$ideal of R such that $I \subseteq \bigcup_{\alpha \in \triangle} J_\alpha$ and $a \notin \bigcup_{\alpha \in \triangle} J_\alpha$. Hence by the Zorn's lemma, $\mathfrak{F}$ has a maximal element M, say. That means M is a $\theta-$ideal, $I \subseteq M$ and $a \notin M$. Suppose $x, y \in R$ such that $x \notin M$ and $y \notin M$. Then $M \subset M \vee (x] \subseteq (M \vee (x])^{\theta}$ and $M \subset M \vee (y] \subseteq (M \vee (y])^{\theta}$. By the maximality of M, we get that $a \in (M \vee (x])^{\theta} \cap (M \vee (y])^{\theta} = (M \vee (x \wedge y])^{\theta}$. If $x \wedge y \in [0]_{\theta}$, we get that $x \wedge y \in [0]_{\theta} \subseteq M$. Hence $a \in M$, which is a contradiction. Hence M is a $\theta-$prime ideal. Therefore for any $a \notin I$, there exists a $\theta-$prime ideal M of an ADL R such that $I \subseteq M$ and $a \notin M$. Thus $a \notin I_0$. Hence $I_0 \subseteq I$. Therefore $I_0 = I$. ☐

Corollary 3.16. *[17]* $[0]_{\theta} = \cap\{P : P$ *is a* $\theta-$*prime ideal* $\}$.

Corollary 3.17. *[17] If* θ *is the smallest congruence on* R*, then we have* $\{0\} = \cap\{P : P$ *is a* $\theta-$*prime ideal*$\}$.

Corollary 3.18. *[17] Let* θ *be a congruence relation on an ADL* R*. If* $a \notin [0]_{\theta}$ *then there exist a* $\theta-$*prime ideal* P *of* R *such that* $a \notin P$.

Theorem 3.2.4. *[17] Let* θ *be a congruence on* R*. Suppose* I *is a* $\theta-$*ideal and* F *is a filter of* R *such that* $I \cap F = \phi$*. Then there exist a* $\theta-$*prime ideal* P *of* R *such that* $I \subseteq P$ *and* $F \cap P = \phi$.

Proof. Let I be a $\theta-$ideal and F, a filter of R such that $I \cap F = \phi$. Consider $\Im = \{J : J$ is a $\theta-$ideal, $I \subseteq J$ and $J \cap F = \phi\}$. Clearly $I \in \Im$
Let $\{J_i : i \in \triangle\}$ be a chain of $\theta-$ideals in $\Im$. Clearly, $\bigcup_{i\in\triangle} J_i$ is a $\theta-$ideal such that $I \subseteq \bigcup_{i\in\triangle} J_i$ and $(\bigcup_{i\in\triangle} J_i) \cap F = \phi$. Let M be a maximal element of $\Im$. suppose $x, y \in R$ such that $x \notin M$ and $y \notin M$. Then $M \subset M \vee (x] \subseteq \{M \vee (x]\}^{\theta}$ and $M \subset M \vee (y] \subseteq \{M \vee (y]\}^{\theta}$. By the maximality of M, we get that $\{M \vee (x]\}^{\theta} \cap F \neq \phi$ and $\{M \vee (y]\}^{\theta} \cap F \neq \phi$. Choose $a \in \{M \vee (x]\}^{\theta} \cap F$ and $b \in (M \vee (y]\}^{\theta} \cap F$. Hence $a \wedge b \in \{M \vee (x]\}^{\theta} \cap \{M \vee (y]\}^{\theta} = \{M \vee (x \wedge y]\}^{\theta}$ If $x \wedge y \in [0]_{\theta}$, then $x \wedge y \in [0]_{\theta} \subseteq M$. Since M is a $\theta-$prime ideal of R. ☐

Bibliography

[1] M. Atiyah and I. Macdonald, *Introduction to commutative Algebra*, Published by Reading, MA; Menlo Park, CA; London; Amsterdam; Don Mills, Ontario; Sydney, Addison-Wesley Publishing Company, 1967.

[2] A. Badawi, *On semi commutative π regular ring*, Comm. Algebra,Emory and Henry collage, 1994,151-157.

[3] F. Balogun, *A study of Derivations on latices*, Acta. Math. Acad. Sci. Hungar, vol. 4, no. 11, pp. 14-19, 2014.

[4] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publications, U.S.A, 1967.

[5] K. Border, *Preliminary Notes on Lattices*, Lecture notes, California Institute of Technology, 2001.

[6] Y. Cho, *Some results on π - regularity and s-unitality*. Korean J. Math, Pusan. Korea,2009, No.4, pp.314-347

[7] R. Conover, *A First Course in Topology*, Dover Books on Mathematics,Menlo Park, California . London . Don Mills, Ontario , Courier Corporation, ADDISON-WESLEY PUBLISHING COMPANY,2014, .

[8] B. Davey and Priestly, *Introduction to Lattices and Order*, New York. U.S.A. Australia. Madrid. Spain. South Africa,printed in the united kingdom at the University press, Cambridge, 1990.

[9] L.Ferrari, *On derivations of lattices*. Pure Mathematics and Application, vol. 12, no. 4,2001, pp 365-382.

[10] J. Gallian, *Contemporary Abstract Algebra*, Boston. New York, Houghton Mifflin Company, fourth edition, 1998.

[11] K. Goodearl, *Von Neumann Regular Rings*, Pltman, London - San Franclsco - Melbourne, second edition, Krleger, Malabar, Fl, 1991.

[12] G. Gratzer, *The Congruences of a Finite Lattice*, Birkhauser Verlag, Boston, 2005.

[13] G. Gratzer, *General Lattice Theory*, Academic Press, New York, Sanfransisco, Birkhäuser verlag. Boston, 1998.

[14] T. Khan, *Distributive Lattices*, submitted to center European University, Budapest, Hungary, 2011.

[15] S. Lane, G. Birkhoff, *Algebra*, AMS Chelsea, 1999.

[16] Y. Pawar and I. Shaikh, *Congruence Relation On Almost Distributive Lattices*, Southeast Asian Bulletine of Mathematics, 2012.

[17] N. Rafi, *Ideals of Almost Distributive Lattices with respect to a Congruence*, IJSIMR, India, Department of mathematics, 2014.

[18] N.Rafi, R.Banbaru and G. Rao, *θ_{-} Filter in Almost Distributive Lattices*, Accepted for publication in International Journal of Scientific and Innovative Mathematical Research, ISSN 2347-307X (Print) and ISSN 2347-3142 (Online), 2013.

[19] N. Rafi, R. Bandaru and G. Rao, *d_congruences of Almost Distributive Lattices*, Bulletin of the international mathematical virtual institute ISSN, 2015 .

[20] G. Rao and K. BABU, *The Theory Of Derivations In Almost Distributive Lattices*, Bulletin of the international mathematical virtual institute, 2017

[21] S. Roman, *Lattices and Ordered Sets*, Springer, U.S.A, 2008.

[22] E. Posner, *Derivations in prime rings*. Proc. Amer. Math. Soc, nol. 8, pp.1093-1100, 1957.

[23] U. Swamy, *Birkhoff center of an almost distributive lattice*, International Journal of Algebra, vol. 3, 2009.

[24] U. Swamy and G. Rao. Almost Distributive Lattices.Austral. J. Aust. Math. Soc. (series A), 1981.

[25] U. Swamy, C. Santhi and R. Chudamani, *On Almost Boolean Algebras and Rings*, P.V.P Siddhartha Institute of Technology, Vijayawada, Andhra Pradesh, India, 2016.

[26] J. Whitesitt, *Boolean Algebra and it's Applications*, Dover publications, INC, Mineola, New York, Courier Corporation,2012.