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Oscillation of delay diffrential equations with a middle term

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Declaration

I declare that the master thesis entitled "*Oscillation of delay diffrential equations with a middle term*" is my own work, and hereby certify that unless stated, all work contained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgment is made in the text.

Manal H. Juneidi.

Signature :_________________ *Date* :______________

Dedication

To my beloved family especially my parents, and to all my friends for their endless love and support.

Manal Hamdi Juneidi

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Abstract

This thesis is concerned with the oscillation of delay differential equations with a middle term. This type of equation appears in several applications in physical and engineering systems. The existence of middle term adds extra difficulties to the study of oscillation theory for delay differential equations. Indeed, the middle term can change the qualitative behavior of solutions. Recently, considerable attention have been devoted to study the oscillation of delay differential equations with a middle term.

In this thesis we present several theorems for the oscillation of third and fourth order delay differential equations with a middle term. Linear and nonlinear equations are considered, as well as several examples are given. In general, the oscillation results of such equations are deduced by using two approaches. The first is applying a suitable Riccati type transformation to reduce the studied equation to a first order Riccati inequality. The second is applying suitable comparison principles.

Contents

Chapter 1

Introduction

Delay differential equations (DDE) are differential equations in which the highest derivative of the unknown function at a certain time depends on past values of the function and/or its lower derivatives.

The equation

$$y^{(n)}(t) + f\Big(t, y(t), y\big(\tau(t)\big)\Big) = 0,$$

where $\tau, f \in C([0,\infty), \mathbb{R}_+)$, and $\tau(t) < t$, is an n-th order DDE.

DDE have a wide range of applications in applied sciences and engineering systems. In particular, they are used to model many time processes in biology, economy, control and electrodynamics fields, see [8, 40].

Usually, solving DDE is difficult. So, the main interests in the researches is to give a characterization of the qualitative properties of the solution, such as existence and uniqueness of solution, boundedness, oscillation, asymptotic behavior, as well as existence of nonoscillatory solutions.

A solution of functional differential equation is called *oscillatory* if it has arbitrarily large number zeros on $[T_y, \infty)$, for some $T_y > 0$. Otherwise, it is said to be *nonoscillatory.* A functional differential equation is said to be *oscillatory* if all its solutions are oscillatory.

In literature, a considerable efforts have been devoted to the qualitative behavior of DDE, see [1, 3, 5, 12, 25, 30–32, 35–38, 41] and the references cited therein. Most of the studies consider a DDE with one derivative only.

In the last years, an increasing interest has been devoted to the oscillation theory of DDE with two different derivatives, the term that contains a lower derivative is called a *Middle Term (Damping)*, see [10, 11, 24, 42].

The DDE with middle terms are used to model several applied and engineering problems. For instance, the beam deflection in linear theory of elasticity is represented by the binomial fourth order equation

$$y^{(4)}(t) + q(t)y(t) = 0,$$

where $y(t)$ approximates the shape of a beam deflected from the equilibrium due to some external forces. While, the motion of a sufficiently long beam is governed by the trinomial fourth order DDE

$$y^{(4)}(t) + p(t)y'(t) + q(t)y\big(\tau(t)\big) = 0, \qquad \tau(t) \leq t,$$

where the middle term is incorporated to control the slope of the beam, see [24].

The existence of the middle term adds extra difficulties to the study of the solution, also, it may change the qualitative behavior of solutions. For instance, the second order binomial differential equation

$$y''(t) + y(t) = 0,$$

has oscillatory solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$, while the second order trinomial differential equation

$$y''(t) + 2y'(t) + y(t) = 0,$$

has nonoscillatory solutions $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$, which converge to zero as t goes to infinity.
Moreover, the fourth order differential equation

$$y^{(4)}(t) - e^{2\pi} y(t - 2\pi) = 0,$$

has a nonoscillatory solution $y(t) = e^t$, while the equation

$$y^{(4)}(t) + (1 + e^{2\pi})y'(t) - e^{2\pi} y(t - 2\pi) = 0,$$

has an oscillatory solution $y(t) = \sin t$.

In recent years, the oscillation theory of DDE with a middle term have been received intensive attention. Dzurina, Baculikova, and Jadlovska have presented many papers on the oscillation of fourth order DDE, see [9–11, 13]. Aktas,Tiryaki, and Zafer have worked on third order DDE, see [4, 16, 42]. There are also works on the oscillation results of second order DDE, see [12, 39].

In this thesis we are interested in the oscillation of third and fourth order DDE with middle term. Linear and nonlinear equations are considered. Mainly two approaches are used. In first approach *Riccati type transformation* is applied to reduce the studied equation to a first order Riccati Inequality. In the other technique we make a comparison with a lower order differential equation where the oscillation of the desired equation can be deduced from it.

In Chapter 2 we present results on the oscillation and the asymptotic behavior of fourth order linear DDEs with positive and negative middle terms, given by

$$\Big(r_3(t)\Big(r_2(t)(r_1(t)y\prime(t))'\Big)'\Big)' + \delta p(t)y'(t) + q(t)y(\tau(t)) = 0,$$

where $\delta = \pm 1$, $r_1, r_2, r_3 \in C([t_0, \infty), \mathbb{R}^+)$, $p, q, \tau \in C([t_0, \infty), \mathbb{R})$ such that $p(t) \geq 0, q(t) > 0$, and $\tau(t) \leq t$.

Chapter 3 is devoted to the oscillation results for fourth order nonlinear DDE of the form

$$y^{(4)}(t) + p(t)y^{(n)}(t) + q(t)f(y(\tau(t))) = 0,$$

where $n = 1, 2,\ f \in C(\mathbb{R}, \mathbb{R})$ and $p, q, \tau \in C([t_0, \infty), \mathbb{R})$.

In Chapter 4 we discuss the oscillation of the third order nonlinear DDE of the form

$$\Big(r_2(t)\Big(r_1(t)y'(t)\Big)'\Big)' + p(t)y'(t) + q(t)f\Big(y\Big(\tau(t)\Big)\Big) = 0,$$

where $r_1, r_2 \in C([t_0, \infty), \mathbb{R}^+)$, $p, q, \tau \in C([t_0, \infty), \mathbb{R})$ such that $p(t) \geq 0, q(t) > 0$, and $\tau(t) \leq t$, by using Riccati transformation.
In addition, we presents more oscillation results for a general form of it, given by

$$\Big(r_2(t)\Big(r_1(t)\Big(y'(t)\Big)^{\gamma}\Big)'\Big)' + p(t)\Big(y'(t)\Big)^{\gamma} + q(t)f\Big(y\big(\tau(t)\big)\Big) = 0,$$

wher γ is the quotient of two positive odd integers.

Chapter 2

Oscillation of fourth order linear DDE with a middle term

In this chapter we present oscillation theorems for the linear fourth-order delay differential equations, with positive and negative middle terms, by relating it to an auxiliary linear third-order diffrential equation.

The study of linear fourth-order differential equations has started with the vibrating rod problem of mathematical physics, see [8]. Such equations and other higher-order differential equations appeared in various fields of science and engineering systems. For example, the problem of beam deflection in linear theory of elasticity is given by the linear fourth-order equation

$$y^{(4)}(t) + q(t)y(t) = 0,$$

where t-axis represents the axis of symmetry and $y(t)$ approximates the shape of a beam, deflected from the equalibrium due to some external forces.

Also, the oscillatory movements of muscles is an important appliction of fourth order DDE, see [40].

Many authors have considered fourth order linear DDE and its oscillatory behavior, see [10, 11, 13].

2.1 Oscillation of fourth-order linear DEE with a positive first order middle term

This section is concerned with the linear fourth-order trinomial differential equation with delay argument of the form

$$\Big(r_3(t)\Big(r_2(t)(r_1(t)y'(t))'\Big)'\Big)' + p(t)y'(t) + q(t)y(\tau(t)) = 0 \tag{2.1}$$

under the assumption that the auxiliary third-order differential equation

$$\Big(r_3(t)\Big(r_2(t)z'(t)\Big)'\Big)' + \frac{p(t)}{r_1(t)}z(t) = 0 \tag{2.2}$$

is nonoscillatory.

Throughout the section, the following hypotheses will be made:

(H_1) $p, q \in C([t_0, \infty), \mathbb{R})$ such that $p(t) \geq 0$, $q(t) > 0$, for all $t \geq t_0$.

(H_2) $\tau \in C([t_0, \infty), \mathbb{R})$ such that $\tau(t) \leq t$ for all $t \geq t_0$ and $\lim\limits_{t\to\infty} \tau(t) = \infty$.

(H_3) $r_i(t) \in C([t_0, \infty), \mathbb{R}), r_i(t) > 0, \int\limits_{t_0}^{\infty} \frac{ds}{r_i(s)} = \infty$, $i = 1, 2, 3$.

(H_4) $\lim\limits_{t\to\infty} \frac{r_{3(t)}}{r_1(t)} > 0$.

A solution to (2.1) is a function $y \in C([\tau(T_y), \infty))$, $T_y \in [t_0, \infty)$ with the property $r_1y', r_2(r_1y')', r_3(r_2(r_1y')')' \in C^1([T_y, \infty))$ and satisfies (2.1) on $[T_y, \infty)$. We are intrested only in those solutions $y(t)$ of (2.1) which satisfy $\sup\{|y(t)| : t \geqslant T\} > 0$ for all $T \geq T_y$.

The main results of this section are given in theorems 2.14 and 2.19. These results are due to Dzurina et al. [10, 11].
The authors in [10] provided oscillation criteria for the solutions of the equation (2.24), then in [11] they improved the results to meet a more general form given by equation (2.1).

Classification of nonoscillatory solutions

The existence of the middle term adds difficulties to the study of the behavior of the solution. To get over it, the following operators are given to rewrite the trinomial equation (2.1) into a binomial form.

$$L_0 y(t) = y(t),$$

$$L_i y(t) = r_i (L_{i-1} y(t))', \quad i = 1, 2, 3$$

and

$$L_4 y(t) = (L_3 y(t))'.$$

Rewrite equation (2.1) using these operators, it becomes

$$L_4 y(t) + \frac{p(t)}{r_1(t)} L_1 y(t) + q(t) y(\tau(t)) = 0$$

During this chapter we assume that all the functional inequalities hold eventually, that is, they are satisfied for all t large enough.

In order to study the asymptotic properties of equation (2.1) we need to determine the sign of particular quasi-derivatives $L_i y(t)$. To do so, we make use of Kiguradze'z Lemma and its generalized form.

Lemma 2.1 (Kiguradze's lemma). *[29] Let $y^{(n)}(t) \in C^n([t_0, \infty), \mathbb{R}^+)$. If $y^{(n)}(t)$ is eventually of one sign, and not identically zero for all large t, then there exist a $t_x \geq t_0$ and an integer m, $0 \geq m \geq n$ with $m+n$ even for $y^{(n)}(t) \geq 0$, or $m+n$ odd for $y_{(n)}(t) \leq 0$ such that for every $t \geq t_x$,*

$$m > 0 \quad implies \quad y^{(k)}(t) > 0, \quad k = 0, 1, ..., m-1.$$

$$m \leq n-1 \quad implies \quad (-1)^{m+k} y^{(k)}(t) > 0, \quad k = m, m+1, ..., n-1.$$

This lemma was extended (generalized) for more general forms, see [29].

Consider the linear differential equation

$$L_n y(t) + q(t) y(t) = 0,$$

where $n \geq 2$ and

$$L_n = r_n(t) \frac{\mathrm{d}}{\mathrm{d}t} r_{n-1}(t) \frac{\mathrm{d}}{\mathrm{d}t} \quad . \, . \, . \quad \frac{\mathrm{d}}{\mathrm{d}t} r_1(t) \frac{\mathrm{d}}{\mathrm{d}t} r_0(t).$$

Let r_i, $0 \leq i \leq n$, and q be positive continuous functions on $[a, \infty)$, and consider the notaion

$$L_0(y; r_0)(t) = r_0(t) y(t),$$

$$L_j(y; r_0, \quad . \, . \, . \quad , r_j(t)) = r_j \frac{\mathrm{d}}{\mathrm{d}t} L^{j-1}(y; r_0, \quad . \, . \, . \quad , r_{j-1})(t), \quad 1 \leq j \leq n.$$

The differential operator L_n can be rewritten as

$$L_n = L_n(\ . \ ; r_0 \quad . \, . \, . \quad , r_n).$$

The domain $\mathfrak{D}(L_n)$ of L_n is the set of all functions $y \in ([a,\infty),\mathbb{R})$ such that $L_j(y;r_0, \ \ldots \ ,r_j)(t) \in C[a,\infty)$ for $0 \le j \le n$.

Lemma 2.2 (Generalized Kiguradze's lemma). *[29] Suppose*

$$\int_a^\infty \frac{1}{r_i(t)}\,\mathrm{d}t = \infty.$$

If $y \in \mathfrak{D}(L_n)$ satisfies $y(t)L_ny(t) < 0$ or $(y(t)L_ny(t) > 0)$ on $[t_0,\infty)$, then there exist an odd or (even) integer l, $0 \le l \le n$, and a $t_1 > t_0$ such that

$$y(t)L_j(y;r_0,\ \ldots\ ,r_j)(t) > 0 \quad on \quad [t_1,\infty) \quad for \quad 1 \le j \le l,$$

$$(-1)^{j-1}y(t)L_j(y;r_0,\ \ldots\ ,r_j)(t) > 0 \quad on \quad [t_1,\infty) \quad for \quad l+1 \le j \le n.$$

By Kiguradze's Lemma 2.1, when (2.1) equals

$$L_4y(t) + q(t)y(\tau(t)) = 0, \tag{2.3}$$

the set N of nonoscillatory solutions can be decomposed as

$$N = N_1 \cup N_3,$$

where for the nonoscillatory, say positive solution $y(t)$, we have

$$L_4y(t) < 0,$$

from (2.3), since $q(t)$ and $y(t)$ are positive.

Then by Generalized Kiguradze's Lemma 2.2

$$y(t) \in N_1 \Leftrightarrow L_1y(t) > 0, \quad L_2y(t) < 0, \quad \text{and} \quad L_3y(t) > 0,$$

for $l = 1$, or

$$y(t) \in N_3 \Leftrightarrow L_1 y(t) > 0, \quad L_2 y(t) > 0, \quad \text{and} \quad L_3 y(t) > 0,$$

for $l = 3$.

However, this is not applicable when $p(t)$ does not vanish identically. To treat with the presence of the middle term, we use an associated binomial form of (2.1) that deduces the result on the signs of $L_i y(t)$, $i = 1, 2, 3, 4$.

Let $z(t)$ be a positive solution for (2.2), since $p(t) \geq 0$ and $r_1(t) \geq 0$, then

$$\Big(r_3(t)\big(r_2(t)z'(t)\big)'\Big)' < 0$$

By Kiguradze's Lemma 2.2 and the assumption (H_3) we note that equation (2.2) always admits a decreasing solution $z(t)$ satisfying

$$z(t) > 0, \quad r_2 z'(t) < 0, \quad r_3(t)(r_2(t)z'(t))' > 0, \tag{2.4}$$

for $l = 0$, while increasing solutions such that

$$z(t) > 0, \quad r_2 z'(t) > 0, \quad r_3(t)(r_2(t)z'(t))' > 0, \tag{2.5}$$

for $l = 2$, exists only if (2.2) is nonoscillatory.

The formal adjoint to (2.2), which is given by

$$\left(r_2(t)\left(r_3(t)\sigma'(t)\right)'\right)' - \frac{p(t)}{r_1(t)}\sigma(t) = 0, \tag{2.6}$$

helps us to decide whether (2.2) is oscillatory or not. According to [6] the solutions of (2.2) are nonoscillatory if and only if the solutions of (2.6) are also nonoscillatory.

In the following lemma we rewrite the linear differential operator

$$L_y = \left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)' + p(t)y'(t) \tag{2.7}$$

in terms of a positive solution of (2.2).

Lemma 2.3. *Let $z(t)$ be a positive solution of (2.2). Then the operator (2.7) can be written as*

$$L_y = \left(\frac{r_3(t)}{z(t)}\left(r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\right)' + r_3(t)\left(r_2(t)z'(t)\right)'\left(\frac{r_1(t)}{z(t)}y'(t)\right)'. \tag{2.8}$$

Proof. By differentiating the term

$$\left(\frac{r_1(t)}{z(t)}y'(t)\right)' = \frac{z(t)\left(r_1(t)y'(t)\right)' - r_1(t)y'(t)z'(t)}{z^2(t)},$$

and substituting it in the right-hand side of (2.8), we get

$$\left(\frac{r_3(t)}{z(t)}\left(r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\right)' + r_3(t)\left(r_2(t)z'(t)\right)'\left(\frac{r_1(t)}{z(t)}y'(t)\right)'$$

$$\begin{aligned}
&= \left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)' - r_3(t)\left(r_2(t)z'(t)\right)'\left(\frac{r_1(t)}{z(t)}y'(t)\right)' + \\
&\qquad - \left(r_3(t)\left(r_2(t)z'(t)\right)'\right)'\frac{r_1(t)}{z(t)}y'(t) + r_3(t)\left(r_2(t)z'(t)\right)'\left(\frac{r_1(t)}{z(t)}y'(t)\right)' \\
&= \left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)' + p(t)y'(t) \\
&= L_y.
\end{aligned}$$

□

Lemma 2.4. *Let $z(t)$ be a positive solution of (2.2) and let the equation*

$$\left(\frac{r_3(t)}{z(t)}v'(t)\right)' + \left(\frac{r_3(t)(r_2(t)z'(t))'}{r_2(t)z^2(t)}\right)v(t) = 0 \tag{2.9}$$

possess a positive solution. Then the operator (2.7) can be written as

$$L_y = \frac{1}{v(t)}\left[\frac{r_3(t)v^2(t)}{z(t)}\left(\frac{r_2(t)z^2(t)}{v(t)}\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\right]' \tag{2.10}$$

Proof. Differentiate the term

$$\frac{r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'}{v(t)}$$

and substitute it in the right-hand side of (2.10), we get

$$\frac{1}{v(t)}\left[\frac{r_3(t)}{z(t)}\left(r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'v(t) - r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\frac{r_3(t)}{z(t)}v'(t)\right]'.$$

Then differentiate the terms

$$\left[\frac{r_3(t)}{z(t)}\left(r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\right] * v(t)$$

and

$$\left(r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right) * \left(\frac{r_3(t)}{z(t)}v'(t)\right),$$

we get

$$\begin{aligned}
&\left[\frac{r_3(t)}{z(t)}\left(r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\right]' + \left[r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right]'\frac{r_3(t)v'(t)}{z(t)v(t)} + \\
&- \left[r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right]'\frac{r_3(t)v'(t)}{z(t)v(t)} - \frac{r_2(t)z^2(t)}{v(t)}\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\left(\frac{r_3(t)}{z(t)}v'(t)\right)' \\
&= \left(\frac{r_3(t)}{z(t)}\left(r_2(t)z^2(t)\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\right)' - \frac{r_2(t)z^2(t)}{v(t)}\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\left(\frac{r_3(t)}{z(t)}v'(t)\right)'.
\end{aligned}$$

By (2.7) and (2.8), this becomes

$$\left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)' + p(t)y'(t) - r_3(t)\left(r_2(t)z'(t)\right)'\left(\frac{r_1(t)}{z(t)}y'(t)\right)' + \\ - \frac{r_2(t)z^2(t)}{v(t)}\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\left(\frac{r_3(t)}{z(t)}v'(t)\right)'$$

$$= \left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)' + p(t)y'(t) + \\ - \frac{r_2(t)z^2(t)}{v(t)}\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\left[\left(\frac{r_3(t)}{z(t)}v'(t)\right)' + \frac{r_3(t)\left(r_2(t)z'(t)\right)'}{r_2(t)z^2(t)}v(t)\right].$$

But $v(t)$ is a solution of (2.9), so this reduces the above equation to

$$\left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)' + p(t)y'(t),$$

this equation by (2.7) is L_y, i.e.,

$$L_y = \left(r_3(t)\left(r_2\left(t)(r_1(t)y'(t)\right)'\right)'\right)' + p(t)y'(t).$$

□

Lemma 2.3 and Lemma 2.4 allow us to write (2.1) into its binomial form

$$\frac{1}{v(t)}\left(\frac{r_3(t)v^2(t)}{z(t)}\left(\frac{r_2(t)z^2(t)}{v(t)}\left(\frac{r_1(t)}{z(t)}y'(t)\right)'\right)'\right)' + q(t)y\left(\tau(t)\right) = 0, \tag{2.11}$$

where $z(t)$ and $v(t)$ are positive solutions of (2.2) and (2.9), respectively.

Now, we will derive criterion for (2.9) to have a positive solution.

Lemma 2.5. *If $z(t)$ is a positive decreasing solution of (2.2), and $z_*(t)$ is any solution of (2.2), then*

$$v(t) = r_2(t)[z(t){z_*}'(t) - z'(t)z_*(t)] \tag{2.12}$$

is a solution of (2.9)

Proof. By substituting (2.12) in (2.9) we get

$$\left[\frac{r_3(t)}{z(t)}\Big(r_2(t)\big(z(t){z_*}'(t)-z'(t)z_*(t)\big)\Big)'\right]'+\frac{r_3(t)\big(r_2(t)z'(t)\big)'}{z^2(t)}\Big(z(t){z_*}'(t)-z'(t)z_*(t)\Big)$$

$$= \left[\frac{r_3(t)}{z(t)}\Big(r_2(t){z_*}'(t)z(t)\Big)'\right]' - \left[\frac{r_3(t)}{z(t)}\Big(r_2(t)z'(t)z_*(t)\Big)'\right]' + \frac{r_3(t)\big(r_2(t)z'(t)\big)'{z_*}'(t)}{z(t)} + \\ - \frac{r_3(t)\big(r_2(t)z'(t)\big)'z'(t)z_*(t)}{z^2(t)}$$

$$= \left[r_3(t)\Big(r_2(t){z_*}'(t)\Big)'z(t)\right]' - \left[\frac{r_3(t)}{z(t)}\Big(r_2(t)z'(t)\Big)'z_*\right]' + \frac{r_3(t)\big(r_2(t)z'(t)\big)'{z_*}'(t)}{z(t)} + \\ - \frac{r_3(t)\big(r_2(t)z'(t)\big)'z'(t)z_*(t)}{z^2(t)}$$

$$= \left[r_3(t)\Big(r_2(t){z_*}'(t)\Big)'z(t)\right]' - \left[\frac{r_3(t)}{z(t)}\Big(r_2(t)z'(t)\Big)'\right]'z_*(t) - \frac{r_3(t)\big(r_2(t)z'(t)\big)'z'(t)z_*(t)}{z^2(t)}$$

Since $z(t)$ and $z_*(t)$ are solutions to (2.2), the previous equality becomes

$$-\frac{p(t)}{r_1(t)}z_*(t) - \frac{z(t)\Big(r_3(t)\big(r_2(t)z'(t)\big)'\Big)' - r_3(t)\big(r_2(t)z'(t)\big)'z'(t)}{z^2(t)}z_*(t) + \\ - \frac{r_3(t)\big(r_2(t)z'(t)\big)'z'(t)z_*(t)}{z^2(t)}$$

$$= -\frac{p(t)}{r_1(t)} z_*(t) + \frac{\Big(r_3(t)\big(r_2(t)z'(t)\big)'\Big)'}{z^2(t)} z_*(t)$$

$$= -\frac{p(t)}{r_1(t)} z_*(t) + \frac{p(t)}{r_1(t)z(t)} z(t)z_*(t) = 0.$$

This completes the proof. □

Lemma 2.6. *Let (2.2) be nonoscillatory and $z(t)$ be its positive decreasing solution. Then (2.9) admits a nonoscillatory solution $v(t)$ such that*

$$v(t) > 0, \quad v'(t) > 0, \quad \text{and} \quad \left(\frac{r_3(t)}{z(t)} v'(t)\right)' < 0. \tag{2.13}$$

Proof. See [10]. □

Remark 2.7. [7] The condition

$$\int_{t_0}^{\infty} \frac{p(t)}{r_1(t)} \int_{t_0}^{t} \frac{1}{r_2(s)} \int_{t_0}^{s} \frac{1}{r_3(u)} \mathrm{d}u \, \mathrm{d}s \, \mathrm{d}t < \infty \tag{2.14}$$

is sufficient for (2.2) to be nonoscillatory.

Remark 2.8. The equation (2.11) is said to be in the canonical form if the following conditions are satisfied

$$\int_{t_0}^{\infty} \frac{z(s)}{r_3(s)v^2(s)} \mathrm{d}s = \infty, \tag{2.15}$$

$$\int_{t_0}^{\infty} \frac{v(s)}{r_2(s)z^2(s)} \mathrm{d}s = \infty, \tag{2.16}$$

$$\int_{t_0}^{\infty} \frac{z(s)}{r_1(s)} \mathrm{d}s = \infty. \tag{2.17}$$

Lemma 2.9. *Let (2.2) be nonoscillatory. Then there exist positive solutions $z(t)$ and $v(t)$ of (2.2) and (2.9), respectively, such that (2.15), (2.16), and (2.17) are satisfied.*

Proof. Let $z(t)$ be a positive decreasing solution of (2.2). By Lemma 2.6, there exists a positive solution $v(t)$ of (2.9).

Suppose that $v(t)$ does not meet condition (2.15).

Let

$$v_*(t) = v(t) \int_t^\infty \frac{z(s)}{r_3(s)v^2(s)} \mathrm{d}s.$$

Differentiate $v_*(t)$ and multiply it by $\dfrac{r_3(t)}{z(t)}$, we get

$$\frac{r_3(t)}{z(t)} {v_*}'(t) = -\frac{1}{v(t)} + \frac{r_3(t)}{z(t)} v'(t) \int_t^\infty \frac{z(s)}{r_3(s)v^2(s)} \mathrm{d}s.$$

Again, differentiate the previous equation, we get

$$\left(\frac{r_3(t)}{z(t)} {v_*}'(t)\right)' = \frac{v'(t)}{v^2(t)} - \frac{r_3(t)}{z(t)} v'(t) \frac{z(t)}{r_3(t)v^2(t)} + \left(\frac{r_3(t)}{z(t)} v'(t)\right)' \int_t^\infty \frac{z(s)}{r_3(s)v^2(s)} \mathrm{d}s$$

$$= \left(\frac{r_3(t)}{z(t)} v'(t)\right)' \int_t^\infty \frac{z(s)}{r_3(s)v^2(s)} \, \mathrm{d}s.$$

Since $v(t)$ is a solution of (2.9), then

$$\left(\frac{r_3(t)}{z(t)} {v_*}'(t)\right)' = -\left(\frac{r_3(t)\Big(r_2(t)z'(t)\Big)'}{r_2(t)z^2(t)}\right) v(t) \int_t^\infty \frac{z(s)}{r_3(s)v^2(s)} \mathrm{d}s$$

$$= -\left(\frac{r_3(t)\Big(r_2(t)z'(t)\Big)'}{r_2(t)z^2(t)}\right) v_*(t).$$

This means that $v_*(t)$ is also a positive solution of (2.9).

Now, let

$$V(t) = \int_t^\infty \frac{z(s)}{r_3(s)v^2(s)} \, \mathrm{d}s$$

By our assumption, $V(t)$ converges, then

$$\lim_{t\to\infty} V(t) = 0$$

and

$$\begin{aligned}
\int_{t_0}^{\infty} \frac{z(t)}{r_3(t)v_*^2(t)}\,\mathrm{d}t &= \int_{t}^{\infty} \frac{z(t)}{r_3(t)v^2(t)} \left(\int_{t}^{\infty} \frac{z(s)}{r_3(s)v^2(s)}\,\mathrm{d}s \right)^{-2} \mathrm{d}t \\
&= -\int_{t_0}^{\infty} \frac{V'(t)}{V^2(t)}\,\mathrm{d}t \\
&= \lim_{t\to\infty} \left(\frac{1}{V(t)} - \frac{1}{V(t_0)} \right) = \infty,
\end{aligned}$$

i.e., $v_*(t)$ satisfies (2.15).

By (H_4) and (2.13), there exist a positive number m such that $\dfrac{r_3(t)}{r_1(t)} v_*^2(t) > m$, and from the previous equality, we get

$$\int_{t_0}^{\infty} \frac{z(t)}{r_1(t)}\,\mathrm{d}t = \int_{t_0}^{\infty} \frac{z(t)}{r_3(t)v_*^2(t)} \frac{r_3(t)}{r_1(t)} v_*^2(t)\,\mathrm{d}t > m \int_{t_0}^{\infty} \frac{z(t)}{r_3(t)v_*^2(t)}\,\mathrm{d}t = \infty,$$

i.e., $v_*(t)$ satisfies (2.17).

Since $z(t)$ is positive decreasing solution of (2.2), then $z(t) < c$, where c is positive. Also, $v_*(t) > d$, where d is positive, since $v(t)$ is a positive increasing solution of (2.9). Now, from (H_3)

$$\int_{t_0}^{\infty} \frac{v_*(t)}{r_2(t)z^2(t)}\,\mathrm{d}t > \frac{d}{c^2} \int_{t_0}^{\infty} \frac{1}{r_2(t)}\,\mathrm{d}t = \infty,$$

i.e., $v_*(t)$ satisfies (2.16).

$\square$

Remark 2.10. If $z(t)$ is a solution of (2.2) such that condition (2.17) holds, we can release assumption (H_4).

The canonical representation of (2.11) is guaranteed by Lemma 2.9.

Assume that $y(t)$ is a positive solution of (2.1), and (2.2) is nonoscillatory such that $z(t)$ and $v(t)$ are the needed solutions of (2.2) and (2.9), respectively, then either

$$y'(t) > 0, \quad \left(\frac{r_1(t)}{z(t)}y'(t)\right)' < 0, \quad \text{and} \quad \left(\frac{r_2(t)z^2(t)}{v(t)}\big(\frac{r_1(t)}{z(t)}y'(t)\big)'\right)' > 0,$$

or

$$y'(t) > 0, \quad \left(\frac{r_1(t)}{z(t)}y'(t)\right)' > 0, \quad \text{and} \quad \left(\frac{r_2(t)z^2(t)}{v(t)}\big(\frac{r_1(t)}{z(t)}y'(t)\big)'\right)' > 0,$$

eventually.

Now, we could state the final result on the sign properties of possible nonoscillatory solutions for (2.1).

Lemma 2.11. *Let* (2.2) *be nonoscillatory. Then any positive solution* $y(t)$ *of* (2.1) *satisfies either*

$$y(t) \in N_1 \Leftrightarrow L_1y(t) > 0, \quad L_2y(t) < 0, \quad L_3y(t) > 0, \quad L_4y(t) < 0,$$

$$y(t) \in N_3 \Leftrightarrow L_1y(t) > 0, \quad L_2y(t) > 0, \quad L_3y(t) > 0, \quad L_4y(t) < 0,$$

eventually.

Proof. Let $y(t)$ be an eventually positive solution of (2.1). Since $L_1y(t) > 0$, then $y'(t) > 0$. From (2.1) we conclude that $L_4y(t) < 0$.
By Generalized Kiguradze's Lemma 2.2, $y(t) \in N_1$ when $l = 1$, or $y(t) \in N_3$ when $l = 3$. $\square$

In order to establish the main results of this section, the following functions are defined as

$$J_1(t) = \int_{t_1}^{t} \frac{\mathrm{d}s}{r_3(s)},$$

$$J_k(t) = \int_{t_1}^{t} \frac{1}{r_{4-k}(s)} J_{k-1}(s)\mathrm{d}s, \qquad k = 2, 3,$$

$$R_i(t) = \int_{t_1}^{t} \frac{1}{r_i(s)}, \qquad i = 1, 2,$$

$$I_2(t) = \int_{t_1}^{t} \frac{1}{r_1(s)} R_2(s)\mathrm{d}s,$$

where t_1 is sufficiently large.

Theorem 2.12. *Assume that* (2.2) *is nonoscillatory. Let* $y(t)$ *be a positive solution of* (2.1)*. If*
(i) $y(t) \in \mathrm{N}_1$, *then* $\dfrac{y(t)}{R_1(t)}$ *is decreasing.*
(ii) $y(t) \in \mathrm{N}_3$, *then* $\dfrac{y(t)}{J_3(t)}$ *is decreasing and* $L_1 y(t) \geq J_2(t) L_3 y(t)$.

Proof. Suppose that $y(t)$ is a positive solution of (2.1) such that $y(t) \in N_1$. Since $y(t)$ is positive and increasing, by integrating $y'(t)$ from t_1 to t, we get

$$y(t) > y(t) - y(t_1) = \int_{t_1}^{t} \frac{L_1 y(s)}{r_1(s)}\, \mathrm{d}s$$

$$\geq L_1 y(t) \int_{t_1}^{t} \frac{1}{r_1(s)}\, \mathrm{d}s = L_1 y(t) R_1(t),$$

i.e.,

$$y(t) > r_1(t) y'(t) R_1(t),$$

then

$$\frac{y'(t)R_1(t) - \dfrac{y(t)}{r_1(t)}}{R_1^2(t)} = \left(\frac{y(t)}{R_1(t)}\right)' < 0,$$

i.e., $\dfrac{y(t)}{R_1(t)}$ is decreasing.

Now, suppose $y(t) \in N_3$. Since

$$\Big(L_2y(t)\Big)' = \frac{L_3y(t)}{r_3(t)},$$

then, by integrating this equation from t_1 to t, we get

$$L_2y(t) - L_2y(t_1) = \int\limits_{t_1}^{t} \frac{L_3y(s)}{r_3(s)}\, \mathrm{d}s.$$

But $L_2y(t) > 0$, then

$$\begin{aligned} L_2y(t) &> \int\limits_{t_1}^{t} \frac{L_3y(s)}{r_3(s)}\, \mathrm{d}s \\ &> L_3y(t) \int\limits_{t_1}^{t} \frac{1}{r_3(s)}\, \mathrm{d}s \\ &= L_3y(t)J_1(t) \\ &= r_3(t)L_2'y(t)J_1(t) \end{aligned}$$

then

$$\frac{\Big(L_2y(t)\Big)'J_1(t) - \dfrac{L_2y(t)}{r_3(t)}}{J_1^2(t)} = \left(\frac{L_2y(t)}{J_1(t)}\right)' < 0,$$

i.e., $\dfrac{L_2y(t)}{J_1(t)}$ is decreasing.

Also,

$$\Big(L_1y(t)\Big)' = \frac{L_2y(t)}{r_2(t)}.$$

By integrating the previous equation from t_1 to t, we get

$$L_1y(t) - L_1y(t_1) = \int\limits_{t_1}^{t} \frac{L_2y(s)J_1(s)}{J_1(s)r_2(s)}\, \mathrm{d}s.$$

Since $L_1y(t) > 0$, then

$$L_1y(t) > \frac{L_2y(t)J_2(t)}{J_1(t)}$$

but $L_2y(t) > L_3y(t)J_1(t)$, then $L_1y(t) \geq J_2(t)L_3y(t)$ and

$$L_1y(t)J_1(t) > r_2(t)\Big(L_1y(t)\Big)'J_2(t),$$

i.e.,

$$\frac{\Big(L_1y(t)\Big)'J_2(t) - \dfrac{1}{r_2(t)}L_1y(t)J_1(t)}{J_2^2(t)} = \left(\frac{L_1y(t)}{J_2(t)}\right)' < 0$$

Also,

$$y(t) - y(t_1) = \int\limits_{t_1}^{t} \frac{r_1(s)y'(s)J_2(s)}{r_1(s)J_2(s)}\, \mathrm{d}s.$$

But $y(t) > 0$ and increasing, then

$$\begin{aligned} y(t) &> \frac{L_1y(t)J_3(t)}{J_2(t)}, \\ &> \frac{r_1(t)y'(t)J_3(t)}{J_2(t)}, \end{aligned}$$

i.e.,

$$\frac{y'(t)J_3(t) - \dfrac{1}{r_1(t)}y(t)J_2(t)}{J_3^2(t)} = \left(\frac{y(t)}{J_3(t)}\right)' < 0.$$

So, we conclude that $\dfrac{y(t)}{J_3(t)}$ is decreasing. Thus, the proof is complete. $\square$

Define the function

$$Q(t)=\frac{1}{r_1(t)}\Bigg(\int_t^{\tau^{-1}(t)}\frac{1}{r_2(s)}\int_s^{\tau^{-1}(t)}\frac{1}{r_3(v)}\mathrm{d}v\mathrm{d}s\int_{\tau^{-1}(t)}^{\infty}q(s)\mathrm{d}s\Bigg),$$

where τ^{-1} is the inverse function of τ.

Theorem 2.13. *Assume that* (2.2) *is nonoscillatory. Let $y(t)$ be a positive solution of* (2.1)*. If*
(i) $y(t)\in N_1$, then $y'(t)\geq Q(t)y(t)$.

(ii) $y(t)\in N_3$, then $y'(t)\geq \dfrac{1}{r_1(t)R_1(t)}y(t)$.

Proof. Let $y(t)$ be a positive solution of (2.1) such that $y(t)\in N_1$. Since $L_2y(t)<0$, then for any $u>t$

$$\begin{aligned}-L_2y(t)&\geq L_2y(u)-L_2y(t)\\&=\int_t^u\frac{L_3y(s)}{r_3(s)}\,\mathrm{d}s,\end{aligned}$$

i.e.,

$$-L_2y(t)\geq L_3y(u)\int_t^u\frac{1}{r_3(s)}\,\mathrm{d}s.$$

Multiplying the previous inequality by $\dfrac{1}{r_2(t)}$ and then integrating from t to u, we get

$$\begin{aligned}L_1y(t)-L_1y(u)&\geq\int_t^u\frac{L_3(u)}{r_2(s)}\int_s^u\frac{1}{r_3(v)}\,\mathrm{d}v\mathrm{d}s\\&\geq L_3(u)\int_t^u\frac{1}{r_2(s)}\int_s^u\frac{1}{r_3(v)}\,\mathrm{d}v\mathrm{d}s.\end{aligned}$$

But $L_1y(t)>0$, then

$$L_1y(t)\geq L_3(u)\int_t^u\frac{1}{r_2(s)}\int_s^u\frac{1}{r_3(v)}\,\mathrm{d}v\mathrm{d}s.$$

Also, integrate (2.1) from u to t, we get

$$L_3 y(u) \geq \int_u^\infty p(s)y'(s)\ \mathrm{d}s + \int_u^\infty q(s)y(\tau(s))\ \mathrm{d}s$$

$$\geq y(\tau(u)) \int_u^\infty q(s)\ \mathrm{d}s.$$

From the previous two inequalities and by setting $u = \tau^{-1}(t)$ and letting $t \to \infty$, we get

$$y'(t) \geq \frac{y(t)}{r_1(t)} \int_t^{\tau^{-1}(t)} \frac{1}{r_2(s)} \int_s^{\tau^{-1}(t)} \frac{1}{r_3(v)}\ \mathrm{d}v\mathrm{d}s \int_{\tau^{-1}(t)}^\infty q(s)\ \mathrm{d}s,$$

i.e., $y'(t) \geq Q(t)y(t)$.

Now, suppose that $y(t) \in N_3$ and since $L_1 y(t) \to \infty$ as $t \to \infty$, then

$$y(t) = y(t_1) + \int_{t_1}^t \frac{L_1 y(s)}{r_1(s)}\ \mathrm{d}s$$

$$\leq y(t_1) + L_1 y(t) \int_{t_1}^t \frac{1}{r_1(s)}\ \mathrm{d}s$$

$$= L_1 y(t) \int_{t_0}^t \frac{1}{r_1(s)}\ \mathrm{d}s + y(t_1) - L_1 y(t) \int_{t_0}^{t_1} \frac{1}{r_1(s)}\ \mathrm{d}s$$

$$\leq L_1 y(t) \int_{t_0}^t \frac{1}{r_1(s)}\ \mathrm{d}s$$

$$= r_1(t)y'(t)R_1(t),$$

i.e., $y'(t) \geq \dfrac{1}{r_1(t)R_1(t)} y(t)$. $\square$

Now, we apply these results to obtain a criterion for (2.2). We denote

$$\begin{aligned} Q_1(t) &= p(t)Q(t) + q(t)\frac{R_1(\tau(t))}{R_1(t)}. \\ Q_2(t) &= \frac{p(t)}{r_1(t)R_1(t)} + q(t)\frac{J_3(\tau(t))}{J_3(t)}. \end{aligned}$$

Theorem 2.14. *Assume that* (2.2) *is nonoscillatory and there exists a positive continuously differentiable function* $\rho(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_1}^{\infty} \left(\frac{\rho(v)}{r_2(v)} \int_{v}^{\infty} \frac{1}{r_3(u)} \int_{u}^{\infty} Q_1(s)\mathrm{d}s\,\mathrm{d}u - \frac{r_1(v)(\rho'(v))^2}{4\rho(v)} \right)\mathrm{d}v = \infty, \tag{2.18}$$

and a positive continuously differentiable function $\gamma(t)$ *such that*

$$\limsup_{t\to\infty} \int_{t_1}^{\infty} \left(Q_2(v)\gamma(s) - \frac{r_1(s)(\gamma'(s))^2}{4\gamma(s)J_2(s)} \right)\mathrm{d}s = \infty. \tag{2.19}$$

Then (2.1) *is oscillatory.*

Proof. Let $y(t)$ be a positive solution of (2.1). Then we have 2 cases.

Case 1:- $y(t) \in N_1$.

By Theorems 2.12 and 2.13, since $\tau(t) \le t$, then

$$\frac{y\big(\tau(t)\big)}{R_1\big(\tau(t)\big)} \ge \frac{y(t)}{R_1(t)},$$

and

$$y'(t) \ge Q(t)y(t).$$

By substituting these inequalities in (2.1), we get

$$0 = L_4 y(t) + p(t)y'(t) + q(t)y(\tau(t))$$

$$\geq L_4y(t) + p(t)Q(t)y(t) + q(t)\frac{R_1(t)}{R_1(\tau(t))}y(t)$$

$$= L_4y(t) + Q_1(t)y(t).$$

By integrating

$$L_4y(t) + Q_1(t)y(t) \leq 0$$

from t to ∞, we get

$$L_3y(t) \geq \int_t^\infty Q_1(s)y(s)\ \mathrm{d}s$$

$$\geq y(t)\int_t^\infty Q_1(s)\ \mathrm{d}s.$$

Again, by integrating from t to ∞, we get

$$L_2y(t) + y(t)\int_t^\infty \frac{1}{r_3(u)}\int_u^\infty Q_1(s)\ \mathrm{d}s\mathrm{d}u \leq 0. \tag{2.20}$$

Now, let

$$\omega(t) = \rho(t)\frac{L_1y(t)}{y(t)} > 0.$$

By differentiating and using (2.20), we get

$$\omega'(t) = \rho(t)\frac{y(t)\frac{L_2y(t)}{r_2(t)} - L_1y(t)y'(t)}{y^2(t)} + \rho'(t)\frac{L_1y(t)}{y(t)}$$

$$= \frac{\rho(t)}{r_2(t)}\frac{L_2y(t)}{y(t)} - \rho(t)\frac{\omega(t)}{\rho(t)}\frac{\omega(t)}{r_1(t)\rho(t)} + \frac{\rho'(t)}{\rho(t)}\omega(t)$$

$$\leq -\frac{\rho(t)}{r_2(t)}\int_t^\infty \frac{1}{r_3(u)}\int_u^\infty Q_1(s)\ \mathrm{d}s\mathrm{d}u - \frac{1}{r_1(t)\rho(t)}\omega^2(t) + \frac{\rho'(t)}{\rho(t)}\omega(t).$$

By completing the square, we get

$$\omega'(t) \leq -\frac{\rho(t)}{r_2(t)} \int_t^{\infty} \frac{1}{r_3(u)} \int_u^{\infty} Q_1(s) \, \mathrm{d}s\mathrm{d}u + \frac{r_1(t)\big(\rho'(t)\big)^2}{4\rho(t)}.$$

The integration of this inequality from t_1 to t yields

$$\int_{t_1}^{t} \left[\frac{\rho(v)}{r_2(v)} \int_v^{\infty} \frac{1}{r_3(u)} \int_u^{\infty} Q_1(s) \, \mathrm{d}s\mathrm{d}u - \frac{r_1(v)\big(\rho'(v)\big)^2}{4\rho(v)} \right] \mathrm{d}v \leq \omega(t_1),$$

which contradicts the assumption (2.18) as $t \to \infty$.

Case 2:- $y(t) \in N_3$.

By Theorems 2.13 and 2.12, since $\tau(t) \leq t$, then

$$\frac{y(\tau(t))}{J_3(\tau(t))} \geq \frac{y(t)}{J_3(t)},$$

$$L_1 y(t) \geq J_2(t) L_3 y(t),$$

and

$$y'(t) \geq \frac{y(t)}{r_1(t) R_1(t)}.$$

By substituting these inequalities in (2.1), we get

$$\begin{aligned} 0 &= L_4 y(t) + p(t) y'(t) + q(t) y(\tau(t)) \\ &\geq L_4 y(t) + \frac{p(t) y(t)}{r_1(t) R_1(t)} + \frac{q(t) J_3(\tau(t))}{J_3(t)} y(t) \\ &= L_4 y(t) + Q_2(t) y(t). \end{aligned}$$

Now, let

$$\omega_*(t) = \gamma(t) \frac{L_3 y(t)}{y(t)} > 0,$$

then

$$\begin{aligned}
\omega_*'(t) &= \gamma(t)\frac{y(t)L_4y(t) - L_3y(t)y'(t)}{y^2(t)} + \gamma'(t)\frac{L_3y(t)}{y(t)} \\
&= \gamma(t)\frac{L_4y(t)}{y(t)} - \gamma(t)\frac{L_3y(t)}{y(t)}\frac{y'(t)}{y(t)} + \frac{\gamma'(t)}{\gamma(t)}\omega_*(t) \\
&\leq -\gamma(t)Q_2(t) - \omega_*(t)\frac{J_2(t)}{r_1(t)}\frac{L_3y(t)}{y(t)} + \frac{\gamma'(t)}{\gamma(t)}\omega_*(t) \\
&= -\gamma(t)Q_2(t) - \frac{J_2(t)}{r_1(t)\gamma(t)}\omega_*^2(t) + \frac{\gamma'(t)}{\gamma(t)}\omega_*(t).
\end{aligned}$$

By completing the square, we get

$$\omega_*'(t) \leq -\gamma(t)Q_2(t) + \frac{r_1(t)\big(\gamma'(t)\big)^2}{4\gamma(t)J_2(t)}. \tag{2.21}$$

By integrating the last inequality from t_1 to t, we get

$$\int_{t_1}^{t}\left[\gamma(s)Q_2(s) - \frac{r_1(s)\big(\gamma'(s)\big)^2}{4\gamma(s)J_2(s)}\right]\mathrm{d}s \leq \omega_*(t_1),$$

which contradicts (2.19) as $t \to \infty$. The proof is complete. $\square$

Remark 2.15. Setting

$$\rho(t) = R_1(t) \quad \textit{and} \quad \gamma(t) = J_3(t)$$

in Theorem 2.14, then (2.1) is oscillatory if the conditions

$$\lim_{t\to\infty} sup \int_{t_1}^{\infty}\left(\frac{R_1(v)}{r_2(v)}\int_v^{\infty}\frac{1}{r_3(u)}\int_u^{\infty}Q_1(s)\mathrm{d}s\,\mathrm{d}u - \frac{1}{4r_1(v)R_1(v)}\right)\mathrm{d}v = \infty \tag{2.22}$$

and

$$\lim_{t\to\infty} sup \int_{t_1}^{\infty}\left(\frac{p(s)J_3(s)}{r_1(s)R_1(s)} + q(s)J_3(\tau(s)) - \frac{J_2(s)}{4r_1(s)J_3(s)}\right)\mathrm{d}s = \infty \tag{2.23}$$

are satisfied.

Remark 2.16. In case $r_i(t) = 1$ for $i = 1, 2, 3$, equation (2.1) becomes

$$y^{(4)}(t) + p(t)y'(t) + q(t)y(\tau(t)) = 0, \tag{2.24}$$

and (2.2) becomes

$$z'''(t) + p(t)z(t) = 0. \tag{2.25}$$

and (2.9) becomes

$$\left(\frac{1}{z(t)}v'(t)\right)' + \frac{z''(t)}{z^2(t)}v(t) = 0. \tag{2.26}$$

This case is widely studied in Dzurina et al. [11]. Next we only highlight the results that only applicable for the special case form (2.24).

Lemma 2.17. *[26] Assume that*

$$\limsup_{t\to\infty} t^3 p(t) < \frac{2}{3\sqrt{3}}, \tag{2.27}$$

then all solutions of (2.25) *are nonoscillatory.*

The following lemma give criterion for the oscillation of first order DDE's.

Lemma 2.18. *[28] Consider the DDE*

$$x'(t) + p(t)x(t) = 0, \qquad t \geq t_0$$

where p is a function of nonnegative real numbers, and τ is a function of positive real numbers such that

$$\tau(t) \leq t \qquad \textit{and} \quad \lim_{t\to\infty} \tau(t) = \infty.$$

This equation is oscillatory when

$$\liminf_{t\to\infty} \int\limits_{\tau(t)}^{t} p(s)\, \mathrm{d}s > \frac{1}{e}. \tag{2.28}$$

To obtain oscillation of (2.24), we eliminate the cases where it has nonoscillatory solutions.

Notation

Let

$$Q_1(t) = \left(\frac{v(t)}{z^2(t)} \int_{t_1}^{\tau(t)} z(s) \, \mathrm{d}s \right) \left(\int_t^{\infty} \frac{z(u)}{v^2(u)} \int_u^{\infty} v(s)q(s) \, \mathrm{d}s\mathrm{d}u \right) \tag{2.29}$$

and

$$Q_2(t) = v(t)q(t) \int_{t_1}^{\tau(t)} z(s_1) \int_{t_1}^{s_1} \frac{v(u)}{z^2(u)} \int_{t_1}^{u} \frac{z(s)}{v^2(s)} \, \mathrm{d}s\mathrm{d}u\mathrm{d}s_1. \tag{2.30}$$

Theorem 2.19. *Let $\tau(t) \in C^1([t_0, \infty))$, $\tau'(t) > 0$. Assume that both first-order delay differential equations*

$$x'(t) + Q_1(t)x(\tau(t)) = 0 \tag{2.31}$$

and

$$x'(t) + Q_2(t)x(\tau(t)) = 0. \tag{2.32}$$

are oscillatory. Then (2.24) is oscillatory.

Proof. See [11]. □

The next corollary follows directly from Lemma 2.18.

Corollary 2.20. *Let $\tau(t) \in C^1([t_0, \infty))$, $\tau'(t) > 0$. Assume that for $i = 1, 2$*

$$\liminf_{t \to \infty} \int_{\tau(t)}^{t} Q_i(s)\mathrm{d}s > \frac{1}{e} \tag{2.33}$$

hold. Then (2.24) is oscillatory.

Corollary 2.21. *Let $\tau(t) \in C^1([t_0, \infty))$, $\tau'(t) > 0$. Assume that for $i = 1, 2$*

$$\liminf_{t\to\infty} \int_t^\infty Q_i(s)\mathrm{d}s = \infty \tag{2.34}$$

holds. Then (2.24) is oscillatory.

Example 2.1. *Consider the fourth-order DEE*

$$y^{(4)}(t) + \frac{0.370875}{t^3}y'(t) + \frac{a}{t^4}y(\lambda t) = 0, \qquad a > 0, \qquad \lambda \in (0, 1), \qquad t \geq 1. \tag{2.35}$$

where

$$p(t) = \frac{0.231}{t^3}, \qquad q(t) = \frac{a}{t^4} \qquad \textit{and} \qquad \tau(t) = \lambda t$$

are positive and continuous on $(0, \infty)$, and

$$\tau'(t) = \lambda > 0, \qquad \lambda t < t, \qquad \textit{and} \qquad \lim_{t\to\infty} \lambda t = \infty.$$

Its auxiliary equation is

$$z'''(t) + \frac{0.370875}{t^3}z(t) = 0 \tag{2.36}$$

with positive solution $z(t) = t^{-0.15}$.

The equation (2.9) becomes

$$\left(t^{0.15}\, v'(t)\right)' + \frac{0.1725}{t^{1.85}}v(t) = 0 \tag{2.37}$$

with positive solution $v(t) = t^{\alpha}$, where $\alpha = \frac{17-\sqrt{13}}{40}$.

From equations (2.29) and (2.30), we get

$$Q_1(t) = \frac{a\lambda^{0.85}}{0.85(3-\alpha)(2.15+\alpha)t} - \frac{K}{t^{1.85}}, \quad K \in \mathbb{R} \tag{2.38}$$

and

$$Q_2(t) = \frac{a\lambda^{3-\alpha}}{(3-\alpha)(2.15-\alpha)(0.85-2\alpha)t} - \frac{K_1}{t^{1.85-2\alpha}} - \frac{K_2}{t^{3.15-\alpha}} - \frac{K_2}{t^{4-\alpha}}, \quad K_i \in \mathbb{R}, \tag{2.39}$$

When

$$a\lambda^{0.85} \ln \frac{1}{\lambda} > \frac{0.85(3-\alpha)(2.15+\alpha)}{e} \tag{2.40}$$

and

$$a\lambda^{3-\alpha} \ln \frac{1}{\lambda} > \frac{(3-\alpha)(2.15-\alpha)(0.85-2\alpha)}{e} \tag{2.41}$$

conditions (2.33) *and* (2.34) *are satisfied. By the last corollary (2.35) is oscillatory. When* $\lambda = 0.5$ *and* $a = 6$, (2.35) *is oscillatory.*

Example 2.2. *Let*

$$\left(t^{1/3}\left(t^{2/3}y'(t)\right)''\right)' + ay'(t) + \frac{b}{t^3}y(\lambda t) = 0, \tag{2.42}$$

here we have

$$r_1(t) = t^{2/3}, \qquad r_2(t) = 1, \qquad r_3(t) = t^{1/3},$$

and

$$p(t) = a, \qquad q(t) = \frac{b}{t^3}, \qquad \tau(t) = \lambda t,$$

where $a = 0.19433$, $b > 0$, *and* $\lambda \in (0,1)$.

Its auxiliary equation is given by

$$\left(t^{1/3}z''(t)\right)' + \frac{a}{t^{2/3}}z(t) = 0,$$

has a solution $z(t) = t^{-0.1}$, *i.e, it is nonoscillatory.*

Here we have

$$Q(t) = \frac{b}{2}\lambda^{1/3}\left(\frac{2}{5} - \lambda + \frac{3}{5}\lambda^{5/3}\right)t^{-1},$$

$$R_1(t) = 3t^{1/3},$$

$$J_1(t) = \frac{3}{2}t^{2/3},$$

$$J_2(t) = \frac{9}{10}t^{5/3},$$

$$J_3(t) = \frac{9}{20}t^{2},$$

and

$$Q_1(t) = \left[\frac{ab}{2}\lambda^{1/3}\left(\frac{2}{5} - \lambda + \frac{3}{5}\lambda^{5/3}\right)\right]t^{-3}.$$

The conditions (2.22) *and* (2.23) *are satisfied when*

$$b\lambda^{1/3}\left[a\left(\frac{4}{5} - 2\lambda + \frac{6}{5}\lambda^{5/3}\right) + 2\right] > \frac{4}{27},$$

and

$$a + b\lambda^2 > \frac{10}{9},$$

respectively. By Remark 2.15, the equation (2.42) *is oscillatory when these two inequalities are satisfied simultaneously. For* $\lambda = 0.25$ *and* $b = 4$, (2.42) *is oscillatory.*

2.2 Oscillation of fourth-order linear DDE with a negative first-order middle term

In this section we study the oscillation of the fourth order linear DDE with a negative middle term of the form

$$y^{(4)}(t) - p(t)y'(t) + q(t)y(\tau(t)) = 0, \tag{2.43}$$

under the assumption that all solutions of the auxiliary third order differential equation

$$z'''(t) - p(t)z(t) = 0, \tag{2.44}$$

are oscillatory.

Throughout this section, the following hypothesis are made:

(H_1) $p, q, \in C([t_0, \infty], \mathbb{R}^+)$ for all $t \geq t_0$.

(H_2) $\tau \in C^1([t_0, \infty], \mathbb{R})$ such that $\tau(t) \leq t$, $\tau'(t) \geq 0$ for all $t \geq t_0$, and $\lim_{t\to\infty} \tau(t) = \infty$.

A function $y(t)$ is a solution of (2.43) if $y(t) \in ([t_0, \infty), \mathbb{R})$, 4-times continuously differentiable on $[t_0, \infty)$, and satisfies (2.43) for $t \geq t_0$. We are only interested in solutions satisfy $\sup\{y(t) : t \geq T\} > 0$ for all $T \geq t_0$.

The results of this section are due to Dzurina et al. [13] which is an extention to [10].

As we did in the previous section, we rewrite (2.43) into its equivalent binomial form by using positive solutions of (2.44) and its associated second order differential equation

$$\left(\frac{1}{z(t)}v'(t)\right)' + \frac{z''(t)}{z^2(t)}v(t) = 0. \tag{2.45}$$

Consider the linear differential operator

$$Ly(t) = y^{(4)}(t) - p(t)y'(t),$$

the next lemma allows us to rewrite $Ly(t)$ in terms of positive solutions $z(t)$ and $v(t)$ of (2.44) and (2.45), respectively.

Lemma 2.22. *Let $z(t)$ and $v(t)$ be positive solutions of equations (2.44) and (2.45), respectively. Then, (2.43) can be written as*

$$\left(\frac{v^2(t)}{z(t)}\left(\frac{z^2(t)}{v(t)}\left(\frac{1}{z(t)}y'(t)\right)'\right)'\right)' + v(t)q(t)y(\tau(t)) = 0. \tag{2.46}$$

Proof. Similar to the proof of Lemma 2.3 and Lemma 2.4, we can show that

$$Ly(t) = \frac{1}{v(t)}\left(\frac{v^2(t)}{z(t)}\left(\frac{z^2(t)}{v(t)}\left(\frac{1}{z(t)}y'(t)\right)'\right)'\right)',$$

and the proof is complete. □

The equation (2.46) is in canonical form, see [43], if

$$\int_{t_0}^{\infty} z(t)\ \mathrm{d}t = \infty, \tag{2.47}$$

$$\int_{t_0}^{\infty} \frac{z(t)}{v^2(t)}\ \mathrm{d}t = \infty, \tag{2.48}$$

$$\int_{t_0}^{\infty} \frac{v(t)}{z^2(t)}\ \mathrm{d}t = \infty. \tag{2.49}$$

To guarantee the existence of positive solutions of (2.44) and (2.45) such that (2.46) is in canonical form, we will introduce the sufficient conditions to do so .

By Kiguradze's Lemma 2.1, the set N of nonoscillatory solutions, say positive, of (2.44) can be decomposed as

$$N = N_1 \cup N_3,$$

where the solution

$$z(t) \in N_1 \iff z(t) > 0, \quad z' > 0, \quad z''(t) < 0, \quad \text{and} \quad z'''(t) > 0,$$

is said to be of degree 1, while the solution

$$z(t) \in N_3 \iff z(t) > 0, z' > 0, \quad z''(t) > 0, \quad \textit{and} \quad z'''(t) > 0,$$

is of degree 3.

Lemma 2.23. *[26] assume that*

$$\lim_{t\to\infty} \sup t^3 p(t) < \frac{2}{3\sqrt{3}}. \tag{2.50}$$

Then all solutions of (2.44) *are nonoscillatory and, moreover,* (2.44) *possesses a couple of linearly independent solutions of degree* 1.

It is acceptable to only work with $z(t) \in N_1$, so, we always assume that (2.50) is satisfied. By Kigurdaze's lemma 2.1, the equation (2.45) always possesses positive solutions

$$v(t) \in N_0 \iff v(t) > 0, \quad v'(t) > 0, \quad \left(\frac{v'(t)}{z(t)}\right)' > 0,$$
$$v(t) \in N_2 \iff v(t) > 0, \quad v'(t) < 0, \quad \left(\frac{v'(t)}{z(t)}\right)' > 0.$$

We will only consider $v(t) \in N_0$, and we say $v(t)$ is associated to $z(t)$.

Lemma 2.24. *Let $z(t) \in N_1$, be a positive solution of* (2.44) *and let $v(t)$ be associated to $z(t)$. Then, both* (2.47) *and* (2.48) *are satisfied.*

Proof. Since $z(t) \in N_1$, then $z(t) > 0$ and $z'(t) > 0$, so (2.47) is satisfied. Also $v(t) > 0$ and $v'(t) > 0$, implies (2.48). □

However, when $z(t) \in N_1$ and $v(t)$ is associated to $z(t)$, the condition (2.49) fails. As a result, we look for another solution $z_*(t) \in N_1$ of (2.44) such that condition (2.49) is satisfied.

Lemma 2.25. *Assume that* $z(t) \in N_1$ *is a positive solution of* (2.44) *with associated function* $v(t)$ *such that*

$$\int_{t_0}^{\infty} \frac{v(t)}{z^2(t)}\,\mathrm{d}t < \infty.$$

Then,

$$z_*(t) = z(t) \int_{t}^{\infty} \frac{v(s)}{z^2(s)}\,\mathrm{d}s \tag{2.51}$$

is another solution of degree 1 *of* (2.44)*;* $v(t)$ *is also associated to* $z_*(t)$*; for* $z_*(t)$ *and* $v(t)$*, the condition* (2.49) *is satisfied.*

Proof. By direct computation

$$z_*'(t) = z'(t) \int_{t}^{\infty} \frac{v(s)}{z^2(s)}\,\mathrm{d}s - \frac{v(t)}{z(t)},$$

$$z_*''(t) = z''(t) \int_{t}^{\infty} \frac{v(s)}{z^2(t)}\,\mathrm{d}s - \frac{v'(t)}{z(t)}, \tag{2.52}$$

$$z_*'''(t) = z'''(t) \int_{t}^{\infty} \frac{v(s)}{z^2(t)}\,\mathrm{d}s - \left(\left(\frac{v'(t)}{z(t)}\right)' + \frac{z''(t)v(t)}{z^2(t)}\right),$$

by (2.45)

$$z_*'''(t) = z'''(t) \int_{t}^{\infty} \frac{v(s)}{z^2(t)}\,\mathrm{d}s,$$

and by (2.44)

$$z_*'''(t) = p(t)z(t) \int_{t}^{\infty} \frac{v(s)}{z^2(s)}\,\mathrm{d}s,$$

$$= p(t)z_*(t).$$

We proved that $z_*(t)$ is a positive solution of (2.44), so, either $z_*(t) \in N_1$ or $z_*(t) \in N_3$.

Now, from (2.51) and (2.45)

$$\left(\frac{v'(t)}{z_*(t)}\right)' = \left(\frac{v'(t)}{z(t)}\right)'\left(\int_t^\infty \frac{v(s)}{z^2(s)}\,\mathrm{d}s\right)^{-1} + \left(\frac{v(t)v'(t)}{z^3(t)}\right)\left(\int_t^\infty \frac{v(s)}{z^2(s)}\,\mathrm{d}s\right)^{-2}$$

$$= -\left(\frac{z''(t)v(t)}{z^2(t)}\right)\left(\int_t^\infty \frac{v(s)}{z^2(s)}\,\mathrm{d}s\right)^{-1} + \left(\frac{v(t)v'(t)}{z^3(t)}\right)\left(\int_t^\infty \frac{v(s)}{z^2(s)}\,\mathrm{d}s\right)^{-2}$$

$$= -\frac{v(t)}{z^2(t)}\left(\int_t^\infty \frac{v(s)}{z^2(s)}\,\mathrm{d}s\right)^{-2}\left(z''(t)\int_t^\infty \frac{v(s)}{z^2(s)}\,\mathrm{d}s - \frac{v'(t)}{z(t)}\right).$$

$$= -\frac{z_*''(t)v(t)}{z_*^2(t)},$$

i.e.,

$$\left(\frac{v'(t)}{z_*(t)}\right)' = -\frac{z_*''(t)v(t)}{z_*^2(t)},$$

which means that $v(t)$ is associated to $z_*(t)$.

Moreover, the previous equality implies that

$$z_*(t)v''(t) - v'(t)z_*'(t) + z_*''(t)v(t) = 0,$$

that is,

$$\left(\frac{{z_*}'(t)}{v(t)}\right)' = \frac{v(t){z_*}''(t) - v'(t)z_*'(t)}{v^2(t)} = -\frac{v''(t)z_*(t)}{v^2(t)} < 0$$

which means $z_*'' < 0$, so $z_*(t) \in N_1$.

To verify (2.49), denote

$$V(t) = \int_t^\infty \frac{v(s)}{z^2(s)}\,\mathrm{d}s.$$

So,

$$\lim_{t\to\infty} V(t) = 0,$$

and

$$\begin{aligned}
\int_{t_0}^{\infty} \frac{v(t)}{z_*^2(t)}\,\mathrm{d}t &= \int_{t0}^{\infty} \frac{v(t)}{z^2(t)}\left(\int_{t}^{\infty} \frac{v(s)}{z^2(s)}\,\mathrm{d}s\right)^{-2} \mathrm{d}t \\
&= -\int_{t_0}^{\infty} \frac{V'(t)}{V^2(t)}\,\mathrm{d}t \\
&= \lim_{t\to\infty}\left(\frac{1}{V(t)} - \frac{1}{V(t_0)}\right) \\
&= \infty.
\end{aligned}$$

□

Remark 2.26. If $z(t) \in N_1$ does not satisfy (2.49), then $z_*(t)$ defined by (2.51) does it and, moreover,

$$\lim_{t\to\infty} \frac{z_*(t)}{z(t)} = 0.$$

So, if $z_1(t)$ and $z_2(t)$ are solutions of degree 1 of (2.44), then to achieve the canonical form of (2.46), we consider the solution $z_1(t)$ such that

$$\lim_{t\to\infty} \frac{z_1(t)}{z_2(t)} = 0. \tag{2.53}$$

Definition 2.27. The solution $z_1(t) \in N_1$ of (2.44), which satisfies (2.53), is called a **principal solution** of (2.44).

Corollary 2.28. *Let* (2.50) *hold,* $z_1(t) \in N_1$ *be a principal solution of* (2.44) *and* $v(t)$ *its associated function. Then,* (2.43) *has an equivalent binomial representation* (2.46) *in such a way that* (2.46) *is in canonical form.*

After we ensure the canonical form of (2.46), we will use it to study the properties of (2.43).

Without loss of generality, we will assume nonoscillatory solutions of (2.43) to be positive.

Lemma 2.29. *Let (2.50) hold, $z(t) \in N_1$ be a principal solution of (2.44) and $v(t)$ its associated function. If $y(t)$ is an eventually positive solution of (2.43), then*

$$\left(\frac{v^2(t)}{z(t)}\left(\frac{z^2(t)}{v(t)}\left(\frac{y'(t)}{z(t)}\right)'\right)'\right)' < 0,$$

and either

$$y(t) \in \tilde{N}_1 \iff y(t) > 0, \quad \left(\frac{y'(t)}{z(t)}\right)' < 0, \quad \left(\frac{z^2(t)}{v(t)}\left(\frac{y'(t)}{z(t)}\right)'\right)' > 0,$$

or

$$y(t) \in \tilde{N}_3 \iff y(t) > 0, \quad \left(\frac{y'(t)}{z(t)}\right)' > 0, \quad \left(\frac{z^2(t)}{v(t)}\left(\frac{y'(t)}{z(t)}\right)'\right)' > 0,$$

eventually.

This lemma is based on a modification of Kiguradze's Lemma 2.1 for a canonical form of (2.46) and the idea that a solution of (2.43) is also a solution of (2.46).

According to this lemma, the set of all positive solutions of (2.43) can be decomposed as:

$$N = \tilde{N}_1 \cup \tilde{N}_3.$$

The oscillation criteria for (2.43) are obtained by eliminating the cases of having nonoscillatory solutions.

We define the following functions

$$Q_1(t) = \left(\frac{v(t)}{z^2(t)}\int_{t_1}^{\tau(t)} z(s)\,\mathrm{d}s\right)\left(\int_t^\infty \frac{z(u)}{v^2(u)}\int_u^\infty v(s)q(s)\,\mathrm{d}s\mathrm{d}u\right),$$

$$Q_2(t) = v(t)q(t) \int_{t_1}^{\tau(t)} z(s_1) \int_{t_1}^{s_1} \frac{v(u)}{z^2(u)} \int_{t_1}^{u} \frac{z(s)}{v_2(s)} \, \mathrm{d}s\mathrm{d}u\mathrm{d}s_1,$$

where $t_1 \in [t_0, \infty)$ is large enough, and we use first-order DDE's to get the comparison results which allows us to study the oscillation of (2.43).

Theorem 2.30. *[13] Let (2.50) hold, $z(t) \in N_1$ be a principal solution of (2.44) and $v(t)$ its associated function. Assume that both first-order DEE's*

$$x'(t) + Q_1(t)x(\tau(t)) = 0, \tag{2.54}$$

$$x'(t) + Q_2(t)x(\tau(t)) = 0 \tag{2.55}$$

are oscillatory. Then, (2.43) is oscillatory.

Corollary 2.31. *Let (2.50) hold, $z(t) \in N_1$ be a principal solution of (2.44) and $v(t)$ its associated function. Assume for $i = 1, 2$,*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} Q_i(s) \, \mathrm{d}s > \frac{1}{e}. \tag{2.56}$$

Then, (2.43) is oscillatory.

Lemma 2.32. *let $z(t) \in N_1$ be a positive solution of (2.44) and*

$$\int_{t}^{\infty} s^2 p(s) \, \mathrm{d}s = \infty. \tag{2.57}$$

Then,

$$\frac{z(t)}{t} \tag{2.58}$$

is nonincreasing.

Proof.

Assume $z(t) \in N_1$ be a positive solution of (2.44).

Then $z'(t)$ is bounded, i.e, $\lim_{t\to\infty} z'(t) = l < \infty$.

Claim: $l = 0$.

Suppose not, i.e, $z'(t) \geq l > 0$. Integrating both sides from t_1 to t and since $z(t)$ is increasing, we get

$$z(t) \geq l(t - t_1). \tag{2.59}$$

Integrating (2.44) from t to ∞ and since $z''(t) < 0$, we get

$$-z''(t) \geq \int_t^\infty p(s)z(s)\ \mathrm{d}s.$$

By (2.59), we have

$$-\,z''(t) \geq l \int_t^\infty (s - t_1)p(s)\ \mathrm{d}s.$$

Integrating the above inequality from t_1 to ∞ and since $z''(t) < 0$, we get

$$z'(t_1) \geq l \int_{t_1}^\infty \int_t^\infty (s - t_1)p(s)\ \mathrm{d}s\mathrm{d}t,$$

which is a contradiction with (2.57), so our claim is true.

Since $z(t)$ is increasing , then

$$z'(t) \leq \frac{z(t_1)}{t_1},$$

and

$$\begin{aligned} z(t) &= z(t_1) + \int_{t_1}^t z'(s)\ \mathrm{d}s \\ &\geq z(t_1) + tz'(t) - t_1 z'(t) \\ &\geq tz'(t). \end{aligned}$$

Then,

$$tz'(t) - z(t) \leq 0$$

implies

$$\left(\frac{z(t)}{t}\right)' = \frac{tz'(t) - z(t)}{t^2} \leq 0.$$

The proof is complete.

□

Lemma 2.33. *Let*

$$\int_t^\infty \int_u^\infty p(s)\mathrm{d}s\mathrm{d}u < \infty, \tag{2.60}$$

and $v(t) \in \mathrm{N}_0$ *be a positive solution of* (2.45)*. Then,* $v(t)\varphi(t)$ *is nondecreasing, where*

$$\varphi(t) = \exp\left(\int_{t_1}^t \int_v^\infty \int_u^\infty p(s)\ \mathrm{d}s\mathrm{d}u\mathrm{d}v\right).$$

Proof.

Assume $v(t) \in N_0$ be a positive solution of (2.45). From (2.45), $v(t)$ satisfies

$$-z'(t)v'(t) + z(t)v''(t) + z''(t)v(t) = 0.$$

Differentiating the previous equation and divide it by $z(t)$, we have

$$z'''(t)v(t) + z(t)v'''(t) = 0.$$

From (2.44) $p(t) = \dfrac{z'''(t)}{z(t)}$, so $v(t)$ satisfies the equation

$$v'''(t) + p(t)v(t) = 0.$$

Integrating it twice from t to ∞ and since $v(t)$ is increasing, we have

$$v'(t) \geq -v(t) \int_t^\infty \int_u^\infty p(s)\ \mathrm{d}s\mathrm{d}u.$$

Now,

$$\Big(v(t)\varphi(t)\Big)' = v'(t)\varphi(t) + v(t)\varphi'(t).$$

By the previous inequality, we get

$$\Big(v(t)\varphi(t)\Big)' \geq v(t)\Bigg(\varphi'(t) - \varphi(t)\Bigg(\int\limits_t^\infty \int\limits_u^\infty p(s)\ \mathrm{d}s\mathrm{d}u\Bigg)\Bigg) = 0$$

which means $v(t)\varphi(t)$ is nondecreasing. □

By defining the following function,

$$\tilde{Q}_1(t) = \frac{\tau^2(t) - t_1^2}{2t}\Bigg(\int\limits_t^\infty \varphi(u) \int\limits_u^\infty \frac{q(s)}{\varphi(s)}\ \mathrm{d}s\mathrm{d}u\Bigg),$$

$$\tilde{Q}_2(t) = \frac{q(t)}{\varphi(t)}\Bigg(\int\limits_{t_1}^{\tau(t)} \int\limits_{t_1}^{s_1} \frac{1}{u\varphi(u)} \int\limits_{t_1}^{u} s\varphi^2(s)\ \mathrm{d}s\mathrm{d}u\mathrm{d}s_1\Bigg),$$

where $\varphi(t)$ is as in Lemma 2.33 and $t_1 \in [t_0, \infty)$ is large enough, we obtain another result which does not require solving auxiliary equations.

Theorem 2.34. *[13] Let (2.50),(2.57), and (2.60) hold. Assume that both first-order DDE's*

$$x'(t) + \tilde{Q}_1 x(\tau(t)) = 0 \tag{2.61}$$

and

$$x'(t) + \tilde{Q}_2 x(\tau(t)) = 0. \tag{2.62}$$

are oscillatory. Then, (2.43) is oscillatory.

Corollary 2.35. *Let (2.50),(2.57), and (2.60) hold. Assume that for $i = 1, 2$*

$$\liminf_{t\to\infty} \int_{\tau(t)}^t \tilde{Q}_i(s)\ \mathrm{d}s > \frac{1}{e}. \tag{2.63}$$

Then, (2.43) is oscillatory.

Theorem 2.34 and Corollary 2.35 allow us to study the oscillation of (2.43) without using auxiliary solutions $z(t)$ and $v(t)$, but they are partly weaker.

Next we will produce another technique uses the same idea including the term $p(t)$ but more precise than Corollary 2.35.

Define the function

$$\tilde{p}(t) = \max\left\{ \frac{2tp(t)}{\tau^2(t) - t_1^2}, \frac{tp(t)\phi(t)}{\int_{t_1}^{\tau(t)} s\phi(s)\mathrm{d}s} \right\},$$

where

$$\phi(t) = \int_{t_1}^{t}\int_{t_1}^{u} s\varphi^2(s)\mathrm{d}s\mathrm{d}u,$$

where $\varphi(t)$ is as in Lemma 2.33 and $t_1 \in [t_0, \infty)$ is large enough. We assume that

$$q(t) - \tilde{p}(t) > 0 \tag{2.64}$$

holds eventually.

Theorem 2.36. *[13] Let (2.50),(2.57), (2.60), and (2.64) hold. Assume that the fourth-order DDE*

$$x^{(4)}(t) + \big(q(t) - \tilde{p}(t)x(\tau(t))\big) = 0$$

is oscillatory. Then, (2.43) is oscillatory.

Theorem 2.37. *Let (2.50),(2.57), (2.60), and (2.64) hold. Assume that the first-order DDE*

$$x'(t) + \lambda_0\tilde{Q}(t)x(\tau(t)) = 0, \tag{2.65}$$

where

$$\tilde{Q}(t) = \frac{\tau^3(t)}{3!}\big(q(t) - \tilde{p}(t)\big),$$

is oscillatory for some constant $\lambda_0 \in (0,1)$. Then, (2.43) is oscillatory.

Corollary 2.38. *Let* (2.50),(2.57), (2.60), *and* (2.64) *hold. Assume that*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \tilde{Q}(s)\,\mathrm{d}s > \frac{1}{e}.$$

Then, (2.43) *is oscillatory.*

Example 2.3. *Consider the Fourth-order DEE*

$$y^{(4)}(t) - \frac{0.171}{t^3} y'(t) + \frac{b}{t^4} y(\lambda t) = 0,$$

where

$$b > 0, \qquad \lambda \in (0,1), \qquad t \geq 1.$$

The auxiliary equation (2.44) *takes the form*

$$z'''(t) - \frac{0.171}{t^3} z(t) = 0$$

with positive solutions

$$z_1(t) = t^{0.1} \quad \text{and} \quad z_2(t) = t^{\frac{2.9-\sqrt{1.75}}{2}}$$

of degree 1, *while*

$$z(t) = t^{\frac{2.9+\sqrt{1.75}}{2}}$$

of degree 3.

Since

$$\lim_{t\to\infty} \frac{z_1(t)}{z_2(t)} = 0,$$

then to achieve the canonical form, we consider $z_1(t) = t^{0.1}$ *for which the equation* (2.45) *reduces to*

$$\left(\frac{v'(t)}{t^{0.1}}\right)' - \frac{0.171}{t^3}v(t) = 0$$

with positive solutions

$$v_1(t) = t^{\frac{1.1+\sqrt{1.75}}{2}} \qquad v_1(t) \in N_1$$

and

$$v_2(t) = t^{\alpha}, \quad \text{where} \quad \alpha = \frac{1.1+\sqrt{1.75}}{2}$$

is associated to $z_1(t)$.

By direct computation we have

$$Q_1(t) \approx \frac{b\lambda^{1.1}}{1.1(3-\alpha)(1.9+\alpha)t},$$

and

$$Q_2(t) \approx \frac{b\lambda^{3-\alpha}}{(3-\alpha)(1.9-\alpha)(1.1-2\alpha)t}.$$

Condition (2.56) *from Corollary 2.33 yields*

$$b\lambda^{1.1}\ln\frac{1}{\lambda} > \frac{1.1(3-\alpha)(1.9+\alpha)}{e}$$

and

$$b\lambda^{3-\alpha}\ln\frac{1}{\lambda} > \frac{(3-\alpha)(1.9-\alpha)(1.1-2\alpha)}{e}.$$

We conclude from Corollary 2.33 that the given forth-order DDE is oscillatory if the previous two inequalities hold simultanuously.

When $\lambda = 0.7$ *and* $b > 23.5446$ *the results is satisfied.*

Example 2.4. *Consider the forth-order DDE*

$$y^{(4)}(t) - \frac{a}{t^3}y'(t) + \frac{b}{t^4}y(\lambda t) = 0, \tag{2.66}$$

where

$$a < \frac{2}{3\sqrt{3}} \qquad b > 0, \qquad \lambda \in (0,1), \qquad t \geq 1.$$

since

$$\lim_{t\to\infty} \sup t^3 p(t) = a < \frac{2}{3\sqrt{3}},$$

$$\int s^2 p(s)\ \mathrm{d}s = a \ln s \to \infty \quad \text{as } s \to \infty,$$

and

$$\int_t^\infty \int_u^\infty p(s)\ \mathrm{d}s\mathrm{d}u = \frac{a}{t} < \infty,$$

then conditions (2.50), (2.57), *and* (2.60) *are satisfied.*

We have

$$\varphi(t) = t^{a/2},$$

then

$$\tilde{Q}_1 = \frac{b\lambda^2}{4\left(3+\frac{a}{2}\right)t},$$

and

$$\tilde{Q}_2 = \frac{b\lambda^{(3+\frac{a}{2})}}{\left(3+\frac{a}{2}\right)\left(2+\frac{a}{2}\right)\left(2+a\right)t}.$$

By Corollary (2.35), *the given fourth-order DEE is oscillatory if*

$$\frac{b\lambda^2}{4\left(3+\frac{a}{2}\right)} \ln\frac{1}{\lambda} > \frac{1}{e}$$

and

$$\frac{b\lambda^{(3+\frac{a}{2})}}{\left(3+\frac{a}{2}\right)\left(2+\frac{a}{2}\right)\left(2+a\right)}\ln\frac{1}{\lambda}>\frac{1}{e}$$

hold simultanuously.

When $a = 0.171$, $\lambda = 0.7$, *and* $b > 43.309$ *the result is satisfied.*

Example 2.5. *Consider the equation* (2.66) *from the previous example , conditions* (2.50), (2.57), *and* (2.60) *are satisfied.*

Also, we have

$$\varphi(t) = t^{\frac{a}{2}}, \qquad \phi(t) = \frac{t^{\frac{a}{2}}}{(a+2)(a+3)},$$

$$\tilde{p}(t) = \frac{a(a+5)}{\lambda^{a+5}t^4}, \qquad \text{and} \qquad \tilde{Q}(t) = \left(b - \frac{a(a+5)}{\lambda^{a+5}}\right)\frac{\lambda^3}{3!t},$$

then condition 2.64 *is satisfied since* $\lambda \in (0, 1)$.

By Corollary 2.38, *the equation* (2.66) *is oscillatory when*

$$\left(b - \frac{a(a+5)}{\lambda^{a+5}}\right)\frac{\lambda^3}{3!}\ln\frac{1}{\lambda} > \frac{1}{e}.$$

When $a = 0.171$, $\lambda = 0.7$, *and* $b > 27.5757$ *the result is satisfied.*

Chapter 3

Oscillation of fourth order non-linear DDE with a middle term

In this chapter we present oscillation theorems for the non-linear fourth-order delay differential equations, with first order and second order middle terms, by relating it to an auxiliary linear third-order diffrential equation.

Many authors have considered fourth order nonlinear DDE and its oscillatory behavior, see [5, 17, 24].

3.1 Oscillation of fourth order non-linear DDE with a first order middle term

In this section we study the oscillation of the forth order non-linear DDE's of the form

$$y^{(4)}(t) + p(t)y'(t) + q(t)f(y(\tau(t))) = 0, \tag{3.1}$$

by relating it to the third order differential equation

$$z'''(t) + p(t)z(t) = 0. \tag{3.2}$$

Throughout this section the following hypothesis are made:

(H_1) $t \in [t_0, \infty)$ where t_0 is a non-negative constant.

(H_2) p is a non-negative function in $C([t_0, \infty), \mathbb{R})$ such that it does not vanish identically on any $[T, \infty) \subseteq [t_0, \infty)$.

(H_3) q is a positive function in $[t_0, \infty)$.

(H_4) $\tau \in C^1([t_0, \infty), \mathbb{R})$ and $0 < \tau(t) < t$, $\tau'(t) \geq 0$ for $t \in [t_0, \infty)$ and $\lim_{t \to +\infty} \tau(t) = \infty$.

(H_5) $f \in C(\mathbb{R}, \mathbb{R})$ and $\dfrac{f(u)}{u} \geq k > 0$ for $u \neq 0$.

We are only interested in the solutions that belong to $t \in [t_0, \infty)$ such that $\sup\{|y(t)| : T \leq t < \infty\} > 0$ for any $T \in [t_0, \infty)$.

The results of this section are due to Hou et. al. [24].

Lemma 3.1. *Assume that $x \in C^n([t_0, \infty), \mathbb{R})$ such that $x(t) > 0$ for $t \geq t_0$ and $x^n(t)$ is non-positive for $t \geq t_0$ and does not vanish identically on any $[T, \infty) \subseteq [t_0, \infty)$.*
If n is even (odd), then there exists $l \in \{1, 3, \ \ldots, n-1\}$ ($l \in \{0, 2, \ \ldots, n-1\}$) such that for all sufficientlly large t, $x(t)x^j(t) > 0$ for $j = 0, 1, \ \ldots, l$ and $(-1)^{n+j-1}x(t)x^j(t) > 0$ for $j = l+1, l+2, \ \ldots, n-1$.
Furthermore,

$$|x'(\tau(t))| \geq \frac{\tau^{l-1}(t)(t - \tau(t))^{n-l-1}}{2^{l-1}(l-1)!(n-l-1)!} \left| x^{(n-1)}(t) \right| \tag{3.3}$$

for all sufficiently large t, where τ is as in H_4.

Proof. See [24]. □

Lemma 3.2. *Suppose (3.2) has an eventually positive increasing solution on $[t_0, \infty)$ and y is a nonoscillatory solution of (3.1) on $[t_0, \infty)$. Then there exists $t_1 \in [t_0, \infty)$ such that $y(t)y'(t) > 0$ or $y(t)y'(t) < 0$ for $t \geq t_1$.*

Proof. See [24]. □

The following theorem states the main result of this section.

Theorem 3.3. *suppose that*

1. *equation (3.2) has an eventually positive increasing solution z on $[t_0, \infty)$,*

2. *there is a differentiable function $\rho : [t_0, \infty) \to (0, \infty)$ such that*

$$\limsup_{t\to\infty} \int_T^t \left\{ k\rho(s)q(s) - \frac{(l-1)!(3-l)!(\rho'(s))^2}{2^{3-l}\tau^{l-1}(s)(s-\tau(s))^{3-l}\tau'(s)\rho(s)} \right\} \mathrm{d}s = \infty, \quad l = 1, 3, \tag{3.4}$$

for every $T \geq t_0$.

3. $$\limsup_{t\to\infty} \int_T^t \int_T^s \{kq(u) - p(u)\} \,\mathrm{d}u\mathrm{d}s = \infty. \tag{3.5}$$

Then every solution y of (3.1) either oscillates or converges to 0.

Proof.
Let y be a solution of (3.1) that does not converge to 0. By way of contradiction, assume that y is nonoscillatory. Without loss of generality, assume $y(t) > 0$ and $y(\tau(t)) > 0$ for $t \geq t_0$.

From Lemma 3.2, there exists $t_1 \geq t_0$ such that $y'(t) > 0$ or $y'(t) < 0$ for $t \geq t_1$.

Case 1:

When $y'(t) > 0$ for $t \geq t_1$, then from (3.1) and since $p \geq 0$, $q > 0$, and $y(\tau(t)) > 0$, $\dfrac{f(y(\tau(t)))}{y(\tau(t))} \geq k > 0$ implies $f(y(\tau(t))) > 0$ we have

$$y^{(4)}(t) = -(p(t)y'(t) + q(t)f(y(\tau(t)))) < 0, \qquad t \geq t_1.$$

By Lemma 3.2, $y'''(t) > 0$ and (3.3) holds for $t \geq t_1$.

Let

$$w(t) = \frac{\rho(t)y'''(t)}{y(\tau(t))} > 0, \qquad t \geq t_1,$$

then

$$w'(t) = \frac{\rho(t)y^{(4)}}{y(\tau(t))} + \frac{\rho'(t)y'''(t)}{y(\tau(t))} - \frac{\rho(t)y'''(t)y'(\tau(t))\tau'(t)}{y^2(\tau(t))}.$$

Multiplying (3.1) by $\dfrac{\rho(t)}{y(\tau(t))}$ and by (H_5), we get

$$w'(t) \leq -k\rho(t)q(t) + \frac{\rho'(t)w(t)}{\rho(t)} - \frac{\rho(t)y'''(t)y'(\tau(t))\tau'(t)}{y^2(\tau(t))}.$$

From (3.2)

$$w'(t) \leq -k\rho(t)q(t) + \frac{\rho'(t)w(t)}{\rho(t)} - \frac{\tau^{l-1}(t)(t-\tau(t))^{3-l}\tau'(t)w^2(t)}{2^{l-1}(l-1)!(3-l)!\rho(t)}$$

$$= -k\rho(t)q(t) - \frac{\tau^{l-1}(t)(t-\tau(t))^{3-l}\tau'(t)}{2^{l-1}(l-1)!(3-l)!\rho(t)}\left(w - \frac{2^{l-1}(l-1)!(3-l)!\rho'(t)}{2\tau^{l-1}(t)(t-\tau(t))^{3-l}\tau'(t)}\right)^2 + \frac{(l-1)!(3-l)!(\rho'(t))^2}{2^{3-l}\tau^{l-1}(t)(t-\tau(t))^{3-l}\tau'(t)\rho(t)}$$

$$\leq -\left(k\rho(t)q(t) - \frac{(l-1)!(3-l)!(\rho'(t))^2}{2^{3-l}\tau^{l-1}(t)(t-\tau(t))^{3-l}\tau'(t)\rho(t)}\right), \qquad t \geq t_0.$$

By integrating the last inequality from t_1 to t

$$\int_{t_1}^{t} \left(k\rho(t)q(t) - \frac{(l-1)!(3-l)!(\rho'(t))^2}{2^{3-l}\tau^{l-1}(t)(t-\tau(t))^{3-l}\tau'(t)\rho(t)} \right) \mathrm{d}s \leq -\int_{t_1}^{t} w'(s)\ \mathrm{d}s$$
$$= w(t_1) - w(t)$$
$$\leq w(t_1)$$

for all $t \geq t_1$, which is a contradiction to our assumption (3.4).

Case 2:

Suppose $y'(t) < 0$ for $t \geq t_1$. Since y is eventually positive, eventually decreasing, and does not converge to 0, we have

1. $y''(t)$ cannot be eventually nonpositive,
2. $\lim_{t\to\infty} y(t) = \lambda > 0$,
3. $\lim_{t\to\infty} y'(t) = 0$,
4. there is $\alpha \geq t_0$ such that $-y'(t) < \lambda < y(t)$ for $t \geq \alpha$.

From (1), either $y''(t)$ is oscillatory or eventually nonnegative.
If $y''(t)$ is oscillatory, then $y'''(t)$ is also oscillatory. Hence there exists some number $\beta \geq \alpha$ such that $y'''(\beta) = 0$ and $y(\tau(t)) \geq \lambda$ for $t \geq \beta$.

By integrating (3.1) from β to t and from (3.5), we have

$$y'''(t) = -\int_{\beta}^{t} y(\tau(s))q(s)\frac{f(y(\tau(s)))}{y(\tau(s))}\ \mathrm{d}s - \int_{\beta}^{t} p(s)y'(s)\ \mathrm{d}s$$

$$\leq -\lambda \int_{\beta}^{t} \Big(kq(s) - p(s) \Big)\ \mathrm{d}s.$$

Again, integrate the last inequality from β to t we get

$$y''(t) \leq y''(\beta) - \lambda \int_{\beta}^{t} \int_{\beta}^{s} \Big(kq(u) - p(u) \Big) \, \mathrm{d}u\mathrm{d}s \tag{3.6}$$

But the right-hand side goes to $-\infty$ as t goes to ∞, which implies that $y''(t)$ is eventually negative. This contradicts our assumption on $y''(t)$.

As a consequence, we must have $y(t) > 0$, $y'(t) < 0$, and $y''(t) \geq 0$ for $t \geq \alpha > t_0$.

We have three cases for $y''(t)$:

i. If $y'''(t) \geq 0$ and does not vanish identically for $t \geq \alpha$, then $y'(t) > 0$ for all large t, which is a contradiction.
If $y'''(t) \equiv 0$ for $t \geq \alpha$, then $y'(t)$ is a linear function. Since $\lim_{t\to\infty} y'(t) = 0$, we conclude that $y'(t) = 0$, which is a contradiction to our assumption that $y'(t) < 0$.

ii. If $y'''(t) \leq 0$ for $t \geq \alpha$, then by integrating (3.1) from α to t, we get

$$y'''(\alpha) - y'''(t) = \int_{\alpha}^{t} q(s) f(y(\tau(s))) \, \mathrm{d}s + \int_{\alpha}^{t} p(s) y'(s) \, \mathrm{d}s$$

$$\geq \lambda \int_{\alpha}^{t} \Big(kq(s) - p(s) \Big) \, \mathrm{d}s.$$

Again, integrate (3.1) from α to t, we get

$$0 \geq -y''(t) \geq -y''(\alpha) + \lambda \int_{\alpha}^{t} \int_{\alpha}^{s} \Big(kq(u) - p(u) \Big) \, \mathrm{d}u\mathrm{d}s,$$

which contradicts (3.5).

iii. If $y'''(t)$ oscillates, then (3.6) holds so that $y''(t)$ is eventually negative. This is a contradiction to our assumption on $y''(t)$. $\square$

Remark 3.4. When $p(t)$ is differentiable, the condition (3.5) could be replaced by

$$\limsup_{t\to\infty} \int_T^t \Big(kq(u) - p'(u)\Big)\, \mathrm{d}u = \infty \tag{3.7}$$

for every $T \geq 0$.

Note that

$$\int_\gamma^t q(s)f(y(\tau(s)))\, \mathrm{d}s + \int_\gamma^t p(s)y'(s)\, \mathrm{d}s$$

$$= \int_\gamma^t y(\tau(s))q(s)\frac{f(y(\tau(s)))}{y(\tau(s))}\, \mathrm{d}s + p(s)y(s)|_\gamma^t - \int_\gamma^t p'(s)y(s)\, \mathrm{d}s$$

$$\geq \lambda \int_\gamma^t \Big(kq(s) - p'(s)\Big)\, \mathrm{d}s - p(\gamma)y(\gamma),$$

where $\gamma = \alpha$ or β.

Example 3.1. *Consider the fourth-order DDE*

$$y^{(4)}(t)+\frac{1}{t^3(\ln t-1)}y'(t)+\frac{1}{t^2}\left(1-\frac{1}{t\ln t-1)}\right)\left(y\left(t-\frac{3\pi}{2}\right)+1\right)y\left(t-\frac{3\pi}{2}\right) = 0, \quad t > \frac{3\pi}{2}. \tag{3.8}$$

Here,

$$p(t) = \frac{1}{t^3(\ln t - 1)}, \qquad q(t) = \frac{1}{t^2}\left(1 - \frac{1}{t\ln t - 1)}\right),$$

and $f(u) = u^2 + u$ *with* $k = 1$. *Its auxiliary equation is*

$$z'''(t) + \frac{1}{t^3(\ln t - 1)}z(t) = 0,$$

with eventually positive increasing solution

$$z(t) = t\big(\ln t - 1\big).$$

By choosing $\rho(t) = t$ *and for large* T, *condition* (3.4) *becomes*

$$\limsup_{t\to\infty} \int\limits_T^t \left[\frac{1}{s} - \frac{1}{s^2(\ln s - 1)} - \frac{(l-1)!(3-l)!}{(3\pi/2)^{3-l}s(s-3\pi/2)^{l-1}}\right] \mathrm{d}s = \infty, \qquad l = 1, 3.$$

Also, condition (3.5) *becomes*

$$\limsup_{t\to\infty} \int\limits_T^t \int\limits_T^s \left(\frac{1}{u^2}\left(1 - \frac{1}{u\ln u - 1)}\right) - \frac{1}{u^3(\ln u - 1)}\right) \mathrm{d}u\mathrm{d}s = \infty.$$

By Theorem 3.3, any solution of this fourth-order DDE is oscillatory or convergent.

3.2 Oscillation of fourth order non-linear DDE with a second order middle term

In this section we study the oscillation of the fourth order non-linear DDE's of the form

$$y^{(4)}(t) + p(t)y''(t) + q(t)f\Big(y\big(\tau(t)\big)\Big) = 0, \quad t \geq t_0 > 0 \tag{3.9}$$

by relating it to the second order differential equation

$$z''(t) + p(t)z(t) = 0. \tag{3.10}$$

Throughout this section the following hypothesis are assumed:

(H_1) $p(t), q(t) \in C([t_0, \infty), \mathbb{R}^+)$.

(H_2) $\tau(t) \in C^1([t_0, \infty), \mathbb{R}^+)$ such that $\tau(t) < t$, $\tau'(t) \geq 0$ and $\lim\limits_{t\to\infty} \tau(t) = \infty$.

(H_3) $f \in C(\mathbb{R}, \mathbb{R})$ such that $uf(u) > 0$ and $\dfrac{f(u)}{u^\beta} \geq k > 0$ for $u \neq 0$, where k is a constant and β is the ratio of positive odd integers.

We are only interested in the solutions that belong to $[t_0, \infty)$ such that $\sup\{|y(t)| : t_1 \leq t < \infty\} > 0$ for $t_1 \in [t_0, \infty)$.

The results of this section are due to Akin et. al. [17].

Lemma 3.5. *[18, 19] Every eventually positive solution $y(t)$ of equation (3.9)is one of the following types:*

1. *Type (a):*

$$y(t) > 0, \quad y'(t) > 0 \ \text{ and } \ y''(t) < 0 \ \text{ for large } t,$$

2. *Type (b):*

$$y(t) > 0, \quad y'(t) > 0, \quad y''(t) > 0 \ \text{ and } \ y'''(t) > 0 \ \text{ for large } t,$$

3. *Type (c):*

$$y''(t) \ \text{ changes sign eventually.}$$

Moreover, if equation (3.10) is nonoscillatory, then $y(t)$ is of Type (a) or Type (b) and if equation (3.10) is oscillatory, then $y(t)$ is of Type (a) or Type (c).

Lemma 3.6. *[17] Let $\beta \leq 1$ and equation (3.10) be nonoscillatory. If*

$$\int_t^\infty \Big(p(s) + \tau^{2\beta}(s) q(s)\Big) \, \mathrm{d}s = \infty, \tag{3.11}$$

then equation (3.9) has no solution of Type (b), i.e., every eventually positive solution of (3.9) is of Type (a).

Lemma 3.7. *[17] Let $\beta \leq 1$ and equation (3.10) be nonoscillatory. If for every positive constant c, the first order delay equation*

$$x'(t) + c^{\beta} k q(t) \tau^{3\beta}(t) x^{\beta}\Big(\tau(t)\Big) = 0, \tag{3.12}$$

is oscillatory, then equation (3.9) has no solution of Type (b), i.e., every eventually positive solution of (3.9) is of Type (a).

Corollary 3.8. *Let $\beta \leq 1$ and equation (3.10) be nonoscillatory. If for every positive constant c,*

$$\liminf_{t\to\infty} \int\limits_{\tau(t)}^{t} q(s)\tau^{3\beta}(s)\ \mathrm{d}s > \frac{1}{c^{\beta} k e}, \tag{3.13}$$

then equation (3.9)has no solution of Type (b).

Lemma 3.9. *[17] Let $\beta \leq 1$ and equation (3.10) be (non)oscillatory. If there exist a function $\rho(t) \in C^1([t_0, \infty), \mathbb{R})$ such that $\tau(t) \leq \rho(t) \leq t$, $\rho'(t) \geq 0$ for $t \geq t_0$ such that the second order inequality*

$$w''(t) \geq P(t) w\big(\rho(t)\big), \tag{3.14}$$

where

$$P(t) = cq(t)\tau^{\beta}(t)\big(\rho(t) - \tau(t)\big)^{\beta} - p(t) > 0$$

for some constant $c > 0$, has no positive bounded solutions, then equation (3.9) has no solution of Type (a).

The next two lemma's are given to see when the second order delay differential inequality (3.14) has no bounded positive solutions.

Lemma 3.10. *[17] If*

$$\limsup_{t\to\infty} \int\limits_{\rho(t)}^{t} \big(\rho(t) - \rho(s)\big) P(s)\ \mathrm{d}s > 1, \tag{3.15}$$

for positive P, then inequality (3.14) has no positive bounded solutions.

Lemma 3.11. *[17] If*

$$\limsup_{t\to\infty} \int\limits_{\rho(t)}^{t} \left(\int\limits_{u}^{t} P(s)\,\mathrm{d}s \right) \mathrm{d}u > 1, \tag{3.16}$$

then inequality (3.14) has no positive bounded solutions.

Now, we are ready to give the main results of this section.

Theorem 3.12. *Let $\beta \leq 1$ and equation (3.10) be nonoscillatory. If condition (3.11) (or (3.12)) holds and either condition (3.16) or (3.15) hold, then equation (3.9) is oscillatory.*

Proof. Let $y(t)$ be an eventually positive solution of equation (3.9).
Since equation (3.10) is nonoscillatory, then by Lemma 3.5, $y(t)$ is either of Type (a) or of Type (b).
Moreover, condition (3.11) (or condtion (3.12) holds, then by Lemma 3.6 (or Lemma 3.7), equation 3.9 has no solution of Type (b).

Also, condition (3.15) (or condition (3.16)) is satisfied, then by Lemma 3.10 (or Lemma 3.11), the inequality (3.14) has no positive bounded solutions, hence, by Lemma 3.9, we conclude that equation (3.9) has no solution of Type (a).
From this we conclude that equation (3.9) is oscillatory. □

Theorem 3.13. *Let $\beta \leq 1$ and equation (3.10) be oscillatory. If condition (3.15) (or condition (3.16)) holds, then every solution $y(t)$ of equation (3.9) is oscillatory or $y''(t)$ is oscillatory.*

Proof. Let $y(t)$ be an eventually positive solution of equation (3.9).
Since equation (3.10) is oscillatory, then by Lemma 3.5, $y(t)$ is either of Type (a) or of Type (c).

Also, condition (3.15) (or condition (3.16)) is satisfied, then by Lemma 3.10 (or Lemma 3.11), the inequality (3.14) has no positive bounded solutions, so, by Lemma 3.9, we conclude that equation (3.9) has no solution of Type (a). From this we conclude that $y(t)$ is of Type (c), i.e., $y(t)$ is oscillatory or $y''(t)$ is oscillatory. □

Example 3.2. *Consider the fourth order DDE*

$$y^{(4)}(t) + \frac{1}{4t^2}y''(t) + \left(1 - \frac{1}{4t^2}\right)y(t-\pi) = 0, \tag{3.17}$$

where

$$p(t) = \frac{1}{4t^2}, \qquad q(t) = 1 - \frac{1}{4t^2}, \qquad \tau(t) = t - \pi.$$

Let $\rho(t) = t - \frac{\pi}{2}$ with $\beta = 1$. The auxiliary equation, which is given by

$$z''(t) + \frac{1}{4t^2}z(t) = 0, \tag{3.18}$$

is nonoscillatory with $z(t) = \sqrt{t}$. Also, the conditions (3.11) and (3.15) are satisfied . By Theorem 3.12, equation (3.17) is oscillatory. One such solution is $y(t) = \sin t$.

Example 3.3. *Consider the fourth order DDE*

$$y^{(4)}(t) + 2y''(t) + y(t - 2\pi) = 0, \tag{3.19}$$

where

$$p(t) = 2, \qquad q(t) = 1, \qquad \tau(t) = t - 2\pi.$$

Let $\rho(t) = t - \pi$ with $\beta = 1$. The auxiliary equation, wich is given by

$$z''(t) + 2z(t) = 0, \tag{3.20}$$

is oscillatory with $z_1(t) = \sin\sqrt{2}t$ and $z_2(t) = \cos\sqrt{2}t$. Also, the condition (3.15) is satisfied. By Theorem 3.13, equation (3.19) is oscillatory. One such solution $y(t) = \sin t$.

Chapter 4

Oscillation of third order nonlinear DDE with a middle term

The study of third order differential equations has started by Hanan [22] and Lazer [31] and then many authors have considered different classes of them with different techniques to obtain results about the behavior of their solutions, see [20, 21, 33, 35, 37].

The third order differential equations have important applications in different feilds of applied mathematics and physics. For example, they are used to study the feedback nuclear reactor problem and the entry-flow phenomenon. The propagation of electrical pulses in the nerve of a squid, which approximated by the famous *Nagumo's equation* , is also an application on the third order differential equations. See [27, 34, 44].

In this chapter we presents oscillation theorems for third order delay differential equations of the form

$$\left(r_2(t)\left(r_1(t)\left(y'(t)\right)^{\gamma}\right)'\right)' + p(t)\left(y'(t)\right)^{\gamma} + q(t)f\left(y\left(\tau(t)\right)\right) = 0, \tag{4.1}$$

where $r_1(t), r_2(t), p(t), q(t) \in C([t_0, \infty), \mathbb{R})$, $t_0 \geq 0$, $f \in C(-\infty, \infty)$, by relating it to a second order differential equation and using some operators to rewrite it into its binomial form, which ease the study of it.

The first section of this chapter deal with the case of equation (4.23) when $\gamma = 1$. *Riccati transformation* was used to provide some sufficient conditions to see when the solution of

$$\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)' + p(t)y'(t) + q(t)f\left(y\left(\tau(t)\right)\right) = 0, \tag{4.2}$$

oscillates or converges.

4.1 Oscillation of third order DDE with a middle term using Riccati-type transformation

This section is concerned with the non-linear third-order DDE of the form

$$\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)' + p(t)y'(t) + q(t)f\left(y\left(\tau(t)\right)\right) = 0, \tag{4.3}$$

under the assumption that all solutions of the auxiliary second-order differential equation

$$\left(r_2(t)z'(t)\right)' + \frac{p(t)}{r_1(t)}z(t) = 0, \tag{4.4}$$

are nonoscillatory.

Throughout the section, the following hypotheses will be made:

(H_1) $p, q, \in C([t_0, \infty], \mathbb{R})$ such that $p(t) \geq 0$, $q(t) > 0$, for all $t \geq t_0 \geq 0$.

(H_2) $\tau \in C^1([t_0, \infty], \mathbb{R})$ such that $0 < \tau(t) \leq t$, $\tau'(t) \geq 0$ for all $t \geq t_0 \geq 0$ and $\lim_{t\to\infty} \tau(t) = \infty$.

(H_3) $r_i(t) > 0$ for $i = 1, 2$. $r_1(t) \in C^1([t_0, \infty], \mathbb{R})$, and $r_2(t) \in C([t_0, \infty], \mathbb{R})$.

(H_4) $f \in C(\mathbb{R}, \mathbb{R})$ such that $\dfrac{f(u)}{u} \geq k > 0$ for $u \neq 0$.

A solution of equation (4.3) is a function $y(t)$ such that $y(t) \in C[t_y, \infty)$, $r_1(t)y'(t) \in C^1[t_y, \infty)$, and $r_2(t)\Big(r_1(t)y'(t)\Big)' \in C^1[t_y, \infty)$, where $y(t)$ satisfies equation (4.3) on $[t_y, \infty)$ for every $t \geq t_y \geq t_0$.

We will only consider those solutions such that the condition $\sup\{|y(t)| : t_1 \leq t < \infty\} > 0$ is satisfied on $[t_0, \infty)$.

In this section we make use of *Riccati Transformation* to establish some sufficient conditions to see that any solution of the third order DDE (4.3) oscillates or converges to zero. The results of this section are due to Grace et al. [16] and [42]. In [16], Grace improves the results given by Aktas et al. in [42] and [4].

Consider the following operators

$$L_0 y(t) = y(t),$$

$$L_i y(t) = r_i(L_{i-1}y(t))', \quad i = 1, 2,$$

and

$$L_3 y(t) = (L_2 y(t))',$$

we can rewrite equation (4.3) as

$$L_3y(t) + p(t)y'(t) + q(t)f\Big(y\Big(\tau(t)\Big)\Big) = 0.$$

Lemma 4.1. *If $y(t)$ is a solution of (4.3), then $z(t) = -y(t)$ is a solution of the equation*

$$L_3z(t) + p(t)z'(t) + q(t)f^*\Big(z\Big(\tau(t)\Big)\Big) = 0,$$

where $f^*(z) = -f(-z)$ and $zf^*(z) > 0$ for $z \neq 0$.

Proof. By direct computation,

$$L_3z(t) + p(t)z'(t) + q(t)f^*\Big(z\Big(\tau(t)\Big)\Big) = L_3y(t) - p(t)y'(t) - q(t)f\Big(y\Big(\tau(t)\Big)\Big)$$

but $y(t)$ is a solution of (4.3), then

$$L_3y(t) + p(t)y'(t) + q(t)f\Big(y\Big(\tau(t)\Big)\Big) = 0.$$

□

According to Lemma 4.1 we can only consider positive solutions, for large t, of equation (4.3).

Notation

We define the functions

$$R_i(t, t_1) = \int_{t_1}^{t} \frac{1}{r_i(s)}\,\mathrm{d}s \tag{4.5}$$

for $i = 1, 2$ and $t_0 \leq t_1 \leq t < 0$.

We assume that

$$R_i(t, t_0) \to \infty \quad \text{as} \quad t \to \infty. \tag{4.6}$$

Lemma 4.2. *Suppose that* (4.4) *is nonoscillatory. If* $y(t)$ *is a nonoscillatory solution of* (4.3) *on* $[t_1, \infty)$, $t_1 \geq t_0$, *then there exists a* $t_2 \in [t_1, \infty)$ *such that*

$$y(t)L_1y(t) > 0$$

or

$$y(t)L_1y(t) < 0$$

for $t \geq t_2$.

Proof. Let $y(t)$ be a nonoscillatory solution of (4.3) on $[t_1, \infty)$.
We may take $y(t) > 0$ and $y(\tau(t)) > 0$ for $t \geq t_2 \geq t_1$. By (H_4) and since $y(\tau(t)) > 0$, we have $f\Big(y\big(\tau(t)\big)\Big) > 0$.

Consider the following second order nonhomogeneuos delay differential equation

$$\Big(r_2(t)x'(t)\Big)' + \frac{p(t)}{r_1(t)}x(t) = q(t)f\Big(y\big(\tau(t)\big)\Big), \quad t \geq t_0. \tag{4.7}$$

For $x(t) = -L_1y(t)$, we have

$$\begin{aligned}\bigg(r_2(t)\Big(-L_1y(t)\Big)'\bigg)' - \frac{p(t)}{r_1(t)}L_1y(t) &= -\big(r_2(t)L_2y(t)\big)' - \frac{p(t)}{r_1(t)}r_1(t)y'(t)\\ &= -L_3y(t) - p(t)y'(t).\end{aligned}$$

Since $y(t)$ is a solution of (4.3), then

$$\bigg(r_2(t)\Big(-L_1y(t)\Big)'\bigg)' - \frac{p(t)}{r_1(t)}L_1y(t) = q(t)f\Big(y\big(\tau(t)\big)\Big),$$

i.e, $x(t) = -L_1y(t)$ is a solution of (4.7).

Claim:- All solutions of (4.7) are nonoscillatory.
Let $z(t)$ be a solution of (4.4), where $r_2(t)$ and $\dfrac{p(t)}{r_1(t)} \in C([t_2, \infty), \mathbb{R})$ such that

$r_2(t) > 0$ and $\dfrac{p(t)}{r_1(t)} \geq 0$. Without loss of generality, let $z(t) > 0$ for $t \geq t_2$. To contradict, let $x(t)$ be oscillatory solution of (4.7) with consecutive zeros at a and b, $t_2 < a < b$, such that $x'(a) \geq 0$ and $x'(b) \leq 0$.

Multiplying (4.7) by $z(t)$ and (4.4) by $x(t)$,we get

$$r_2(t)x''(t)z(t) + {r_2}'(t)x'(t)z(t) + \frac{p(t)}{r_1(t)}x(t)z(t) = z(t)q(t)f\Big(y\big(\tau(t)\big)\Big),$$

and

$$r_2(t)x(t)z''(t) + {r_2}'(t)x(t)z'(t) + \frac{p(t)}{r_2(t)}x(t)z(t) = 0.$$

Subtract the second equation from the first equation, we get

$$\begin{aligned} r_2(t)x''(t)z(t) + {r_2}'(t)x'(t)z(t) - r_2(t)x(t)z''(t) - \\ {r_2}'(t)x(t)z'(t) = z(t)q(t)f\Big(y\big(\tau(t)\big)\Big), \end{aligned}$$

then add and subtract the term $r_2(t)x'(t)z'(t)$ from this equation, we get

$$\begin{aligned} r_2(t)x''(t)z(t) + r_2(t)x'(t)z'(t) + {r_2}'(t)x'(t)z(t) - r_2(t)x(t)z''(t) \\ - r_2(t)x'(t)z'(t) - {r_2}'(t)x(t)z'(t) = z(t)q(t)f\Big(y\big(\tau(t)\big)\Big), \end{aligned}$$

i.e.,

$$\Big(r_2(t)x'(t)z(t) - r_2(t)x(t)z'(t)\Big)' = z(t)q(t)f\Big(y\big(\tau(t)\big)\Big).$$

Now, integrate this equation from a to b, and since $r_2(t) > 0$, $z(t) > 0$, $x(a) = x(b) = 0$, $x'(a) \geq 0$, and $x'(b) \leq 0$, we get

$$0 \geq \int_a^b z(t)q(t)f\Big(y\big(\tau(t)\big)\Big)\, \mathrm{d}t,$$

which contradicts the fact that $z(t)$, $q(t)$, and $f\Big(y\big(\tau(t)\big)\Big)$ are positive. The proof is complete. $\square$

Consider the second order DDE

$$\Big(r_2(t)x'(t)\Big)' = Q(t)x\Big(\sigma(t)\Big), \tag{4.8}$$

where $r_2(t)$ as in H_3, $Q(t) \in C([t_0,\infty),\mathbb{R}^+)$, and $\sigma(t) \in C^1([t_0,\infty),\mathbb{R})$ such that $\sigma(t) \leq t$, $\sigma'(t) \geq 0$ for $t \geq t_0$ and $\sigma(t) \to \infty$ as $t \to \infty$.

This equation is used in Lemma 4.3 and 4.4 to obtain the results of Theorem 4.5.

Lemma 4.3. *If*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} Q(s)R_2\big(\sigma(t),\sigma(s)\big)\ ds > 1, \tag{4.9}$$

then all bounded solutions of equation (4.8) *are oscillatory.*

Proof. Suppose $x(t)$ is a bounded nonoscillatory solution of equation (4.8). Without loss of generality, let $x(t) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1 \geq t_0$.
From (4.8), $\big(r_2(t)x'(t)\big)' \geq 0$, and either $x'(t) < 0$ or $x'(t) > 0$ for some $t_2 \geq t_1$.

For $x'(t) > 0$, integrate $\big(r_2(t)x'(t)\big)' \geq 0$ from t_1^* to t_1, where $t_1^* \geq t_1$, then there exists a constant $c^* > 0$ such that

$$r_2(t)x'(t) \geq c^* \qquad \text{for} \qquad t \geq t_1^*.$$

Devide this inquality by $r_2(t)$ and take the integration from t_1^* to t, we get

$$x(t) \geq c^* R_1(t, t_1^*).$$

But $R_1(t,t_1^*) \to \infty$ as $t \to \infty$, i.e, $x(t) \to \infty$, which is a contradiction since $x(t)$ is bounded.

Now, for $x'(t) < 0$, and for $v \geq u \geq t_2$, integrate $x'(t)$ from v to u and since $\big(r_2(t)x'(t)\big)' \geq 0$, we get

$$\begin{aligned} x(u) - x(v) &= -\int_u^v x'(s)\ \mathrm{d}s \\ &= -\int_u^v \frac{r_2(s)x'(s)}{r_2(s)}\ \mathrm{d}s \\ &\geq \Big(\int_u^v \frac{1}{r_2(s)}\ \mathrm{d}s\Big)\Big(-r_2(v)x'(v)\Big) \\ &= R_2(v,u)\big(-r_2(v)x'(v)\big). \end{aligned}$$

For $t \geq s \geq t_2$, setting $u = \sigma(s)$ and $v = \sigma(t)$ in the last inequality, we get

$$x\big(\sigma(s)\big) \geq R_2\big(\sigma(t),\sigma(s)\big)\big(-r_2(\sigma(t))x'(\sigma(t))\big). \tag{4.10}$$

By integrating (4.8) from $\sigma(t) \geq t_2$ to t, we get

$$r_2(t)x'(t) - r_2(\sigma(t))x'(\sigma(t)) = \int_{\sigma(t)}^t Q(s)x(\sigma(s))\ \mathrm{d}s.$$

Since $x'(t) < 0$ and $\sigma(t) \leq t$, we get

$$-r_2(\sigma(t))x'(\sigma(t)) \geq \int_{\sigma(t)}^t Q(s)x(\sigma(s))\ \mathrm{d}s.$$

By substituting (4.10) in this inequality, we get

$$-r_2(\sigma(t))x'(\sigma(t)) \geq \left(\int\limits_{\sigma(t)}^{t} Q(s)R_2\big(\sigma(t),\sigma(s)\big) \ \mathrm{d}s \right)\left(-\, r_2\Big(\sigma(t)x'(\sigma(t)\Big) \right),$$

then

$$1 \geq \int\limits_{\sigma(t)}^{t} Q(s)R_2\big(\sigma(t),\sigma(s)\big) \ \mathrm{d}s.$$

By taking the lim sup as $t \to \infty$ of the last inequality, we get a contradiction with the assumption (4.9). □

Lemma 4.4. *If*

$$\limsup_{t\to\infty} \int\limits_{\sigma(t)}^{t} \left(\frac{1}{r_2(u)} \int\limits_{u}^{t} Q(s) \ \mathrm{d}s \right) \mathrm{d}u > 1, \tag{4.11}$$

then all bounded solutions of (4.8) *are oscillatory.*

Proof. See [16]. □

Theorem 4.5. *Let the condition* (4.6) *be satisfied and equation* (4.4) *be nonoscillatory. If there exist two functions* ρ *and* $\sigma \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\tau(t) \leq \sigma(t) < t, \qquad \sigma'(t) \geq 0, \qquad \text{and} \qquad \rho(t) > 0 \qquad \text{for} \qquad t \geq t_0$$

such that

$$\limsup_{t\to\infty} \int\limits_{t_1}^{t} \left[K\rho(s)q(s) - \frac{r_1\Big(\tau(s)\Big)\Big(\rho'(s)r_1(s) - \rho(s)p(s)R_2(\tau(s),t_1)\Big)^2}{4\rho(s)R_2(\tau(s),t_1)\tau'(s)r_1^2(s)} \right] \mathrm{d}s = \infty, \tag{4.12}$$

for all large t *and condition* (4.9) *or* (4.11) *holds with*

$$Q(t) = \left[Kq(t)R_1(\sigma(t),\tau(t)) - \frac{p(t)}{r_1(t)} \right] \geq 0 \qquad \text{for} \qquad t \geq t_1, \tag{4.13}$$

then equation (4.3) *is oscillatory.*

Proof. Let $y(t)$ be a nonoscillatory solution of (4.3). Without loss of generality, let $y(t) > 0$ and $y\big(\tau(t)\big) > 0$ for $t \geq t_1 \geq t_0$. By Lemma 4.2, we have two cases for $L_1 y(t)$.

Case 1:- $L_1 y(t) > 0$, then either $L_2 y(t) > 0$ or $L_2 y(t) < 0$.
If $L_2 y(t) < 0$, then there exists a constant $c^* < 0$ and $t^* \geq t_1$ such that

$$r_2(t)\Big(L_1 y(t)\Big)' = L_2 y(t) \leq c^* \qquad \text{for} \qquad t^* \leq t,$$

i.e.,

$$\Big(L_1 y(t)\Big)' \leq \frac{c^*}{r_2(t)} \qquad \text{for} \qquad t^* \leq t.$$

By integrating this inequality from t^* to t , we get

$$r_1(t) y'(t) \leq c^* \int_{t^*}^{t} \frac{1}{r_2(s)} \, \mathrm{d}s.$$

By condition (4.6), we conclude that $L_1 y(t) \to -\infty$ as $t \to \infty$. Thus there exist constants $c^{**} < 0$ and $t^{**} \geq t^*$ such that

$$y'(t) \leq \frac{c^{**}}{r_1(t)} \qquad \text{for} \qquad t^{**} \leq t.$$

By integrating this inequality from t^{**} to t , we get

$$y(t) \leq c^{**} \int_{t^{**}}^{t} \frac{1}{r_1(s)} \, \mathrm{d}s.$$

By condition (4.6), we conclude that $y(t) \to -\infty$ as $t \to \infty$, which is a contradiction with the fact that $y(t) > 0$ for $t \geq t_1$.

So, we have $L_2 y(t) > 0$, then $L_1 y(t) \geq L_1 y\big(\tau(t)\big)$ for $\tau(t) \leq t$.

Now, integrate $\big(L_1y(t)\big)'$ from t_1 to $\tau(t)$, we get

$$\begin{aligned} L_1y\big(\tau(t)\big) - L_1y(t_1) &= \int_{t_1}^{\tau(t)} \big(L_1y(s)\big)' \,\mathrm{d}s \\ &= \int_{t_1}^{\tau(t)} \frac{L_2y(s)}{r_2(s)} \,\mathrm{d}s \\ &\geq L_2y\big(\tau(t)\big) R_2\big(\tau(t), t_1\big), \end{aligned}$$

From equation (4.3), $L_3y(t) \leq 0$, then $L_2y\big(\tau(t)\big) \geq L_2y(t)$, and since $L_2y(t) > 0$, then $L_1y(t) \geq L_1y\big(\tau(t)\big) > 0$, so, the last inequality becomes

$$L_1y(t) \geq R_2\big(\tau(t), t_1\big) L_2y(t) \qquad \text{for} \qquad t \geq t_1.$$

Now, let us define the function

$$w(t) = \rho(t) \frac{L_2y(t)}{y\big(\tau(t)\big)} \qquad \text{for} \qquad t \geq t_1,$$

then

$$w'(t) = \rho(t) \frac{y\big(\tau(t)\big)\big(L_2y(t)\big)' - L_2y(t) y'\big(\tau(t)\big)\tau'(t)}{y^2\big(\tau(t)\big)} + \rho'(t) \frac{L_2y(t)}{y\big(\tau(t)\big)}.$$

By equation (4.3) and the last inequality we get

$$\begin{aligned} w'(t) &\leq -K\rho(t)q(t) - \rho(t)p(t) \frac{R_2\big(\tau(t), t_1\big)}{r_1(t)} - \frac{w^2(t)}{\rho(t)} \frac{R_2\big(\tau(t), t_1\big)\tau'(t)}{r_1\big(\tau(t)\big)} + \rho'(t) \frac{w(t)}{\rho(t)} \\ &= -K\rho(t)q(t) - \left[w^2(t) \frac{R_2\big(\tau(t), t_1\big)\tau'(t)}{r_1\big(\tau(t)\big)\rho(t)} - w(t) \frac{\rho'(t)}{\rho(t)} + \rho(t) \frac{p(t) R_2\big(\tau(t), t_1\big)}{r_1(t)} \right]. \end{aligned}$$

By completing the square for $w(t)$, we get

$$w'(t) \leq -K\rho(t)q(t) + \frac{r_1\big(\tau(t)\big)\Big(\rho'(t)r_1(t) - \rho(t)p(t)R_2\big(\tau(t),t_1\big)\Big)^2}{4\rho(t)R_2\big(\tau(t),t_1\big)\tau'(t)r_1^2(t)}.$$

By integrating this inequality from t_1 to t we get

$$w(t)-w(t_1) \leq \int_{t_1}^{t} \left[-K\rho(s)q(s)+\frac{r_1\big(\tau(s)\big)\Big(\rho'(s)r_1(s) - \rho(s)p(s)R_2\big(\tau(s),t_1\big)\Big)^2}{4\rho(s)R_2\big(\tau(s),t_1\big)\tau'(s)r_1^2(s)}\right] \mathrm{d}s.$$

Since $w(t) > 0$, this inequality becomes

$$\int_{t_1}^{t} \left[K\rho(s)q(s) - \frac{r_1\big(\tau(s)\big)\Big(\rho'(s)r_1(s) - \rho(s)p(s)R_2\big(\tau(s),t_1\big)\Big)^2}{4\rho(s)R_2\big(\tau(s),t_1\big)\tau'(s)r_1^2(s)}\right] \mathrm{d}s \leq w(t_1),$$

which contradicts the condition (4.12).

Case 2:- $L_1y(t) < 0$, then either $L_2y(t) > 0$ or $L_2y(t) < 0$.
Now, integrate $\Big(L_1y'(t)\Big)'$ from $t_2 \geq t_1$ to t, we get

$$r_1(t)y'(t) - r_1(t_2)y'(t_2) = \int_{t_2}^{t} \frac{L_2y(s)}{r_2(s)} \mathrm{d}s.$$

Since $r_2(t) > 0$, then there exist $c_* > 0$ such that $\int_{t_2}^{t} \frac{1}{r_2(s)} \mathrm{d}s > c_*$. Also $L_1y(t) < 0$, this inequality becomes

$$y'(t) \leq \frac{c_* L_2y(t_2)}{r_1(t)}, \qquad t \geq t_2.$$

Again, integrate this inequality from $t_2 \geq t_1$ to t,and since $y(t) > 0$, we get

$$y(t) \leq c_* L_2y(t_2) \int_{t_2}^{t} \frac{1}{r_1(s)} \mathrm{d}s.$$

By condition (4.5) and since $L_2y(t) < 0$, we conclude that $y(t) < 0$, which is a contradiction.

Now, let $y(t) > 0$, $L_1y(t) < 0$ and $L_2y(t) \geq 0$ for all large t, say $t \geq t_3 \geq t_2$. For $v \geq u \geq t_3$, integrate $y'(t)$ from v to u, we get

$$y(u) - y(v) = \int_v^u \frac{1}{r_1(s)}\big(r_1(s)y'(s)\big)\ \mathrm{d}s$$

$$\geq -L_1y(v)\int_u^v \frac{1}{r_1(s)}\ \mathrm{d}s$$

$$= -L_1y(v)R_1(v,u).$$

Setting $u = \tau(t)$ and $v = \sigma(t)$, and since $y(t) > 0$, we get

$$y\big(\tau(t)\big) \geq x\big(\sigma(t)\big)R_1\big(\sigma(t),\tau(t)\big) \qquad \text{for} \qquad t \geq t_3,$$

where $x(t) = -L_1y(t) > 0$ for $t \geq t_3$.
By (H_4) and the last inequality, we get

$$f\Big(y\big(\tau(t)\big)\Big) \geq ky\big(\tau(t)\big) \geq kx\big(\tau(t)\big)R_1\big(\sigma(t),\tau(t)\big). \tag{4.14}$$

Now,

$$x'(t) = -\Big(L_1y(t)\Big)' = -\frac{L_2y(t)}{r_2(t)},$$

and since $r_2(t) > 0$ and $L_2y(t) \geq 0$, we conclude that $x(t)$ is decreasing.

Substituting $x(t)$ in equation (4.3), we get

$$\Big(r_2(t)x'(t)\Big)' + \frac{p(t)}{r_1(t)}x(t) = q(t)f\Big(y\big(\tau(t)\big)\Big).$$

By (4.14) and since $x(t)$ is decreasing for $\tau(t) \leq \sigma(t) \leq t$, this equation becomes

$$\Big(r_2(t)x'(t)\Big)' + \frac{p(t)}{r_1(t)}x\big(\sigma(t)\big) \geq kq(t)x\big(\sigma(t)\big)R_1\big(\sigma(t),\tau(t)\big),$$

i.e,

$$\Big(r_2(t)x'(t)\Big)' \geq kq(t)x\big(\sigma(t)\big)R_1\big(\sigma(t),\tau(t)\big) - \frac{p(t)}{r_1(t)}x\big(\sigma(t)\big) \qquad \text{for} \qquad t \geq t_3.$$

The rest of the proof is similar to that of Lemma 4.3 and Lemma 4.4 and hence is omitted. □

Remark 4.6. From the proof of Theorem 4.5 we obtain

$$w'(t) \leq -K\rho(t)q(t) + \frac{r_1\big(\tau(t)\big)\Big(P(t,t_1)\Big)^2}{4\rho(t)R_2\big(\tau(t),t_1\big)\tau'(t)r_1^2(t)},$$

where

$$P(t,t_1) = \rho'(t)r_1(t) - \rho(t)p(t)R_2\big(\tau(t),t_1\big).$$

1. If $P(t,t_1) \geq 0$ for $t \geq t_3$, we get

$$\rho'(t)r_1(t) \geq P(t,t_1) \qquad \text{for} \qquad t \geq t_3,$$

   and hence

$$w'(t) \leq -K\rho(t)q(t) + \frac{r_1\big(\tau(t)\big)\Big(\rho'(t)r_1(t)\Big)^2}{4\rho(t)R_2\big(\tau(t),t_1\big)\tau'(t)r_1^2(t)} \qquad \text{for} \qquad t \geq t_3.$$

   Therefore, the condition (4.12) can be replaced by

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[K\rho(s)q(s) - \frac{r_1\Big(\tau(s)\Big)\Big(\rho'(s)r_1(s)\Big)^2}{4\rho(s)R_2(\tau(s),t_1)\tau'(s)r_1^2(s)}\right] \mathrm{d}s = \infty, \qquad (4.15)$$

   for all large t.

2. If $P(t,t_1) \le 0$ for $t \ge t_3$, the condition (4.12) can be replaced by

$$\int_{t_1}^{\infty} \rho(s)q(s)\,\mathrm{d}s = \infty \tag{4.16}$$

for all large t.

3. If $\rho'(t) \le 0$ for $t \ge t_3$, the condition (4.12) can be replaced by

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[K\rho(s)q(s) - \rho(s)\frac{R_2\big(\tau(t),t_1\big)}{r_1(s)} \right] \mathrm{d}s = \infty, \tag{4.17}$$

for all large t.

The following theorem, Theorem 4.7, gives an alternative comparison results of Theorem 4.5 by relating it to a first-order DDE.

Theorem 4.7. *If in Theorem 4.5 the condition (4.12) is replaced by the first-order DDE*

$$w'(t) + \left[\frac{p(t)}{r_1(t)} R_2\big(\tau(t),t_1\big) + Kq(t) \int_{t_1}^{t} \frac{R_2\big(\tau(s),t_1\big)}{r_1(s)}\,\mathrm{d}s \right] w\big(\tau(t)\big) = 0, \tag{4.18}$$

is oscillatory, then the conclusion of Theorem 4.5 holds.

Proof. Let $y(t)$ be a nonoscillatory solution of (4.3) for $t \ge t_1$. Without loss of generality, we may assume that $y(t) > 0$ and $y\big(\tau(t)\big) > 0$ for $t \ge t_1$.
By Lemma 4.2, $L_1y(t) > 0$ or $L_1y(t) < 0$ for $t \ge t_1$.

Case 1 :- $L_1y(t) > 0$ for $t \ge t_1$.
Similar to the proof of Theorem 4.5, we can show that $L_2y(t) > 0$ for $t \ge t_1$, and obtain

$$r_1(t)y'(t) = L_1y(t) \ge R_2\big(\tau(t),t_1\big)L_2y\big(\tau(t)\big) \quad \text{for} \quad t \ge t_1. \tag{4.19}$$

By dividing this inequality by $r_1(t)$ and integrating it from t_1 to t, we get

$$y(t) - y(t_1) \geq \int\limits_{t_1}^{t} \frac{R_2\big(\tau(s), t_1\big) L_2 y\big(\tau(s)\big)}{r_1(s)} \, \mathrm{d}s.$$

Since $y(t) > 0$ and $L_2 y(t)$ is decreasing, we conclude that

$$y(t) \geq L_2 y\big(\tau(t)\big) \int\limits_{t_1}^{t} \frac{R_2\big(\tau(s), t_1\big)}{r_1(s)} \, \mathrm{d}s.$$

By using this inequality and (4.19) in equation (4.3) and letting $w(t) = L_2 y(t) > 0$, we get

$$w'(t) + \frac{p(t)}{r_1(t)} R_2\big(\tau(t), t_1\big) w\big(\tau(t)\big) + K q(t) \left[\int\limits_{t_1}^{t} \frac{R_2\big(\tau(s), t_1\big)}{r_1(s)} \, \mathrm{d}s \right] w\big(\tau(t)\big) \leq 0.$$

This inequality has a positive solution $w(t)$. It follows from Theorem 2.6.3 in [2] that equation (4.18) has a positive solution, which is a contradiction. □

Case 2 :- $L_1 y(t) < 0$ for $t \geq t_1$.

The proof of this case is similar to Theorem 4.5, therefore it is omitted.

Corollary 4.8. *If in Theorem 4.5 the condition (4.12) is replaced by*

$$\liminf_{t \to \infty} \int\limits_{\tau(t)}^{t} \left[\frac{p(u) R_2\big(\tau(u), t_1\big)}{r_1(u)} + K q(u) \int\limits_{t_1}^{u} \frac{R_2\big(\tau(s), t_1\big)}{r_1(s)} \, \mathrm{d}s \Bigg) \right] \mathrm{d}u \geq \frac{1}{e}, \tag{4.20}$$

then the conclusion of Theorem 4.5 holds.

Proof. By Lemma 2.18, the equation (4.18)is oscillatory when the condition (4.20) is satisfied. □

After we discussed the oscillation results of equation (4.3) when equation (4.4) is nonoscillatory, in the next theorem, we will give a result when it is oscillatory.

Theorem 4.9. *Let condition (4.6) holds and equation (4.4) is oscillatory. If there exist a function $\sigma \in C([t_0, \infty), \mathbb{R})$ such that $\tau(t) \leq \sigma(t) \leq t$ and $\sigma'(t) \geq 0$ for $t \geq t_0$ such that (4.9) or (4.11) holds with $Q(t)$ is as in Theorem 4.5, then for every solution $y(t)$ of equation (4.3) either $y(t)$ is oscillatory or $y'(t)$ is oscillatory.*

Proof. Suppose $y(t)$ is a nonoscillatory solution of (4.3) for $t \geq t_1$. Without loss of generality, we may assume that $y(t) > 0$ and $y\big(\tau(t)\big) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$.
Now, we have two cases for $L_1 y(t)$.

Case 1 :- $L_1 y(t) > 0$ for $t \geq t_1$.
In this case, by letting $x(t) = L_1 y(t) > 0$, equation (4.3) becomes

$$\Big(r_2(t)x'(t)\Big)' + \frac{p(t)}{r_1(t)}x(t) + q(t)f\Big(y\big(\tau(t)\big)\Big) = 0.$$

But $q(t)f\Big(y\big(\tau(t)\big)\Big) > 0$ since $q(t) > 0$ and $y\big(\tau(t)\big)$, i.e the previous equation becomes

$$\Big(r_2(t)x'(t)\Big)' + \frac{p(t)}{r_1(t)}x(t) \leq 0 \qquad \text{for} \qquad t \geq t_2 \geq t_1.$$

By Lemma 2.6 in [15], equation (4.4) has a positive solution, which is a contradiction.

Case 2 :- $L_1 y(t) > 0$ for $t \geq t_1$.
The proof of this case is similar to Theorem (4.5), therefore it is omitted. □

Example 4.1. *Consider the third-order DDE*

$$y'''(t) + \frac{1}{2}y'(t) + \frac{1}{2}y(t - \frac{5\pi}{2}) = 0. \tag{4.21}$$

Here we have

$$r_1(t) = r_2(t) = 1, \qquad p(t) = q(t) = \frac{1}{2},$$

$$f(u) = u, \qquad K = 1, \qquad \tau(t) = \left(t - \frac{5\pi}{2}\right).$$

Its auxiliary equation

$$z''(t) + \frac{1}{2}z(t) = 0$$

is oscillatory with general solution $z(t) = c_1 cos\frac{1}{2}t + c_2 sin\frac{1}{2}t$. *Condition* (4.6) *holds since*

$$R_1(t, t_0) = R_2(t, t_0) = t - t_0 \to \infty \quad as \quad t \to \infty.$$

Let $\sigma(t) = t - 2\pi$, *here we have* $R_1\left(t - 2\pi, t - \dfrac{5\pi}{2}\right)$ *and* $Q(t) = \dfrac{\pi}{2}$. *Also condition* (4.11) *is satisfied since*

$$\limsup_{t\to\infty} \int\limits_{t-2\pi}^{t} \int\limits_{u}^{t} Q(s)\ \mathrm{d}s\ \mathrm{d}u = \pi^3 > 1.$$

All hypotheses of Theorem 4.9 are satisfied and therefore every solution $y(t)$ *of equation* (4.21) *is oscillatory or* $y'(t)$ *is oscillatory. One such solution is*
$y(t) = \cos t$.

Example 4.2. *Consider the equation*

$$y'''(t) + e^{-t}y'(t) + \left(1 - e^{-t}\right)y\left(t - \frac{5\pi}{2}\right) = 0. \tag{4.22}$$

Here we have

$$r_1(t) = r_2(t) = 1, \qquad p(t) = e^{-t}, \qquad q(t) = 1 - e^{-t}$$

$$f(u) = u, \qquad \tau(t) = t - \frac{5\pi}{2}, \qquad K = 1.$$

Its auxiliary equation is given by

$$z''(t) + e^{-t}z(t) = 0,$$

which is nonoscillatory. See [23].

In Theorem 4.5, for

$$\rho(t) = 1, \qquad \sigma(t) = t - 2\pi, \qquad Q(t) = (1 - e^{-t})\frac{\pi}{2} - e^{-t},$$

we have

$$\limsup_{t\to\infty} \int_{t-2\pi}^{t} Q(s)(t-s)\ \mathrm{d}s = \pi^3 > 1,$$

and the condition (4.12)*is reduced to*

$$\limsup_{t\to\infty} t = \infty,$$

i.e., the conditions of Theorem 4.5 are fulfilled, therefore, the equation (4.22) *is oscillatory having* $y(t) = \sin t$ *as a solution.*

4.2 More oscillation results for third order nonlinear DDE with a middle term

In this section we will discuss the asymptotic behavior of solutions of the third order nonlinear functional differential equation

$$\left(r_2(t)\left(r_1(t)\left(y'(t)\right)^{\gamma}\right)'\right)' + p(t)\left(y'(t)\right)^{\gamma} + q(t)f\left(y\left(\tau(t)\right)\right) = 0, \tag{4.23}$$

by relating it to the solutions of the auxiliary second order differential equation

$$\left(r_2(t)v'(t)\right)' + \frac{p(t)}{r_1(t)}v(t) = 0. \tag{4.24}$$

Throughout the section, the following hypotheses are assumed:

(H_1) $p, q, \in C([t_0, \infty], \mathbb{R})$ such that $p(t) \geq 0$, $q(t) > 0$, for all $t \geq t_0 \geq 0$.

(H_2) $\tau \in C^1([t_0,\infty],\mathbb{R})$ such that $0 < \tau(t) \le t$, $\tau'(t) > 0$ for all $t \ge t_0 \ge 0$ and $\lim_{t\to\infty} \tau(t) = \infty$.

(H_3) $r_i(t) > 0$, $r_i(t) \in C([t_0,\infty],\mathbb{R})$ for $i = 1, 2$,

and $\int\limits_{t_0}^{t} \frac{1}{r_2(s)}\,\mathrm{d}s \to \infty$ as $t \to \infty$.

(H_4) $f \in C(\infty,\infty)$ such that $xf(x) > 0$, $f'(x) \ge 0$ for $x \ne 0$, and $-f(-xy) \ge f(xy) \ge f(x)f(y)$ for $xy > 0$.

(H_5) γ is the quotient of two positive odd integers.

A solution of equation (4.23) is a function $y(t)$ such that $r_2(t)\Big(r_1(t)\big(y'(t)\big)^{\gamma}\Big)' \in C^1[t_y,\infty)$ where $y(t)$ satisfies equation (4.23) on $[t_y,\infty)$ for every $t \ge t_y \ge t_0$.

We only consider those solutions such that the condition $\sup\{|y(t)| : t_1 \le t < \infty\} > 0$ is satisfied on $[t_0,\infty)$.

The results of this section are due to Hou et. al. [14].

Note: We say that the equation (4.23) has the *property* (P) if all of its nonoscillatory solutions $y(t)$ satisfy the condition

$$y(t)y'(t) < 0.$$

Theorem 4.10. *Let* (4.24) *possess a positive solution* $v(t)$*. Then the operator*

$$Ly(t) = \left(r_2(t)\left(r_1(t)\left(y'(t)\right)^{\gamma}\right)'\right)' + p(t)\left(y'(t)\right)^{\gamma}$$

can be represented as

$$Ly(t) = \frac{1}{v(t)}\left(r_2(t)v^2(t)\left(\frac{r_1(t)}{v(t)}\left(y'(t)\right)^{\gamma}\right)'\right)'.$$

Proof. Differentiate $Ly(t)$ and use equation (4.24) to get the result. $\square$

Corollary 4.11. *If $v(t)$ is a positive solution of (4.24), then (4.23) can be written as the binomial equation*

$$\left(r_2(t)v^2(t)\left(\frac{r_1(t)}{v(t)}\left(y'(t)\right)^{\gamma}\right)'\right)' + q(t)v(t)f\Big((\tau(t))\Big) = 0. \tag{4.25}$$

Note: Equation (4.25) is said to be in the canonical form if the following conditions are satisfied

$$\int_t^{\infty} \frac{1}{r_2(s)v^2(s)}\, \mathrm{d}s = \infty, \tag{4.26}$$

$$\int_t^{\infty} \left(\frac{v(s)}{r_1(s)}\right)^{\frac{1}{\gamma}} \mathrm{d}s = \infty. \tag{4.27}$$

The next two lemma's are given to insure the existence of a nonoscillatory solution of (4.24).

Lemma 4.12. *Assume that*

$$\frac{r_2(t)}{r_1(t)}p(t)\int_{t_0}^{t} r_2(s)\, \mathrm{d}s \leq \frac{1}{4}, \qquad for \quad t \geq t_0. \tag{4.28}$$

Then (4.24) possesses a positive solution.

Lemma 4.13. *[14] Assume that (4.28) is fulfilled, then (4.24) always possesses a nonoscillatory solution satisfying (4.26)*

The Next lemma is a direct conclusion of Lemma 4.12 and 4.13.

Lemma 4.14. *Let (4.28) hold. Then the trinomial equation (4.23) can be written in its binomial form (4.25). Moreover, if (4.28) is satisfied, then (4.25) is in canonical form.*

The Generalized Kiguraze's Lemma 2.2 is used to study nonoscillatory solutions of (4.23) in the following lemma.

Lemma 4.15. *Let (4.28) hold and assume that $v(t)$ is such a positive solution of (4.24) that satisfies (4.26). If (4.27) holds, then every positive solution of (4.25) is either of degree 0, that is*

$$\begin{aligned}
\frac{r_1(t)}{v(t)}\Big(y'(t)\Big)^{\gamma} &< 0,\\
r_2(t)v^2(t)\left(\frac{r_1(t)}{v(t)}\Big(y'(t)\Big)^{\gamma}\right)' &> 0,\\
\left(r_2(t)v^2(t)\left(\frac{r_1(t)}{v(t)}\Big(y'(t)\Big)^{\gamma}\right)'\right)' &< 0,
\end{aligned}$$

or of degree 2, that is,

$$\begin{aligned}
\frac{r_1(t)}{v(t)}\Big(y'(t)\Big)^{\gamma} &> 0,\\
r_2(t)v^2(t)\left(\frac{r_1(t)}{v(t)}\Big(y'(t)\Big)^{\gamma}\right)' &> 0,\\
\left(r_2(t)v^2(t)\left(\frac{r_1(t)}{v(t)}\Big(y'(t)\Big)^{\gamma}\right)'\right)' &< 0,
\end{aligned}$$

The classes of positive solutions of (4.25) are denoted by N_0 and N_2, respectively. The set N of all positive solutions of (4.25), when it is in the canonical form, has the following decomposition

$$N = N_0 \cup N_2.$$

Lemma 4.16. *[38] If $y(t)$ is a positive and strictly decreasing solution of the integral inequality*

$$y(t) \geq \int_t^{\infty} \frac{(s-t)^{n-1}}{(n-1)!} f(s; y[\sigma_1(s), ..., y[\sigma_m(s)]) \,\mathrm{d}s,$$

then there exists a positive solution $x(t)$ of the differential equation

$$(-1)^n x^{(n)}(t) = f(t; y[\sigma_1(t), ..., y[\sigma_m(t)]), \quad t \geq t_0, \quad \textit{for} \quad f \in C([t_0, \infty), \mathbb{R}^+], \quad , f' > 0$$

for

$$\sigma_j \in C[t_0, \infty) \quad \textit{and} \quad \sigma_j(t) < t \quad \textit{where} \quad \lim_{t\to\infty} \sigma_j(t) = \infty, \quad \textit{for} \quad j = 1, ..., m,$$

such that $x(t) \leq y(t)$ for all large t and satisfying

$$\lim_{t\to\infty} x^i = 0 \quad \textit{monotonically for} \quad i = 1, ..., n-1.$$

Next, we presents criteria for property (P) of (4.23).

Theorem 4.17. *Let (4.28) hold and assume that $v(t)$ is such positive solution of (4.24) that satisfies (4.26) and (4.27). If the first order nonlinear differential equation*

$$z'(t) + Q(t) f\left(z^{\frac{1}{\gamma}}\left(\tau(t)\right)\right) = 0, \tag{4.29}$$

where,

$$Q(t) = q(t)v(t) f\left(\int_{t_1}^{\tau(t)} \left(\frac{v(s)}{r_1(s)} \int_{t_1}^{s} \frac{1}{r_2(u)v^2(u)} \,\mathrm{d}u\right)^{\frac{1}{\gamma}} \mathrm{d}s\right), \tag{4.30}$$

is oscillatory, then (4.23) has property (P).

Proof. Let (4.23) has an eventually positive solution $y(t)$. Then $y(t)$ is also solution of (4.25).

By Lemma 4.15, $y(t)$ is either of degree 0 or 2. If $y(t) \in N_2$, then

$$z(t) = r_2(t)v^2(t)\left(\frac{r_1(t)}{v(t)}\Big(y'(t)\Big)^{\gamma}\right)' > 0$$

is decreasing.

Integrate the term

$$\left(\frac{r_1(t)}{v(t)}\Big(y'(t)\Big)^{\gamma}\right)' = \frac{z(t)}{r_2(t)v^2(t)}$$

from t_1 to t, and since $z(t) > 0$, $z'(t) < 0$, we get

$$\frac{r_1(t)}{v(t)}\Big(y'(t)\Big)^{\gamma} \geq \int_{t_1}^{t} \frac{z(u)}{r_2(u)v^2(u)}\,\mathrm{d}u \geq z(t)\int_{t_1}^{t} \frac{1}{r_2(u)v^2(u)}\,\mathrm{d}u.$$

From the above inequality, we have

$$y'(t) \geq \left(\frac{v(t)}{r_1(t)}z(t)\int_{t_1}^{t} \frac{1}{r_2(u)v^2(u)}\,\mathrm{d}u\right)^{\frac{1}{\gamma}}.$$

Again, integrate this inequality from t_1 to t. Since $y(t) > 0$ and $z(t) < 0$, we have

$$y(t) \geq \int_{t_1}^{t}\left(\frac{v(s)}{r_1(s)}z(s)\int_{t_1}^{s} \frac{1}{r_2(u)v^2(u)}\,\mathrm{d}u\right)^{\frac{1}{\gamma}}\mathrm{d}s.$$

$$\geq z^{\frac{1}{\gamma}}(t)\int_{t_1}^{t}\left(\frac{v(s)}{r_1(s)}\int_{t_1}^{s} \frac{1}{r_2(u)v^2(u)}\,\mathrm{d}u\right)^{\dfrac{1}{\gamma}}\mathrm{d}s$$

Since f is increasing, and $q(t), v(t) > 0$, then

$$q(t)v(t)f\Big(y\big(\tau(t)\big)\Big) \geq q(t)v(t)f\left(z^{\frac{1}{\gamma}}\big(\tau(t)\big)\int_{t_1}^{\tau(t)}\left(\frac{v(s)}{r_1(s)}\int_{t_1}^{s} \frac{1}{r_2(u)v^2(u)}\,\mathrm{d}u\right)^{\frac{1}{\gamma}}\mathrm{d}s\right).$$

By equation (4.25) and (H_4) , the last inequality becomes

$$-z'(t) \geq q(t)v(t)f\left(\int_{t_1}^{\tau(t)}\left(\frac{v(s)}{r_1(s)}\int_{t_1}^{s}\frac{1}{r_2(u)v^2(u)}\,\mathrm{d}u\right)^{\frac{1}{\gamma}}\mathrm{d}s\right)f\left(z^{\frac{1}{\gamma}}(\tau(t))\right).$$

Therefore, $z(t)$ is a positive solution of the differential inequality

$$z'(t) + Q(t)f\left(z^{\frac{1}{\gamma}}(\tau(t))\right) \leq 0.$$

By Lemma 4.16, the equation (4.29) also has a positive solution, which contradicts our assumption that it is oscillatory. Hence, $y(t)$ is of degree 0, from which, we can conclude that equation (4.23) has property (P). □

Corollary 4.18. *Let* (4.28) *hold and assume that* $v(t)$ *is such positive solution of* (4.24) *that* (4.26) *and* (4.27) *are satisfied. Let* $f(u) = u^\gamma$*. If*

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} q(u)v(u)\left(\int_{t_1}^{\tau(u)}\left(\frac{v(s)}{r_1(s)}\int_{t_1}^{s}\frac{1}{r_2(x)v^2(x)}\,\mathrm{d}x\right)^{\frac{1}{\gamma}}\mathrm{d}s\right)^{\gamma}\mathrm{d}u > \frac{1}{e}, \tag{4.31}$$

then (4.23) *has property* (p).

Proof. It is well-known that (4.29) is oscillatory when condition (4.31) is satisfied. □

Now, we are ready to provide criteria to see when the solution of (4.23) oscillates and when it converges to zero.

Lemma 4.19. *Assume that equation* (4.23) *possesses property* P*. If*

$$\int_{t_0}^{\infty}\left(\frac{v(u)}{r_1(u)}\int_{u}^{\infty}\frac{1}{r_2(s)v^2(s)}\int_{s}^{\infty}v(x)q(x)\,\mathrm{d}x\,\mathrm{d}s\right)^{\frac{1}{\gamma}}\mathrm{d}u = \infty, \tag{4.32}$$

then every nonoscillatory solution of (4.23) *tends to zero as* $t \to \infty$.

Proof. Let $y(t)$ be an eventually positive solution of (4.23). Since (4.23) possesses property P and $y(t) > 0$, then $y'(t) < 0$. Also, $y(t)$ is a solution of (4.25) of degree 0.

Now, $y(t)$ is positive and decreasing, we conclude that

$$\lim_{t\to\infty} y(t) = l \geq 0.$$

We want to show that $l = 0$. Suppose not, i.e. $l > 0$. From (4.25), since f is increasing, we have

$$-\left(r_2(t)v^2(t)\left(\frac{r_1(t)}{v(t)}\left(y'(t)\right)^{\gamma}\right)'\right)' = v(t)q(t)f\left(y\left(\tau(t)\right)\right)$$

$$\geq f(l)v(t)q(t).$$

Integrate this inequality from t to ∞ and since $y(t)$ is of degree 0, we get

$$(r_2(t)v^2(t)\left(\frac{r_1(t)}{v(t)}\left(y'(t)\right)^{\gamma}\right)' \geq f(l)\int_t^{\infty} v(x)q(x)\ \mathrm{d}x,$$

then

$$\left(\frac{r_1(t)}{v(t)}\left(y'(t)\right)^{\gamma}\right)' \geq \frac{f(l)}{r_2(t)v^2(t)}\int_t^{\infty} v(x)q(x)\ \mathrm{d}x.$$

Again, integrate the last inequality from t to ∞, we get

$$-\frac{r_1(t)}{v(t)}\left(y'(t)\right)^{\gamma} \geq f(l)\int_t^{\infty}\frac{1}{r_2(s)v^2(s)}\int_s^{\infty} v(x)q(x)\ \mathrm{d}x\mathrm{d}s,$$

then

$$y'(t) \geq f^{\frac{1}{\gamma}}(l)\left(\frac{v(t)}{r_1(t)}\int_t^{\infty}\frac{1}{r_2(s)v^2(s)}\int_s^{\infty} v(x)q(x)\ \mathrm{d}x\mathrm{d}s\right)^{\frac{1}{\gamma}}.$$

The integration of the last inequality from t_1 to t yields

$$y(t) \leq y(t_1) - f^{\frac{1}{\gamma}}(l) \int_{t_1}^{t} \left(\frac{v(u)}{r_1(u)} \int_{t}^{\infty} \frac{1}{r_2(s)v^2(s)} \int_{s}^{\infty} v(x)q(x) \, \mathrm{d}x\mathrm{d}s \right)^{\frac{1}{\gamma}} \mathrm{d}u.$$

By (4.32), let $t \to \infty$, we conclude that $\lim_{t\to\infty} y(t) = -\infty$, which contradicts that $y(t) > 0$. Hence, we conclude that $l = 0$. □

Theorem 4.20. *Let (4.28) holds and assume that $v(t)$ is such positive solution of (4.24) that (4.26) and (4.27) are satisfied. Suppose that there exists a function $\zeta \in C^1([t_0, \infty))$ such that*

$$\zeta'(t) \geq 0, \qquad \zeta(t) > t, \qquad \eta(t) = \tau(\zeta(\zeta(t))) < t.$$

If both the first order delay differential equations (4.29) and

$$z'(t) + \left(\frac{v(t)}{r_1(t)} \int_{t}^{\zeta(t)} \frac{1}{r_2(s)v^2(s)} \int_{t}^{\zeta(s)} v(x)q(x) \, \mathrm{d}x\mathrm{d}s \right)^{\frac{1}{\gamma}} f^{\frac{1}{\gamma}}(z(\eta(t)) = 0 \tag{4.33}$$

are oscillatory, then (4.23) is oscillatory.

Proof. Let $y(t)$ be an eventually positive solution of (4.23). By Lemma 4.15, $y(t)$ is either of degree 0 or 2. From the proof of Theorem 4.17, we conclude that $y(t)$ is of degree 0 (i.e., by Theorem 4.17 $y'(t) < 0$).

Integrate equation (4.25) from t to $\zeta(t)$, and since $y(t)$ is of degree 0 and decreasing, we get

$$r_2(t)v^2(t)\left(\frac{r_1(t)}{v(t)} \Big(y'(t)\Big)^{\gamma} \right)' \geq \int_{t}^{\zeta(t)} v(x)q(x)f\Big(y\big(\tau(t)\big)\Big) \, \mathrm{d}x$$

$$\geq f\Big(y\Big(\tau\big(\zeta(t)\big)\Big)\Big) \int_{t}^{\zeta(t)} v(x)q(x) \, \mathrm{d}x,$$

i.e.,

$$\left(\frac{r_1(t)}{v(t)}\Big(y'(t)\Big)^{\gamma}\right)' \geq \frac{f\Big(y\Big(\tau\big(\zeta(t)\big)\Big)\Big)}{r_2(t)v^2(t)} \int\limits_{t}^{\zeta(t)} v(x)q(x)\ \mathrm{d}x.$$

Again, integrate the last inequality from from t to $\zeta(t)$, and since $y(t)$ is of degree 0, and $\eta(t) < t$, we conclude

$$-\frac{r_1(t)}{v(t)}\Big(y'(t)\Big)^{\gamma} \geq \int\limits_{t}^{\zeta(t)} \frac{f\Big(y\Big(\tau\big(\zeta(s)\big)\Big)\Big)}{r_2(s)v^2(s)} \int\limits_{s}^{\zeta(s)} v(x)q(x)\ \mathrm{d}x\ \mathrm{d}s$$

$$\geq f\Big(y\big(\eta(t)\big)\Big) \int\limits_{t}^{\zeta(t)} \frac{1}{r_2(s)v^2(s)} \int\limits_{s}^{\zeta(s)} v(x)q(x)\ \mathrm{d}x\ \mathrm{d}s,$$

i.e.,

$$-y'(t) \geq \left(\frac{v(t)}{r_1(t)} f\Big(y\big(\eta(t)\big)\Big) \int\limits_{t}^{\zeta(t)} \frac{1}{r_2(s)v^2(s)} \int\limits_{s}^{\zeta(s)} v(x)q(x)\ \mathrm{d}x\ \mathrm{d}s\right)^{\frac{1}{\gamma}}.$$

Now, integrate this inequality from t to ∞, and since $y(t) > 0$, we get

$$y(t) \geq \int\limits_{t}^{\infty} \left(\frac{v(u)}{r_1(u)} f\Big(y\big(\eta(u)\big)\Big) \int\limits_{u}^{\zeta(u)} \frac{1}{r_2(s)v^2(s)} \int\limits_{s}^{\zeta(s)} v(x)q(x)\ \mathrm{d}x\ \mathrm{d}s\right)^{\frac{1}{\gamma}} \mathrm{d}u.$$

Let $z(t)$ equals the right-hand side of the above inequality. Then $y(t) \geq z(t) > 0$, and by direct computation of $z'(t)$ we conclude that

$$0 = z'(t) + \left(\frac{v(t)}{r_1(t)} \int\limits_{t}^{\zeta(t)} \frac{1}{r_2(s)v^2(s)} \int\limits_{s}^{\zeta(s)} v(x)q(x)\ \mathrm{d}x\ \mathrm{d}s\right)^{\frac{1}{\gamma}} f^{\frac{1}{\gamma}}\Big(y\big(\eta(t)\big)\Big)$$

$$\geq z'(t) + \left(\frac{v(t)}{r_1(t)} \int\limits_{t}^{\zeta(t)} \frac{1}{r_2(s)v^2(s)} \int\limits_{s}^{\zeta(s)} v(x)q(x)\ \mathrm{d}x\ \mathrm{d}s\right)^{\frac{1}{\gamma}} f^{\frac{1}{\gamma}}\Big(z\big(\eta(t)\big)\Big).$$

This inequality has a positive solution $z(t)$. By Theorem 1 of Philos [38], the equation (4.33) has also a positive solution $z(t)$, which contradicts our assumption. Hence, $y(t)$ is neither of degree 0 nor 2, i.e., (4.23) is oscillatory. $\square$

Corollary 4.21. *Let (4.28) holds and assume that $v(t)$ is such positive solution of (4.24) that (4.26) and (4.27) are satisfied. Let $f(u) = u^\gamma$. Suppose that there exists a function $\zeta \in C^1([t_0, \infty))$ such that*

$$\zeta'(t) \geq 0, \qquad \zeta(t) > t, \qquad \eta(t) = \tau(\zeta(\zeta(t))) < t.$$

If, moreover, (4.31) is satisfied and

$$\liminf_{t\to\infty} \int_{\eta(t)}^{t} \frac{v(u)}{r_1(u)} \int_{u}^{\zeta(u)} \frac{1}{r_2(s)v^2(s)} \int_{s}^{\zeta(s)} v(x)q(x)\, \mathrm{d}x\, \mathrm{d}s \Bigg)^{\frac{1}{\gamma}} \mathrm{d}u > \frac{1}{e} \tag{4.34}$$

holds, then (4.23) is oscillatory.

Proof. By Lemma 2.18, the equation (4.33) is oscillatory when the condition (4.34) is satisfied. Moreover, the condition (4.31) implies that the equation (4.29) is oscillatory. By Theorem 4.20, we get the result. $\square$

Example 4.3. *[14] Consider the differential equation*

$$\left(t^{\frac{1}{4}})\left(y'(t)\right)^{\frac{1}{3}}\right)'' + \frac{3}{16t^{\frac{7}{4}}}\left(y'(t)\right)^{\frac{1}{3}} + \frac{a}{t^{\frac{25}{12}}}y^{\frac{1}{3}})(\lambda t) = 0, \quad \lambda \in (0,1). \tag{4.35}$$

Here we have

$$r_1(t) = t^{\frac{1}{4}}, \qquad r_2(t) = 1, \qquad p(t) = \frac{3}{16t^{\frac{7}{4}}}$$

$$q(t) = \frac{a}{t^{\frac{25}{12}}}, \qquad \tau(t) = \lambda t, \qquad \gamma = \frac{1}{3}.$$

The auxiliary equation is given by

$$v''(t) + \frac{3}{16t^2}v(t) = 0 \tag{4.36}$$

with positive solutions $v_1(t) = t^{\frac{1}{4}}$ *and* $v_2(t) = t^{\frac{3}{4}}$. *The solution* $v_1(t)$ *satisfies the conditions* (4.26) *and* (4.27) *while* $v_2(t)$ *does not, so, we take the solution* $v(t) = t^{\frac{1}{4}}$.

By condition (4.31) *in Corollary 4.18 , the equation* (4.35) *has property (P) when*

$$a\sqrt[3]{\frac{16}{5}}\lambda^{\frac{5}{6}}\ln\left(\frac{1}{\lambda}\right) > \frac{1}{e}.$$

Moreover, condition (4.32) *holds, then, by Lemma 4.19, every solution of* (4.35) *tends to zero as* $t \to \infty$.

By Condition (4.34) *in Corollary 4.21, take* $\zeta(t) = \alpha t$, *where* $\alpha = \dfrac{(1+\sqrt{\lambda})}{2\lambda}$, *the equation* (4.35) *is oscillatory when*

$$\frac{18a}{5}(1-\alpha^{\frac{-5}{6}})(1-\alpha^{\frac{-1}{3}})\ln\left(\frac{1}{\lambda\alpha^2}\right) > \frac{1}{e}.$$

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