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Research Article

A New Family of Optimal Fourth-Order Iterative Methods for Solving Nonlinear Equations With Applications

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A new family of fourth-order iterative methods for solving nonlinear equations is proposed using the weight function procedure. This family is optimal in the sense of the Kung–Traub conjecture, as it requires three function evaluations per iteration. Due to its flexible structure, the new family offers a variety of options, demonstrating that it includes several well-known and recent methods as special cases. In particular, three new specific methods are designed to achieve better results compared to existing methods within the same family. Various nonlinear functions and engineering problems are used to illustrate the performance of these new specific methods, comparing them with existing ones. Furthermore, the analysis of complex dynamics and basins of attraction shows that the newly proposed methods yield the best results, with wider sets of initial points that lead to convergence.

Keywords: basins of attraction; engineering applications; iterative methods; nonlinear equations; optimal methods; weight function

1. Introduction

Solving nonlinear equations efficiently is a fundamental challenge in numerical analysis, with a wide range of applications in engineering and the applied sciences. For instance, nonlinear equations arise in chemical equilibrium problems, radioactive transfer, and positioning systems. Several applications are discussed in [1–7]. See also Section 5 for specific applied problems.

Over the years, a large number of iterative methods have been developed for approximating the root of a nonlinear equation $f(x) = 0$. Newton's method is the best known and most widely used algorithm defined as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, \cdots \tag{1}$$

The convergence order of this method is two for simple roots.

In recent years, numerous multistep iterative methods have been proposed to achieve a higher order of convergence, see [2, 4, 5, 7–18] and the references therein. It seems that using the multistep technique aims to avoid higher-order derivatives in schemes.

In addition to the order of convergence, the efficiency index (EI) is important in rating different iterative methods. Indeed, the EI is defined as $p^{1/d}$, where p is the order of convergence and d is the total number of new function evaluations per iteration. According to the Kung–Traub conjecture [19], the order of convergence of any multipoint method without memory cannot exceed the bound $p = 2^{d-1}$, the *optimal order*. In this sense, Newton's method is considered optimal, as it only requires two function evaluations. In recent literature, several fourth-order optimal iterative methods have been introduced [2, 7, 8, 10–14, 16].

In this paper, a new family of optimal fourth-order iterative methods is developed using the weight function technique. Each iteration requires one function evaluation and two first derivative evaluations. Therefore, the family is considered optimal according to the Kung–Traub conjecture, achieving an EI of $EI = 1.587$.

The proposed family offers flexibility in its formulation and the selection of weight functions. It is demonstrated that several well-known and recent methods can be considered as special cases of the proposed family. Additionally, three new specific methods are derived from the family, showing good performance.

The organization of this paper is as follows: Section 2 covers some preliminaries. Section 3 presents the construction and convergence analysis of the new family of optimal fourth-order methods. In Section 4, several well-known schemes are presented as specific cases within the proposed family, and three new methods are derived from it. Section 5 provides numerical results for these methods in the real domain, along with comparisons. Additionally, five engineering problems are considered to validate the applicability of the new methods. Finally, Section 6 is devoted to study the stability of these specific methods using the basins of attraction technique.

2. Preliminaries

Definition 1 (see [20]). Suppose $\{x_n\}$ is a sequence that converges to a limit α. If there exists an integer constant $p \geq 1$ and a nonzero constant C such that

$$\lim_{n \longrightarrow \infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = C$$

then p is called the order of convergence and C is called the asymptotic error constant.

Let $e_n = x_n - \alpha$ is the error in the n^{th} iteration. The equation

$$e_{n+1} = C e_n^p + \mathcal{O}\left(e_n^{p+1}\right)$$

is called the error equation for the method, p being the order of convergence, see [17].

Definition 2 (see [17]). Let α be a root of the function $f(x)$ and suppose that x_{n-1}, x_n, and x_{n+1} are three consecutive iterations closer to the root α. Then, the computational order of convergence can be computed by using the formula

$$p \approx \frac{\ln |(x_{n+1} - \alpha)/(x_n - \alpha)|}{\ln |(x_n - \alpha)/(x_{n-1} - \alpha)|}$$

3. Construction of the New Family and Convergence Analysis

Assume that α is a simple root of $f(x) = 0$, where $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is a sufficiently differentiable function. A two-step family of fourth-order iterative methods is introduced as follows:

$$y_n = x_n - \frac{2}{3} \frac{f(x_n)}{f'(x_n)} \tag{2}$$

$$x_{n+1} = x_n - \tau \frac{f(x_n)}{f'(x_n)} - G(\eta_n) \frac{f(x_n)}{A f'(x_n) + B f'(y_n)} \tag{3}$$

where τ, A, and B are parameters and G is a weight function in terms of

$$\eta_n = 1 - \frac{f'(y_n)}{f'(x_n)} \tag{4}$$

The first step of this family represents Jarratt's method step [21]. The parameters τ, A, and B are chosen arbitrarily. Then, the weight function G is designed to achieve fourth-order convergence, as shown in Theorem 1. In the existing schemes, τ can be either 0 or 2/3. If $\tau = 0$, the first two terms on the right-hand side of Equation (3) reduce to x_n, while for $\tau = 2/3$, they are equivalent to y_n.

Theorem 1. *Let $\alpha \in I$ be a simple root of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to α, and the weight function $G(\eta)$ satisfies*

$$\begin{aligned} G(0) &= (1 - \tau)(A + B), G'(0) \\ &= \frac{3A - B}{4} + \tau B, G''(0) \\ &= \frac{9A + 3B}{4} \end{aligned} \tag{5}$$

then the scheme (3) converges to α with order of convergence four and satisfies the error equation

$$\begin{aligned} e_{n+1} = {} & \left[\left(405A + 189B - 32G'''(0) \right) \frac{c_2^3}{81(A + B)} - c_2 c_3 + \frac{1}{9} c_4 \right] e_n^4 \\ & + \mathcal{O}\left(e_n^5\right) \end{aligned}$$

where $c_j = f^{(j)}(\alpha)/j! f'(\alpha), j = 2, 3, \cdots$. Provided $A + B \neq 0$ and $|G'''(0)| < \infty$.

Proof 1. Using Taylor's expansion for $f(x_n)$ and $f'(x_n)$ about α, we have

$$f(x_n) = f'(\alpha) \left[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 \right] + \mathcal{O}\left(e_n^5\right) \tag{6}$$

$$f'(x_n) = f'(\alpha) \left[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 \right] + \mathcal{O}\left(e_n^5\right) \tag{7}$$

where $e_n = x_n - \alpha$.

From Equation (6) and Equation (7), we get

$$\begin{aligned} \frac{f(x_n)}{f'(x_n)} = {} & e_n - c_2 e_n^2 + \left(2c_2^2 - 2c_3\right) e_n^3 \\ & + \left(-4c_2^3 + 7c_2 c_3 - 3c_4\right) e_n^4 + \mathcal{O}\left(e_n^5\right) \end{aligned} \tag{8}$$

After subtracting α from both sides of (2) and making

Table 1: The test functions and their simple roots.

Function	**Root**
$f_1(x) = \cos x - x$	0.7390851332151606416553 12
$f_2(x) = x^3 + 4x^2 - 10$	1.365230013414096845760807
$f_3(x) = \sin^2 x - x^2 + 1$	1.404491648215341226035087
$f_4(x) = x^2 - e^x - 3x + 2$	0.257530285439860760455367
$f_5(x) = e^x \sin x + \ln\left(1 + x^2\right)$	0
$f_6(x) = e^{x^2+7x-30} - 1$	3
$f_7(x) = \sqrt{x^2 + x + 3} - 2\sin(x-2) - x^2 + 1$	2
$f_8(x) = \cos\left(\frac{\pi x}{2}\right) + \frac{\ln\left(x^2 + 2x + 2\right)}{x^2 + 1}$	−1

the substitution $e_n = x_n - \alpha$, upon inserting (8), we obtain the following

$$y_n - \alpha = e_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} = \frac{1}{3}e_n + \frac{2}{3}c_2 e_n^2 + \frac{4}{3}\left(c_3 - {c_2}^2\right)e_n^3 + \frac{2}{3}\left(4{c_2}^3 - 7c_2c_3 + 3c_4\right)e_n^4 + \mathcal{O}\left(e^5\right) \tag{9}$$

Expanding $f'(y_n)$ about α and using Equation (9), we obtain

$$f'(y_n) = f'(\alpha)\left[1 + \frac{2}{3}c_2 e_n + \frac{1}{3}\left(c3 + 4c_2^2\right)e_n^2 + \left(\frac{4}{27}c_4 + 4c_2c_3 - \frac{8}{3}c_2^3\right)e_n^3 + \left(\frac{44}{9}c_2c_4 - \frac{32}{3}c_2^2c_3 + \frac{8}{3}c_3^2 + \frac{16}{3}c_2^4 + \frac{5}{81}c_5\right)e_n^4\right] + \mathcal{O}\left(e_n^5\right) \tag{10}$$

From Equations (6), (7), and (10), we have

$$\begin{aligned}\frac{f(x_n)}{Af'(x_n) + Bf'(y_n)} = {} & \frac{e_n}{A+B} + (-3A + B)\frac{c_2 e_n^2}{3(A+B)^2} + \left[18\left(c_2^2 - c_3\right)A^2 - 12B\left(c_2^2 + c_3\right)A - B^2\left(14c_2^2 - 6c_3\right)\right]\frac{e_n^3}{9(A+B)^3} \\ & + \left[\left(-108c_2^3 + 189c_2c_3 - 81c_4\right)A^3 + 144\left(c_2^3 + \frac{7}{16}c_2c_3 - \frac{139}{144}c_4\right)BA^2 + 276\left(c_2^3 - \frac{83}{92}c_2c_3 - \frac{35}{276}c_4\right)AB^2 \right. \\ & \left. + 88\left(c_2^3 - \frac{123}{88}c_2c_3 + \frac{23}{88}c_4\right)B^3\right]\frac{e_n^4}{27(A+B)^4} + \mathcal{O}\left(e_n^5\right)\end{aligned} \tag{11}$$

Using Equations (7) and (10), we can write the weight function variable η given in Equation (4) as

$$\begin{aligned}\eta_n = 1 - \frac{f'(y_n)}{f'(x_n)} = {} & \frac{4}{3}c_2 e_n + \left(\frac{8}{3}c_3 - 4c_2^2\right)e_n^2 + \left(\frac{104}{27}c_4 - \frac{40}{3}c_2c_3 + \frac{32}{3}c_2^3\right)e_n^3 \\ & + \left(-\frac{484}{27}c_2c_4 + \frac{148}{3}c_3c_2^2 - \frac{32}{3}c_3^2 - \frac{80}{3}c_2^4 + \frac{400}{81}c_5\right)e_n^4 + \mathcal{O}\left(e^5\right)\end{aligned} \tag{12}$$

Then, expanding the weight function $G(\eta_n)$ around zero, we get

$$\begin{aligned}G(\eta_n) = {} & G(0) + G'(0)\eta_n + G''(0)\frac{\eta_n^2}{2} + G'''(0)\frac{\eta_n^3}{6} + G^{(4)}(0)\frac{\eta_n^4}{24} + \cdots = G(0) + \frac{4}{3}G'(0)c_2 e_n \\ & + \left[\left(-36G'(0) + 8G''(0)\right)\frac{c_2^2}{9} + \frac{8}{3}G'(0)c_3\right]e_n^2 \\ & + \left[\left(864G'(0) - 432G''(0) + 32G'''(0)\right)\frac{c_2^3}{81} - 40c_3\left(G'(0) - \frac{4}{15}G''(0)\right)\frac{c_2}{3} + \frac{104}{27}G'(0)c_4\right]e_n^3 + \cdots.\end{aligned} \tag{13}$$

Table 2: Numerical results for test functions $f_1(x)$ and $f_2(x)$.

Function	Method	x_0	N	$\lvert x_N - x_{N-1}\rvert$	$\lvert f(x_N)\rvert$	ρ	Time
f_1	NM	0.5	7	1.57×10^{-78}	9.15×10^{-157}	2.00	0.0011
	VNM		5	5.00×10^{-114}	3.18×10^{-342}	3.00	0.0014
	JM		4	2.77×10^{-72}	2.31×10^{-288}	3.99	0.0017
	CLND		4	9.97×10^{-68}	5.66×10^{-270}	3.99	0.0021
	SBM		4	1.06×10^{-66}	8.10×10^{-266}	3.99	0.0019
	PM1		4	2.28×10^{-68}	1.47×10^{-272}	3.99	0.0026
	M2		4	2.43×10^{-74}	1.17×10^{-296}	3.99	0.0021
	AMF1		4	6.37×10^{-76}	4.80×10^{-303}	3.99	0.0026
	ZM1		4	2.92×10^{-80}	2.92×10^{-320}	4.00	0.0024
	ZM2		4	6.15×10^{-86}	1.51×10^{-343}	3.99	0.0022
	ZM3		4	1.13×10^{-86}	2.80×10^{-346}	4.00	0.0019
f_1	NM	1.7	7	3.25×10^{-65}	3.92×10^{-130}	2.00	0.0011
	VNM		5	1.70×10^{-65}	1.24×10^{-196}	3.00	0.0013
	JM		5	3.53×10^{-198}	6.14×10^{-792}	4.00	0.0022
	CLND		5	4.15×10^{-194}	1.70×10^{-775}	4.00	0.0031
	SBM		5	5.57×10^{-193}	6.12×10^{-771}	4.00	0.0026
	PM1		5	8.87×10^{-195}	3.35×10^{-778}	4.00	0.0036
	M2		5	6.51×10^{-200}	6.00×10^{-799}	4.00	0.0023
	AMF1		5	2.63×10^{-201}	1.41×10^{-804}	4.00	0.0034
	ZM1		4	3.47×10^{-51}	3.11×10^{-204}	3.99	0.0025
	ZM2		4	8.90×10^{-53}	6.63×10^{-211}	4.00	0.0020
	ZM3		4	9.91×10^{-52}	1.63×10^{-206}	4.00	0.0019
f_2	NM	0.8	8	1.59×10^{-59}	2.06×10^{-117}	2.00	0.0017
	VNM		6	1.71×10^{-105}	2.23×10^{-314}	3.00	0.0021
	JM		5	3.50×10^{-130}	2.19×10^{-518}	4.00	0.0026
	CLND		5	6.26×10^{-74}	5.21×10^{-293}	3.99	0.0035
	SBM		5	9.23×10^{-77}	2.94×10^{-304}	3.99	0.0030
	PM1		5	1.17×10^{-82}	5.61×10^{-328}	3.99	0.0044
	M2		5	1.27×10^{-169}	2.08×10^{-676}	4.00	0.0027
	AMF1		5	3.02×10^{-194}	2.96×10^{-775}	4.00	0.0034
	ZM1		5	1.15×10^{-157}	8.61×10^{-629}	4.00	0.0036
	ZM2		5	2.20×10^{-73}	3.87×10^{-291}	4.00	0.0029
	ZM3		5	2.82×10^{-99}	6.18×10^{-395}	4.00	0.0028

Table 2: Continued.

Function	Method	x_0	N	$\lvert x_N - x_{N-1}\rvert$	$\lvert f(x_N)\rvert$	ρ	Time
f_2	NM	2.5	8	3.21×10^{-53}	8.32×10^{-105}	2.00	0.0017
	VNM		6	6.28×10^{-96}	1.11×10^{-285}	3.00	0.0018
	JM		5	3.32×10^{-114}	1.76×10^{-454}	4.00	0.0023
	CLND		5	3.40×10^{-95}	4.53×10^{-378}	4.00	0.0025
	SBM		5	2.77×10^{-90}	2.37×10^{-358}	4.00	0.0027
	PM1		5	1.34×10^{-98}	9.78×10^{-392}	4.00	0.0038
	M2		5	1.53×10^{-127}	4.45×10^{-508}	4.00	0.0024
	AMF1		5	1.53×10^{-148}	1.93×10^{-592}	4.00	0.0034
	ZM1		4	6.52×10^{-54}	8.87×10^{-214}	4.00	0.0027
	ZM2		5	1.18×10^{-177}	3.16×10^{-708}	4.00	0.0029
	ZM3		4	1.21×10^{-51}	2.13×10^{-204}	3.99	0.0021

Because the least order in Equation (11) is one, we expand $G(\eta_n)$ to the third order.

Finally, according to Equations (8), (11), and (13), the error equation of Scheme (3) is

$$\begin{aligned} e_{n+1} &= x_n - \alpha - \tau\frac{f(x_n)}{f'(x_n)} - G(\eta_n)\frac{f(x_n)}{Af'(x_n)+Bf'(y_n)} \\ &= [(1-\tau)(A+B)-G(0)]\frac{e_n}{A+B} + \sum_{i=2}^{4}\xi_i e_n^i + \mathcal{O}\left(e_n^5\right) \end{aligned} \tag{14}$$

where $\xi_i = \xi_i(\tau, A, B, c_2, \cdots c_4, G(0), G'(0), G''(0), G'''(0))$.

To obtain a fourth-order convergence, we need the coefficients of e_n, e_n^2 and e_n^3 in Equation (14) to be zero. Obviously, if the first condition in (5) holds, the first-order error disappears. To simplify the calculations, we substitute $G(0) = (1-\tau)(A+B)$, to get

$$e_{n+1} = \left[4\tau B + 3A - B - 4G'(0)\right] c_2 \frac{e_n^2}{3(A+B)} \sum_{i=3}^{4}\gamma_i e_n^i ++\mathcal{O}\left(e_n^5\right) \tag{15}$$

where $\gamma_i = \gamma_i(\tau, A, B, c_2, \cdots c_4, G'(0), G''(0), G'''(0))$.

By applying the second condition in (5), the second error vanishes in the error equation (15). Thus, substituting $G'(0) = (3A-B)/4 + \tau B$ leads to

$$e_{n+1} = \left[9A + 3B - 4G''(0)\right]\frac{2c_2^2 e_n^3}{9(A+B)} + \mu e_n^4 + \mathcal{O}\left(e_n^5\right) \tag{16}$$

where $\mu = \mu(A, B, c_2, \cdots c_4, G(0), G''(0), G'''(0))$.

To eliminate the third-order error, it is necessary that $G''(0) = (9A+3B)/4$. Consequently, we establish the third condition in (5), leading to the following error equation:

$$\begin{aligned} e_{n+1} = &\left[\left(405A + 189B - 32G'''(0)\right)\frac{c_2^3}{81(A+B)} - c_2c_3 + \frac{1}{9}c_4\right]e_n^4 \\ &+ \mathcal{O}\left(e_n^5\right) \end{aligned}$$

This ends the proof of the theorem. □

4. Special Cases of the Proposed Family

Many specific methods can be derived from Family (3) by varying the parameters τ, A, and B. Furthermore, several weight functions can be chosen for a specific set of values for τ, A, and B. Here, some well-known methods are listed as special cases of Family (3):

1. If $\tau = 0$, $A = 1$, and $B = 0$, the conditions in (5) are satisfied when

$$G(0) = 1, G'(0) = \frac{3}{4}, G''(0) = \frac{9}{4}$$

Assuming the weight function $G(\eta) = (4-3\eta)/(4-6\eta)$ yields the well-known Jarratt's method (JM) [21], as represented by the iterative expression

$$x_{n+1} = x_n - \left[\frac{3f'(y_n) + f'(x_n)}{6f'(y_n) - 2f'(x_n)}\right]\frac{f(x_n)}{f'(x_n)} \tag{17}$$

Another option for the weight function is $G(\eta) = 16/(16 - 12\eta - 9\eta^2)$, leading to the method of Chun et al. (CLND) [8]

$$x_{n+1} = x_n - \frac{16f'(x_n)f(x_n)}{-5f'^2(x_n) + 30f'(x_n)f'(y_n) - 9f'^2(y_n)} \tag{18}$$

TABLE 3: Numerical results for test functions $f_3(x)$ and $f_4(x)$.

Function	Method	x_0	N	$\lvert x_N - x_{N-1} \rvert$	$\lvert f(x_N) \rvert$	ρ	Time
f_3	NM	1.0	8	4.21×10^{-51}	3.44×10^{-101}	2.00	0.0026
	VNM		6	3.59×10^{-89}	7.54×10^{-266}	3.00	0.0030
	JM		5	5.38×10^{-111}	8.28×10^{-442}	4.00	0.0037
	CLND		6	2.84×10^{-176}	1.43×10^{-702}	4.00	0.0047
	SBM		5	1.21×10^{-55}	5.55×10^{-220}	3.99	0.0044
	PM1		5	1.26×10^{-59}	4.92×10^{-236}	3.99	0.0062
	M2		5	1.79×10^{-159}	6.06×10^{-636}	4.00	0.0042
	AMF1		5	1.38×10^{-175}	1.13×10^{-700}	4.00	0.0048
	ZM1		5	2.30×10^{-139}	5.79×10^{-556}	4.00	0.0056
	ZM2		5	2.95×10^{-75}	6.98×10^{-299}	4.00	0.0041
	ZM3		5	2.29×10^{-62}	1.38×10^{-247}	4.00	0.0038
f_3	NM	2.5	9	2.75×10^{-96}	1.47×10^{-191}	2.00	0.0026
	VNM		6	8.11×10^{-104}	8.71×10^{-310}	3.00	0.0033
	JM		5	5.09×10^{-88}	6.66×10^{-350}	4.00	0.0037
	CLND		5	3.85×10^{-78}	4.77×10^{-310}	3.99	0.0042
	SBM		5	1.61×10^{-75}	1.73×10^{-299}	3.99	0.0045
	PM1		5	8.08×10^{-80}	8.36×10^{-317}	3.99	0.0061
	M2		5	6.65×10^{-94}	1.16×10^{-373}	4.00	0.0042
	AMF1		5	5.28×10^{-101}	2.44×10^{-402}	4.00	0.0047
	ZM1		5	9.94×10^{-111}	2.01×10^{-441}	4.00	0.0048
	ZM2		5	3.53×10^{-97}	1.43×10^{-386}	4.00	0.0045
	ZM3		5	2.02×10^{-102}	8.34×10^{-408}	4.00	0.0038
f_4	NM	-0.5	7	4.06×10^{-68}	5.82×10^{-136}	2.00	0.0012
	VNM		5	2.01×10^{-70}	1.15×10^{-210}	3.00	0.0014
	JM		4	1.42×10^{-59}	9.38×10^{-238}	4.00	0.0019
	CLND		4	7.60×10^{-68}	6.68×10^{-271}	4.00	0.0021
	SBM		4	1.08×10^{-78}	2.61×10^{-314}	4.00	0.0023
	PM1		4	3.41×10^{-65}	2.77×10^{-260}	4.00	0.0034
	M2		4	7.05×10^{-58}	5.94×10^{-231}	4.00	0.0023
	AMF1		4	1.02×10^{-56}	2.73×10^{-226}	3.99	0.0027
	ZM1		4	5.01×10^{-55}	1.65×10^{-219}	3.99	0.0026
	ZM2		4	4.04×10^{-54}	7.46×10^{-216}	3.99	0.0023
	ZM3		4	1.35×10^{-54}	8.98×10^{-218}	3.99	0.0020

Table 3: Continued.

Function	Method	x_0	N	$\lvert x_N - x_{N-1}\rvert$	$\lvert f(x_N)\rvert$	ρ	Time
f_4	NM	1.5	7	1.06×10^{-57}	3.94×10^{-115}	2.00	0.0014
	VNM		6	8.79×10^{-146}	9.57×10^{-437}	3.00	0.0016
	JM		5	4.17×10^{-149}	6.97×10^{-596}	4.00	0.0023
	CLND		5	8.18×10^{-157}	8.92×10^{-627}	4.00	0.0025
	SBM		5	3.83×10^{-160}	4.06×10^{-640}	4.00	0.0028
	PM1		5	5.56×10^{-155}	1.96×10^{-619}	4.00	0.0039
	M2		5	1.98×10^{-146}	3.71×10^{-585}	4.00	0.0025
	AMF1		5	3.43×10^{-144}	3.42×10^{-576}	4.00	0.0033
	ZM1		5	1.20×10^{-140}	5.46×10^{-562}	4.00	0.0034
	ZM2		5	1.94×10^{-139}	3.95×10^{-557}	4.00	0.0029
	ZM3		5	4.27×10^{-140}	8.92×10^{-560}	4.00	0.0024

If we take another $G(\eta) = -(1 + 3\eta)/8 + 9/(8 - 8\eta)$, Scheme (3) becomes

$$x_{n+1} = x_n - \left[-\frac{1}{2} + \frac{9}{8}\frac{f'(x_n)}{f'(y_n)} + \frac{3}{8}\frac{f'(y_n)}{f'(x_n)}\right]\frac{f(x_n)}{f'(x_n)} \tag{19}$$

which is the method proposed by Sharma and Bahl (SBM) [16].

2. Let $\tau = 0$, $A = 1$, and $B = 3$, then the conditions in (5) hold if

$$G(0) = 4, G'(0) = 0, G''(0) = \frac{9}{2}$$

Assume $G(\eta) = [1 + 5\eta^2/16][4 + \eta^2/(1 - \eta)^2]$ leads to the following method:

$$x_{n+1} = x_n - 4\left[1 + \frac{5}{16}\left(1 - \frac{f'(y_n)}{f'(x_n)}\right)\right] \cdot \left[1 + \frac{1}{4}\left(\frac{f'(x_n) - f'(y_n)}{f'(y_n)}\right)^2\right]\frac{f(x_n)}{f'(x_n) + 3f'(y_n)} \tag{20}$$

This is the method introduced by Madhu and Jayaraman (PM1) [13].

3. Suppose $\tau = 0$, $A = 0$, and $B = 1$, according to (5)

$$G(0) = 1, G'(0) = -\frac{1}{4}, G''(0) = \frac{3}{4}$$

Taking $G(\eta) = 16(1 - \eta)^2/(3(1 - \eta)^2 + 22(1 - \eta) - 9)$, then the resulting scheme is given as

$$x_{n+1} = x_n - \left[\frac{16f'^2(y_n)}{-9f'^2(x_n) + 22f'(x_n)f'(y_n) + 3f'^2(y_n)}\right]\frac{f(x_n)}{f'(y_n)} \tag{21}$$

This is a special method of the family of fourth-order iterative methods introduced by Ozban and Kaya (M2) [14]. In fact, the family proposed in [14] can be considered as a special case of our general family.

4. If $\tau = 2/3$, $A = 1$ and $B = 1$, the weight function $G(\eta)$ satisfies the conditions in (5) when

$$G(0) = \frac{2}{3}, G'(0) = \frac{7}{6}, G''(0) = 3$$

By selecting the weight function as $G(\eta) = (85(1 - \eta) - 41(1 - \eta)^2)/(66 - 120\eta)$, we obtain the Khirallah and Alkhomsan method (AMF1) [12].

$$x_{n+1} = y_n - \left[\frac{85f'(y_n)f'(x_n) - 41f'^2(y_n)}{-54f'^2(x_n) + 120f'(y_n)f'(x_n)}\right]\frac{f(x_n)}{f'(x_n) + f'(y_n)} \tag{22}$$

Alternatively, by using different values for τ, A, and B, we introduce three new specific methods within the proposed family (Equation (3)) as outlined below:

i. Assuming $\tau = 2/3$, $A = -1$ and $B = 2$, then the weight function $G(\eta)$ satisfies the conditions in (5) when

$$G(0) = \frac{1}{3}, G'(0) = \frac{1}{12}, G''(0) = -\frac{3}{4}$$

Table 4: Numerical results for test functions $f_5(x)$ and $f_6(x)$.

Function	Method	x_0	N	$\lvert x_N - x_{N-1}\rvert$	$\lvert f(x_N)\rvert$	ρ	Time
f_5	NM	1.0	10	2.47×10^{-94}	1.22×10^{-187}	2.00	0.0041
	VNM		7	1.75×10^{-137}	2.24×10^{-410}	3.00	0.0059
	JM		6	2.91×10^{-182}	5.22×10^{-726}	4.00	0.0068
	CLND		6	2.89×10^{-151}	1.06×10^{-601}	4.00	0.0068
	SBM		6	1.62×10^{-142}	1.24×10^{-566}	4.00	0.0064
	PM1		6	3.74×10^{-157}	2.69×10^{-625}	4.00	0.0083
	M2		5	5.32×10^{-51}	3.69×10^{-201}	3.99	0.0054
	AMF1		5	5.76×10^{-57}	3.02×10^{-225}	3.99	0.0069
	ZM1		5	1.30×10^{-79}	2.08×10^{-316}	3.99	0.0071
	ZM2		5	7.28×10^{-70}	1.55×10^{-276}	4.00	0.0068
	ZM3		5	1.59×10^{-82}	1.74×10^{-327}	4.00	0.0061
f_5	NM	1.5	10	2.59×10^{-76}	1.34×10^{-151}	2.00	0.0042
	VNM		7	1.76×10^{-125}	2.29×10^{-374}	3.00	0.0059
	JM		6	2.18×10^{-138}	1.65×10^{-550}	4.00	0.0064
	CLND		6	4.65×10^{-116}	7.14×10^{-461}	4.00	0.0067
	SBM		6	9.25×10^{-110}	1.31×10^{-435}	4.00	0.0071
	PM1		6	2.94×10^{-120}	1.04×10^{-477}	4.00	0.0084
	M2		6	9.50×10^{-152}	3.76×10^{-604}	4.00	0.0059
	AMF1		6	1.93×10^{-168}	3.85×10^{-671}	4.00	0.0072
	ZM1		5	8.70×10^{-55}	4.13×10^{-217}	3.99	0.0065
	ZM2		6	5.71×10^{-199}	5.85×10^{-793}	4.00	0.0078
	ZM3		5	7.12×10^{-52}	7.00×10^{-205}	3.99	0.0060
f_6	NM	2.8	19	2.65×10^{-66}	6.00×10^{-130}	2.00	0.0041
	VNM		div	—	—	—	—
	JM		7	1.30×10^{-91}	3.83×10^{-361}	4.00	0.0046
	CLND		div	—	—	—	—
	SBM		div	—	—	—	—
	PM1		div	—	—	—	—
	M2		18	1.14×10^{-122}	1.90×10^{-486}	4.00	0.0164
	AMF1		6	1.22×10^{-138}	1.63×10^{-549}	4.00	0.0060
	ZM1		6	2.00×10^{-77}	3.73×10^{-304}	4.00	0.0057
	ZM2		7	6.31×10^{-78}	7.23×10^{-306}	4.00	0.0054
	ZM3		8	2.57×10^{-96}	1.42×10^{-379}	4.00	0.0056

Table 4: Continued.

Function	Method	x_0	N	$\lvert x_N - x_{N-1}\rvert$	$\lvert f(x_N)\rvert$	ρ	Time
f_6	NM	3.6	16	4.30×10^{-72}	1.58×10^{-141}	2.00	0.0031
	VNM		11	1.83×10^{-74}	4.62×10^{-219}	3.00	0.0053
	JM		8	5.67×10^{-100}	1.39×10^{-394}	4.00	0.0055
	CLND		9	7.21×10^{-134}	1.37×10^{-529}	4.00	0.0086
	SBM		9	1.47×10^{-92}	2.94×10^{-364}	4.00	0.0060
	PM1		9	1.94×10^{-171}	6.24×10^{-680}	4.00	0.0077
	M2		7	5.37×10^{-58}	9.43×10^{-228}	4.00	0.0058
	AMF1		7	2.56×10^{-146}	3.20×10^{-580}	4.00	0.0067
	ZM1		7	8.85×10^{-159}	1.45×10^{-629}	4.00	0.0078
	ZM2		7	6.39×10^{-119}	7.62×10^{-470}	4.00	0.0051
	ZM3		6	5.30×10^{-58}	2.58×10^{-226}	4.00	0.0042

Taking $G(\eta) = (39\eta^2 + 4\eta - 32)/(36\eta - 96)$, we obtain the following scheme (ZM1)

$$x_{n+1} = y_n + \left[\frac{11f'^2(x_n) - 82f'(x_n)f'(y_n) + 39f'^2(y_n)}{60f'^2(x_n) + 36f'(x_n)f'(y_n)}\right]\frac{f(x_n)}{2f'(y_n) - f'(x_n)} \tag{23}$$

ii. Let $\tau = 2/3$, $A = -4$, and $B = 9$, according to (5).

$$G(0) = \frac{5}{3},\ G'(0) = \frac{3}{4},\ G''(0) = -\frac{9}{4}$$

Selecting the weight function as $G(\eta) = (20 + 39\eta)/(12 + 18\eta)$, the resulting scheme (ZM2) is expressed as follows:

$$x_{n+1} = y_n - \left[\frac{59f'(x_n) - 39f'(y_n)}{30f'(x_n) - 18f'(y_n)}\right]\frac{f(x_n)}{9f'(y_n) - 4f'(x_n)} \tag{24}$$

iii. Suppose $\tau = 1$, $A = -7$, and $B = 15$, by (5).

$$G(0) = 0,\ G'(0) = 6,\ G''(0) = -\frac{9}{2}$$

Taking $G(\eta) = 48\eta/(8 + 3\eta)$, then the corresponding scheme (ZM3) is given as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \left[\frac{48f'(x_n) - 48f'(y_n)}{11f'(x_n) - 3f'(y_n)}\right]\frac{f(x_n)}{15f'(y_n) - 7f'(x_n)} \tag{25}$$

Remark 1. In [11], Jaiswal introduced the following family of fourth-order iterative methods:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \{P(t)\times Q(t)\}\frac{f(x_n)}{f'(x_n)} \end{aligned} \tag{26}$$

where $P(t)$ and $Q(t)$ are weight functions with $t = f'(y_n)/f'(x_n)$. The method (26) is a special case of our general family (Equation (3)), obtained by setting $\tau = 0, A = 1$, and $B = 0$ and using the weight function $G(\eta) = P(1-\eta)Q(1-\eta)$. The conditions for fourth-order convergence derived by Jaiswal [11] for $P(t)$ and $Q(t)$ are equivalent to the conditions (5) for $G(\eta)$.

5. Numerical Examples and Engineering Applications

In this section, we present numerical examples, including some engineering applications, to illustrate the efficiency and applicability of the newly developed methods: ZM1 (23), ZM2 (24), and ZM3 (25). We compare these methods with others from the same fourth-order family (3), namely, JM (17), CLND (18), SBM (19), PM1 (20), M2 (21), and AMF1 (22). Additionally, we compare them with the classical Newton method NM (1) and the third-order method of Weerakoon and Fernando (VNM) [17], which is given by

$$\begin{aligned} \tilde{y}_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(\tilde{y}_n)} \end{aligned} \tag{27}$$

The computations are performed using Maple 2021 with 2000 significant digits, employing the following stopping criterion for the computer programs:

$$\lvert x_n - x_{n-1}\rvert < 10^{-50}.$$

Table 5: Numerical results for test functions $f_7(x)$ and $f_8(x)$.

Function	Method	x_0	N	$\lvert x_N - x_{N-1}\rvert$	$\lvert f(x_N)\rvert$	ρ	Time
f_7	NM	1.0	8	1.06×10^{-69}	1.07×10^{-138}	2.00	0.0079
	VNM		5	4.92×10^{-63}	1.75×10^{-189}	3.00	0.0083
	JM		5	4.24×10^{-102}	2.93×10^{-407}	4.00	0.0101
	CLND		5	9.48×10^{-61}	9.87×10^{-242}	3.99	0.0114
	SBM		5	2.18×10^{-67}	2.98×10^{-268}	3.99	0.0098
	PM1		5	4.56×10^{-70}	5.03×10^{-279}	3.99	0.0133
	M2		5	1.61×10^{-117}	5.41×10^{-469}	4.00	0.0116
	AMF1		5	8.55×10^{-121}	3.86×10^{-482}	4.00	0.0118
	ZM1		5	4.72×10^{-157}	2.90×10^{-627}	4.00	0.0108
	ZM2		5	1.89×10^{-73}	4.94×10^{-293}	4.00	0.0110
	ZM3		5	5.52×10^{-88}	4.65×10^{-351}	4.00	0.0105
f_7	NM	3.0	7	3.30×10^{-63}	1.03×10^{-125}	2.00	0.0073
	VNM		5	1.03×10^{-82}	1.60×10^{-248}	3.00	0.0081
	JM		5	1.01×10^{-191}	9.24×10^{-766}	4.00	0.0093
	CLND		5	7.83×10^{-188}	4.61×10^{-750}	4.00	0.0094
	SBM		5	1.11×10^{-186}	2.04×10^{-745}	4.00	0.0114
	PM1		5	1.66×10^{-188}	8.80×10^{-753}	4.00	0.0172
	M2		5	2.76×10^{-193}	4.63×10^{-772}	4.00	0.0104
	AMF1		5	1.67×10^{-194}	5.62×10^{-777}	4.00	0.0105
	ZM1		5	4.85×10^{-197}	3.21×10^{-787}	4.00	0.0116
	ZM2		5	3.47×10^{-201}	5.68×10^{-804}	4.00	0.0111
	ZM3		5	1.29×10^{-198}	1.38×10^{-793}	4.00	0.0102
f_8	NM	-0.1	8	1.79×10^{-63}	1.61×10^{-126}	2.00	0.0058
	VNM		6	6.16×10^{-85}	2.02×10^{-254}	3.00	0.0095
	JM		5	2.01×10^{-85}	1.59×10^{-340}	3.99	0.0089
	CLND		5	6.96×10^{-80}	3.46×10^{-318}	3.99	0.0091
	SBM		5	1.42×10^{-78}	6.76×10^{-313}	3.99	0.0096
	PM1		5	1.03×10^{-80}	1.58×10^{-321}	3.99	0.0116
	M2		5	1.14×10^{-87}	1.35×10^{-349}	3.99	0.0085
	AMF1		5	2.39×10^{-89}	2.24×10^{-356}	4.00	0.0113
	ZM1		5	1.33×10^{-93}	1.45×10^{-373}	4.00	0.0109
	ZM2		5	2.86×10^{-120}	1.07×10^{-480}	4.00	0.0104
	ZM3		5	3.61×10^{-98}	5.73×10^{-392}	4.00	0.0105

Table 5: Continued.

Function	Method	x_0	N	$\lvert x_N - x_{N-1}\rvert$	$\lvert f(x_N)\rvert$	ρ	Time
f_8	NM	-1.4	8	5.65×10^{-99}	1.60×10^{-197}	2.00	0.0061
	VNM		5	8.87×10^{-73}	6.02×10^{-218}	3.00	0.0073
	JM		5	7.67×10^{-172}	3.37×10^{-686}	4.00	0.0095
	CLND		5	7.68×10^{-147}	5.13×10^{-586}	4.00	0.0089
	SBM		5	1.46×10^{-143}	7.56×10^{-573}	4.00	0.0092
	PM1		5	6.28×10^{-150}	2.15×10^{-598}	4.00	0.0133
	M2		5	2.01×10^{-182}	1.31×10^{-728}	4.00	0.0085
	AMF1		5	1.03×10^{-188}	7.71×10^{-754}	4.00	0.0116
	ZM1		4	6.51×10^{-53}	8.34×10^{-211}	3.99	0.0084
	ZM2		5	3.13×10^{-168}	1.55×10^{-672}	4.00	0.0106
	ZM3		4	2.48×10^{-57}	1.27×10^{-228}	4.00	0.0080

Table 6: Numerical results for different methods applied to the test function f_9.

Method	N	$\lvert x_N - x_{N-1}\rvert$	$\lvert f(x_N)\rvert$	ρ	Time
NM	11	8.13×10^{-61}	4.26×10^{-121}	2.00	0.0027
VNM	8	5.30×10^{-114}	6.57×10^{-340}	3.00	0.0026
JM	6	3.42×10^{-72}	2.40×10^{-285}	3.99	0.0035
CLND	7	1.49×10^{-197}	2.04×10^{-786}	4.00	0.0043
SBM	7	7.25×10^{-175}	1.36×10^{-695}	4.00	0.0042
PM1	6	9.51×10^{-55}	3.02×10^{-215}	3.99	0.0047
M2	6	5.47×10^{-89}	8.75×10^{-353}	4.00	0.0036
AMF1	6	2.38×10^{-119}	1.37×10^{-474}	4.00	0.0045
ZM1	5	3.60×10^{-53}	1.02×10^{-209}	4.00	0.0035
ZM2	6	4.59×10^{-142}	8.98×10^{-565}	4.00	0.0041
ZM3	6	7.72×10^{-176}	4.27×10^{-700}	4.00	0.0037

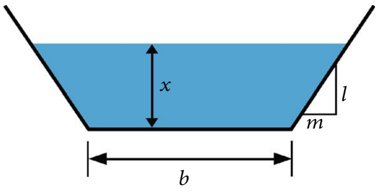


Figure 1: Symmetrical trapezoidal cross-section.

The computational order of convergence ρ is approximated using the formula [9]:

$$\rho = \frac{\ln \lvert (x_{n+1} - x_n)/(x_n - x_{n-1}) \rvert}{\ln \lvert (x_n - x_{n-1})/(x_{n-1} - x_{n-2}) \rvert}$$

Table 1 presents eight commonly used test functions in research, along with their simple roots. Tables 2, 3, 4, and 5 present the following metrics for the iterative methods: the number of iterations (N) at which the stopping criterion is met, the error estimation $\lvert x_N - x_{N-1}\rvert$, the absolute value $\lvert f(x_N)\rvert$, the computational order of convergence (ρ), and the CPU time in seconds. The processing time is computed as the mean of 1000 executions to ensure reasonably accurate values.

For the given test functions and initial estimates, among fourth-order methods, it seems that the new methods ZM1, ZM2, and ZM3 exhibit better accuracy in most cases, with the following ranking: ZM1, followed by ZM3, and then ZM2. Additionally, the methods ZM1 and ZM3 require fewer iterations compared to other methods in several cases. Regarding processing time, the methods JM and ZM3 require the shortest time. In 60% of the cases, the method JM requires less time than ZM3. The number of iterations for ZM3 is less than that of JM in 40% of the given examples. Notably, the method PM1 requires the longest time.

The methods VNM, CLND, SBM, and PM1 fail to converge to a solution even after 50 iterations when attempting

Table 7: Numerical results for different methods applied to the test function f_{10}.

Method	N	$\|x_N - x_{N-1}\|$	$\|f(x_N)\|$	ρ	Time
NM	8	1.99×10^{-78}	5.26×10^{-154}	2.00	0.0041
VNM	6	5.20×10^{-143}	5.80×10^{-426}	3.00	0.0068
JM	5	1.07×10^{-166}	9.66×10^{-664}	4.00	0.0071
CLND	5	2.47×10^{-143}	6.44×10^{-570}	4.00	0.0064
SBM	5	1.55×10^{-137}	1.19×10^{-546}	4.00	0.0069
PM1	5	2.76×10^{-147}	9.05×10^{-586}	4.00	0.0094
M2	5	3.51×10^{-183}	6.09×10^{-730}	4.00	0.0067
AMF1	4	3.03×10^{-52}	1.44×10^{-206}	3.99	0.0063
ZM1	4	3.09×10^{-55}	2.40×10^{-218}	4.00	0.0060
ZM2	4	5.09×10^{-52}	5.80×10^{-205}	4.00	0.0057
ZM3	4	1.53×10^{-53}	2.79×10^{-211}	4.00	0.0057

Table 8: EOS parameters for vapor and liquid water [27].

k	Phase	γ	π(Pa)	C_v(J/kg/K)	q(J/kg)	q'(J/kg/K)
1	Vapor	1.327	0	1.2×10^3	1995×10^3	2.41×10^3
2	Liquid	2.057	1.066×10^9	3.449×10^3	-1994.674×10^3	35.78×10^3

Table 9: Numerical results for different methods applied to the test function f_{11}.

Method	N	$\|x_N - x_{N-1}\|$	$\|f(x_N)\|$	ρ	Time
NM	7	3.97×10^{-59}	5.53×10^{-125}	2.00	0.0019
VNM	5	2.46×10^{-67}	1.16×10^{-211}	3.00	0.0023
JM	4	1.62×10^{-77}	4.76×10^{-324}	3.99	0.0024
CLND	4	1.25×10^{-60}	1.36×10^{-255}	4.00	0.0029
SBM	4	1.91×10^{-57}	1.01×10^{-242}	4.00	0.0029
PM1	4	3.99×10^{-63}	1.10×10^{-265}	4.00	0.0039
M2	4	1.79×10^{-65}	2.80×10^{-275}	3.99	0.0029
AMF1	4	3.16×10^{-61}	4.17×10^{-258}	3.99	0.0034
ZM1	4	1.25×10^{-56}	1.69×10^{-239}	3.99	0.0032
ZM2	4	2.05×10^{-54}	1.87×10^{-230}	3.99	0.0030
ZM3	4	1.50×10^{-55}	4.22×10^{-235}	3.99	0.0027

to solve the equation $f_6(x) = 0$ with an initial estimate of $x_0 = 2.8$. However, a comprehensive understanding of method stability cannot be achieved solely by comparing iterative methods for one or more initial estimates. In Section 6, we investigate the stability of the aforementioned methods using basins of attraction.

Now, we discuss the applicability of the new methods to various problems in engineering and applied sciences.

5.1. Application 1: The van der Waals Equation.

5.1. Application 1: The van der Waals Equation. The well-known van der Waals equation is an equation of state for real gases, expressed as follows (see, e.g., [22]):

$$p = \frac{\mathrm{RT}}{V_m - b} - \frac{a}{V_m^2} \tag{28}$$

where p is the pressure, T is the absolute temperature, V_m is the molar volume, R is the universal gas constant equal to $0.082057\,\mathrm{dm^3 atm K^{-1} mol^{-1}}$, and a and b are specific parameters for each gas that represent molecular attractions and molecular repulsions, respectively.

Table 10: Numerical results for different methods applied to the test function f_{12}.

Method	N	$\lvert x_N - x_{N-1}\rvert$	$\lvert f(x_N)\rvert$	ρ	Time
NM	7	1.82×10^{-70}	5.78×10^{-142}	2.00	0.0011
VNM	5	1.05×10^{-66}	2.97×10^{-201}	3.00	0.0012
JM	5	2.18×10^{-166}	6.19×10^{-667}	4.00	0.0019
CLND	5	2.93×10^{-159}	2.06×10^{-638}	4.00	0.0021
SBM	5	7.32×10^{-158}	8.02×10^{-633}	4.00	0.0023
PM1	5	2.80×10^{-160}	1.70×10^{-642}	4.00	0.0032
M2	5	1.31×10^{-168}	8.02×10^{-676}	4.00	0.0021
AMF1	5	8.66×10^{-170}	1.51×10^{-680}	4.00	0.0027
ZM1	5	1.10×10^{-172}	3.87×10^{-692}	4.00	0.0027
ZM2	5	1.20×10^{-191}	5.38×10^{-768}	4.00	0.0022
ZM3	5	7.99×10^{-177}	1.08×10^{-708}	4.00	0.0020

Equation (28) can be transformed into a nonlinear equation for V_m

$$f(V_m) = V_m^3 - \left(b + \frac{\mathrm{RT}}{p}\right)V_m^2 + \frac{a}{p}V_m^2 - \frac{ab}{p} = 0$$

In particular, consider the CO_2 gas, where the parameters are $a = 3.6073\,\mathrm{dm}^6\mathrm{atmmol}^{-2}$ and $b = 0.042816\,\mathrm{dm}^3\mathrm{mol}^{-1}$. To find the molar volume of CO_2 at $T = 500K$ and $p = 100$ atm, we obtain the following nonlinear equation for the molar volume:

$$f_9(x) = x^3 - 0.45310x^2 + 0.036073x - 1.5445 \times 10^{-3} = 0$$

This equation has three roots, one of which is real and two are complex. The physically acceptable root is $x = 0.36608793125752938862097 2$. We start with an initial guess of $x_0 = 1.0$. Numerical results for different methods are listed in Table 6.

5.2. Application 2: Flow Depth in Trapezoidal Open Channels. The water flow, Q, in an open channel under uniform flow conditions is governed by Manning's equation [23, 24].

$$Q = \frac{AC}{n}R^{2/3}S^{1/2} \tag{29}$$

where C is 1.0 for SI units and 1.486 for BG units, A is the cross-sectional area of the flow, n is the Manning roughness coefficient, S is the longitudinal channel slope, and R is the hydraulic radius, defined as

$$R = \frac{A}{P}$$

where P is the wetted perimeter.

Consider a trapezoidal-shaped channel with bed width b and a side slope of m horizontal to 1 vertical (see Figure 1). If x represents the depth of the water in the channel, then

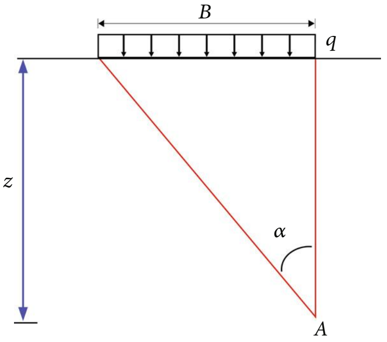


Figure 2: Vertical stress due to a strip load at a point A.

$$A = (b + \mathrm{m}x)x, \text{ and } P = b + 2x\sqrt{1 + m^2}$$

Substituting these values in (29), we get

$$Q = \frac{C\sqrt{S}}{n}\frac{\left(bx + mx^2\right)^{5/3}}{\left(b + 2x\sqrt{1 + m^2}\right)^{2/3}} \tag{30}$$

Assigning specific values to the parameters, as given in [23]: $b = 10$ft, $m = 2$, $S = 0.0006$, $n = 0.016$, and given a flow rate of $Q = 225ft^3/s$, we can determine the depth of the water in the channel using the following equation:

$$f_{10}(x) = 98.9027\left(10 + 2\sqrt{5}x\right)^{2/3} - \left(10x + 2x^2\right)^{5/3} = 0 \tag{31}$$

The solution to this equation is $x = 3.406284331340969211128446$. Table 7 shows the numerical results for an initial guess of $x_0 = 4.5$.

5.3. Application 3: Saturation Curve in Modeling Phase Transition for Two-Phase Flows. In modeling phase transitions for compressible two-phase flows with diffuse interfaces, the authors assume that the Gibbs free energies are

Table 11: Numerical results for different methods applied to the test function f_{13}.

Method	N	$\lvert x_N - x_{N-1}\rvert$	$\lvert f(x_N)\rvert$	ρ	Time
NM	7	2.55×10^{-78}	1.53×10^{-156}	2.00	0.0028
VNM	5	8.45×10^{-125}	1.96×10^{-374}	3.00	0.0037
JM	4	1.97×10^{-69}	1.50×10^{-276}	3.99	0.0033
CLND	4	3.60×10^{-67}	2.44×10^{-267}	3.99	0.0045
SBM	4	1.65×10^{-66}	1.19×10^{-264}	3.99	0.0036
PM1	4	1.46×10^{-67}	6.17×10^{-269}	3.99	0.0048
M2	4	2.18×10^{-70}	1.92×10^{-280}	3.99	0.0041
AMF1	4	3.59×10^{-71}	1.24×10^{-283}	3.99	0.0051
ZM1	4	7.89×10^{-73}	2.11×10^{-290}	3.99	0.0049
ZM2	4	2.06×10^{-75}	4.80×10^{-301}	4.00	0.0040
ZM3	4	8.56×10^{-74}	2.30×10^{-294}	4.00	0.0036

Table 12: Complex polynomials and their roots.

Function	Roots
$g_1(z) = z^3 - 1$	$1, -0.5 \pm 0.866025i$
$g_2(z) = z^3 + z + i$	$-0.56228 - 0.662359i, 1.32472i, 0.56228 - 0.662359i$
$g_3(z) = z^4 - 1$	$\pm 1, \pm i$
$g_4(z) = z^4 - z + i$	$-0.759845 + 0.592595i, -0.532605 - 1.08829i, 0.181924 + 0.732098i, 1.11052 - 0.236405i$
$g_5(z) = z^5 - 1$	$1, -0.809017 \pm 0.587785i, 0.309017 \pm 0.951057i$
$g_6(z) = z^6 - 1$	$\pm 1, -0.5 \pm 0.866025i, 0.5 \pm 0.866025i$

Table 13: Quantifying black points in different iterative methods: individual test cases and average across all cases.

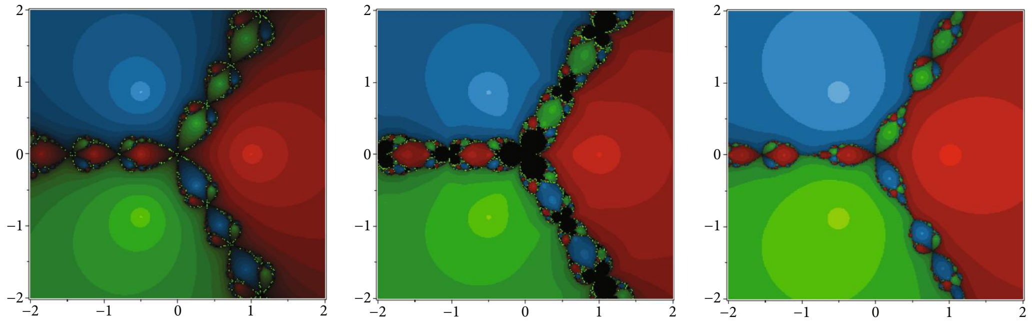
Method	g_1	g_2	g_3	g_4	g_5	g_6	Average
NM	535	212	5000	79	8591	12752	4528
VNM	6636	410	18141	959	19383	21623	11192
JM	0	364	640	0	351	1558	486
CLND	11336	4914	23024	5249	19201	20062	13964
SBM	978	75	8508	397	12283	16946	6531
PM1	928	111	8560	295	12386	16252	6422
M2	3354	109	8848	274	11616	14570	6462
AMF1	0	19	648	0	145	876	281
ZM1	0	16	640	0	68	480	201
ZM2	0	40	648	0	93	662	241
ZM3	0	27	648	0	96	660	239

equal. This equality establishes a direct relationship between the saturation pressure and temperature, see [25, 26].

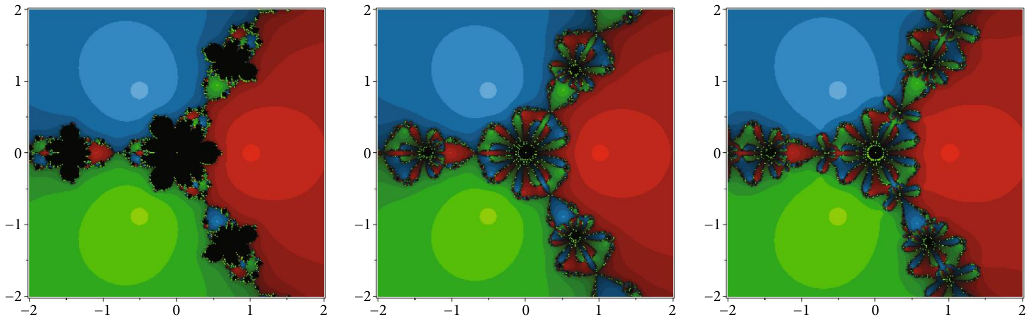
Let g_k denotes the Gibbs free energy of phase k, where $k = 1$ represents the vapor phase and $k = 2$ represents the liquid phase. Using the stiffened gas equation of state (SG-EOS) [25, 26], g_k can be expressed as

$$g_k = \left(\gamma_k C_{vk} - q_k'\right) T_k - T_k C_{vk} \ln \frac{T_k^{\gamma_k}}{\left(p_k + \pi_k\right)^{(\gamma_k - 1)}} + q_k$$

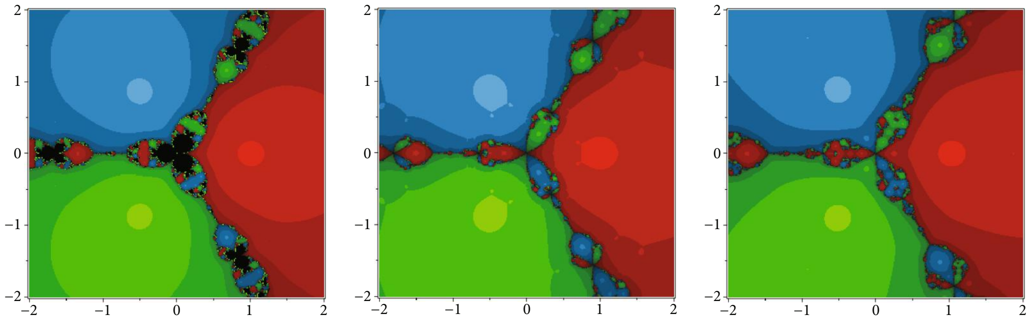
where T_k is the temperature, p_k is the pressure, and C_{vk} is the specific heat capacity at constant volume. The parameters γ_k,



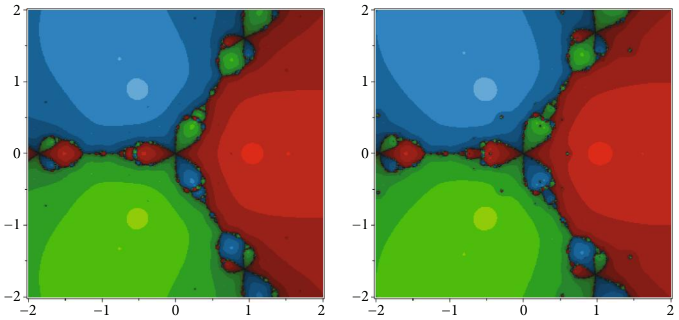
(a) NM (b) VNM (c) JM



(d) CLND (e) SBM (f) PM1



(g) M2 (h) AMF1 (i) ZM1



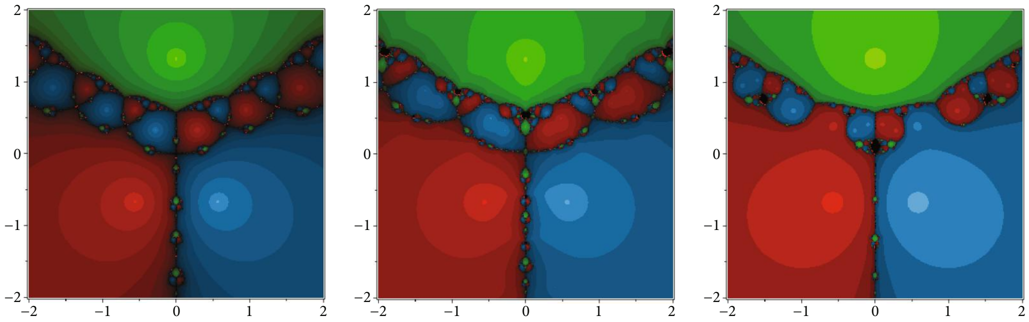
(j) ZM2 (k) ZM3

FIGURE 3: Basins of attraction associated with $g_1(z) = z^3 - 1$.

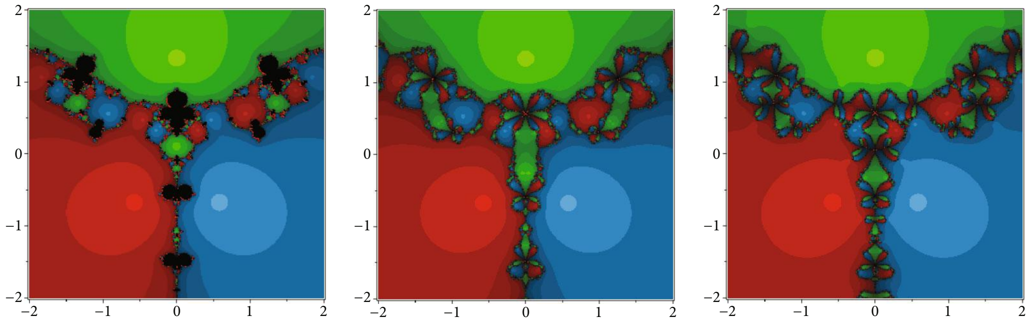
π_k, q_k, and q_k' are characteristic constants of the thermodynamic behavior of the fluid.

At the saturation curve, we have an equilibrium pressure p and an equilibrium temperature T. By the equality of the two Gibbs free energies g_1 and g_2, we obtain

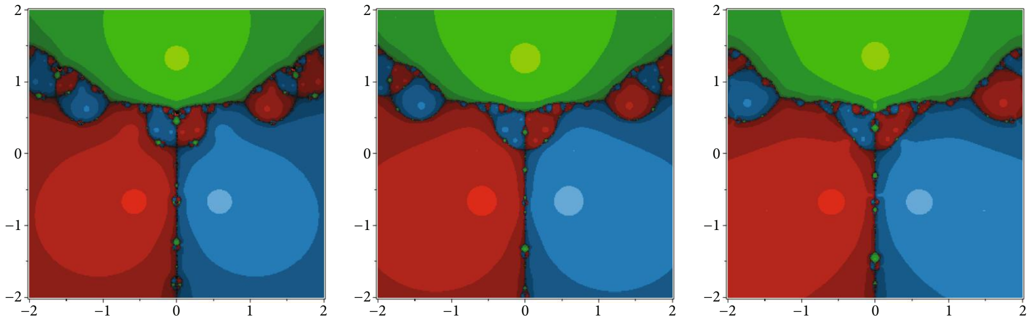
$$\begin{aligned}&\left(\gamma_1 C_{v1} - q_1'\right) T - T C_{v1} \ln \frac{T^{\gamma_1}}{(p+\pi_1)^{(\gamma_1 - 1)}} + q_1 \\ &= \left(\gamma_2 C_{v2} - q_2'\right) T - T C_{v2} \ln \frac{T^{\gamma_2}}{(p+\pi_2)^{(\gamma_2 - 1)}} + q_2.\end{aligned}$$



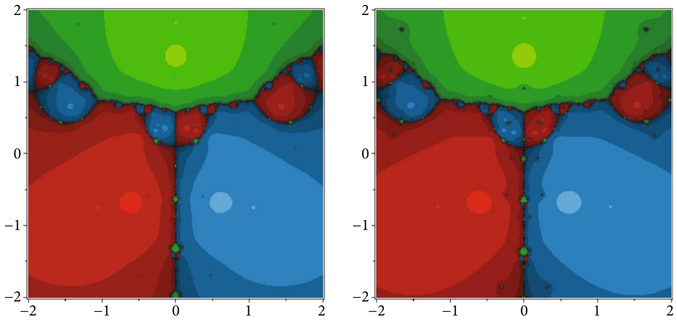
(a) NM (b) VNM (c) JM



(d) CLND (e) SBM (f) PM1



(g) M2 (h) AMF1 (i) ZM1



(j) ZM2 (k) ZM3

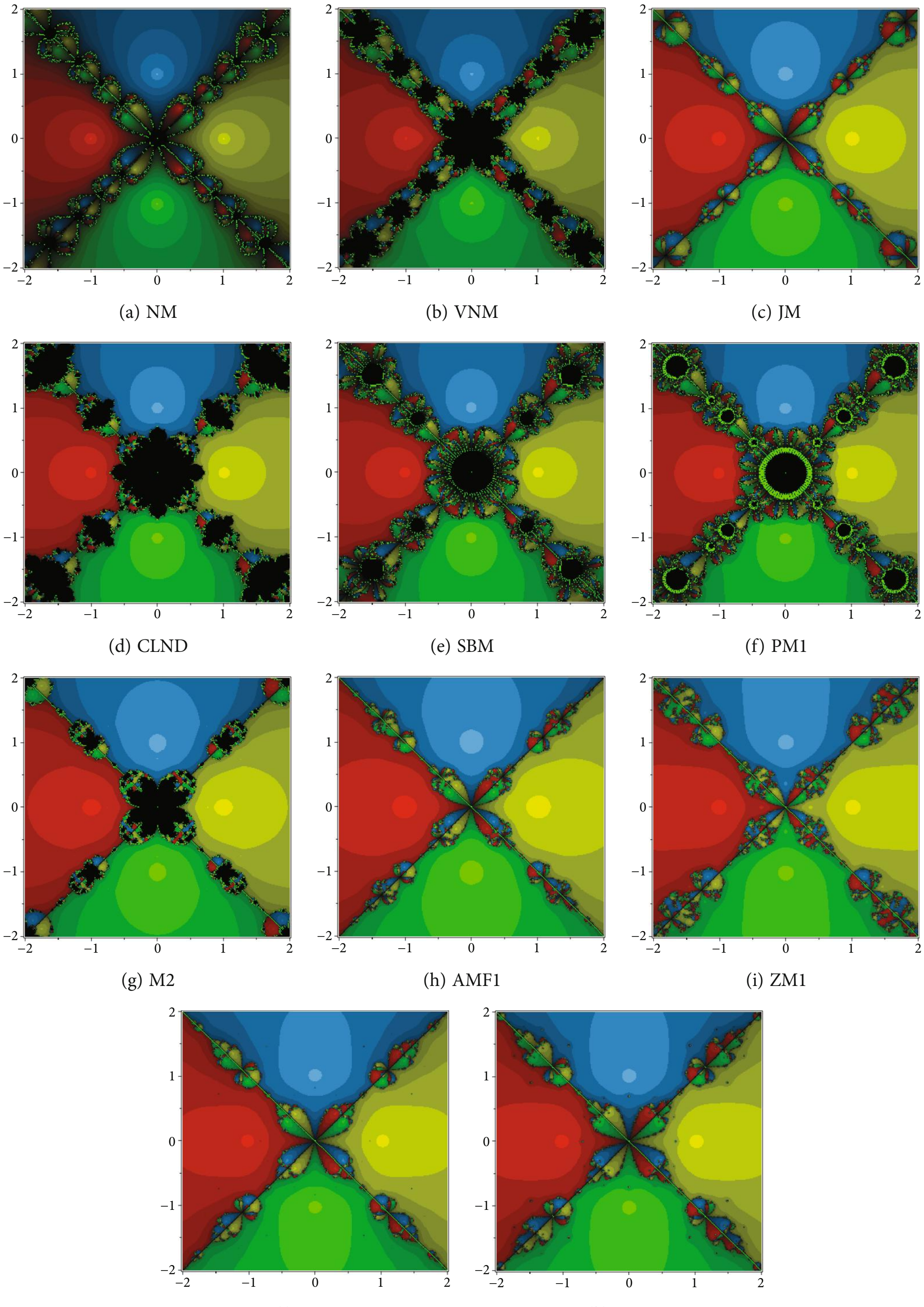
Figure 4: Basins of attraction associated with $g_2(z) = z^3 + z + i$.

This nonlinear equation can be used to compute the saturation pressure in terms of the saturation temperature.

Specifically, for water, we use the parameters for the SG-EOS from Zein, Hantke, and Warnecke [27], as shown in Table 8. For example, when computing the saturation pressure at a saturation temperature of $T_{\text{sat}} = 300K$, the following equation is obtained:

$$f_{11}(x) = -\ln(x) - 184.8879519 + 9.290502039 \ln \cdot \left(x + 1.066 \times 10^9\right) = 0$$

The root of this equation is $x = 3772.191350758676295585854$. Using an initial guess of $x_0 = 3000$, the numerical results are shown in Table 9.


(a) NM (b) VNM (c) JM

(d) CLND (e) SBM (f) PM1

(g) M2 (h) AMF1 (i) ZM1

(j) ZM2 (k) ZM3

Figure 5: Basins of attraction associated with $g_3(z) = z^4 - 1$.

5.4. Application 4: Planck's Radiation Law. Planck's function can be expressed as the spectral emittance of a blackbody in thermal equilibrium at a given temperature as follows (see, e.g., [28]):

$$I(\lambda) = \frac{2\pi hc^2}{\lambda^5\left(e^{hc/\lambda kT} - 1\right)} \quad (32)$$

where λ is the wavelength, h is Planck's constant, c is the speed of light in vacuum, T is the absolute temperature of the blackbody, and k is the Boltzmann constant.

To find the value of λ at which $I(\lambda)$ is maximum, we take the derivative of Equation (32) and set it equal to zero, and we obtain

$$\frac{dI}{d\lambda} = -\frac{10\pi hc^2}{\lambda^6\left(e^{hc/\lambda kT} - 1\right)} + \frac{2\pi h^2 c^3 e^{hc/\lambda kT}}{\lambda^7 kT\left(e^{hc/\lambda kT} - 1\right)^2} = 0 \quad (33)$$

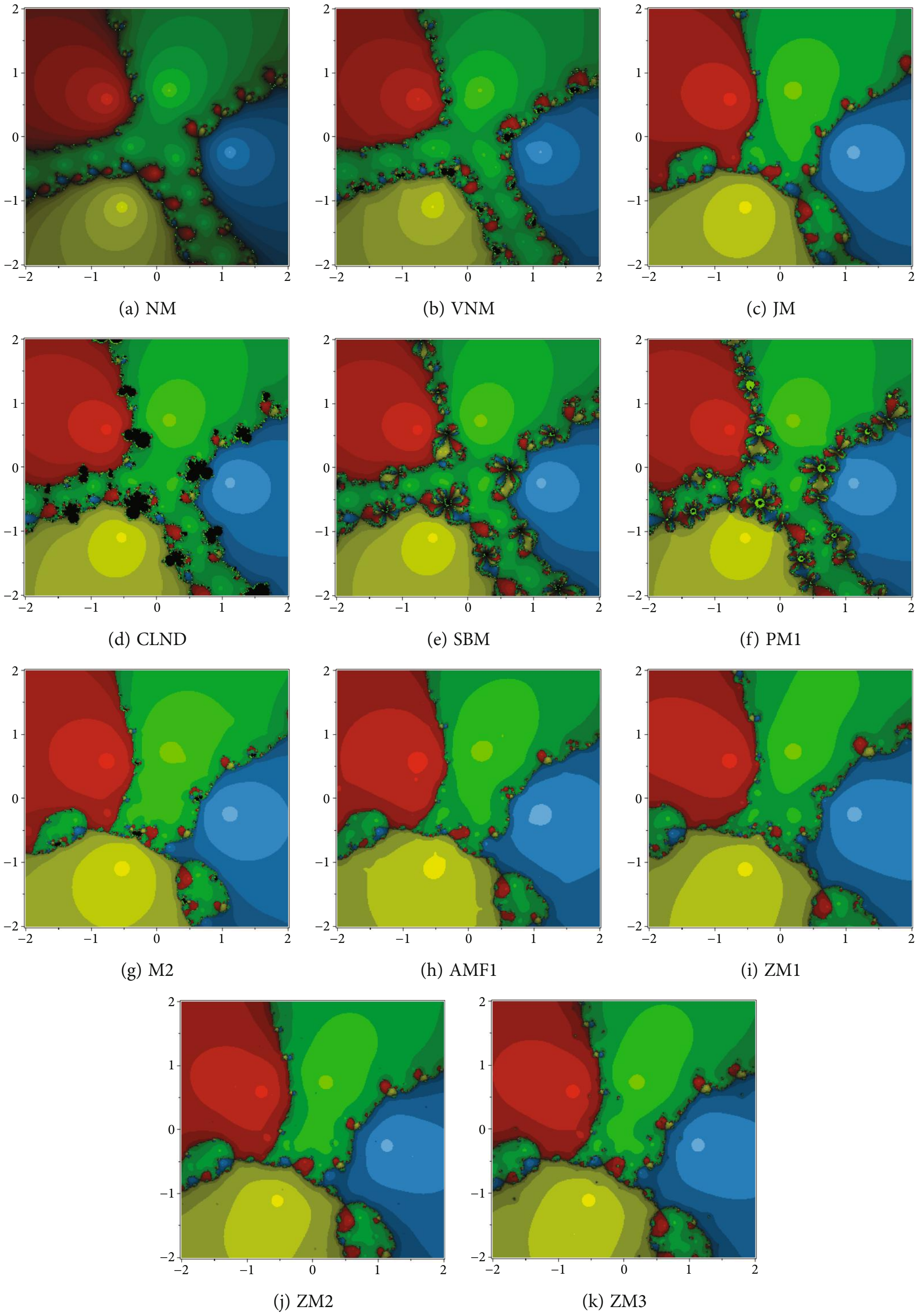

(a) NM
(b) VNM
(c) JM
(d) CLND
(e) SBM
(f) PM1
(g) M2
(h) AMF1
(i) ZM1
(j) ZM2
(k) ZM3

Figure 6: Basins of attraction associated with $g_4(z) = z^4 - z + i$.

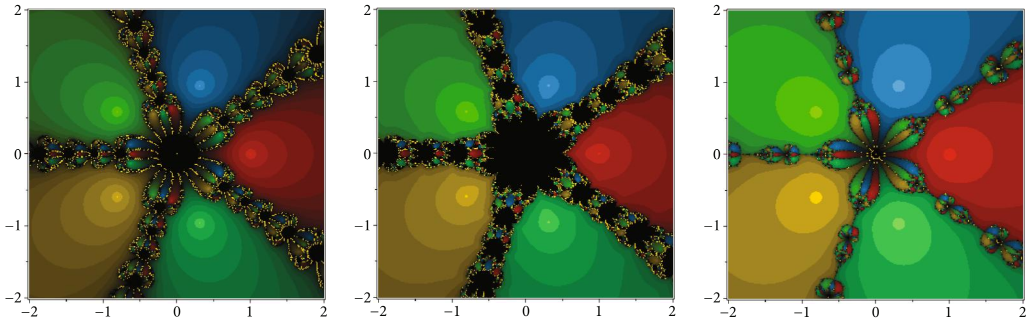
For simplicity, let $x = hc/\lambda kT$, Equation (33) becomes

$$\frac{2\pi k^6 T^6 x^6 (5e^{-x} + x - 5)}{h^5 c^4 (e^x - 1)^2} = 0$$

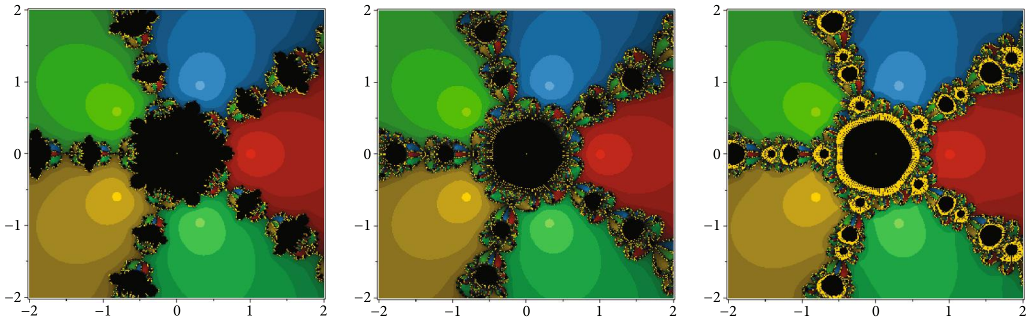
Thus, we have to solve the equation

$$f_{12}(x) = 5e^{-x} + x - 5 = 0$$

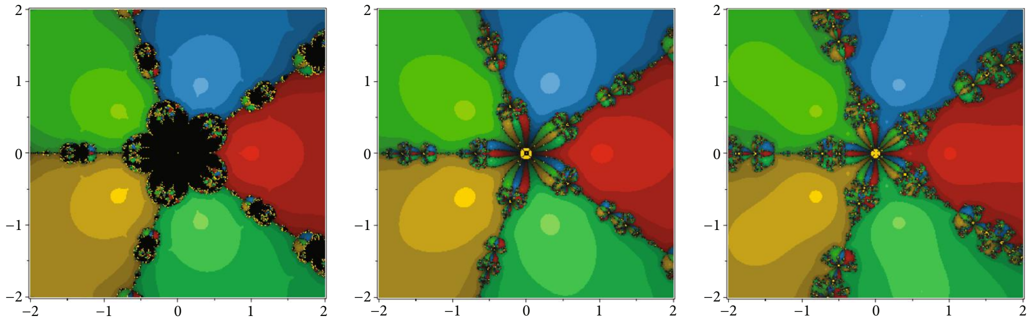
One solution of this equation is $x = 0$, but this solution is not



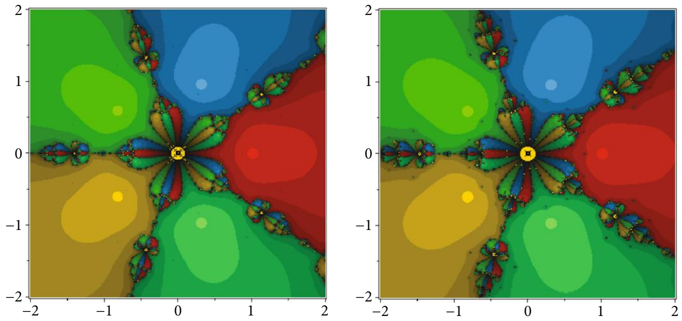
(a) NM (b) VNM (c) JM



(d) CLND (e) SBM (f) PM1



(g) M2 (h) AMF1 (i) ZM1



(j) ZM2 (k) ZM3

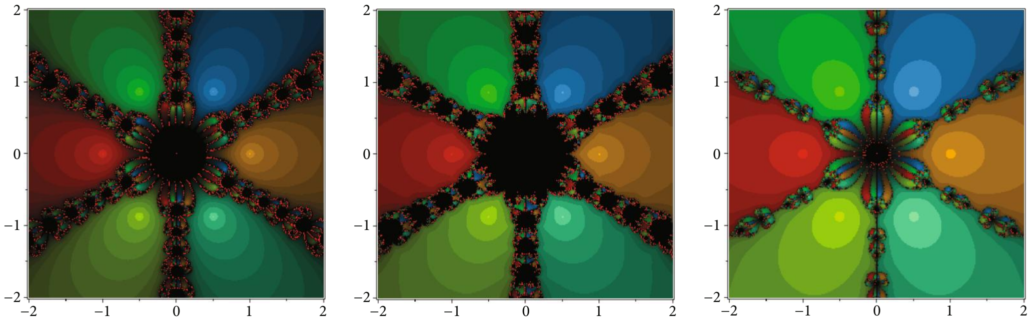
FIGURE 7: Basins of attraction associated with $g_5(z) = z^5 - 1$.

physically meaningful. Another solution is $x = 4.965114231744276303698759$, which is the physically acceptable root. Numerical results of different iterative methods converging to this root, starting with $x_0 = 3.0$, are given in Table 10.

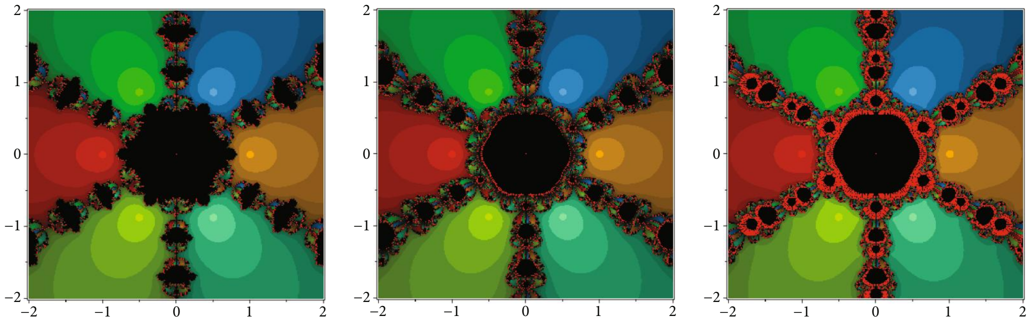
5.5. Application 5: Vertical Stress due to a Strip Load.

Figure 2 shows an infinite strip of width B, subjected to a uniformly distributed load of intensity q (force per unit length). The vertical stress in an elastic soil medium at a point A, situated at a depth z beneath the edge of the strip, can be determined using Boussinesq's formula [29, 30].

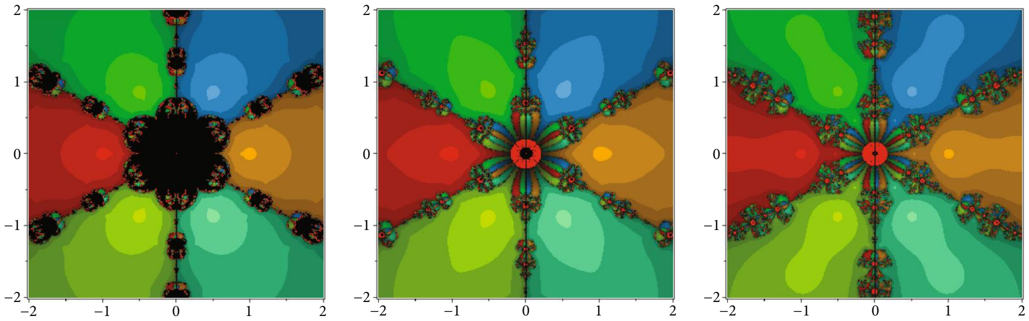
$$\sigma_z = \frac{q}{\pi}(\alpha + \sin\alpha\cos\alpha)$$



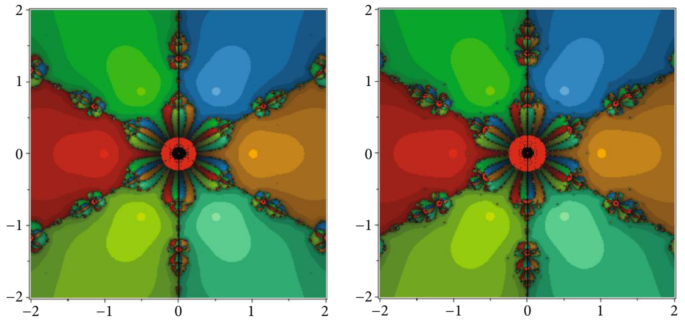
(a) NM (b) VNM (c) JM



(d) CLND (e) SBM (f) PM1



(g) M2 (h) AMF1 (i) ZM1



(j) ZM2 (k) ZM3

FIGURE 8: Basins of attraction associated with $g_6(z) = z^6 - 1$.

Now, to compute the value of angle α at which the vertical stress is 25% of the load intensity q, we get the following equation [4, 29]:

$$f_{13}(x) = \frac{x + \sin x \cos x}{\pi} - \frac{1}{4} = 0$$

We choose an initial guess of $x_0 = 0.25$. The numerical results are presented in Table 11 with the desired root being $x = 0.41585559678986798878800 5$.

Tables 6, 7, 9, 10, and 11 demonstrate the applicability and validity of the new methods in various engineering and applied science problems.

Among fourth-order methods, ZM1 generally demonstrates superior accuracy, requiring fewer iterations in three out of five cases. ZM2 and ZM3 closely follow in performance ranking. In terms of processing time, JM is the fastest, followed by ZM3 and then M2.

6. Basins of Attraction

The basins of attraction provides valuable insights into the stability and reliability of iterative methods. This approach illustrates how an iterative method converges from various initial estimates within a specified region in the complex plane, allowing for graphical comparisons among different methods.

To plot the basins of attraction, we select a rectangular region D within the complex plane that contains all the roots of the given complex polynomial $g(z)$. For each initial guess $z_0 \in D$, we assign a color corresponding to the root to which the iterative method converges. The color intensity is determined by the number of iterations needed for convergence: brighter colors represent fewer iterations, whereas darker colors indicate a greater number of iterations.

The point $z_0 \in D$ is identified as divergent and marked in black if the method does not converge to a root within a tolerance of 10^{-3} after a maximum of 20 iterations. The count of these divergent points is referred to as the number of black points.

In Table 12, six complex polynomials are listed along with their roots as test cases. The computations are carried out over the region $D = [-2, 2] \times [-2, 2]$, using a grid of 320×320 points.

We compare the outcomes of our newly proposed methods, namely, ZM1 (23), ZM2 (24), and ZM3 (25), with those of existing methods: NM (1), VNM (27), JM (17), CLND (18), SBM (19), PM1 (20), M2 (21), and AMF1 (22). Table 13 presents the count of black points for each test case, along with the average value across all cases for each method.

Figure 3 illustrates the basins of attraction of different methods applied to the function $g_1(z)$. It is observed that the methods AMF1, ZM1, ZM2, ZM3, and JM perform very well, with no black points. The methods NM, SBM, and PM1 follow, exhibiting some chaotic behavior. The methods VNM, CLND, and M2 display larger black zones, indicating sensitivity to the initial guess in this test case.

Figure 4 represents the basins of attraction for different methods applied to the function $g_2(z)$. The method CLND exhibits the largest black zone. The other methods perform well, with ZM1, AMF1, ZM3, and ZM2 being the best, followed by M2, SBM, and PM1. The methods NM, JM, and VNM display some chaotic behavior.

Figure 5 displays the basins of attraction of different methods when applied to the function $g_3(z)$. In this case, the methods ZM1, AMF1, ZM3, ZM2, and JM perform well with fewer divergent points. NM shows some chaotic behavior. The methods SBM, PM1, M2, and VNM display larger black zones, with CLND having the largest black zone.

Figure 6 illustrates the basins of attraction of different methods when applied to the function $g_4(z)$. It is observed that the methods ZM1, ZM3, AMF1, ZM2, and JM perform very well. NM follows with a small number black points. The methods M2, PM1, SBM, and VNM exhibit some chaotic behavior. CLND clearly shows the largest black zone.

Figure 7 displays the basins of attraction of different methods when applied to the function $g_5(z)$. The figure illustrates that the methods ZM1, ZM2, ZM3, and AMF1 perform very well with few black points. Following these is the method JM. The methods NM, M2, SBM, PM1, CLND, and VNM exhibit larger black zones, indicating sensitivity to the initial guess in this test case.

Figure 8 shows the basins of attraction of different methods when applied to the function $g_6(z)$. We see that the methods ZM1, ZM2, ZM3, and AMF1 perform well with the fewest black points. The method JM exhibits some chaotic behavior. The methods VNM, CLND, SBM, PM1, M2, and NM exhibit larger black zones, highlighting their sensitivity to the initial guess in this test case.

From Table 13, as well as Figures 3, 4, 5, 6, 7, and 8, we conclude that our new methods, ZM1, ZM2, and ZM3, exhibit good dynamical behavior, that is, they encompass broader sets of initial points that lead to convergence. Indeed, in terms of the number of black points, we rank the methods as follows: ZM1 as the best, followed by ZM3, then ZM2, and then, AMF1.

7. Conclusion

In this paper, we have proposed a new family of optimal fourth-order iterative methods for solving nonlinear equations. The construction of this family involves a weight function and three arbitrary parameters, allowing flexibility in choice and resulting in various specific schemes. It is shown that several well-known and recent schemes are considered as special cases within this family. Additionally, we have derived three new specific methods, namely, ZM1, ZM2, and ZM3, from this general family. The analysis of basins of attraction in the complex plane demonstrates that the newly developed methods ZM1, ZM2, and ZM3 exhibit superior stability compared to other known fourth-order methods within the same family. Furthermore, the application to various test functions and engineering problems has demonstrated the validity and applicability of the newly proposed methods compared to others.

Data Availability Statement

All of the generated data is included in the article.

Conflicts of Interest

The author declares no conflicts of interest.

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