Palestine Polytechnic University
Deanship of Graduate Studies and Scientific Research
Master program of Mathmatics

Zero Forcing And Maximum Nullity Of Graphs

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M.Sc. Thesis

Hebron- palestine
June, 2020

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M.Sc. Thesis
Submitted to the Department of Mathematics at Palestine Polytechnic University as a partial fulfillment of the requirement for the degree of Master of Mathematics.
Hebron- palestine

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2020

Declaration

I declare that the Master Thesis entitled **Zero Forcing And Maximum Nullity Of Graphs** is my original work, and hereby certify that unless stated, all work contained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgement is made in the text.

Hadil Izzat Abughiyatha

Signature: ______________________________ Date: ____________________

Dedication

To my parents,
To my brothers and sisters.

Hadil Izzat Abughiyatha

Acknowledgment

I am very grateful to my supervisor Dr. Mohammad Adam for all his help and support.

I am very grateful to my external referee Dr. Khalidah Nazzal for her useful comments.

I am very grateful to my internal referee Dr. Iyad Alhribat for his useful advice.

In addition, I want to thank the members of the department of mathematics at the Palestine Polytechnic University.

Abstract

The zero forcing game on a simple, undirected graph G is based on the so-called color change rule which can be stated as if some vertices on G are colored black while others are white and vertex v is a black vertex with exactly one white neighbor u, then change the color of vertex u to be black vertex. A zero forcing set is a subset of black vertices such that if we apply the color change rule, we have all vertices on G are black vertices. A new parameter so-called zero forcing number $Z(G)$ which is the minimum size of a zero forcing set among all zero forcing sets of G. The maximum nullity $M(G)$ of a graph G is to determine the largest nullity over all symmetric matrices whose (i, j)th entry, $i \neq j$, is nonzero when $\{i, j\}$ are connected with an edge in G and is zero otherwise. The zero forcing number $Z(G)$ is helpful to study the $M(G)$. Indeed, it is shown that $Z(G)$ is upper bound on $M(G)$. We conclude this thesis by studying the maximum nullity of symmetric matrices that represent of trees with a fixed number of negative eigenvalues.

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Chapter 1

Introduction

1.1 Preliminary

In this section, we will review some basic linear algebra. We start with the definition of vector space.

Definition 1.1.1. [23] *A set of elements V is said to be a* vector space *over a scalar field $\mathbb{F}$ if an addition operation is defined between any two elements of V and a scalar multiplication operation is defined between any element of $\mathbb{F}$ and any vector in V. Moreover, if u, v, and w are vectors in V, and if a and b are any two scalars in $\mathbb{F}$, then these ten axioms must hold.*

Closure properties:

(c1) $u + v$ *is a vector in* V.

(c2) av *is a vector in* V.

Properties of addition:

(a1) $u + v = v + u$.

(a2) $u + (v + w) = (u + v) + w$.

(a3) *There is a vector* $\boldsymbol{0}$ *in* V *such that* $v + \boldsymbol{0} = v$ *for all* v *in* V.

(a4) *Given a vector* v *in* V, *there is a vector* $-v$ *in* V *such that* $v + (-v) = \boldsymbol{0}$.

Properties of scalar multiplication:

(m1) $a(bv) = (ab)v$.

(m2) $a(u + v) = au + av$.

(m3) $(a + b)v = av + bv$.

(m4) $1v = v$ *for all* v *in* V.

Next, we give the definition of linearly dependent and linearly independent.

Definition 1.1.2. *Let V be a vector space, and let $\{v_1, v_2, \cdots, v_p\}$ be a set of vectors in V. This set is* linearly dependent *if there are scalars $a_1, a_2, \cdots, a_p$, not all of which are zeros, such that*

$$a_1v_1 + a_2v_2 + \cdots + a_pv_p = \boldsymbol{0}. \tag{1.1}$$

The set $\{v_1, v_2, \cdots, v_p\}$ is linearly independent *if it is not linearly dependent; that is, the only scalars for which* (1.1) *holds are the scalars $a_1 = a_2 = \cdots = a_p = 0$.*

To define a dimension of a vector space we need the two following definitions.

Definition 1.1.3. *If S is a nonempty subset of the vector space V, then* $\mathrm{sp}(S)$, *the* linear span *of S, is the set of all linear combinations of finite sets of elements of S.*

Definition 1.1.4. *Let V be a vector space, and let $B = \{v_1, v_2, \cdots, v_p\}$ be a spanning set for V. If B is linearly independent, then B is a* basis *for V.*

In the following we present the definition of the dimension of a give vector space.

Definition 1.1.5. *Let V be a vector space. If V has a basis $B = \{v_1, v_2, \cdots, v_n\}$, then V has* dimension n, *and we write* $\dim(V) = n$.

A nullspace or kernal and rank of a matrix is defined in the following definition.

Definition 1.1.6. *The* nullspace (kernal) *of the matrix A, denoted* $\mathrm{N}(A)$ $(\ker(A))$, *is the set of all n-dimensional vectors x such that $Ax = \boldsymbol{0}$. The dimension of the* $\ker(A)$ *is called the* nullity (corank) *of A.*

Definition 1.1.7. *The* row space *of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by rows of A. The dimension of the row space is called the* rank *of the matrix A.*

Next, we give the definition of eigenvalue and eigenvector.

Definition 1.1.8. *An element $\lambda \in \mathbb{C}$ is an* eigenvalue *of a matrix $A \in \mathbb{R}^{n\times n}$ if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$. The vector x is said to be an* eigenvector *of A corresponding to the eigenvalue λ.*

A symmetric matrix is defined in the following definition.

Definition 1.1.9. *A square matrix A is called* symmetric *if $a_{ij} = a_{ji}$ for all i and j, or if $A = A^T$.*

For a square matrix A, $\sigma(A)$ denotes the spectrum of A i.e., the set of all its eigenvalue λ. For $\lambda \in \sigma(A)$, $\mathrm{mult}_A(\lambda)$ denotes the multiplicity of the eigenvalue λ in $\sigma(A)$. If A is symmetric, then $\ker(A)$ is the multiplicity of the zero eigenvalues of A and $\mathrm{rank}(A)$ is the number of nonzero eigenvalues of A.

Definition 1.1.10. *A square matrix is called* lower triangular *if all the entries above the main diagonal are zero.*

Now we give the definition of positive semidefinite and positive definite.

Definition 1.1.11. *A square symmetric matrix H is said to be* positive semidefinite (psd) *if $v^T Hv \geq 0, \forall v \in \mathbb{R}^n$, or all eigenvalue are zeros or positive. The* positive definite (pd) *if the inequality holds with equality only for vectors $v = \boldsymbol{0}$, or all eigenvalues are positive.*

A square matrix D is said to be an *invertible matrix* if and only if there exists another square matrix D^{-1} such that $DD^{-1} = D^{-1}D = I$, where I is an identity matrix. For two square real matrices A and B are said to be congruent if there exists an invertible square matrix C such that $B = CAC^T$. The following theorem is Sylvester's law of inertia.

Theorem 1.1.12. [20] (Sylvester's law of inertia) *The two symmetric matrices A and B are congruent if and only if they have the same inertia, i.e., the number of positive, negative, and zero eigenvalues.*

Definition 1.1.13. *An $n \times n$ matrix A is called* nonsingular *if there exists an $n \times n$ matrix B such that*

$$AB = BA = I$$

A singular *matrix is a square matrix which is not invertible.*

For $n \times m$ matrix $A, \alpha \subseteq \{1, 2, \cdots, n\}$, and $\beta \subseteq \{1, 2, \cdots, m\}$, $A[\alpha|\beta]$ is the submatrix of A lying in the rows indexed by α and the columns indexed by β. And $A(\alpha|\beta)$ is the submatrix obtained from A by deleting the rows indexed by α and columns indexed by β. If A is square matrix (which means that $\alpha = \beta$), then the principal submatrix $A[\alpha|\alpha]$ is also can be written as $A[\alpha]$, and the complementary principal submatrix to $A(\alpha)$. If $\alpha = \{i\}$, then $A(\alpha)$ we can abbreviate it by $A(i)$, or A_{ii}. In the special case when α or β is a contiguous index set, say for example, $\alpha = \{3, 4, 5, 6, 7\}$, we abbreviate the notation $A[\{3, 4, 5, 6, 7\}|\beta]$ by $A[2, 3, 4, 5, 6|\beta]$. If rank and nullity of matrix A equal 0 is denoted by $A[\emptyset]$ and we define $A[\emptyset]$ to be a positive definite matrix.

Example 1.1.14. *Let A be the following matrix*

$$A = \begin{bmatrix} 4 & 1 & 6 & 0 & 3 & 3 \\ 1 & 0 & 5 & 0 & 5 & 7 \\ 9 & 1 & 8 & 1 & 1 & 10 \\ 2 & 4 & 1 & 5 & 11 & 3 \\ 6 & 7 & 2 & 1 & 10 & 5 \end{bmatrix}.$$

Assume $\alpha = \{1, 2, 3\}$, *and* $\beta = \{2, 5\}$, *then we have*

$$A[\alpha|\beta] = \begin{bmatrix} 1 & 3 \\ 0 & 5 \\ 1 & 1 \end{bmatrix},$$

and

$$A(\alpha|\beta) = A[\alpha^c|\beta^c] = \begin{bmatrix} 2 & 1 & 5 & 3 \\ 6 & 2 & 1 & 5 \end{bmatrix}.$$

The following theorem states that the eigenvalues of a real symmetric matrix interlace with those of any principal submatrix.

Theorem 1.1.15. [21] (Cauchy Interlace Theorem). *Let* A *be a symmetric matrix of order* n*, and let* B *be a principal submatrix of* A *of order* $n - q$*. If* $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{n-1} \le \lambda_n$ *lists the eigenvalues of* A *and* $\mu_1 \le \mu_2 \le \cdots \le \mu_{n-q}$ *the eigenvalues of* B*, then* $\lambda_i \le \mu_i \le \lambda_{i+q}$ *for* $i = 1, 2, \cdots, n - q$.

A direct sum of matrices is defined in the following definition.

Definition 1.1.16. *For matrices of any dimension, say* $H_1, H_2, \cdots, H_n$ *the* direct sum *of* $H_1, \cdots, H_n$*, denoted by* $H_1 \oplus \cdots \oplus H_n$*, is given by*

$$= H_1 \oplus H_2 \oplus \cdots \oplus H_n = \begin{bmatrix} H_1 & 0 & \cdots & 0 \\ 0 & H_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H_n \end{bmatrix}$$

Definition 1.1.17. *A* J *is an* $m \times n$ *matrix where all entries equal one's* (1).

A partition of a set is defined in the following definition.

Definition 1.1.18. *A collection of nonempty sets* $\{U_1, \cdots, U_n\}$ *is a* partition *of a set* X *if:*

1. $X = U_1 \cup U_2 \cup \cdots \cup U_n$.
2. X *is pairwise disjoint:* $U_i, U_j \in X : U_i \cap U_j = \phi$ *when* $i \neq j$.

1.2 Graph Theory

A *graph* is a pair $G = (V, E)$, where:

- V is a set of vertices (usually $\{\ v_1, v_2, \cdots, v_n\ \}$).
- E is a set of edges (an edge is a two-element subset of vertices).

The *order* of a graph G is the number of vertices in $V(G)$ and is denoted by $|G|$. A simple undirected graph is a graph without loops or multiple edge. In this thesis each graph is finite, simple, undirected and has nonempty vertex set. The number of edges incident with a vertex $v \in V(G)$ is called the *degree* of that vertex. It is denoted as $\deg(v)$. We define, $\Delta(G) = \max_v\ \deg(v)$. A *leaf*, or *pendant*, is a vertex with degree 1. A vertex with degree 0 is called an *isolated vertex.* An *empty graph* on n vertices consists of an n vertices with no edges i.e., is a graph with n isolated vertices. Let v, u are distinct vertices on a graph G, if $\{u, v\} \in E$ (or $\{v, u\} \in E$), then u and v are said to be *adjacent* or *neighbors* and is denoted by $u \sim v$ (or $v \sim u$). Write $u \nsim v$ (or $v \nsim u$) if $\{u, v\} \notin E$ (or $\{v, u\} \notin E$), then u and v are *nonadjacent* or *non neighbors.* The set of all adjacent (or neighbors) of v is denoted by $N(v)$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Then G_1 and G_2 are called *isomorphic* if there is a bijective function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in $G_2, \forall\ a, b \in V_1$. A graph $G' = (V', E')$ is a *s*ubgraph of graph $G = (V, E)$ if

- $V' \subseteq V$, and
- $E' \subseteq E$.

If H is a subgraph of G we write $H \subseteq G$. The subgraph G' is called an *induced subgraph* of G if all the edges between the vertices in V' from E are in E' and is denoted by $G[V']$. An *independent set* is a set $\mathcal{S}$ of vertices such that any two vertices in $\mathcal{S}$ are nonadjacent. A *path* is a graph $P_n = (V, E)$, where $V = \{v_1, \cdots, v_n\}$ and $E = \{\{v_i, v_{i+1}\} : i = 1, \cdots, n-1\}$. A *connected graph* is a graph that has a path from any vertex to any other vertex. A graph that is not connected is said to be *disconnected.* A *cycle* is a graph $C_n = (V, E)$ where $V = \{v_1, \cdots, v_n\}$ and $E = \{\{v_i, v_{i+1}\} : i = 1, \cdots, n-1 \cup \{\{v_n, v_1\}\}$, or in other words, some of vertices (at least 3) connected in a closed path. The graph obtained from C_{n-1} by joining each vertex to a new vertex v is the *wheel* on n vertices and is denoted by W_n. The *components* of a graph G are its maximal connected subgraphs. A *tree* is connected graph that has no cycles and is denoted by T. A graph without cycles is also called a *forest*. The components of a forest are trees. The *length* of a path or a cycle is the number of edges. A graph is said to be *complete* if and only if any pair of vertices are connected by an edge. The complete graph on n vertices is denoted by K_n or $K_n = (V, E)$, where $V = \{v_1, \cdots, v_n\}$ and $E = \{\{v_i, v_j\} : 1 \leq i < j \leq n\}$.

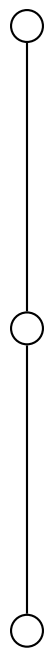
We denoted by $P(G)$ the *path cover number*, namly, the minimum number of vertex disjoint paths, occurring as induced subgraphs of G, that cover all the vertices of G. $\Gamma(G)$ is the maximum of $p - q$ such that the deletion of q vertices from G results in p paths, when q vertices achieve this maximum we say that q is an *optimal*.

The following graph operations are used to construct families of graphs:

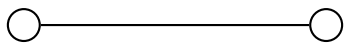
1. Let $G = (V, E)$ be a graph of order n. Then the *complement graph* $\overline{G}$ is the graph with $V(\overline{G}) = V(G)$ such that there will be an edge between two vertices a, b in $\overline{G}$, if and only if there is no edge between a, b in G.
2. Suppose G and H are graphs with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $V(H) = \{u_1, u_2, \cdots, u_m\}$. We define the *cartesian product* $G\Box H$ to be the graph with vertex set $V(G\Box H) = V(G) \times V(H) = \{(v_i, u_j) \mid v_i \in V(G), u_j \in V(H)\}$, and $e = \{v_i, u_j\}\{v_k, u_l\}$ is an edge of $G\Box H$ if and only if either:
   (a) $i = k$ and $\{u_j, u_l\} \in E(H)$, or
   (b) $j = l$ and $\{v_i, v_k\} \in E(G)$.

   It can also be defined as drawing copies of graph G vertically as much as the order of H and horizontally connect the vertices so that the graph H appears according to the order of G. The following example illustrates the definition on $P_3\Box K_2$.

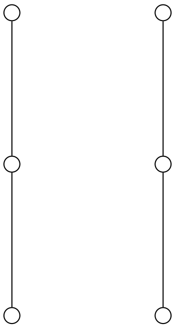
   Example 1.2.1. *The following graph represents P_3.*



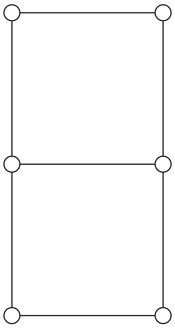
And the following graph represents K_2.



To draw $P_3 \Box K_2$. First, we draw vertically two copies of P_3 (according the order of K_2) as the following figure shown.



Then horizontally we connected the vertices of two copies of P_3 as its connected in K_2 (note that we have three copies of K_2 according to the order of P_3) as shown in the following figure.



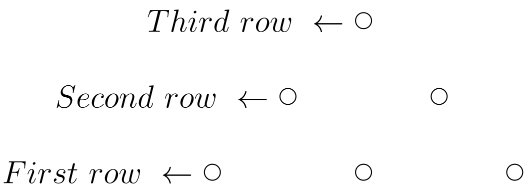
3. The *Strong product* $G \boxtimes H$ of graphs G and H has the vertex set $V(G \boxtimes H) = V(G) \times V(H)$ and $e = \{v_i, u_j\}\{v_k, u_l\}$ is an edge of $G \boxtimes H$ if and only if
   (a) $i = k$ and $\{u_j, u_l\} \in E(H)$, or
   (b) $j = l$ and $\{v_i, v_k\} \in E(G)$, or
   (c) $\{v_i, v_k\} \in E(G)$ and $\{u_j, u_l\} \in E(H)$.

4. The *corona* $G \circ H$ of two graphs G (with n vertices and s edges) and H (with m vertices and t edges), is defined as the graph obtained by taking one copy of G and n copies of H, and joining all the vertices in the ith copy of H to the ith vertex

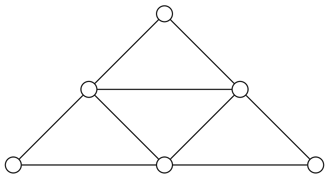
of G where $i = \{1, \cdots, n\}$. It follows from definition of the corona that $G \circ H$ has $nm + n$ vertices and $s + nt + nm$ edges.

The nth *hypercube* Q_n is defined inductively by $Q_1 := K_2$ and $Q_n := Q_{n-1} \Box K_2$. Hypercube graph of order n has 2^n vertices. The nth *supertriangle* $\mathcal{T}_n$ is an equilaternal triangular grid with n vertices on each side. The order of T_n is $\frac{1}{2}n(n+1)$. The following example illustrates the definition on $\mathcal{T}_3$.

Example 1.2.2. *To draw* $\mathcal{T}_3$. *First, we need to draw six vertices in three rows because* $n = 3$ *and they are drawn descendingly, that means the first row consist of three vertices* $(n = 3)$. *In the second row draw two vertices* $(3 - 1 = 2)$. *In the last row draw one vertex* $(3 - 2 = 1)$ *as follows in the following figure.*



Then connected between the vertices as follows in the following figure and we will get $\mathcal{T}_3$.



When deleting a vertex from a graph, you must also delete all edges adjacent to that vertex. The result $G[V \backslash \{v\}]$ of deleting a vertex v is also denoted by $G - v$. We denote by $G - e$ the subgraph of G obtained by deleting edge e, we do not delete the endpoints of that edge. A *cut vertex* is a vertex whose removal increases the number of components. Clearly if v is a cut vertex of a connected graph, $G - v$ is disconnected. A *cut edge* is a edge whose removal either disconnects the connected graph or increases the number of components.

An induced subgraph G' of a graph G is a *clique* if G' has an edge between every pair of vertices of G'. A set of subgraphs of G, each of which is a clique and such that every edge of G is contained in at least one of these clique, is called a *clique covering* of

G. The *clique covering number* of G, denoted by $cc(G)$ is the smallest number of cliques in a clique covering of G.

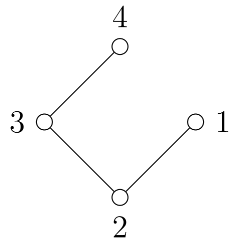
1.3 Graphs and Matrices

The simple graphs and matrices are used interchangeably. For example, graphs are used to describe the zero and non-zero pattern of a matrix and matrices are used to describe the edge and non-edge of the graph.

Definition 1.3.1. *Let $G = (V, E)$ be a graph and assume that $V = \{v_1, \cdots, v_n\}$. The* adjacency matrix *of G is an $n \times n$ matrix $A(G)$ defined as:*

$$a_{ij} = \begin{cases} 1 & \textit{if } \{v_i, v_j\} \in E \\ 0 & \textit{otherwise.} \end{cases}$$

For example let G be the graph in the following figure.



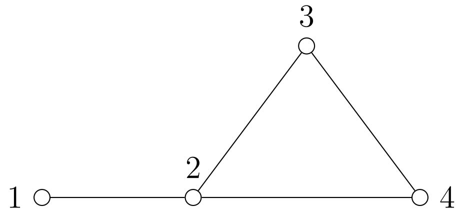
The following adjacency symmetric matrix A that associated to the graph G is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

For the following symmetric matrix B.

$$B = \begin{bmatrix} 2 & 4 & 0 & 0 \\ 4 & 1 & 6 & 3 \\ 0 & 6 & 3 & 9 \\ 0 & 3 & 9 & 0 \end{bmatrix}.$$

The graph that is corresponding to B is



Define $S_n(R)$ to be the set of all real symmetric $n \times n$ matrices. The graph of $A \in S_n(R)$, denoted $\mathcal{G}(A)$, is the graph G with vertices $\{\ 1, \cdots, n\ \}$ and edges $\{\{\ i, j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\ \}$. The diagonal entries of A are ignored in determining $\mathcal{G}(A)$.
The set of *symmetric matrices* of graph G (over $\mathbb{R}$) is defined to be

$$S(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}.$$

Define the *minimum rank* of G over $\mathbb{R}$ as

$$\text{mr}(G) := \min\{\text{rank}(A)|\ A \in S(G)\}.$$

Whereas the *positive semidefinite minimum rank* of G is defined by

$$\text{mr}_+(G) := \ \min\{\text{rank}(A) : A \in S(G),\ \text{A positive semidefinite}\}.$$

Since positive semidefinite minimum rank related of a square matrix A and A is positive semidefinite whereas the minimum rank related of any square matrix that implies to positive semidefinite minimum rank is subset from minimum rank, the following inequality

$$\text{mr}(G) \leq \text{mr}_+(G).$$

'is true.'

The following observation states that the positive semidefinite minimum rank of G is upper bound of minimum rank and lower bound of clique covering number.

Observation 1.3.2. [1], [15] *Since a matrix obtained from a clique covering as a sum of rank* 1 *matrices is positive semidefinite,*

$$\text{mr}(G) \leq \text{mr}_+(G) \leq cc(G).$$

The following are important observations related to minimum rank for the complete graph and path graph.

Observation 1.3.3. [6] $\text{mr}(G) = 1$ *if and only if* $G = K_n$.

Observation 1.3.4. [16] $\text{mr}(G) = n - 1$ *if and only if* $G = P_n$.

1.4 Organization Of The Thesis

The organization of this thesis is as follows:

Chapter 1 is an introduction chapter. It consists of some basic definitions of linear algebra. In addition, we introduce basic concepts of graph theory. We conclude this chapter by presenting the relationship between the symmetric matrices and graphs.

In ***Chapter 2***, the zero forcing number through color change rule is defined. Then the zero forcing number of some interesting families of graphs and some graph operations are studied. After that, the effects on zero forcing number when deleting a vertex or an edge from a graph are presented.

Chapter 3 consists of illustration that the zero forcing number $Z(G)$ is an upper bound of the maximum nullity $M(G)$. Then, the maximum nullity of some interesting families of graphs is presented. We conclude this chapter by studying the maximum nullity of symmetric matrices that represents the trees with a fixed q negative eigenvalue.

Chapter 2

Zero Forcing

In this chapter, a type of graph coloring which defines a graph parameter called the zero forcing number, denoted by $Z(G)$, which is the minimum size of a zero forcing set is presented. Then, the zero forcing number for some interesting families of graphs is determined. After that, the zero forcing parameter with some graph operations is studied. In addition, the effects on zero forcing number when a vertex from a graph G is deleted is introduced. We conclude this chapter by presenting the effects on zero forcing number when an edge from a graph G is deleted.

2.1 Zero Forcing Parameter

In this section, the zero forcing game which is based on the so-called color change rule is defined. From this game we get the so-called zero forcing set and zero forcing number where the later has an interesting application in bounding the maximum nullity of graph as we will see in Chapter 3.

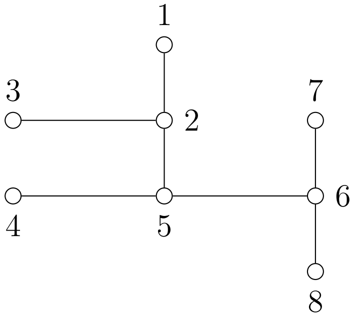
For a given graph G we start with a *coloring* of G, i.e., a set of vertices colored black while the remaining are colored white. There are certain moves by the color change rule which allow white vertices to be changed to black. The *zero forcing number* $Z(G)$ is the minimum size of a black set that eventually allows all vertices to become black [1]. In the following definition it was presented the so-called the color change rule, derived coloring, zero forcing sets, and zero forcing number.

Definition 2.1.1. [1], [5], [8],[22] *For a given graph G and a coloring of G where $u, v \in V(G)$. Some vertices are colored black and the remaining vertices are colored white,* the color change rule *is: If v is a black vertex and has exactly one white neighbor, say*

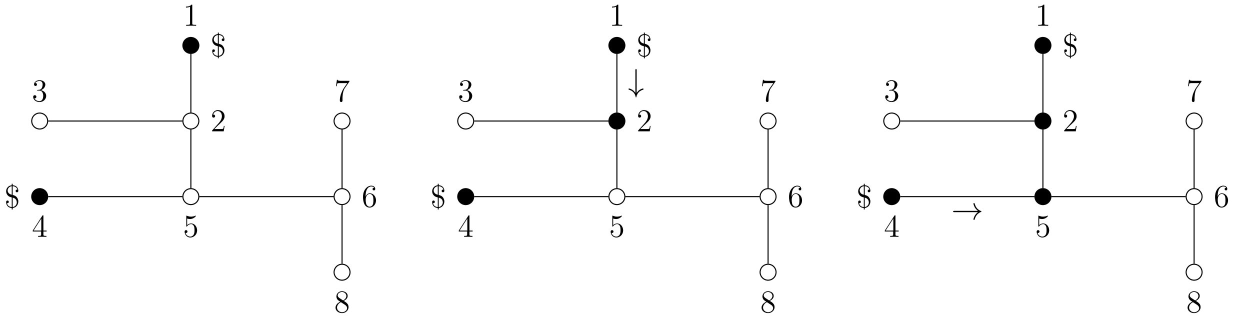
u, then change the color of u to be black. In this case, we say v forces *u to become black and denoted by $v \to u$. Given a coloring of G, the* derived coloring *is the result of applying the color change rule until no more changes are possible. A* zero forcing set *is a subset Z of $V(G)$ such that if we start from a coloring of G such that the vertices in Z are colored black and the others are white, the derived coloring of G is all black. A zero forcing set with the minimum number of vertices is called an* optimal zero forcing set *and the* zero forcing number $Z(G)$ *is the minimum order of zero forcing sets ($|Z|$) over all zero forcing sets $Z \subseteq V(G)$, i.e., is the order of an optimal zero forcing set.*

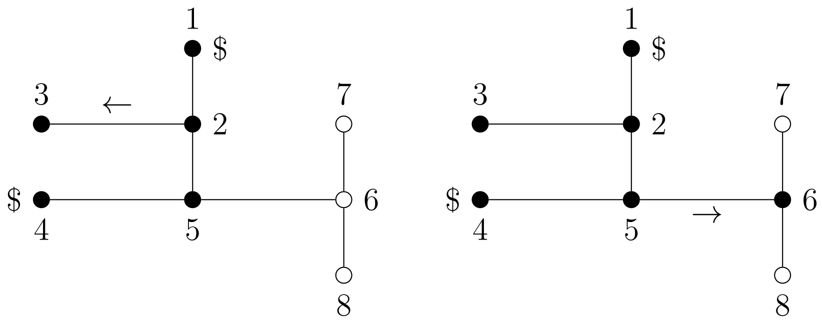
In the following examples we illustrate the definition of zero forcing set.

Example 2.1.2. *For the following tree T,*

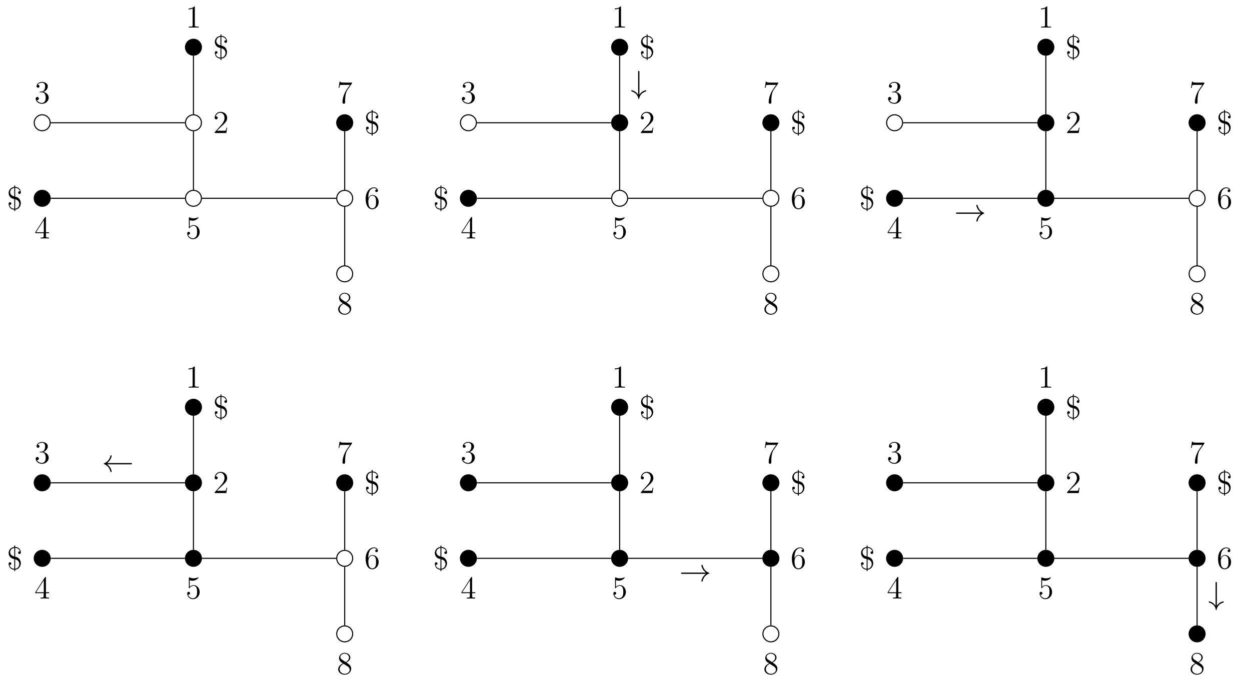


We color the vertices 1 and 4 black. The vertex 1 has one white neighbor so by the color change rule $1 \to 2$, the vertex 4 also has one white neighbor so $4 \to 5$, then vertex 2 can force vertex 3 to become black and $5 \to 6$. Now vertex 6 has two white neighbors therefore, 6 can not forces any vertex. Hence the set of vertices $\{1, 4\}$ can not be a zero forcing set as shown in the following figures.



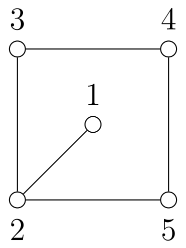


We need to color one more vertex from the neighbors of vertex 6 *so that vertex* 6 *will be able to force the other neighbor. Thus* $\{1, 4, 7\}$ *is a zero forcing set as shown in the following figures.*

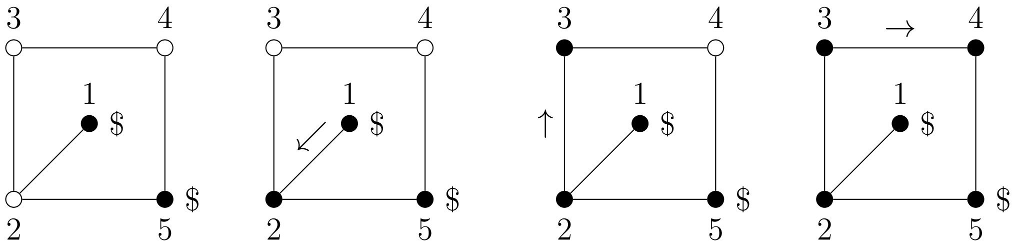


Since we can not color the vertices of T *starting from coloring only two vertices. Therefore,* $Z(G) = 3$.

Example 2.1.3. *For the following graph* G*,*



the set of vertices $\{1, 5\}$ *is a zero forcing set. Indeed,*



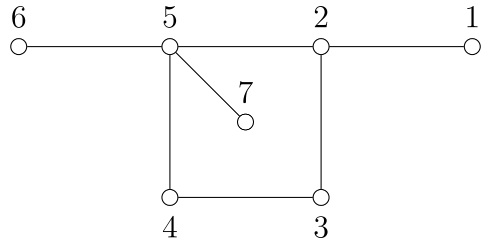
Since we can not color the vertices of G starting from coloring only one vertex because if $Z(G)$ *were* 1*, then applying the color change rule starting from coloring only one vertex, we would reach a vertex with two white neighbors. Therefore,* $Z(G) = 2$*.*

Given an initial coloring of G, the *derived set* is the set of vertices colored black after the color change rule is applied until no more changes are possible. For a given zero forcing set Z, a *chronological list of forces* is a listing of the forces used to construct the derived set in the order they are performed. A *forcing chain* for a chronological list of forces is a sequence of vertices $(v_1, v_2, \cdots, v_k)$ such that for $i = 1, 2, \cdots, k-1, v_i \to v_{i+1}$, and a *maximal forcing chain* is a forcing chain that is not a proper subsequence of any other forcing chain. The collection of *maximal forcing chains* for a chronological list of forces is called the *chain* set of the chronological list of forces, and an *optimal chain* set is a chain set from a chronological list of forces of an optimal zero forcing set. When a chain set contains a chain consisting of a single vertex, we say that the chain set contains the vertex as a *singleton*. Let Z be a zero forcing set of a graph G. A *reversal* of Z is the set of last vertices of the maximal zero forcing chains of a chronological list of forces [27].

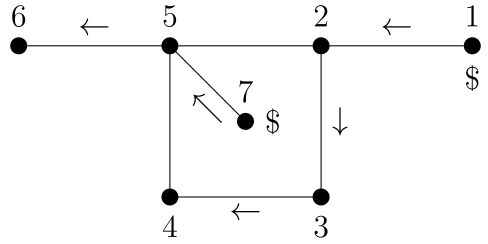
Theorem 2.1.4. [5] *If* Z *is a zero forcing set of* G*, then so is any reversal of* Z*.*

Proof. Let W be the reversal of Z for the list. Write the chronological list of forces in reverse order, reversing each force and call this the reverse chronological list of forces. Now we want to prove the reverse chronological list of forces is a valid list of forces for W. Assume the first force $v \to u$ on the reverse chronological list. Since the last force $u \to v$ of Z was done, so that all neighbors of v except u must be in W and each of them had the white neighbor u and hence did not force any vertex previously in the original chronological list of forces. Therefore, $v \to u$ is a valid force for W. Continue in this manner, we can show the reverse chronological list of forces is a valid list of forces for W. □

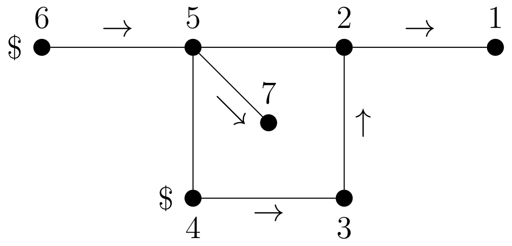
Example 2.1.5. *The following figure represents the graph* G*.*



A zero forcing set for G is Z = $\{1, 7\}$ *as shown in the following figures.*



The vertices $W = \{4, 6\}$ *is the reversal of Z and also zero forcing set of G. Indeed,*



2.2 Zero Forcing Number For Some Interesting Families Of Graphs

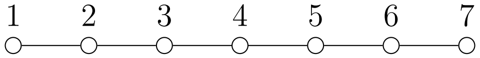
In this section, the zero forcing number for some interesting families of graphs is studied. We start with the paths.

Observation 2.2.1. *An optimal zero forcing set for a path graph* P_n *is one of its pendent vertices, and therefore,* $Z(P_n) = 1$.

Proof. Label the vertices of P_n such that its edges are $\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}$. Color one of the pendent vertices, say vertex 1. The vertex 2 is a unique white neighbor of vertex 1. Hence vertex 1 forces vertex 2. Next vertex 3 is a unique white neighbor of vertex 2, which will force vertex 3. Continue in this manner, we can color all vertices of P_n. Since there is no smaller zero forcing set we have $Z(P_n) = 1$. $\square$

For an illustration of Observation 2.2.1, we present the following example.

Example 2.2.2. *For the following path graph P_7,*



the vertex set $\{1\}$ *is a zero forcing set. Hence* $Z(P_7) = 1$. *Indeed,*

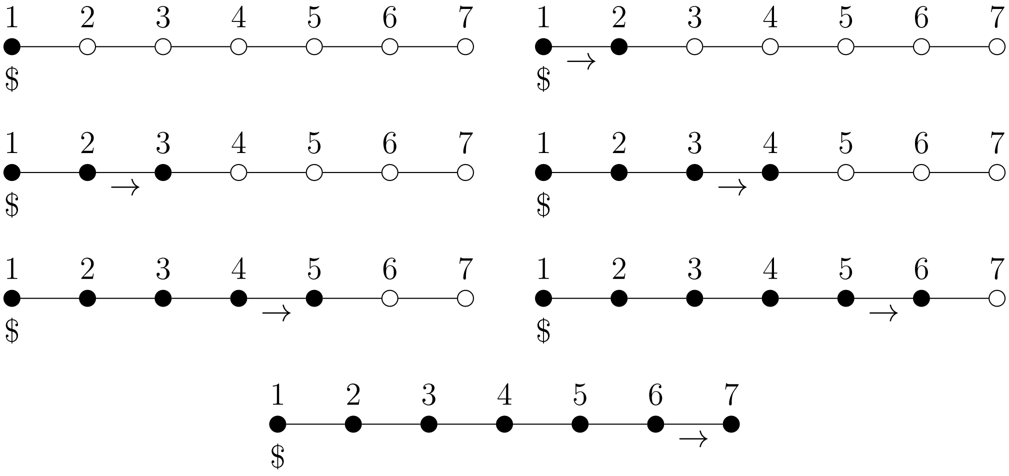


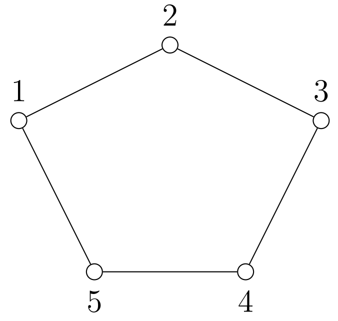
Figure 2.1: Zero forcing set for P_7

Observation 2.2.3. *For the graph C_n, an optimal zero forcing set is any two adjacent vertices and therefore, $Z(C_n) = 2$.*

Proof. Label the vertices of C_n such that its edges are $\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}$. Color two adjacent vertices, say 1 and 2. The vertex 3 is a unique white neighbor of vertex 2. Hence vertex 2 force vertex 3. Next vertex 4 is a unique white neighbor of vertex 3. Which will force vertex 4. Continue in this manner, we can color all the vertices of C_n. Since we can not color the vertices of C_n starting from coloring one vertex we get $Z(C_n) = 2$. $\square$

For an illustration of Observation 2.2.3, we introduce the following example.

Example 2.2.4. *For the following cycle graph C_5,*



the set of vertices $\{1,2\}$ *is a zero forcing set. Indeed,*

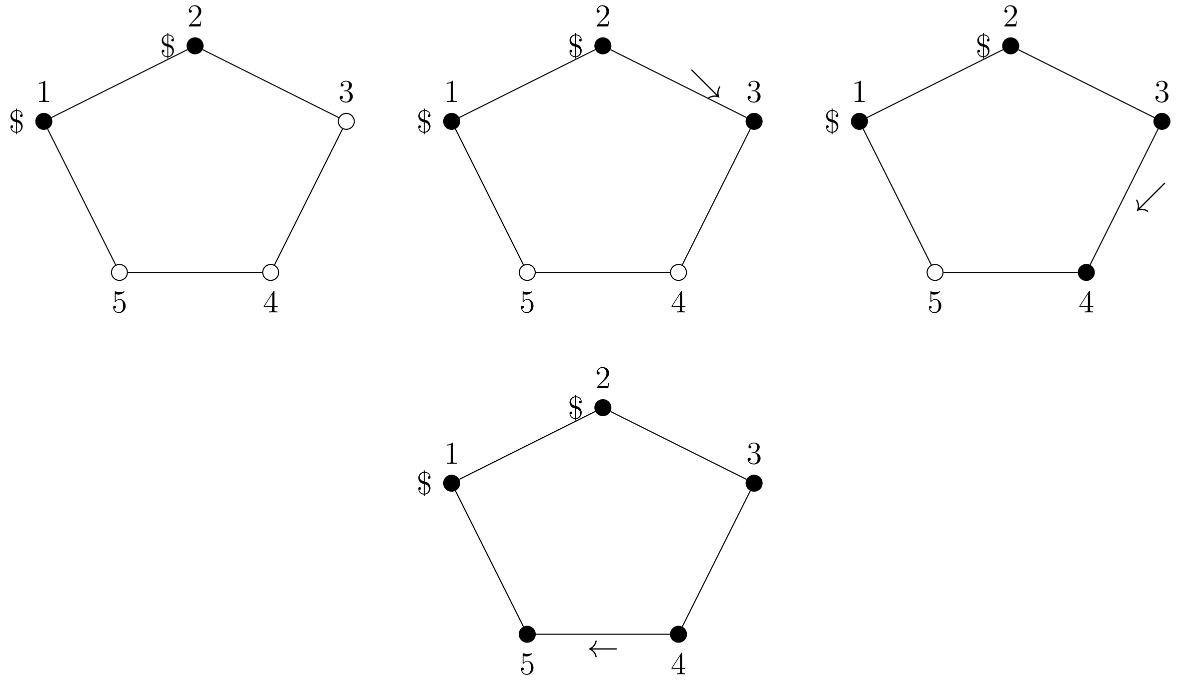


Figure 2.2: Zero forcing set for C_5

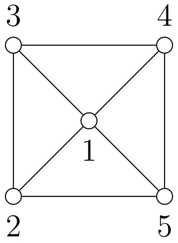
Since we can not color the vertices of C_5 *starting from coloring only one vertex because after coloring only one vertex, we will have two white neighbors and thus we can not apply the color change rule. Therefore,* $Z(C_5) = 2$.

Observation 2.2.5. [14] *For* $n \geqslant 3$, *an optimal zero forcing set for the wheel graph* W_n *is the vertex that has degree* $n-1$ *together with two other adjacent vertices and therefore,* $Z(W_n) = 3$.

Proof. Label the vertices of W_n such that the vertex 1 has degree $n-1$ and $2 \sim 3, 3 \sim 4, \ldots, n-1 \sim n, n \sim 2$. Color the vertex 1 and two adjacent vertices, say 2 and 3. The vertex 4 is a unique white neighbor of vertex 3. Hence vertex 3 forces vertex 4. Next vertex 5 is a unique white neighbor of vertex 4 which will force vertex 5. Continue in this manner, we can color all vertices of W_n. Since we can not color the vertices of W_n starting from coloring only two vertices because every vertex has degree ≥ 3, we conclude that $Z(W_n) = 3$. □

For an illustration of Observation 2.2.5, we present the following example.

Example 2.2.6. *For the following wheel graph W_5,*



the set of vertices $\{1, 2, 3\}$ is a zero forcing set. Indeed,

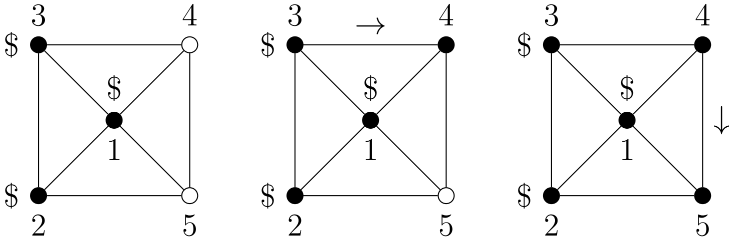


Figure 2.3: Zero forcing set for W_5

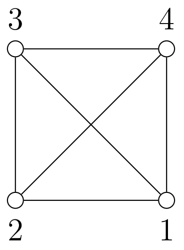
Thus $Z(W_5) \leq 3$. We can not color all vertices of W_5 starting from coloring only two vertices in W_5 because any vertex in W_5 has degree at least 3. Therefore, $Z(W_5) = 3$.

Observation 2.2.7. *For $n \geqslant 2$, an optimal zero forcing set for the complete graph is a set of all vertices except one and therefore, $Z(K_n) = n-1$.*

Proof. Color all vertices except one say $1, 2, \cdots, n-1$. The vertex n is a unique white neighbor of vertex $n-1$. Hence vertex $n-1$ forces vertex n. Since we can not color the vertices of K_n starting from coloring only $n-2$ vertices because every vertex has degree $= n-1$. Thus $Z(K_n) = n-1$. □

For an illustration of Observation 2.2.7, we present the following example.

Example 2.2.8. *For the following complete graph K_4,*



the set of vertices $\{1, 2, 3\}$ is a zero forcing set. Indeed,

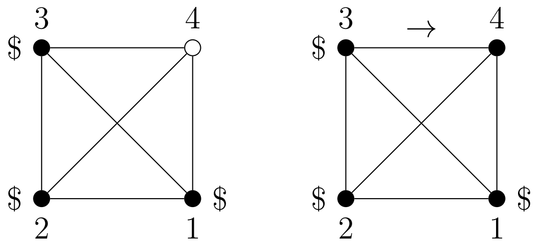
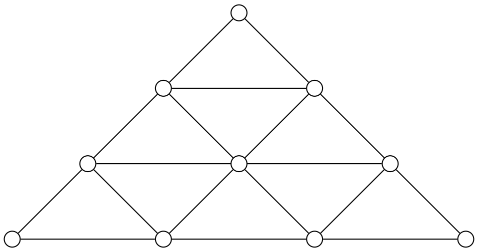


Figure 2.4: Zero forcing set for K_4

Thus $Z(K_4) \leq 3$. Since any vertex in K_4 has degree 3 we conclude that $Z(K_4) \geq 3$. Therefore, $Z(K_4) = 3$.

Observation 2.2.9. [1] *The n vertices on one edge of $\mathcal{T}_n$ are a zero forcing set for the supertriangle $\mathcal{T}_n$ and thus $Z(\mathcal{T}_n) \leqslant n$.*

Example 2.2.10. *The following figure represent $\mathcal{T}_4$.*



A zero forcing set for $\mathcal{T}_4$ is the four vertices, which are shown in the following figures. Indeed,

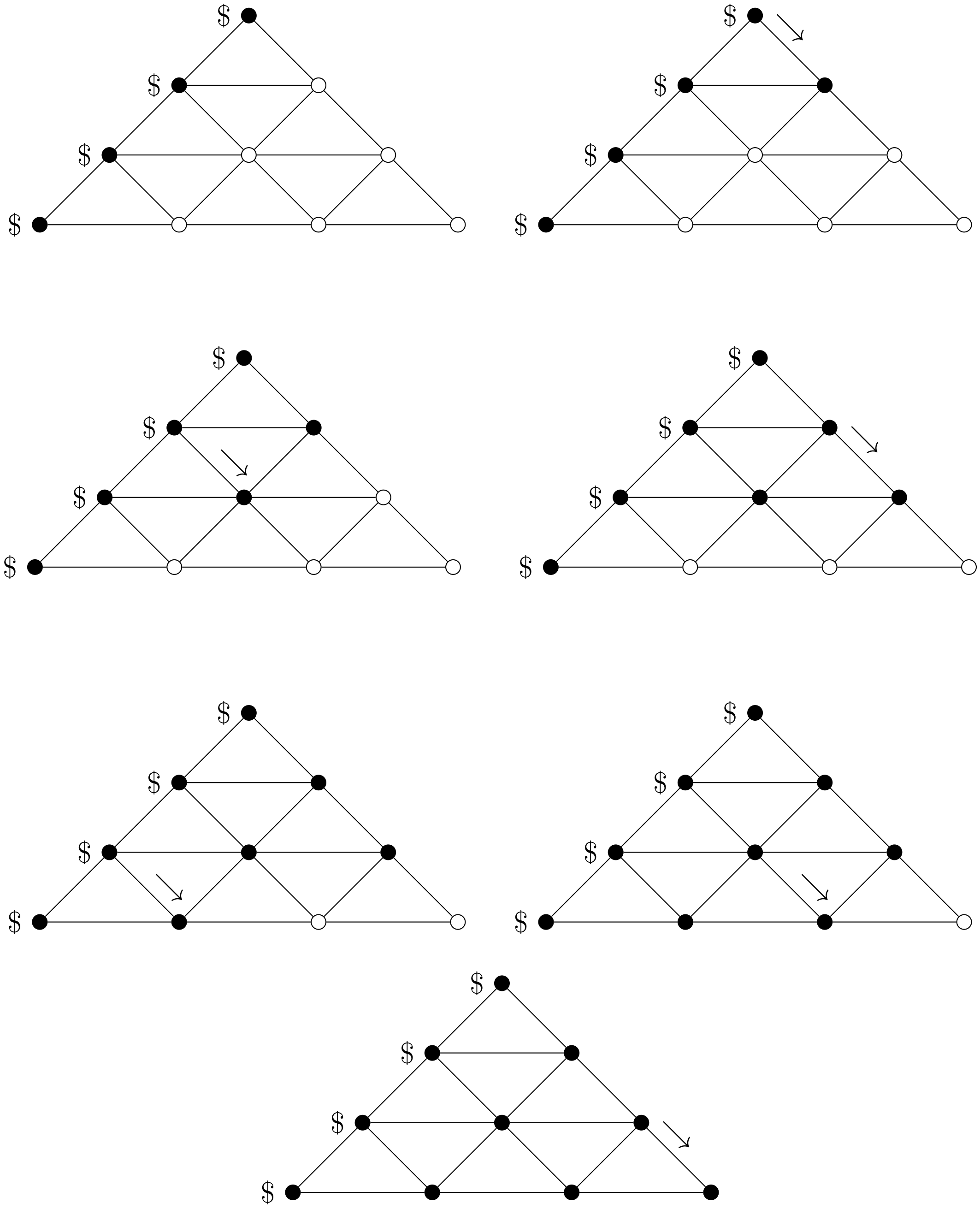


Figure 2.5: Zero forcing set for $\mathcal{T}_4$

Thus $Z(\mathcal{T}_4) \leq 4$. *If there were a zero forcing set with only three vertices, then such a set can not be a subset of the zero forcing set presented in Figure 2.5. By symmetry such a set can not be a subset of the external vertices of the 'largest triangle'. Moreover, not all of the vertices are adjacent since we can not color all of the vertices starting from* *them by only applying the color change rule. Taking all of the possibilities we conclude that* $Z(\mathcal{T}_4) \geq 4$. *Therefore,* $Z(\mathcal{T}_4) = 4$.

2.3 Zero Forcing Parameter And Some Graph Operations

In this section, the interaction between the zero forcing number and some basic graph operations is studied. We start with the cartesian product of two graphs.

Theorem 2.3.1. [1] *For both graphs* G *and* $H, Z(G\Box H) \leqslant \min\{Z(G)|H|, Z(H)|G|\}$.

Proof. Since $G\Box H$ is isomorphic to $H\Box G$, $Z(G\Box H) = Z(H\Box G)$. The zero forcing set of $(G\Box H)$ is set of vertices that associated with zero forcing set in each copy of G. Hence $Z(G\Box H) \leqslant Z(G)|H|$. Similarly, $Z(G\Box H) \leqslant Z(H)|G|$. Thus

$$Z(G\Box H) \leqslant \min\{Z(G)|H|, Z(H)|G|\}.$$

□

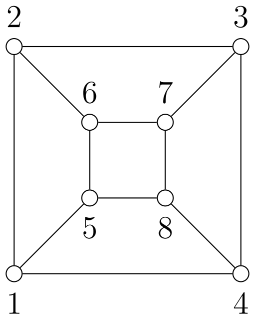
By Observation 2.2.1, we know that $Z(P_n) = 1$. Hence By Theorem 2.3.1, we have

$$Z(G\Box P_n) \leqslant \min\{|G|, Z(G)n\}. \tag{2.1}$$

Another application to Theorem 2.3.1 and since $Q_n = Q_{n-1}\Box P_2$ and $Z(P_2) = 1$. We conclude that

$$Z(Q_n) \leqslant 2^{n-1}. \tag{2.2}$$

Example 2.3.2. *The following figure represents* Q_3.



A zero forcing set for Q_3 *is* $\{1, 2, 5, 6\}$, *as shown in the following figures. Indeed,*

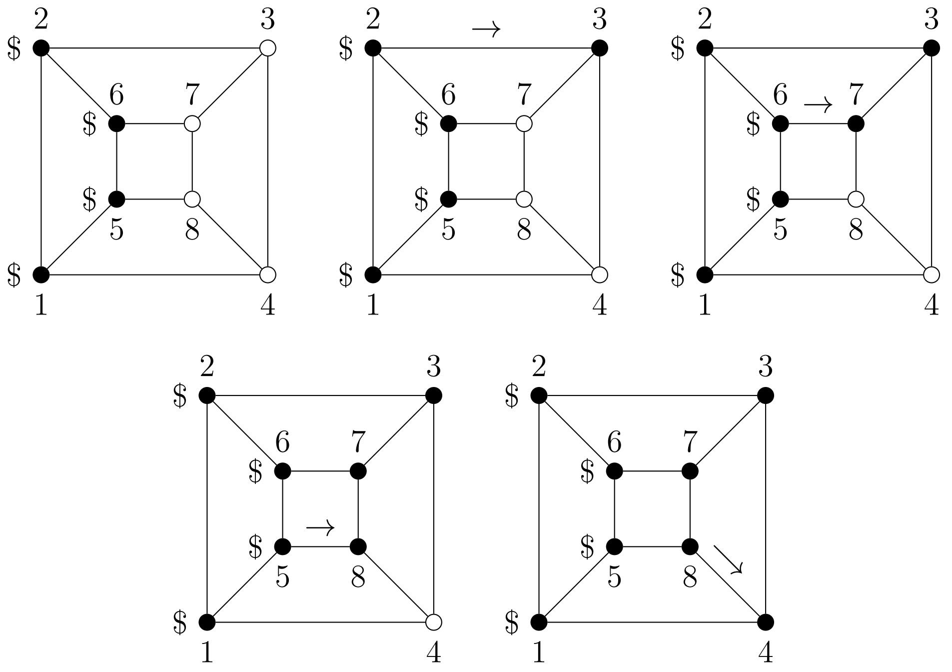


Figure 2.6: Zero forcing set for Q_3

Since we can not color the vertices of Q_3 *starting from coloring three vertices because if* $Z(Q_3)$ *were* 3*, then applying the color change rule starting from coloring three vertices, we would reach a vertex with two white neighbors. Therefore,* $Z(Q_3) = 4$*.*

Proceeding as above, using Observation 2.2.3, Observation 2.2.7, and Theorem 2.3.1, we have

$$Z(G\Box C_t) \leqslant \min\{Z(G)t, 2|G|\}.$$

$$Z(G\Box K_t) \leqslant \min\{Z(G)t, |G|(t-1)\}. \tag{2.3}$$

In the following proposition, we present an upper bound on the zero forcing number of $K_s\Box K_t$ which is better bound than that in (2.3) when $s, t \geq 2$.

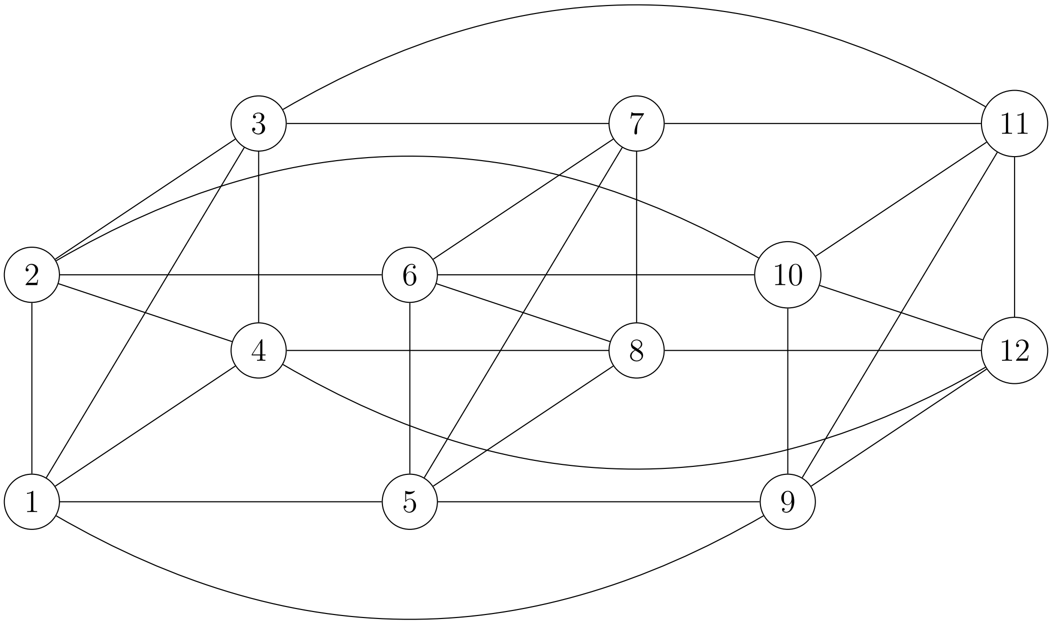
Proposition 2.3.3. [1] $Z(K_s\Box K_t) \leqslant st - s - t + 2$.

Proof. We color all vertices of the first copy of K_s, in the next $(t-2)$ copies of K_s color an optimal zero forcing sets in K_s i.e., color $s-1$ vertices in each copy. Hence we color in total

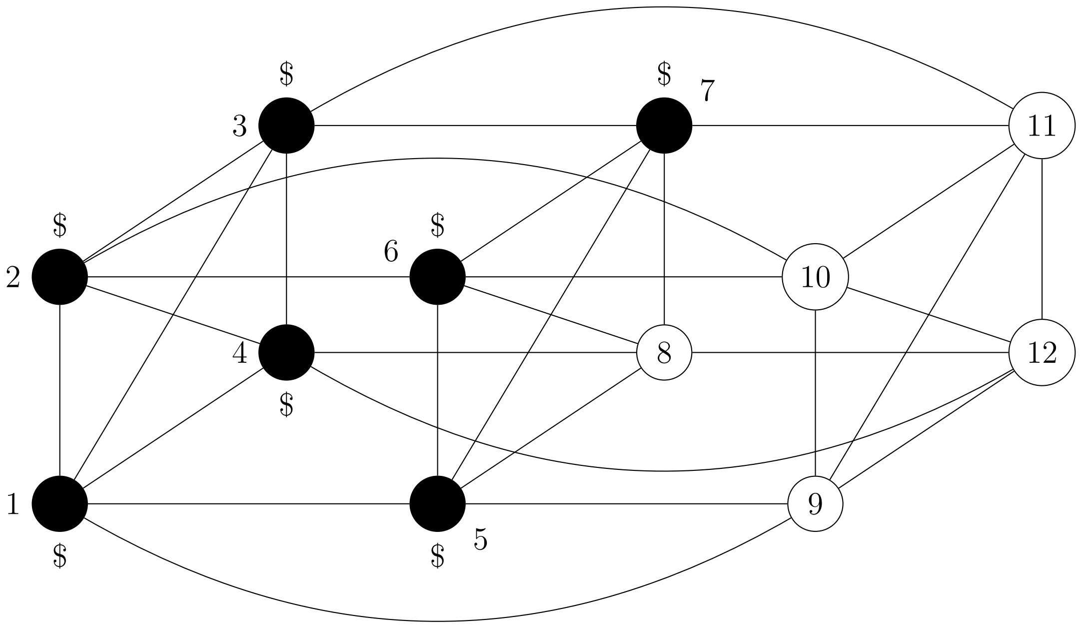
$$s + (t-2)(s-1) = s + st - t - 2s + 2 = st - t - s + 2 \text{ vertices.}$$

Apply the color change rule so that the copy of K_s which all of its vertices are black forces the vertices in the copy of K_s which all of its vertices are white and color the vertices in the this copy to the black according to the zero forcing number of K_s which is equal $s-1$. After that we have all copies of K_s have $s-1$ black vertices. Then we return to the second copy of K_s. The black vertices in this copy can force the white vertices in the same copy to become black. Continue in this manner we can color all vertices of $K_s \Box K_t$. $\Box$

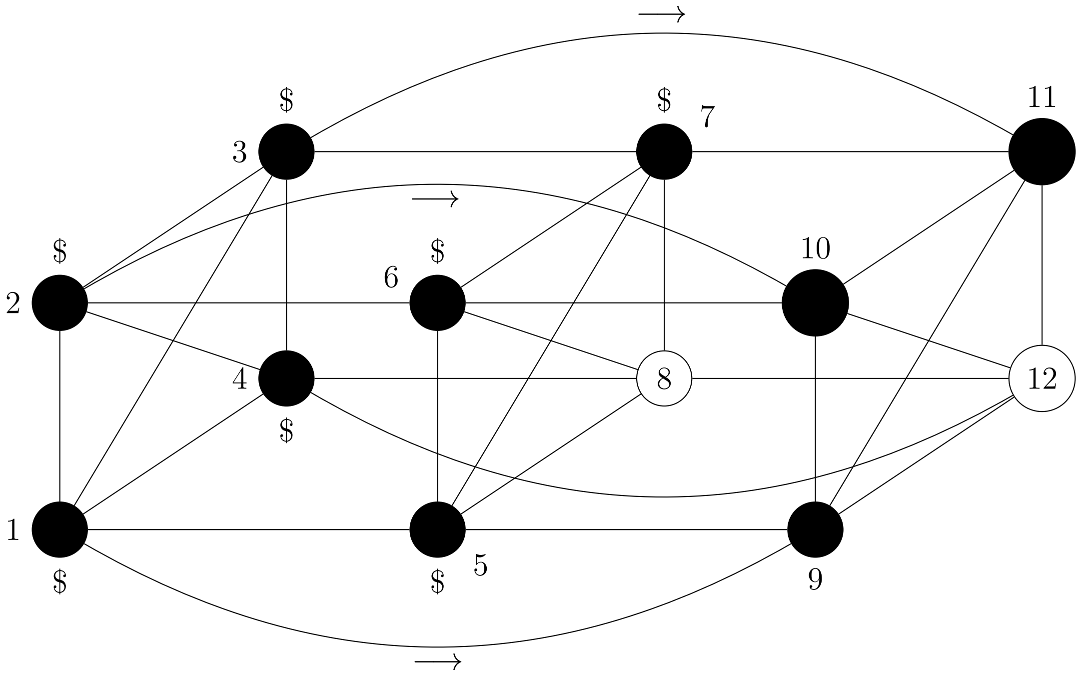
Example 2.3.4. [1] *The following figure represents* $K_4 \Box K_3$.



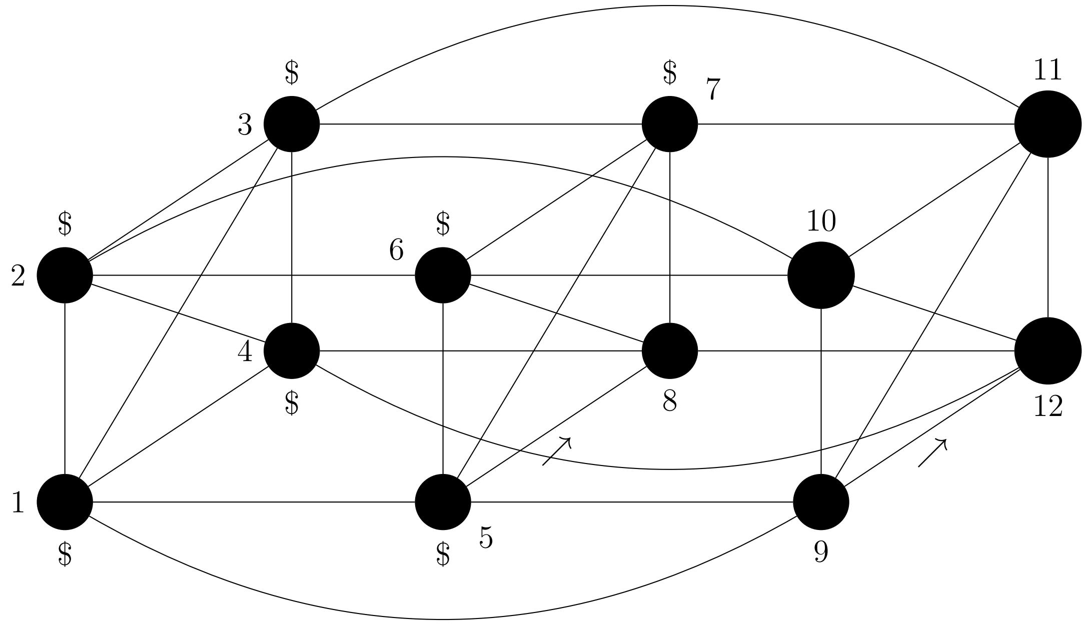
We color all vertices in the first copy of K_4 *and in the second copy of* K_4 *color an optimal zero forcing sets in* K_4 *which is equal to* 3 *vertices. Therefore, a zero forcing set for* $K_4 \Box K_3$ *is* $\{1,2,3,4,5,6,7\}$ *as the following figure shows.*



Applying the color change rule where $1 \to 9, 2 \to 10$ *and* $3 \to 11$ *we have in the last copy* $Z(K_4) = 3$ *vertices are black and shown in the following figure.*



Now we return in the second copy of K_4 *where* $5 \to 8$ *and in the last copy* $9 \to 12$ *as shown following figure.*

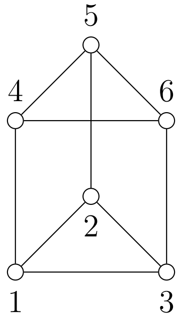


Since we can not color the vertices of $K_4 \Box K_3$ *starting from coloring* 6 *vertices because if* $Z(K_4 \Box K_3)$ *were* 6*, then applying the color change rule starting from coloring six vertices, we would reach a vertex with two white neighbors. Hence* $Z(K_4 \Box K_3) = 7$.

In particular, from Proposition 2.3.3 we have:

$$Z(K_s \Box K_2) \leqslant s.$$

Example 2.3.5. *The following figure represent* $K_3 \Box K_2$.



A zero forcing set for $K_3\square K_2$ *is* $\{1,2,3\}$ *as shown in the following figures.*

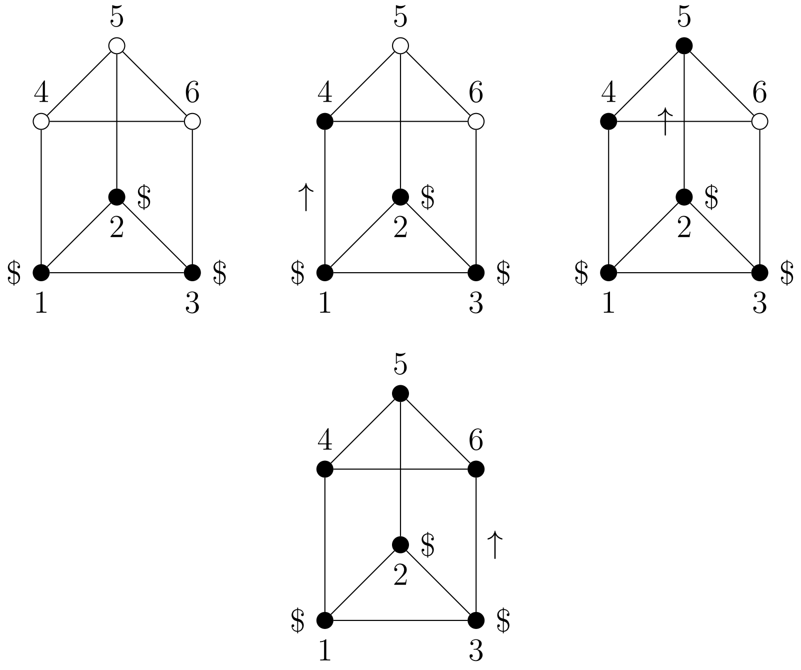


Figure 2.7: Zero forcing set for $K_3\square K_2$

Since we can not color the vertices of $K_3\square K_2$ *starting from coloring* 2 *vertices because every vertex has degree* $=3$*. Hence* $Z(K_3\square K_2)=3$*.*

In the following proposition, it was introduced an upper bound on the zero forcing number for the corona graph.

Proposition 2.3.6. [1] $Z(G\circ H)\leqslant Z(H)|G|+Z(G)|H|-Z(G)Z(H)$*. In particular, for* $t\geq 2, Z(K_t\circ K_s)\leqslant st-1$*.*

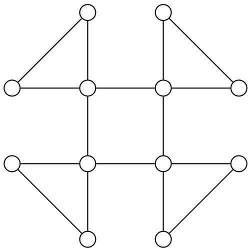
Proof. Choose an optimal zero forcing set Z_G for G. Construct a zero forcing set Z for $G\circ H$ as follows: Color $Z(G)$ copies of H that are corresponding to the vertices in Z_G, and in each of the $|G|-Z_G$ remaining copies of H color an optimal zero forcing set of H. Clearly the order of Z is $Z(H)(|G|-Z(G))+Z(G)|H|=Z(H)|G|+Z(G)|H|-Z(G)Z(H)$. Apply the color change rule focusing on G where we can color the vertices in Z_G black using any vertex in H that is adjacent to the vertex in G. This zero forcing set then turns one more vertex v in G black. Then apply the color change rule in G where each new vertex in G becomes black we apply directly the color change rule in the corresponding copy of H. Repeat in this manner until all vertices become black.

Therefore,

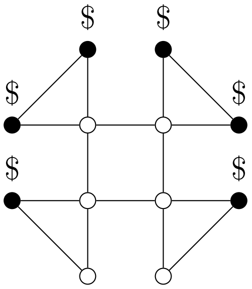
$$Z(G \circ H) \leq Z(H)|G| + Z(G)|H| - Z(G)Z(H).$$

□

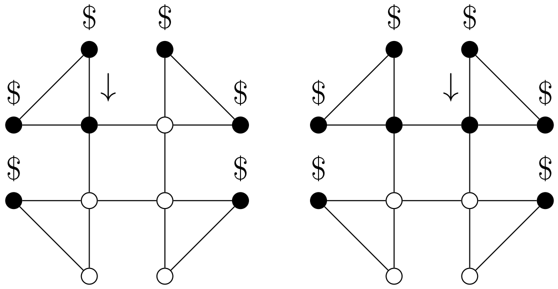
Example 2.3.7. *The following figure illustrates the procedure in $C_4 \circ K_2$ that was presented in the proof of Proposition 2.3.6.*



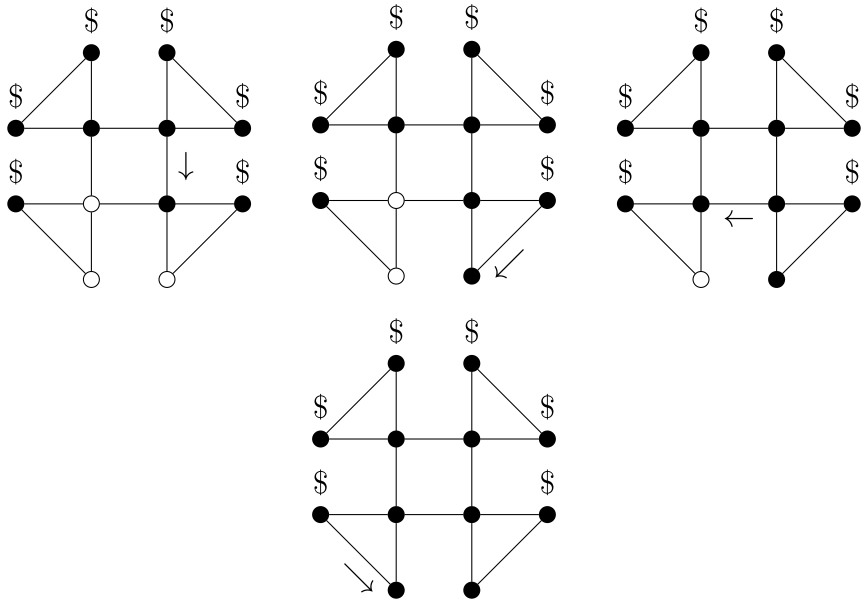
We color $Z(C_4) = 2$ copies of K_2 and color all vertices in that copies because corresponding an optimal zero forcing set of C_4 which is equal 2 vertices. The remaining two copies of K_2 color an optimal zero forcing set of K_2 which is equal 1 vertex. Therefore, a zero forcing set for $C_4 \circ K_2$ consists of six vertices, as shown in the following figures.



Applying the color change rule where the first two copies of K_2 forces two vertices in C_4 as shown in the following figures.

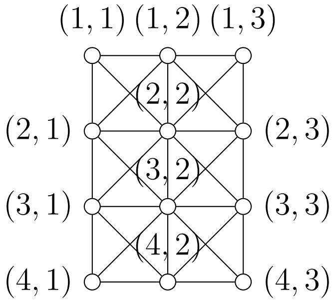


Applying the color change rule to the remaining vertices as shown in the following figures.



Since we can not color the vertices of $C_4 \circ K_2$ starting from coloring 5 vertices because if $Z(C_4 \circ K_2)$ were 5, then applying the color change rule starting from coloring five vertices, we would reach a vertex with two white neighbors. Thus $Z(C_4 \circ K_2) = 6$.

Example 2.3.8. *The following figure represents $P_4 \boxtimes P_3$.*



A zero forcing set for $P_4 \boxtimes P_3$ consist of six vertices which are $\{(1,1),(1,2),(1,3),(2,1),(3,1),(4,1)\}$ as shown in the following figures.

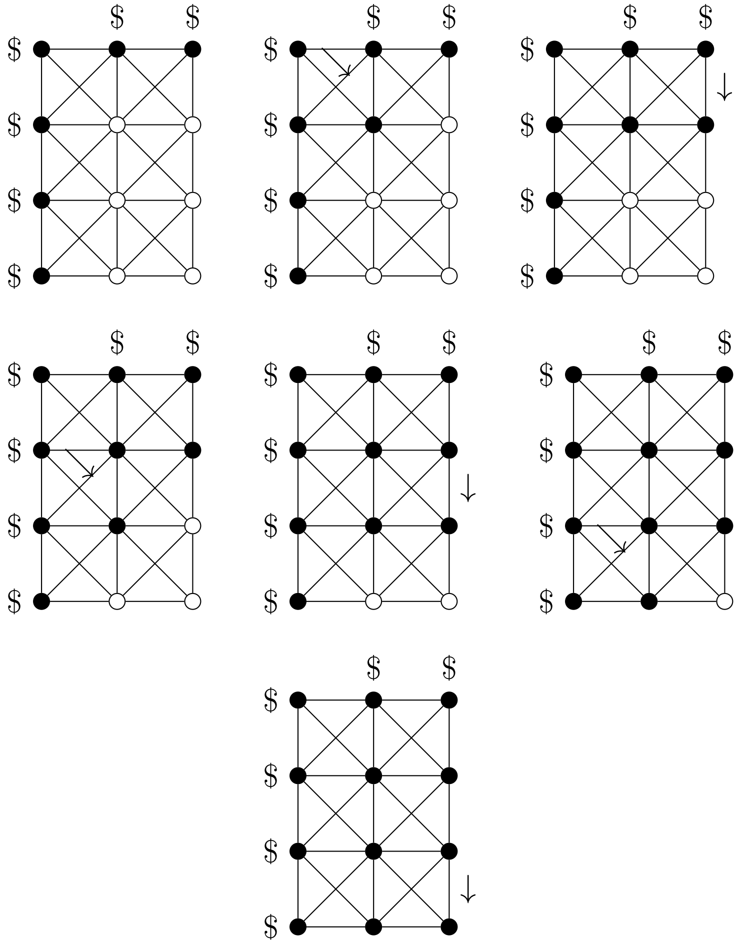


Figure 2.8: Zero forcing set for $P_4 \boxtimes P_3$

Since we can not color the vertices of $P_4 \boxtimes P_3$ *starting from coloring* 5 *vertices because if* $Z(P_4 \boxtimes P_3)$ *were* 5, *then applying the color change rule starting from coloring five vertices, we would reach a vertex with two white neighbors. Thus* $Z(P_4 \boxtimes P_3) = 6$.

In general, $Z = \{(1, j) : 1 \leqslant j \leqslant t\} \cup \{(i, 1) : 1 \leqslant i \leqslant s\}$ is a zero forcing set for the graph $P_s \boxtimes P_t$. Hence [1]

$$Z(P_s \boxtimes P_t) \leqslant s + t - 1.$$

In the following theorem, it was presented the upper bound on $Z(G)$.

Theorem 2.3.9. [3] *Let* G *be a connected graph with* $n > 1$ *vertices and* $\Delta \geq 2$. *Then,*

$$Z(G) \leq \frac{(\Delta - 2)n + 2}{\Delta - 1}. \tag{2.4}$$

Proposition 2.3.10. [17] *If G is a connected graph of order n and maximum degree $\Delta \geq 3$ that is distinct from $K_{\Delta+1}$, then*

$$Z(G) \leq \frac{(\Delta - 1)}{\Delta} n.$$

Proof. We distinguish between the following two cases:
Case 1: $2\Delta \leq n$.
By Theorem 2.3.9, we have

$$Z(G) \leq \frac{(\Delta - 2)n + 2}{\Delta - 1}.$$

We want to prove $\frac{(\Delta-2)n+2}{\Delta-1} \leq \frac{(\Delta-1)}{\Delta} n$.

$$\begin{aligned} 2\Delta &\leq n \\ \Delta^2 n - 2\Delta n + 2\Delta &\leq \Delta^2 n - 2\Delta n + n \\ \Delta[(\Delta - 2)n + 2] &\leq (\Delta - 1)(\Delta - 1)n \\ \frac{(\Delta - 2)n + 2}{\Delta - 1} &\leq \frac{(\Delta - 1)}{\Delta} n. \end{aligned}$$

Hence

$$Z(G) \leq \frac{(\Delta - 1)}{\Delta} n.$$

Case 2: $2\Delta > n$.
Since G is not complete, it contains an induced path vuw of order 3. Hence it is easy to note that the set $V(G) \backslash \{uw\}$ is a zero forcing set. Hence

$$Z(G) \leq n - 2$$

We want to proof that $n - 2 < \frac{(\Delta-1)}{\Delta} n$ in the case $2\Delta > n$.

$$\begin{aligned} 2\Delta &> n \\ \Delta n - 2\Delta &< \Delta n - n \\ \Delta(n - 2) &< (\Delta - 1)n \\ n - 2 &< \frac{(\Delta - 1)}{\Delta} n. \end{aligned}$$

Thus

$$Z(G) \leq \frac{(\Delta - 1)}{\Delta} n.$$

□

2.4 Vertex Spread

This section explores the effect of various vertex operations on the zero forcing number.

Definition 2.4.1. *Let G be a graph and v be a vertex in G. The* zero spread *of G at v is defined as $z_v(G) = Z(G) - Z(G - v)$.*

The following theorem presents a bound on zero spread.

Theorem 2.4.2. [13] *For every graph G and vertex $v \in V(G), |z_v(G)| \leq 1$ or*

$$-1 \leq z_v(G) \leq 1.$$

Proof. If Z' is an optimal zero forcing set for $G - v$, then $Z' \cup \{v\}$ is a zero forcing set of G but it is not necessary to be optimal. Hence

$$Z(G) \leq Z(G - v) + 1,$$

or

$$z_v(G) \leq 1.$$

To prove $z_v(G) \geq -1$, let Z be an optimal zero forcing set for G. Construct a particular chronological list of forces $\mathbb{F}$. If a force $v \to u$ appears in $\mathbb{F}$ for some vertex u, then $Z \cup \{u\}$ is a zero forcing set of $G - v$ with chronological list of forces obtained from $\mathbb{F}$ by deleting $v \to u$; otherwise, Z is a zero forcing set of $G - v$ but it is not necessary to be optimal. Thus

$$Z(G - v) \leq Z(G) + 1,$$

or

$$z_v(G) \geq -1.$$

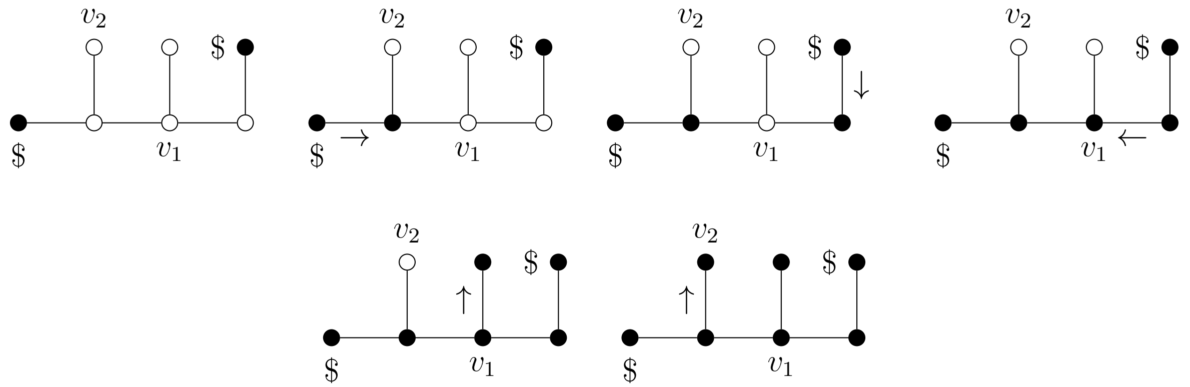
□

The following are examples of all three possible cases in proof of Theorem 2.4.2.

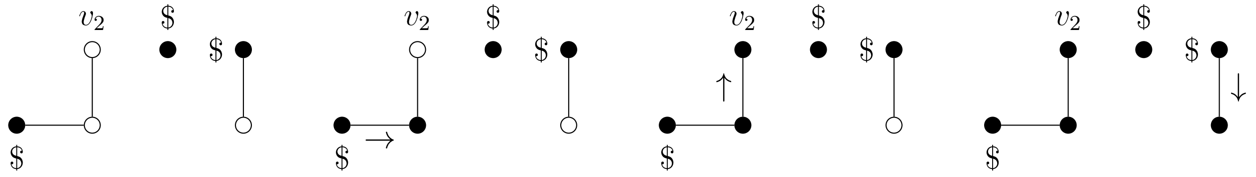
Example 2.4.3. *An example of a graph for which removing a vertex decreases the zero forcing number. Let K_3 be the following complete graph which has $Z(G) = 2$.*

If we remove v, the following graph $K_3 - v$ has $Z(K_3 - v) = 1$. Indeed,

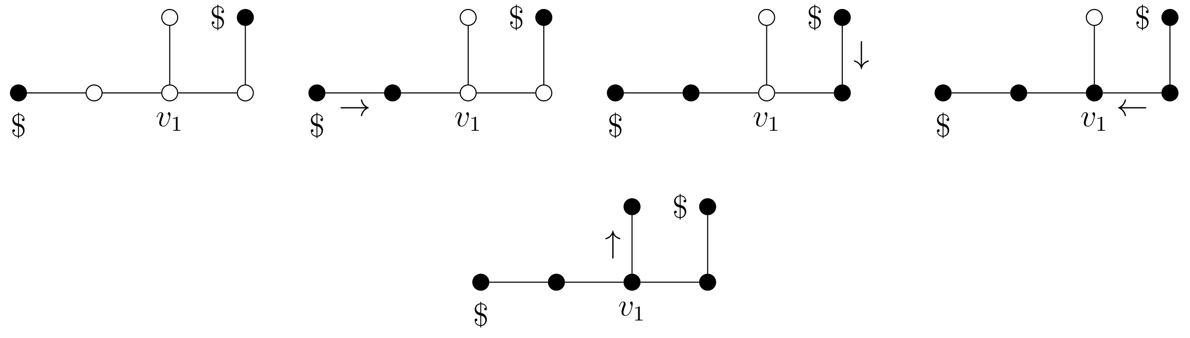
Example 2.4.4. *The following tree T has $Z(T) = 2$, as shown in the following figures.*



If we remove v_1, clearly v_1 in chronological list of forces $\mathbb{F}$. Thus $Z(T - v_1) = 3$, as shown in the following figures.



Now if we remove v_2, clearly v_2 is not in chronological list of forces $\mathbb{F}$. Thus $Z(T - v_2) = Z(T) = 2$, as shown in the following figures.



In the following theorem the zero spread is extended to size greater than one and bounds are proved. First we need to define $z_W(G) = Z(G) - Z(G - W)$ is the zero spread of $W \subseteq V(G)$.

Theorem 2.4.5. [28] *For every graph G and every subset $W \subseteq V(G)$,*

$$-|W| \leq z_W(G) \leq |W|.$$

Proof. Without loss of generality let $W = \{v_1, \cdots, v_k\}$. Define $G_0 := G$ and $G_i := G_{i-1} - v_i, i = 1, \cdots, k$. That implies $G_k = G - W$. For any graph H and any vertex $v \in V(H)$ we have by Theorem 2.4.2,

$$|Z(H) - Z(H - v)| \leq 1.$$

Hence

$$\begin{aligned} |z_W(G)| = |Z(G) - Z(G - W)| &= |\sum_{i=0}^{k-1}(Z(G_i) - Z(G_{i+1}))| \\ &\leq \sum_{i=0}^{k-1} |Z(G_i) - Z(G_{i+1})| \\ &\leq \sum_{i=0}^{k-1} 1 = k = |W|. \end{aligned}$$

Thus $|z_W(G)| \leq |W|$ which is equivalent to

$$-|W| \leq z_W(G) \leq |W|.$$

□

The following lemma shows that the zero forcing number of a graph G with a cut vertex v can be calculated by finding the zero forcing numbers of the connected components of $G - v$.

Lemma 2.4.6. [28] *Let G be a graph with cut vertex $v \in V(G)$. Let $W_1, \cdots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \leq i \leq k$, let $G_i := G[W_i \cup \{v\}]$. Then*

$$Z(G) \geq \sum_{i=1}^{k} Z(G_i) - k + 1.$$

Proof. Suppose Z is an optimal zero forcing set of G. We distinguish between the following two cases:

Case 1: $v \notin Z$.

This means that there is a vertex forced v to become black that we assume it u. Without loss of generality, let $u \in G_1$. A zero forcing set of G_1 is $Z \cap V(G_1)$. Hence $Z(G_1) \leq$

$|Z \cap V(G_1)|$. Also, a zero forcing set of G_i is $(Z \cap V(G_i)) \cup \{v\}, i = 2, \cdots, k$. Hence $Z(G_i) \leq |Z \cap V(G_i)| + 1$. Thus

$$\begin{aligned}\sum_{i=1}^{k} Z(G_i) \leq |Z \cap V(G_1)| + \sum_{i=2}^{k}(|Z \cap V(G_i)| + 1) &= \sum_{i=1}^{k} |Z \cap V(G_i)| + k - 1 \\ &= Z(G) + k - 1.\end{aligned}$$

Therefore,

$$Z(G) \geq \sum_{i=1}^{k} Z(G_i) - k + 1.$$

Case 2: $v \in Z$.

This means that there is a vertex u such that $v \to u$. Without loss of generality, let $u \in G_1$. Now a zero forcing set of G_1 is $Z \cap V(G_1)$. Hence $Z(G_1) \leq |Z \cap V(G_1)|$. Also, a zero forcing set of G_i is $(Z \cap V(G_i)), i = 2, \cdots, k$. Hence $Z(G_i) \leq |Z \cap V(G_i)|$. Therefore,

$$\sum_{i=1}^{k} Z(G_i) \leq \sum_{i=1}^{k}(|Z \cap V(G_i)|) = Z(G).$$

Hence

$$Z(G) \geq \sum_{i=1}^{k} Z(G_i).$$

In either case we have

$$Z(G) \geq \sum_{i=1}^{k} Z(G_i) - k + 1.$$

□

The following theorem shows that if a graph G with an optimal chain set that contain a singleton vertex, the zero spread = 1.

Theorem 2.4.7. [13] *Let G be a graph and $v \in V(G)$. Then there exists an optimal chain set of G that contains v as a singleton if and only if $z_v(G) = 1$.*

Proof. Let G be a graph, v be a vertex in G, and Z be an optimal zero forcing set of G where there exists an optimal chain set of Z with a singleton containing v. Clearly $Z \backslash \{v\}$ is a zero forcing set for $G - v$. Hence

$$Z(G - v) \leq Z(G) - 1,$$

or

$$z_v(G) \geq 1 \tag{2.5}$$

Thus by Theorem 2.4.2 and (2.5) we conclude

$$z_v(G) = 1.$$

Let G be a graph and v be a vertex in G such that $z_v(G) = 1$. Let Z be an optimal zero forcing set for $G - v$ and define $Z' = Z \cup \{v\}$. Clearly Z' is a zero forcing set for G with the same chronological list of forces $\mathbb{F}$ as for Z in $G - v$. Since $z_v(G) = 1$, which implies that

$$\begin{aligned} Z(G) &= Z(G - v) + 1 \\ &= |Z| + 1 \\ &= |Z'|. \end{aligned}$$

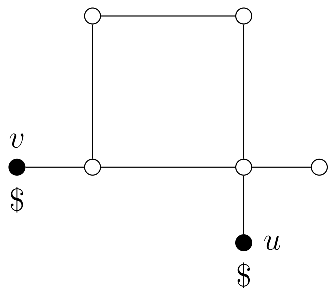
Therefore, Z' is an optimal zero forcing set for G with v is a singleton in the chain set. ☐

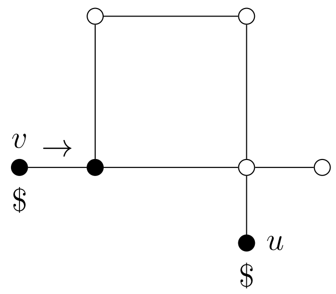
In the following theorem, it was studied the case when v in an optimal zero forcing set which implies that the zero spread is greater than or equal zero.

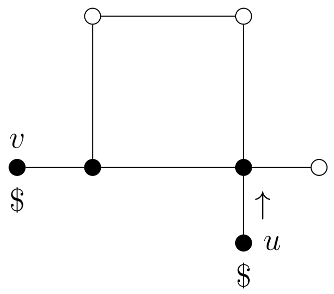
Theorem 2.4.8. [13] *Let G be a graph and $v \in V(G)$. If $z_v(G) = -1$, then $v \notin Z$ for all optimal zero forcing sets Z of G. Equivalently, if $v \in Z$ for some optimal zero forcing set Z of G, then $z_v(G) \geq 0$.*

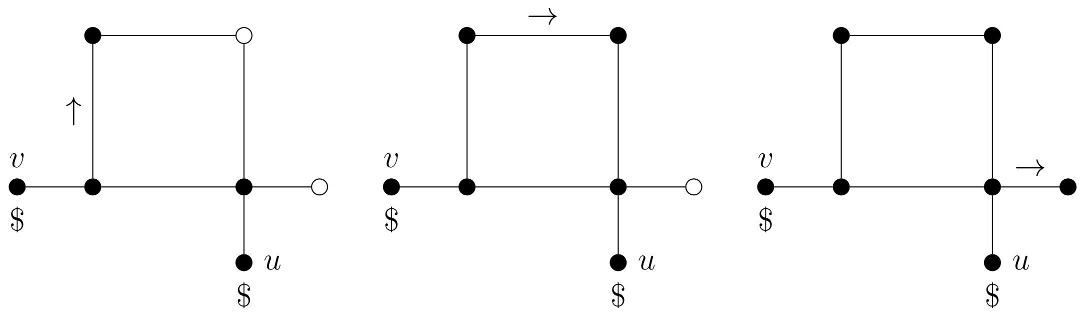
Proof. We prove the second statement. Let Z be an optimal zero forcing set of G where $v \in Z$. Construct a chronological list of forces $\mathbb{F}$. If $v \to w$ appears in $\mathbb{F}$, then define $Z' := Z \backslash \{v\} \cup \{w\}$; if not, define $Z' := Z \backslash \{v\}$. Clearly Z' is a zero forcing set for $G - v$ and $|Z'| \leq |Z|$, so $z_v(G) \geq 0$. ☐

Example 2.4.9. *The following graph G has $Z(G) = 2$ with $Z = \{v, u\}$, as shown in the following figures.*

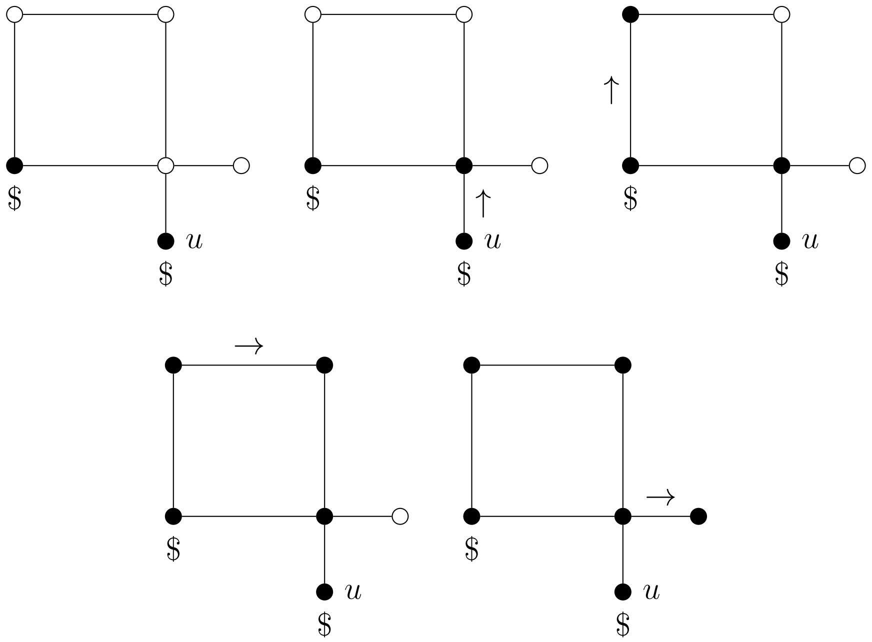






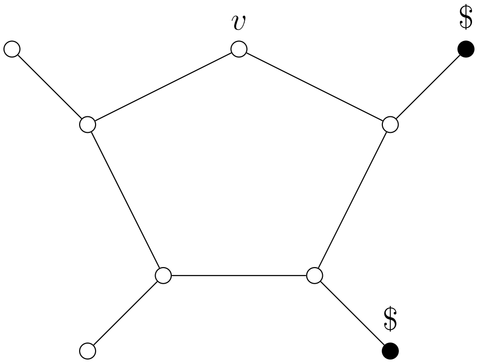


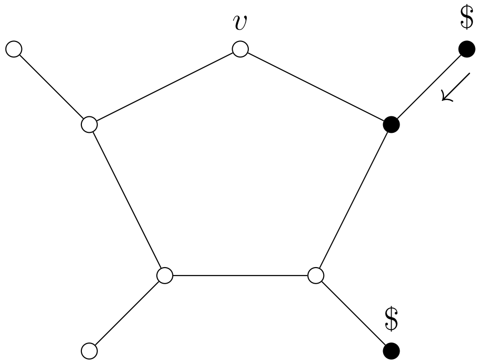
If we remove $v \in Z$, then $Z(G - v) = 2$ (shown in the following figures). Notice that $z_v(G) = 0 \geq 0$.

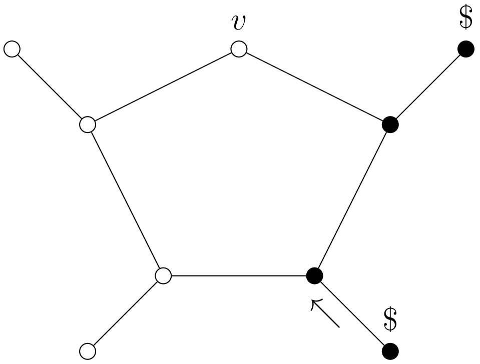


Now a question arises: Is the converse of Theorem 2.4.8, also true. The following example shows a negative answer.

Example 2.4.10. [13] *The following graph G has $Z(G) = 2$, as shown in the following figures.*


v
$
$


v
$
$


v
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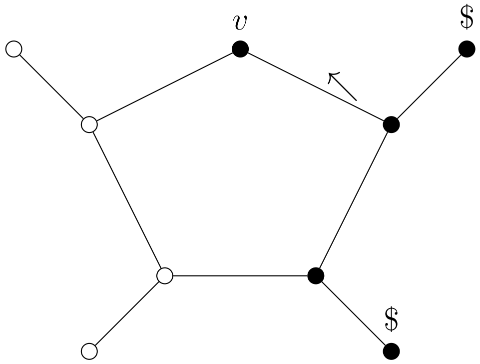
v
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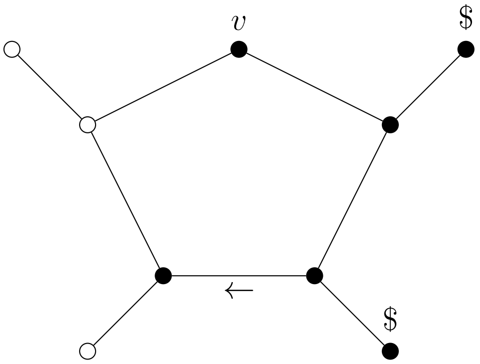
v
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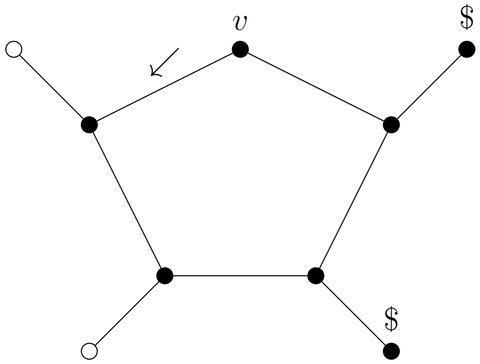
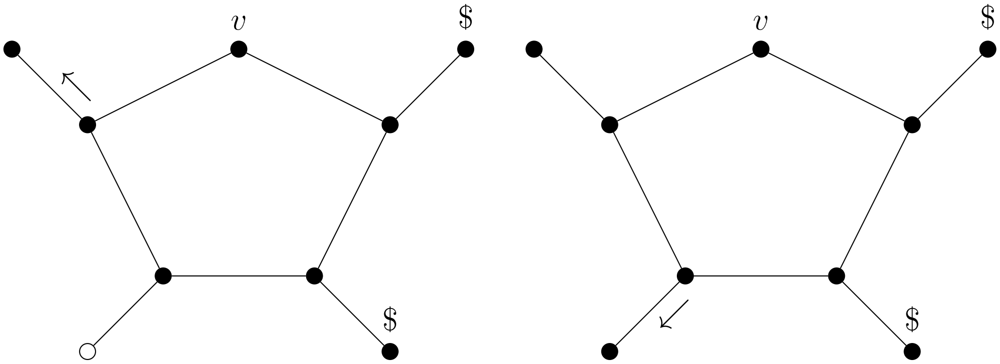
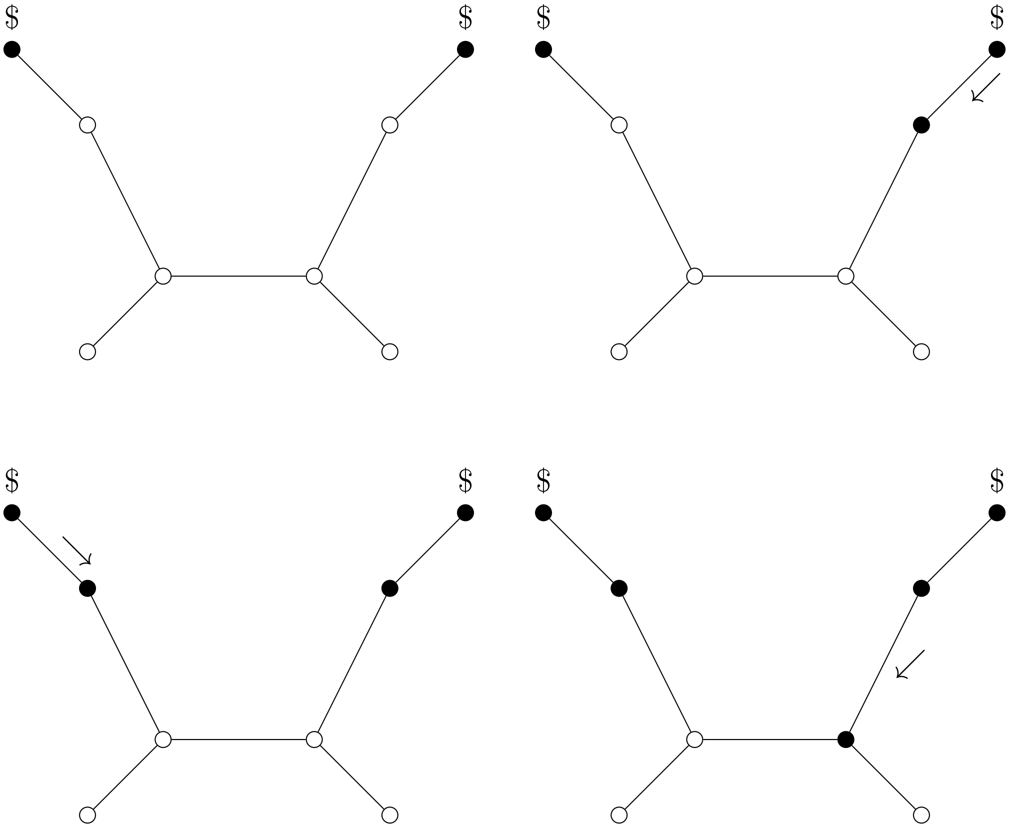
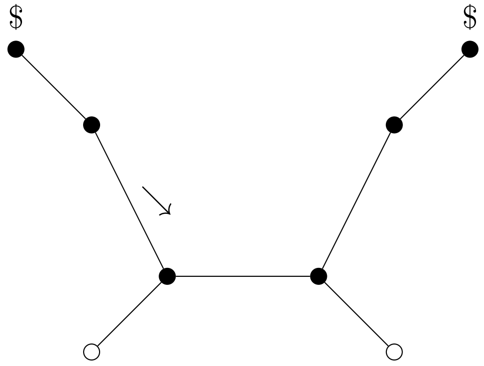
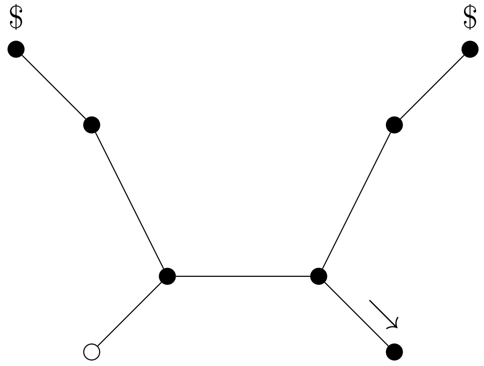

v
$
$

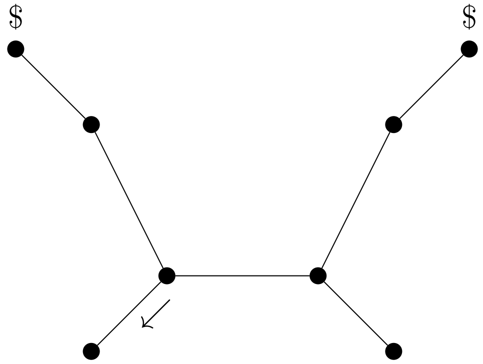


If we remove v*, then* $Z(G - v) = 2$ *(shown in the following figures). Notice that* $z_v(G) = 0 \neq -1$*. Moreover, it is easy to see that* v *is not in any optimal zero forcing set of* G*. Therefore, the converse of Theorem 2.4.8 is not true.*









2.5 Edge Spread

This section explores the effect of various edge operation on the zero forcing number.

Definition 2.5.1. [12],[13] *Let G be a graph and e be an edge in G. The* zero edge spread *of e is $z_e(G) = Z(G) - Z(G - e)$.*

Theorem 2.5.2. [13] *For every graph G and edge e of G,*

$$-1 \le z_e(G) \le 1.$$

Proof. Let G be a graph and $e = \{v, w\}$ be an edge in G. Firstly, let Z be an optimal zero forcing set of $G - e$. If both v and w are in Z, then Z is a zero forcing set for G. Otherwise, without loss of generality let w be black when v is forced. Then $Z \cup \{v\}$ is a zero forcing set of G. In both cases we have

$$Z(G) \le Z(G - e) + 1,$$

or

$$z_e(G) \leq 1.$$

Secondly, let Z be an optimal zero forcing set for G and $e = \{v, w\}$. Write a particular chronological list of forces $\mathbb{F}$. Without loss of generality, assume $w \in Z$ or w is forced before v is forced. If the force $w \to v$ appears in $\mathbb{F}$, then $Z' = Z \cup \{v\}$ is a zero forcing set of $G - e$. Now if the force $w \to v$ does not appear in $\mathbb{F}$, then $Z' = Z$ is a zero forcing set of $G - e$. Hence

$$Z(G - e) \leq Z(G) + 1,$$

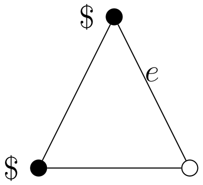
or

$$z_e(G) \geq -1.$$

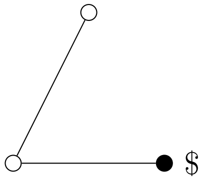
□

The following are examples of all three possible cases in the proof of Theorem 2.5.2.

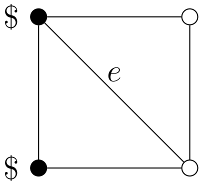
Example 2.5.3. [29] *An example of a graph for which removing an edge decreases the zero forcing number. Let* K_3 *be the following graph which has* $Z(G) = 2$.



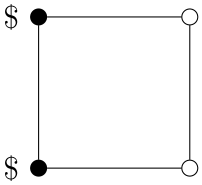
If we delete e*, the following graph* $K_3 - e$ *has* $Z(K_3 - e) = 1$.



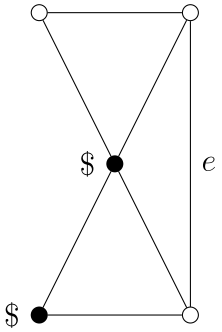
Example 2.5.4. [29] *An example of a graph for which removing an edge does not change the zero forcing number. The following graph* G *has* $Z(G) = 2$.



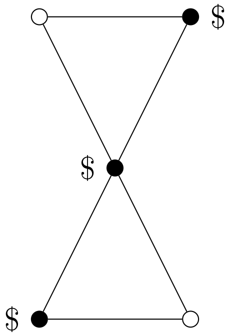
If we remove e, the following graph $G - e$ has $Z(G - e) = 2$.



Example 2.5.5. [29] *An example of a graph for which removing an edge increases the zero forcing number. The following graph G has $Z(G) = 2$.*



If we remove e, the following graph $G - e$ has $Z(G - e) = 3$.



The following theorem states that the zero spread is equal -1 then e is an edge in an optimal zero forcing chain.

Theorem 2.5.6. [13] *Let G be a graph and e be an edge in G. If $z_e(G) = -1$, then for every optimal zero forcing chain set of G, e is an edge in a chain. Equivalently, if there is an optimal zero forcing chain set of G such that e is not an edge in any chain, then $z_e(G) \geq 0$.*

Proof. We prove the second statement. Let Z be an optimal zero forcing set such that for some chronological list of forces, e is not an edge in the optimal chain. Then Z is a zero forcing set for $G-e$ with the same chronological list of forces but it is not necessary to be optimal. Hence

$$Z(G-e) \leq Z(G),$$

or

$$z_e(G) \geq 0.$$

$\square$

Now a question arises: is the converse of the previous theorem also true. The question is answered in [12]. Example 2.5.7 shows the answer is negative.

Example 2.5.7. [12] *The following figure represents a robot graph G with $n = 2$.*

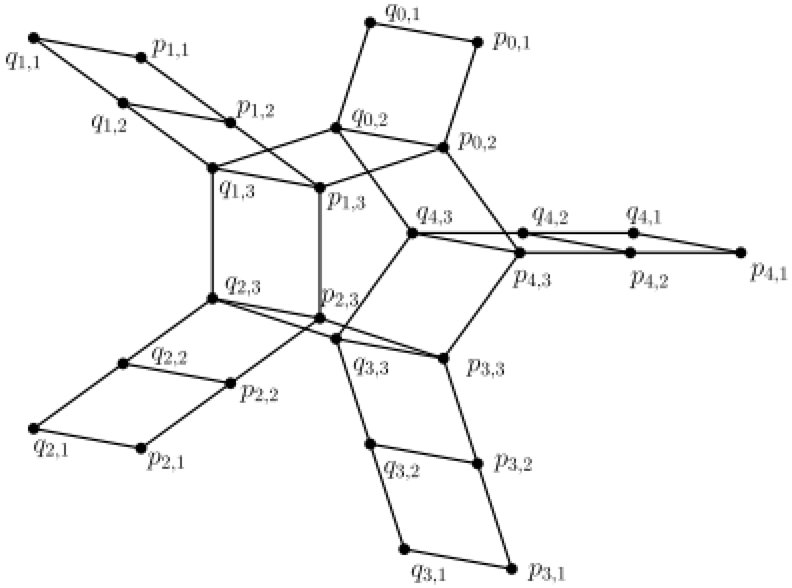


Figure 2.9: Robot graph

In [12] *it was shown that $Z(G) = 5$ and every optimal zero forcing process uses the edge $p_{0,1}q_{0,1}$ to perform a force. Also, the zero forcing number remains* 5 *when the edge $p_{0,1}q_{0,1}$ is deleted.*

The following theorem discusses the zero spread when e is a cut edge of a connected graph G.

Theorem 2.5.8. [27] *Let* $e = \{v_1, v_2\}$ *be a cut edge of a connected graph* G. *Let* G_1 *and* G_2 *be the connected components of* $G - e$ *with* $v_1 \in G_1$ *and* $v_2 \in G_2$. *Then*

$$z_e(G) = \begin{cases} -1 & \textit{if } v_i \textit{ is in an optimal zero forcing set in } G_i \textit{ for } i = 1, 2, \\ 0 & \textit{otherwise.} \end{cases}$$

Chapter 3

Maximum Nullity Of Graphs

In this chapter, we introduce the maximum nullity of a graph which is denoted by $M(G)$. Abound on the maximum nullity which is the zero forcing number is presented. Then, the maximum nullity of some interesting families of graphs is introduced. After that, the maximum nullity of trees is studied. In addition, a formula for minimum rank of vertex-sums of graphs is presented. We conclude this chapter by studying the matrices in $S(T)$ with a fixed number of negative eigenvalues, say q, denoted by $S_q(T)$ and presenting formula that calculate the maximum nullity of matrices in $S_q(T)$, where T is a tree.

3.1 Upper Bound On $M(G)$

For a graph G of order n, the *maximum nullity* (or *maxmum corank*) of G over $\mathbb{R}$ is defined by

$$M(G) := \max\{\text{nullity}(A) : A \in S(G)\},$$

and the *minimum rank* of G over $\mathbb{R}$ is defined by

$$\text{mr}(G) := \min\{\text{rank}(A) |\ A \in S(G)\}.$$

In this section, the zero forcing number $Z(G)$ provides an upper bound for the maximum nullity of a graph is shown. Clearly [15],

$$|G| = \text{mr}(G) + M(G).$$

The following proposition will be used in proving the maximum nullity of a graph is bounded above by the cardinality of any zero forcing set.

Proposition 3.1.1. [1] *If $A \in \mathbb{R}^{n\times n}$, and* $\text{corank}(A) > k$, *then there is a nonzero vector $x \in \ker(A)$ vanishing at any k specified positions. In other words, if W is a set of k*

indices, then there is a nonzero vector $x \in \ker(A)$ *such that* $\mathrm{supp}(x) \cap W = \emptyset$*, where* $\mathrm{supp}(x) = \{i \mid x_i \neq 0\}$.

Proof. Let $1 \leq i_1 < \cdots < i_k \leq n$ and let

$$V_k = \{x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n : x_{i_1} = x_{i_2} = \cdots = x_{i_k} = 0\}.$$

Then $\dim V_k = n - k$. Let $X := \ker(A)$, so $\dim\ X = \mathrm{corank}(A) > k$. Since $V_k + X$ is a subspace of $\mathbb{R}^n$, we have $\dim\ (V_k + X) \leq n$ and $-\dim\ (V_k + X) \geq -n$. Hence

$$\dim(V_k \cap X) = \ \dim(V_k) + \dim(X) - \dim(V_k + X) > n - k + k - n = 0.$$

Therefore, $V_k \cap X \neq \{\mathbf{0}\}$. Hence there exists $\mathbf{0} \neq y \in V_k \cap X$ i.e., $y \in \ker(A)$ such that $\mathrm{supp}(y) \cap \{i_1, \ldots, i_k\} = \emptyset$. □

In the following, a clarification that we need in the proof of Proposition 3.1.2. If $A \in S(G)$ and $x \in \mathbb{R}^n$, then

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} (Ax)_1 \\ (Ax)_2 \\ \vdots \\ (Ax)_u \\ \vdots \\ (Ax)_n \end{bmatrix},$$

where

$$\begin{aligned} (Ax)_u &= \sum_{v=1}^{n} a_{u,v} x_v \\ &= a_{u,u} x_u + \sum_{u \sim v} a_{u,v} x_v + \sum_{u \nsim v} a_{u,v} x_v \\ &= a_{u,u} x_u + \sum_{u \sim v} a_{u,v} x_v, \end{aligned} \tag{3.1}$$

the summand $\sum_{u \nsim v} a_{u,v} x_v = 0$ since $a_{u,v} = 0$ for all v such that $u \nsim v$.

Proposition 3.1.2. [1] *Let* Z *be a zero forcing set of* G *and* $A \in S(G)$*. If* $x \in \ker(A)$ *and* $\mathrm{supp}(x) \cap Z = \emptyset$*, then* $x = \mathbf{0}$*.*

Proof. We distinguish between the following two cases:
Case 1: $Z = V(G)$.
In this case $\text{supp}(x) \cap Z = \emptyset$. Hence $\text{supp}(x) = \emptyset$ which implies that $x = \mathbf{0}$.
Case 2: $Z \neq V(G)$.
Let $N(u) = \{v_1, v_2, \cdots, v_h\}$ and u be colored black (x_u is required to be 0 because $\text{supp}(x) \cap Z = \emptyset$) with exactly one neighbor v_1 colored white (so x_{v_1} is not yet required to be 0 while $x_{v_i} = 0, i = 2, \cdots, h$). Hence by (3.1) we have

$$\begin{aligned}(Ax)_u &= a_{u,u}x_u + \sum_{u \sim v} a_{u,v}x_v = 0 \\ &= \sum_{u \sim v} a_{u,v}x_v \\ &= a_{u,v_1}x_{v_1} + a_{u,v_2}x_{v_2} + \cdots + a_{u,v_h}x_{v_h} \\ &= a_{u,v_1}x_{v_1} = 0.\end{aligned}$$

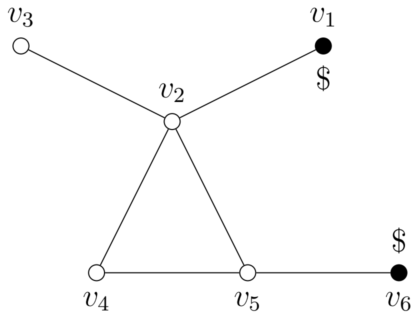
Because $a_{u,v_1} \neq 0$ we have $x_{v_1} = 0$. Similarly each color change step corresponds to requiring another entry in x to be zero. Thus $x = \mathbf{0}$. □

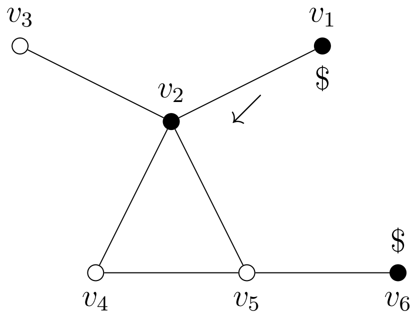
The following examples illustrate the two cases in the proof of Proposition 3.1.2.

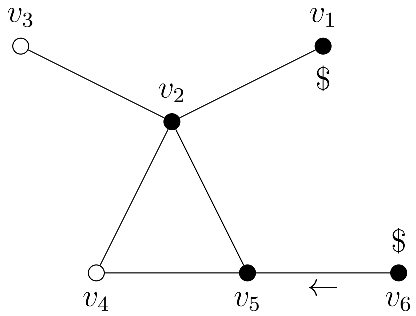
Example 3.1.3. *The following graph G has $Z(G) = 2$.*

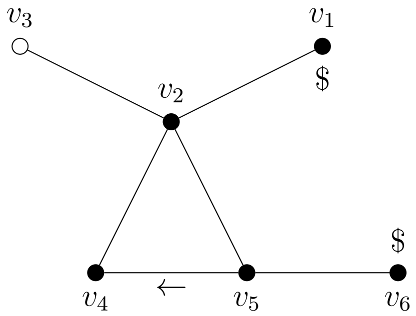
Clearly v_1 and $v_2 \in Z$. Hence we have $x_1 = 0$ and $x_2 = 0$. That implies $x = \mathbf{0}$.

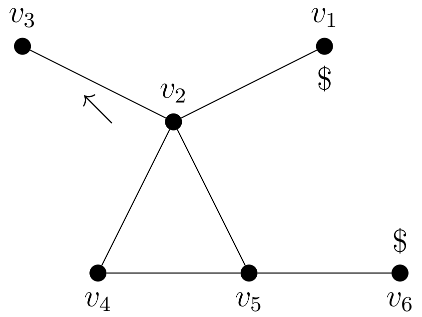
Example 3.1.4. *The following graph G has $Z(G) = 2$.*











For a matrix $A \in S(G)$, A has the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & 0 \\ 0 & a_{23} & a_{33} & 0 & 0 & 0 \\ 0 & a_{24} & 0 & a_{44} & a_{45} & 0 \\ 0 & a_{25} & 0 & a_{45} & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{56} & a_{66} \end{bmatrix}.$$

Clearly $Z = \{v_1, v_6\}$ forms a zero forcing set of the graph G. Let $x \in \ker(A)$ be such *that* $\mathrm{supp}(x) \cap Z = \emptyset$. *Hence* $x_1 = x_6 = 0$. *Thus*

$$Ax = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} & 0 \\ 0 & a_{23} & a_{33} & 0 & 0 & 0 \\ 0 & a_{24} & 0 & a_{44} & a_{45} & 0 \\ 0 & a_{25} & 0 & a_{45} & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{56} & a_{66} \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = \boldsymbol{0}.$$

Note that vertex v_1 *forces vertex* v_2. *By direct calculations we have* $(Ax)_1 = 0 + a_{12}x_2 + 0 + 0 + 0 + 0 = 0$, *since* $a_{12} \neq 0$ *we conclude that* $x_2 = 0$. *Vertex* v_6 *forces vertex* v_5. *Hence* $(Ax)_6 = 0 + 0 + 0 + 0 + a_{56}x_5 + 0 = 0$, *since* $a_{56} \neq 0$. *Hence* $x_5 = 0$. *Vertex* v_5 *forces vertex* v_4 *and so* $(Ax)_4 = 0+0+0+a_{45}x_4+0+0 = 0$, *since* $a_{45} \neq 0$. *Thus* $x_4 = 0$. *Vertex* v_2 *forces vertex* v_3 *which implies that* $(Ax)_2 = 0+0+a_{23}x_3+0+0+0 = 0$, *since* $a_{23} \neq 0$. *Hence* $x_3 = 0$. *Therefore,* $x = \boldsymbol{0}$.

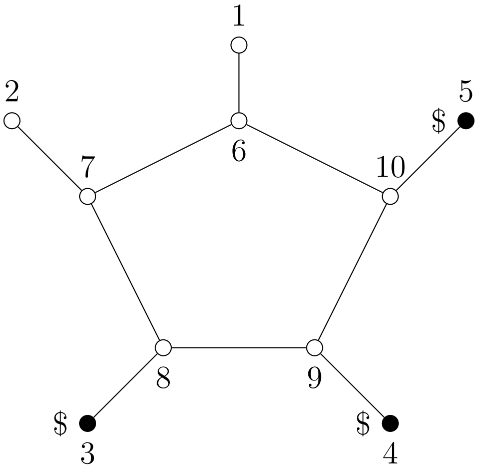
In the following proposition, it was proved that the maximum nullity of a graph is bounded above by the cardinality of any zero forcing set. In particular, it is bounded above by the zero forcing number.

Proposition 3.1.5. [1],[5] *Let* G *be a graph and let* $Z \subseteq V(G)$ *be a zero forcing set. Then* $M(G) \leq |Z|$, *and thus* $M(G) \leq Z(G)$.

Proof. Let $k := |Z|$ and suppose on the contrary that $M(G) > k$ and let $A \in S(G)$ be such that $\mathrm{corank}(A) = M(G)$. Hence $\mathrm{corank}(A) > k$. By Proposition 3.1.1, we can find $\mathbf{0} \neq x \in \ker(A)$ such that x vanishing at these k positions that are corresponding to Z. Hence $\mathrm{supp}(x) \cap Z = \emptyset$. By Proposition 3.1.2, $x = \mathbf{0}$. A contradiction. Therefore, $M(G) \leq |Z|$ and thus $M(G) \leq Z(G)$. ☐

In the following example we show in details that $M(C_5 \circ K_1) < Z(C_5 \circ K_1)$.

Example 3.1.6. [1] *Consider the corona* $C_5 \circ K_1$ *shown in Figure 3.1. The set of vertices* $\{3, 4, 5\}$ *(shown) is a zero forcing set, but there is no smaller zero forcing set. Thus* $Z(C_5 \circ K_1) = 3$.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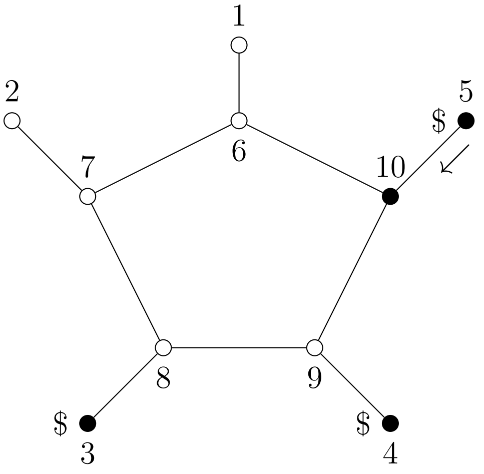
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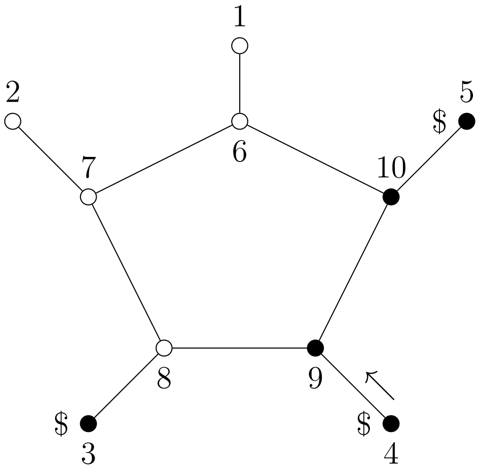
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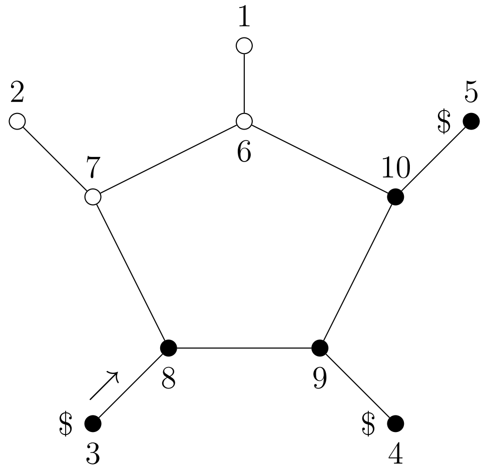
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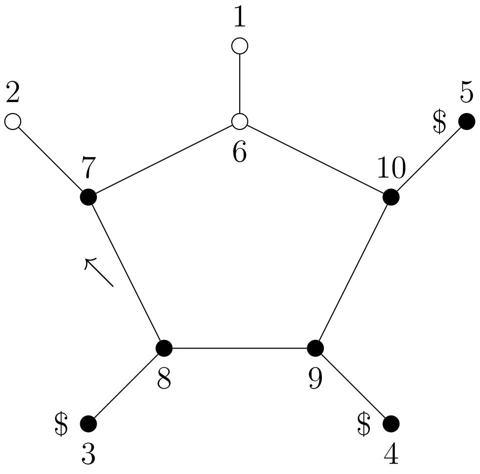
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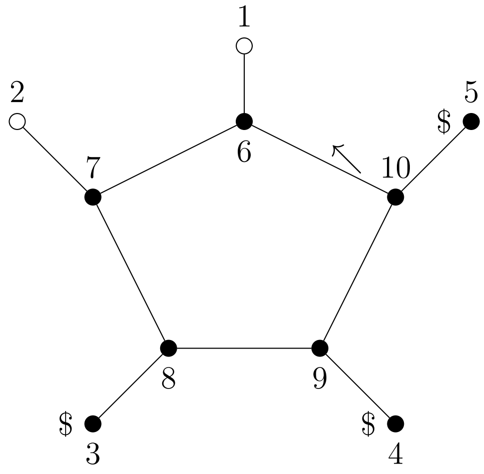
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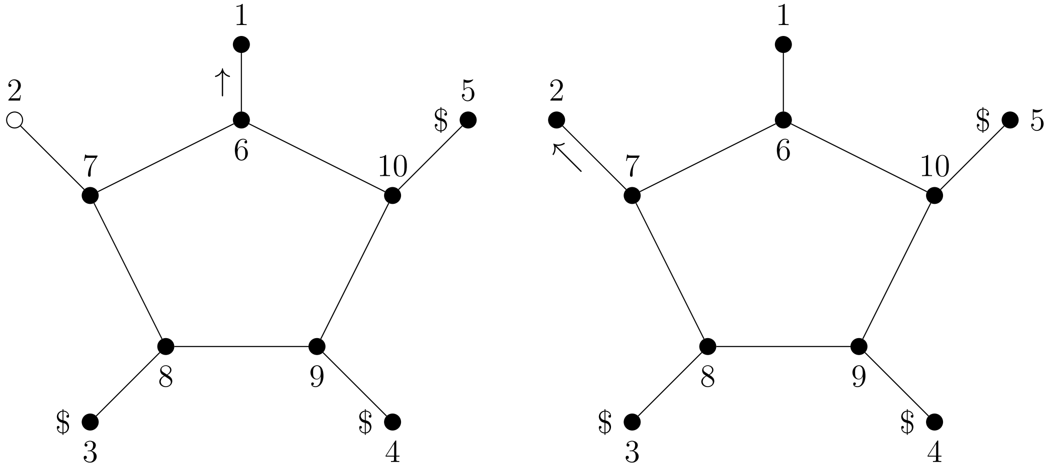


Figure 3.1: Zero forcing set for the corona $C_5 \circ K_1$

In the following we show that $M(C_5 \circ K_1) = 2$. *Let* $A = (a_{ij}) \in S(C_5 \circ K_1)$ *be such that* $\operatorname{null}(A) = M(C_5 \circ K_1)$. *Then we distinguish between the following two cases:*

Case 1: *For all* $i = 1, 2, 3, 4, 5, a_{ii} \neq 0$.

We label the vertices in $C_5 \circ K_1$ *as shown in Figure 3.1. Hence*

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 & a_{16} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 & a_{27} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 & 0 & 0 & a_{38} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 & 0 & 0 & a_{49} & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 & 0 & a_{5,10} \\ a_{16} & 0 & 0 & 0 & 0 & a_{66} & a_{67} & 0 & 0 & a_{6,10} \\ 0 & a_{27} & 0 & 0 & 0 & a_{67} & a_{77} & a_{78} & 0 & 0 \\ 0 & 0 & a_{38} & 0 & 0 & 0 & a_{78} & a_{88} & a_{89} & 0 \\ 0 & 0 & 0 & a_{49} & 0 & 0 & 0 & a_{89} & a_{99} & a_{9,10} \\ 0 & 0 & 0 & 0 & a_{5,10} & a_{6,10} & 0 & 0 & a_{9,10} & a_{10,10} \end{bmatrix}.$$

We apply elementary row operations to make the entry a_{16} *zero. We multiple the first row by* $\frac{-a_{16}}{a_{11}}$ *and add the result to row* 6. *Then apply the same to column* 1 *and column* 6. *Hence we get*

$$B := \left[\begin{array}{c|c} a_{11} & 0 \\ \hline 0 & B_1 \end{array}\right],$$

where $B_1 \in S(G_1)$ *and* G_1 *is given in the following graph:*

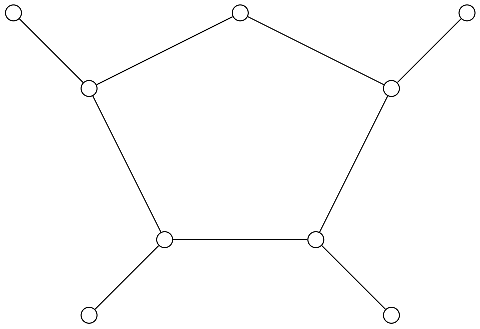


Figure 3.2: The graph G_1

It is easy to note that A *and* B *are congruent. Hence by Theorem 1.1.12*

$$\text{null}(A) = \text{null}(B) = \text{null}(a_{11}) + \text{null}(B_1) = \text{null}(B_1),$$

since $\text{null}(a_{11}) = 0$.

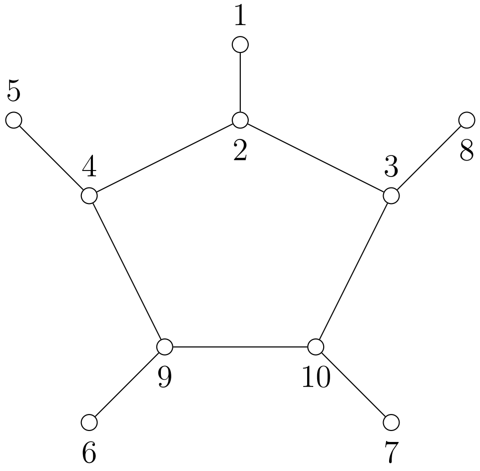
We can continue the same steps to get matrices $B_2, B_3, B_4,$ *and* B_5 *and* G_2, G_3, G_4 *and* $G_5 := C_5$ *where in each step we eliminate the entries* a_{27}, a_{38}, a_{49} *and* $a_{5,10}$. *Hence*

$$M(C_5 \circ K_1) = \text{null}(A) = \text{null}(B) = \text{null}(B_1) = \cdots = \text{null}(B_5) \leq M(C_5) \leq Z(C_5) = 2$$

Hence $M(C_5 \circ K_1) \leq 2$, *since* $C_5 \circ K_1$ *is not the empty graph or the path graph, by Observation 1.3.4 we have* $M(C_5 \circ K_1) = 2$.

Case 2: *For some* $i \in \{1, 2, 3, 4, 5\}, a_{ii} = 0$

Without loss of generality, assume $a_{11} = 0$ *and relabel the vertices of* $C_5 \circ K_1$ *as follows:*



Hence for $A \in S(C_5 \circ K_1)$ *such that* $\text{null}(A) = M(C_5 \circ K_1)$ *looks like*

$$A = \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{12} & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{23} & a_{33} & 0 & 0 & 0 & 0 & a_{38} & 0 & a_{3,10} \\ 0 & a_{24} & 0 & a_{44} & a_{45} & 0 & 0 & 0 & a_{49} & 0 \\ 0 & 0 & 0 & a_{45} & a_{55} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} & 0 & 0 & a_{69} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{77} & 0 & 0 & a_{7,10} \\ 0 & 0 & a_{38} & 0 & 0 & 0 & 0 & a_{88} & 0 & 0 \\ 0 & 0 & 0 & a_{49} & 0 & a_{69} & 0 & 0 & a_{99} & a_{9,10} \\ 0 & 0 & a_{3,10} & 0 & 0 & 0 & a_{7,10} & 0 & a_{9,10} & a_{10,10} \end{bmatrix}$$

We apply elementary row operations to row 3 *and row* 4 *then apply the same column* 3 *and column* 4 *to eliminate the entries* a_{23} *and* a_{24} *by using row* 1 *and column* 1. *Hence* A *is congruent to* D, *where*

$$D := \left[\begin{array}{c|c} C & 0 \\ \hline 0 & R \end{array}\right], C := \begin{bmatrix} 0 & a_{12} \\ a_{12} & a_{22} \end{bmatrix}, \text{and } R := [3, \cdots, 10].$$

It is easy to see that $R \in S(T)$ *and* T *is given in the following graph*

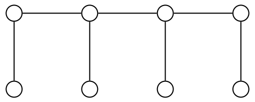
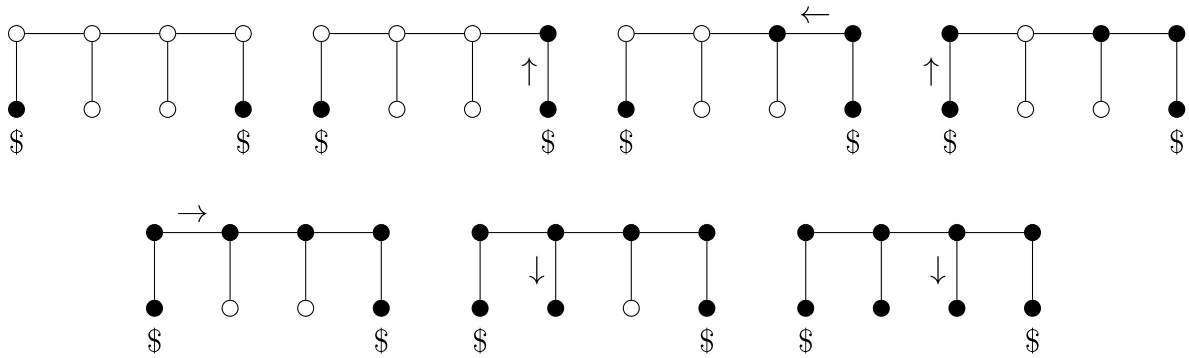


Figure 3.3: The graph T

Hence $\text{null}(D) = \text{null}(R)$ *since* $\text{null}(C) = 0$ *because* $\det(C) \neq 0$. $Z(T) = 2$ *as shown in following figure*



Therefore, by Theorem 1.1.12 and Proposition 3.1.5, we have

$$\mathrm{null}(A) = \mathrm{null}(D) = \mathrm{null}(R) \leq M(T) \leq Z(T) = 2.$$

Since $C_5 \circ K_1$ is not the empty graph or the path graph, by Observation 1.3.4 we have

$$\mathrm{null}(A) = 2.$$

Therefore, In either case we have $M(C_5 \circ K_1) = 2$ and $M(C_5 \circ K_1) < Z(C_5 \circ K_1)$.

3.2 Maximum Nullity of Some Interesting families Of Graphs

In this section, the maximum nullity of some interesting families of graphs are determined. We start with the hypercube Q_n.

Theorem 3.2.1. [1],[22] *For the hypercube Q_n, $M(Q_n) = 2^{n-1} = Z(Q_n)$.*

Proof. By Proposition 3.1.5 and (2.2) we have

$$M(Q_n) \leq Z(Q_n) \leq 2^{n-1}. \tag{3.2}$$

To show $M(Q_n) \geq 2^{n-1}$, define two symmetric matrices H_1 and L_1 as follows

$$H_1 := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } L_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Given H_{n-1} and L_{n-1}, define

$$H_n := \begin{bmatrix} L_{n-1} & I \\ I & L_{n-1} \end{bmatrix} \text{ and } L_n := \frac{1}{\sqrt{2}} \begin{bmatrix} L_{n-1} & I \\ I & -L_{n-1} \end{bmatrix}.$$

Then $H_n, L_n \in S(Q_n)$. Indeed, we use mathematical induction on n. If $n = 1$, then clearly the graph that corresponding to H_1 and L_1 is $K_2 = Q_1$. Thus $H_1, L_1 \in S(K_2) = S(Q_1)$. Assume it is true for H_{n-1} and L_{n-1}. Since $L_{n-1} \in S(Q_{n-1})$, so that the graph that is corresponding to H_n and L_n is two copies of Q_{n-1} and connected the vertices of the interviews to each other in the two copies. Hence $H_n, L_n \in S(Q_{n-1}\Box K_2) = S(Q_n)$. By direct calculation and mathematical induction we can show that $L_n^2 = I$. Define

$$B_nH_n := \begin{bmatrix} I & 0 \\ -L_{n-1} & I \end{bmatrix} \begin{bmatrix} L_{n-1} & I \\ I & L_{n-1} \end{bmatrix} = \begin{bmatrix} L_{n-1} & I \\ 0 & 0 \end{bmatrix} =: A_n.$$

Hence

$$B_nH_nB_n^T = A_n \begin{bmatrix} I & -L_{n-1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} L_{n-1} & 0 \\ 0 & 0 \end{bmatrix} =: D_n$$

By Theorem 1.1.12 and $L_n^2 = I$ we have $\operatorname{rank}(D_n) = \operatorname{rank}(H_n) = 2^{n-1}$. Which implies $\operatorname{mr}(Q_n) \leqslant \operatorname{rank}(H_n) = 2^{n-1}$. Since $\operatorname{mr}(Q_n) + M(Q_n) = 2^n$, we have

$$M(Q_n) \geq 2^{n-1}. \tag{3.3}$$

By (3.2) and (3.3), we have

$$2^{n-1} \leq M(Q_n) \leq Z(Q_n) \leq 2^{n-1}.$$

Therefore,

$$M(Q_n) = Z(Q_n) = 2^{n-1}.$$

□

The following theorem, it was proved that the maximum nullity is equal to the zero forcing number for supertriangle graph $\mathcal{T}_n$.

Proposition 3.2.2. [1] *For the supertriangle $\mathcal{T}_n$, $M(\mathcal{T}_n) = n = Z(\mathcal{T}_n)$.*

Proof. We can cover $\mathcal{T}_n$ by $\frac{1}{2}n(n-1)$ copies of K_3, so by Observation 1.3.2, $\operatorname{mr}(\mathcal{T}_n) \leq cc(\mathcal{T}_n) \leq \frac{1}{2}n(n-1)$. Since $M(\mathcal{T}_n) + \operatorname{mr}(\mathcal{T}_n) = \frac{1}{2}n(n+1)$. We have

$$\begin{aligned} \operatorname{mr}(\mathcal{T}_n) &\leq \frac{1}{2}n(n-1) \\ \frac{1}{2}n(n+1) - M(\mathcal{T}_n) &\leq \frac{1}{2}n(n-1) \\ \frac{1}{2}n(n+1) - \frac{1}{2}n(n-1) &\leq M(\mathcal{T}_n) \end{aligned}$$

$$n \leq M(\mathcal{T}_n). \tag{3.4}$$

Hence By Proposition 3.1.5 and Observation 2.2.9 we conclude that

$$M(\mathcal{T}_n) \leq Z(\mathcal{T}_n) \leq n. \tag{3.5}$$

By (3.4) and (3.5), we have

$$n \leq M(\mathcal{T}_n) \leq Z(\mathcal{T}_n) \leq n.$$

Therefore, $M(\mathcal{T}_n) = Z(\mathcal{T}_n) = n$. □

A *Kronecker product* that can be defined as follows. If A is an $n \times n$ matrix and B is a $m \times m$ matrix, then $A \otimes B$ is the $nm \times nm$ block matrix:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}.$$

In the following we present some facts on the eigenvalues and eigenvectors of the Kronecker product of two matrices that will be used in Theorem 3.2.4, Theorem 3.2.6, and Theorem 3.2.7. Let G be a graph on s vertices, let H be a graph on t vertices, let $A \in S(G)$, and $B \in S(H)$. Then $A \otimes I_t + I_s \otimes B \in S(G \square H)$ [1]. Indeed,

$$A \otimes I_t + I_s \otimes B = \left[\begin{array}{c|c|c|c|c} a_{1,1}I & a_{1,2}I & a_{1,3}I & \cdots & a_{1,n}I \\ \hline a_{1,2}I & a_{2,2}I & a_{2,3}I & \cdots & a_{2,n}I \\ \hline a_{1,3}I & a_{2,3}I & a_{3,3}I & \cdots & a_{3,n}I \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline a_{1,n}I & a_{2,n}I & a_{3,n}I & \cdots & a_{n,n}I \end{array}\right] + \left[\begin{array}{c|c|c|c|c} B & 0 & 0 & \cdots & 0 \\ \hline 0 & B & 0 & \cdots & 0 \\ \hline 0 & 0 & B & \cdots & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & 0 & \cdots & B \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c|c} a_{1,1}I + B & a_{1,2}I & a_{1,3}I & \cdots & a_{1,n}I \\ \hline a_{1,2}I & a_{2,2}I + B & a_{2,3}I & \cdots & a_{2,n}I \\ \hline a_{1,3}I & a_{2,3}I & a_{3,3}I + B & \cdots & a_{3,n}I \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline a_{1,n}I & a_{2,n}I & a_{3,n}I & \cdots & a_{n,n}I + B \end{array}\right] \in S(G \square H).$$

If x is an eigenvector of A for eigenvalue λ ($Ax = \lambda x$) and y is an eigenvector of B for eigenvalue μ ($By = \mu y$), then $x \otimes y$ is an eigenvector of $A \otimes I_m + I_n \otimes B$ for eigenvalue

$\lambda + \mu$. Indeed,

$$\begin{aligned}(A \otimes I_m + I_n \otimes B)(x \otimes y) &= (A \otimes I_m)(x \otimes y) + (I_n \otimes B)(x \otimes y)\\ &= (Ax \otimes I_m y) + (I_n x \otimes By)\\ &= (\lambda x \otimes y) + (x \otimes \mu y)\\ &= \lambda(x \otimes y) + \mu(x \otimes y)\\ &= (\lambda + \mu)(x \otimes y).\end{aligned}$$

The following theorem states that for any connected graph G on n vertices and any distinct real numbers say $\lambda_1, \cdots, \lambda_n$ there exist a matrix $A \in S(G)$ with $\lambda_1, \cdots, \lambda_n$ as its eigenvalues which will be helpful in the next theorems.

Theorem 3.2.3. [25] *Let G be a connected graph on n vertices and let $\lambda_1, \cdots, \lambda_n$ be distinct real numbers. Then there exists a matrix $A \in S(G)$ whose spectrum is $\lambda_1, \cdots, \lambda_n$.*

The following theorem, it was proved that the maximum nullity is equal to the zero forcing number for cartesian product of P_t and any garph G with $|G| \leq t$.

Theorem 3.2.4. [1] *If $|G| \leq t$, then $M(G \square P_t) = |G| = Z(G \square P_t)$.*

Proof. Let $|G| = h$ and let $\lambda_1, \cdots, \lambda_h$ be distinct real numbers. By Theorem 3.2.3, we can choose $A \in S(G)$ with $\lambda_1, \cdots, \lambda_h$ eigenvalues and associated eigenvectors $x_1, \cdots, x_h$. Moreover, there exists $B \in S(P_t)$ having eignvalues $-\lambda_1, \cdots, -\lambda_h, \mu_{h+1}, \cdots, \mu_t$ with associated eigenvectors $y_1, \cdots, y_t$ where the later set of real numbers are distinct. Then $A \otimes I_t + I_h \otimes B$ has at least h eigenvectors, namely $x_i \otimes y_i$, $i = 1, \cdots, h$, for eigenvalues $\lambda_i - \lambda_i = 0$. Thus

$$\text{null}(A \otimes I_t + I_h \otimes B) \geq h. \tag{3.6}$$

Since $(A \otimes I_t + I_h \otimes B) \in S(G \square P_t)$, we conclude by Proposition 3.1.5 and (2.1) that

$$\text{null}(A \otimes I_t + I_h \otimes B) \leq M(G \square P_t) \leq Z(G \square P_t) \leq h. \tag{3.7}$$

By (3.6) and (3.7), we have

$$h \leq \text{null}(A \otimes I_t + I_h \otimes B) \leq M(G \square P_t) \leq Z(G \square P_t) \leq h.$$

Therefore,

$$M(G \square P_t) = Z(G \square P_t) = |G|.$$

□

Since $P_s \Box P_t$ is isomorphic to $P_t \Box P_s$, as a simple result of Theorem 3.2.4, we have $M(P_s \Box P_t) = \min\{s,t\} = Z(P_s \Box P_t)$. The following theorem shows that the maximum nullity is equal to the zero forcing number for cartesian product of P_t and C_s.

Theorem 3.2.5. [10] *If G is a cycle graph on n vertices, then the eigenvalues of $A(C_n)$ are*

$$\lambda_i = 2\cos\frac{2\pi i}{n}, i = 1, \dots, n-1.$$

Theorem 3.2.6. [1] *$M(C_s \Box P_t) = \min\{s, 2t\} = Z(C_s \Box P_t)$.*

Proof. By Proposition 3.1.5 and (2.1) we have

$$M(C_s \Box P_t) \le Z(C_s \Box P_t) \le \min\{s, 2t\}. \tag{3.8}$$

Let $k = \lceil \frac{s}{2} \rceil$ and A be the adjacency matrix of C_s by changing the sign on two symmetrically placed ones. Then by Theorem 3.2.5, A has distinct eigenvalues $\lambda_i = 2\cos\frac{\pi(2i-1)}{s}, i = 1, \dots, k$, each with multiplicity 2, except that if s is odd, we have $\lambda_k = 2\cos\frac{\pi(2\left(\frac{s+1}{2}\right)-1)}{s} = -2$ has multiplicity 1. While A is a real symmetric matrix, each eigenvalue of multiplicity 2 has 2 independent eigenvectors, for eigenvalue λ_i, assume these vectors are x_i, z_i, note that if s is odd there is no z_k. By Theorem 3.2.3, for any distinct real numbers $\mu_1, \cdots, \mu_t$, we can choose $B \in S(P_t)$ with spectrum $\mu_1, \cdots, \mu_t$. Assume $r = \min\{k, t\}$, and choose $B \in S(P_t)$ having eigenvalues $\mu_i = -\lambda_i, i = 1, \cdots, r$ with eigenvectors y_i. Then when $s \le 2t$ we have

$$M(C_s \Box P_t) \ge \text{null}(A \otimes I_t + I_s \otimes B) \ge s = \min\{s, 2t\},$$

and when $2t \le s$ we have

$$M(C_s \Box P_t) \ge \text{null}(A \otimes I_t + I_s \otimes B) \ge 2t = \min\{s, 2t\}.$$

Therefore,

$$M(C_s \Box P_t) \ge \min\{s, 2t\}. \tag{3.9}$$

By (3.8) and (3.9) we have

$$\min\{s, 2t\} \le M(C_s \Box P_t) \le Z(C_s \Box P_t) \le \min\{s, 2t\}.$$

Thus $M(C_s \Box P_t) = Z(C_s \Box P_t) = \min\{s, 2t\}$. □

The following theorem, it was presented a lower bound on the maximum nullity of cartesian product of G and K_t.

Theorem 3.2.7. [1] *For any graph G with at least one edge and any integer $t \geq 2$,*

$$M(G\square K_t) \geq M(G)(t-1) + \zeta,$$

where ζ is the maximum multiplicity of a nonzero eigenvalue in a matrix $A \in S(G)$ such that $\operatorname{rank}(A) = \operatorname{mr}(G)$.

Proof. Choose $A \in S(G)$ such that $\operatorname{null}(A) = M(G)$. Hence A has eigenvalue 0 of multiplicity $M(G)$ and $\lambda \neq 0$ of multiplicity ζ. Since A is a real symmetric matrix, eigenvalue 0 has $M(G)$ independent eigenvectors $x_i, i = 1, \cdots, M(G)$, and eigenvalue λ has ζ independent eigenvectors $z_j, j = 1, \cdots, \zeta$. By Theorem 3.2.3, we choose $B \in S(K_t)$ having eigenvalues 0 with multiplicity $t-1$ and $t-1$ independent eigenvectors $y_k, k = 1, \cdots, t-1$ and $-\lambda$ of multiplicity 1 with eigenvector w (indeed take $B := \frac{-\lambda}{n}J$). Then $A \otimes I_t + I_{|G|} \otimes B$ has at least $M(G)(t-1)+\zeta$ eigenvectors for eigenvalue 0, namely $x_i \otimes y_k, i = 1, \cdots, M(G); k = 1, \cdots, t-1$, and $z_j \otimes w, j = 1, \cdots, \zeta$, thus

$$M(G\square K_t) \geq \operatorname{null}(A \otimes I_t + I_s \otimes B) \geq M(G)(t-1) + \zeta.$$

□

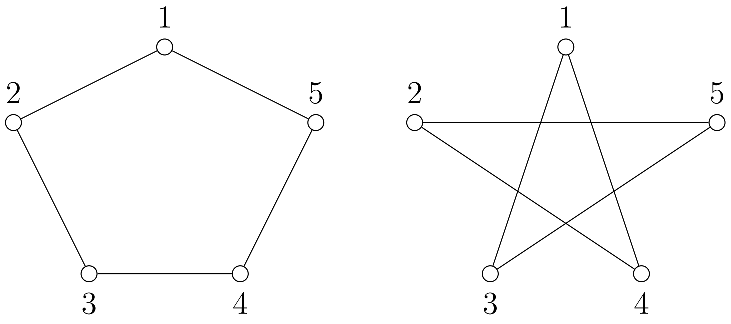
Proceeding as above, using Proposition 3.1.5, Proposition 2.3.3, (2.3), and Theorem 3.2.7 we have

$$M(K_s\square K_t) = st - s - t + 2 = Z(K_s\square K_t), s, t \geq 2. \tag{3.10}$$

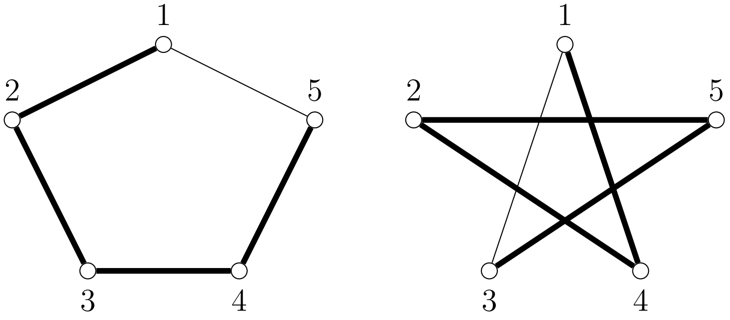
$$M(C_s\square K_t) = 2t = Z(C_s\square K_t), s \geq 4.$$

In the following example, we show that C_5 and $\overline{C_5}$ have P_4 as induced subgraph.

Example 3.2.8. *The following figures represent C_5 and $\overline{C_5}$ respectively.*



Notice that C_5 and $\overline{C_5}$ have P_4 as an induced subgraph as shown in the following figures.



In general, it is easy to note that in every $C_n, n \geq 5, C_n$ and $\overline{C_n}$ have P_4 as an induced subgraph.

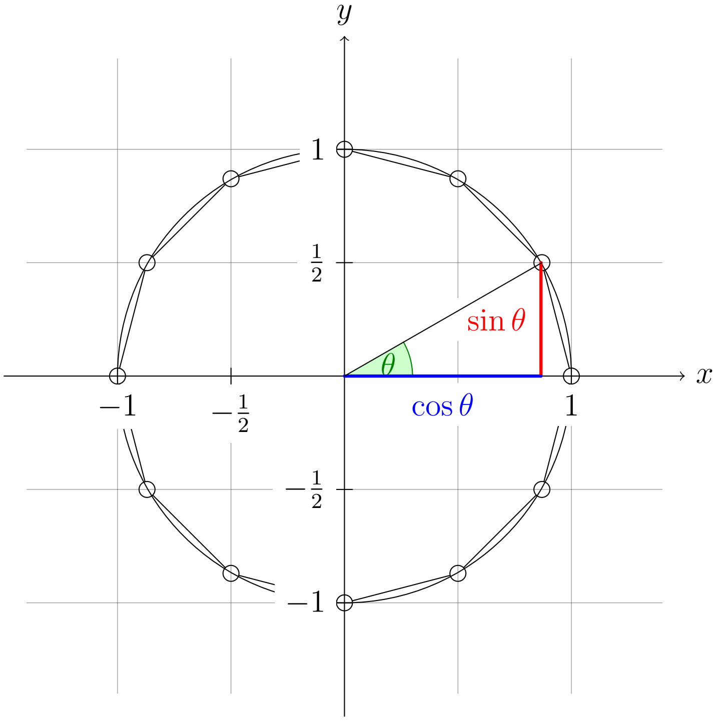
In the following proposition we show that the complement of a cycle has minimum rank equal three.

Proposition 3.2.9. [1] *If* $n \geq 5$, *then* $\mathrm{mr}(\overline{C_n}) = 3$ *and thus* $M(\overline{C_n}) = n - 3$.

Proof. If $n \geq 5$, then C_n contains an induced P_4, therefore $\overline{C_n}$ does, too. In this case, we can find lower triangle submatrix with size 3×3 and its determinate not equal zero. Hence

$$\mathrm{mr}(\overline{C_n}) \geq 3. \tag{3.11}$$

Embed C_n as a regular polygon on the unit circle in $\mathbb{R}^2$ and let $u_1, \cdots, u_n$ be the vectors representing the vertices as shown in the following figure (i.e., $u_i = \cos\theta_i \hat{\imath} + \sin\theta_i \hat{\jmath}$ where $\theta_i = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{2\pi(n-1)}{n}$ or $\frac{-2\pi}{n}, i = 1, 2, 3, \ldots, n$).



Let B be the Gram matrix of these vectors, i.e.,

$$B := \begin{bmatrix} u_1^T \\ \vdots \\ u_n^T \end{bmatrix}_{n\times 2} \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}_{2\times n},$$

$$= \begin{bmatrix} 1 & cos(\theta_1-\theta_2) & cos(\theta_1-\theta_3) & cos(\theta_1-\theta_4) & \cdots & cos(\theta_1-\theta_n) \\ cos(\theta_1-\theta_2) & 1 & cos(\theta_2-\theta_3) & cos(\theta_2-\theta_4) & \cdots & cos(\theta_2-\theta_n) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ cos(\theta_1-\theta_n) & cos(\theta_2-\theta_n) & cos(\theta_3-\theta_n) & cos(\theta_4-\theta_n) & \cdots & 1 \\ \end{bmatrix}.$$

Then $b_{i,i+1} = \cos(2\pi/n)$ and if $1 < |i-j| < n-1$ then $b_{i,j} < b_{i,i+1}$ (indeed, when θ becomes bigger the cosine function becomes smaller). Thus $\operatorname{rank}(B) = 2$. Now, define $D := B - \cos(2\pi/n)J$. Hence

$$\begin{aligned} \operatorname{rank}(D) &\leq \operatorname{rank}(B) + \operatorname{rank}(-\cos(2\pi/n)J) \\ &= 2+1 = 3, \end{aligned}$$

and $D \in S(\overline{C_n})$. Indeed,

$$D = \begin{bmatrix} 1 & cos(\theta_1-\theta_2) & cos(\theta_1-\theta_3) & \cdots & cos(\theta_1-\theta_n) \\ cos(\theta_1-\theta_2) & 1 & cos(\theta_2-\theta_3) & \cdots & cos(\theta_2-\theta_n) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ cos(\theta_1-\theta_n) & cos(\theta_2-\theta_n) & cos(\theta_3-\theta_n) & \cdots & 1 \\ & & & & \end{bmatrix} - \begin{bmatrix} \cos(2\pi/n) & \cdots & \cos(2\pi/n) \\ \vdots & \ddots & \vdots \\ \cos(2\pi/n) & \cdots & \cos(2\pi/n) \end{bmatrix}$$

$$= \begin{bmatrix} d_{1,1} & 0 & d_{1,3} & d_{1,4} & \cdots & d_{1,n-1} & 0 \\ 0 & d_{2,2} & 0 & d_{2,4} & \cdots & d_{2,n-1} & d_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & d_{2,n} & d_{3,n} & d_{4,n} & \cdots & 0 & d_{n,n} \end{bmatrix} \in S(\overline{C_n}),$$

where $d_{i,i} = 1 - \cos(2\pi/n)$ and $d_{i,j} = cos(\theta_i - \theta_j) - \cos(2\pi/n)$. So

$$\operatorname{mr}(\overline{C_n}) \leq 3. \tag{3.12}$$

By (3.11) and (3.12), $\operatorname{mr}(\overline{C_n}) = 3$. □

The following theorem states that the complement of a tree has positive semidefinite minimum rank at most 3.

Theorem 3.2.10. [1] *For any tree T, $\operatorname{mr}_+(\overline{T}) \leq 3$.*

The following theorem gives the minimum rank of $\overline{T}$.

Theorem 3.2.11. [1] *Let T be a tree of order $n \geq 3$. Then*

$$\operatorname{mr}(\overline{T}) = \begin{cases} 3 & \textit{if } P_4 \textit{ is an induced subgraph of } T; \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. By Theorem 3.2.10, we have $\operatorname{mr}(\overline{T}) \leq 3$ since $\operatorname{mr}(\overline{T}) \leq \operatorname{mr}_+(\overline{T})$. Let $|T| = n$. If T contains an induced P_4, $\overline{T}$ does too. Hence

$$\operatorname{mr}(\overline{T}) \geq 3.$$

Therefore,

$$\operatorname{mr}(\overline{T}) = 3.$$

If P_4 is not an induced subgraph in T, any two vertices are connected by a path of length at most two, so $T = K_{1,n-1}$. Since $\overline{K_{1,n-1}} = K_{n-1} \cup K_1$ we conclude that $\operatorname{mr}(\overline{K_{1,n-1}}) \leq 1$ (indeed take $J_{n-1} \oplus 0$). Since $(\overline{K_{1,n-1}})$ is not the empty graph we have $\operatorname{mr}(\overline{K_{1,n-1}}) = 1$. □

Definition 3.2.12. [11] *The graph G on n vertices is called* strongly regular *with parameters (n, k, α, β) (denoted* $\operatorname{srg}(n, k, \alpha, \beta)$*) if*

1. *G is k-regular i.e., every vertex in V has k neighbors*
2. *Each pair of adjacent vertices has exactly α common neighbors*
3. *Each pair of non-adjacent vertices has exactly β common neighbors.*

Example 3.2.13. *The graph shown in Figure 3.4 is* $C_5 = \operatorname{srg}(5, 2, 0, 1)$.

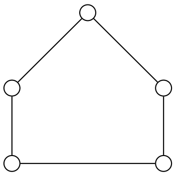


Figure 3.4: The strongly regular graph $\operatorname{srg}(5, 2, 0, 1)$.

Example 3.2.14. *The graph shown in Figure 3.5 is* $\operatorname{srg}(6, 4, 2, 4)$.

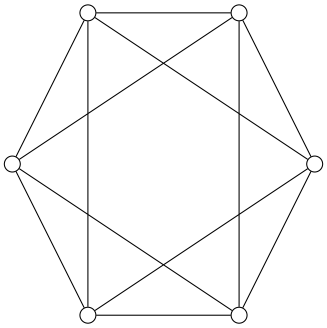


Figure 3.5: The strongly regular graph $\operatorname{srg}(6, 4, 2, 4)$.

A strongly regular graph G is *primitive* if both G and its complement are connected. The following theorem shows that the primitive strongly regular graph has exactly three distinct eigenvalues.

Theorem 3.2.15. [18] *If G is a primitive strongly regular graph with parameters (n, k, α, β) and define*

$$\Upsilon := \sqrt{(\alpha - \beta)^2 + 4(k - \beta)},$$

then the three eigenvalues of $A(G)$ *are*

$$k, \qquad \theta = \frac{1}{2}(\alpha - \beta + \Upsilon), \qquad \tau = \frac{1}{2}(\alpha - \beta - \Upsilon),$$

with respective multiplicities

$$m_k = 1, \qquad m_\theta = -\frac{(n-1)\tau + k}{\theta - \tau}, \qquad m_\tau = \frac{(n-1)\theta + k}{\theta - \tau}.$$

In the following proposition, it was determined a lower bound on the maximum nullity of strongly regular graphs.

Proposition 3.2.16. [1] *Let* $G = \mathrm{srg}(n, k, \alpha, \beta)$ *be a primitive strongly regular graph. Then* $M(G) \geq \lfloor \frac{n}{2} \rfloor$.

Proof. By Theorem 3.2.15, the adjacency matrix $A(G)$ of a strongly regular graph G has exactly three eigenvalues, one of which is k and has multiplicity 1. For the remaining eigenvalues θ (positive eigenvalues) and τ (negative eigenvalues) have in total multiplicity $n-1$. Without loss of generality, assume the multiplicity of θ is m_θ and m_θ is greater than or equal to the multiplicity of τ. Therefore, $m_\theta \geq \lceil \frac{n-1}{2} \rceil$, $A(G) - \theta I$ has 0 eigenvalue with multiplicity m_θ. Clearly $M(G) \geq m_\theta \geq \lceil \frac{n-1}{2} \rceil = \lfloor \frac{n}{2} \rfloor$. Thus,

$$M(G) \geq \lfloor \frac{n}{2} \rfloor.$$

□

Example 3.2.17. *The graph shown in Figure 3.6 is* $K_2 \square K_2 = \mathrm{srg}(4, 2, 0, 2)$. *By* (3.10), $M(K_2 \square K_2) = 2$, *achieves equality of the bound in Proposition 3.2.16, which implies that a translation of the adjacency matrix realizes maximum nullity.*

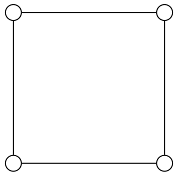


Figure 3.6: The strongly regular graph $\mathrm{srg}(4, 2, 0, 2)$.

Example 3.2.18. *The graph shown in Figure 3.7 is* $K_3 \square K_3 = \mathrm{srg}(9, 4, 1, 2)$. *By* (3.10), $M(K_3 \square K_3) = 5 > \lfloor \frac{9}{2} \rfloor = 4$, *not achieves equality of the bound in Proposition 3.2.16.*

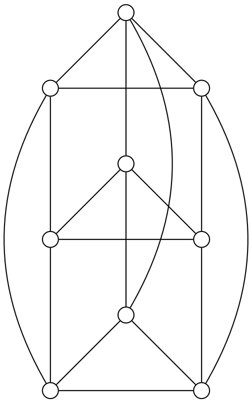


Figure 3.7: The strongly regular graph srg$(9, 4, 1, 2)$.

In the next proposition shows that the so-called *Petersen graph* satisfies $M(P) = Z(P)$.

Proposition 3.2.19. [1] *Let P denote the Petersen graph shown in Figure 3.8. Then $M(P) = 5 = Z(P)$ and so* $\operatorname{mr}(P) = 5$.

Proof. A zero forcing set is the set of five vertices on the outer cycle shown in Figure 3.8. Hence

$$M(P) \leq Z(P) \leq 5. \tag{3.13}$$

The Petersen graph is strongly regular with parameters srg$(10, 3, 0, 1)$, so by Proposition 3.2.16,

$$M(P) \geq \lfloor \frac{10}{2} \rfloor = 5. \tag{3.14}$$

By (3.13) and (3.14) we have

$$M(P) = Z(P) = 5.$$

Therefore,

$$\operatorname{mr}(P) = 5.$$

□

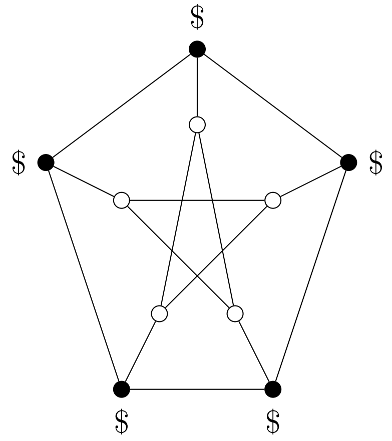


Figure 3.8: Zero forcing set for the Petersen graph

Proposition 3.2.20. [1] *If* $|G| \leq 6$, *then* $Z(G) = M(G)$.

In the following proposition we show the maximum nullity equal zero forcing number for any path, complete, and cycle graphs.

Proposition 3.2.21. *For any path* $P_n, M(P_n) = 1 = Z(P_n)$.

Proof. By Observation 2.2.1, $Z(P_n) = 1$. Hence by Proposition 3.1.5, $M(P_n) \leq Z(P_n) = 1$. Hence $M(P_n) = 0$ or 1. Since P_n is not the empty graph and for any matrix $A \in S(P_n),\ A - \lambda I \in S(P_n)$ for any $\lambda \in \sigma(A)$. Therefore, $M(P_n) = 1$. □

Proposition 3.2.22. *For* $n \geq 2$, *any complete* $K_n, M(K_n) = n - 1 = Z(K_n)$.

Proof. By Observation 2.2.7, $Z(K_n) = n - 1$. To show $M(K_n) = n - 1$, let $A := J_n$. Hence $A \in S(K_n)$ and $\text{null}(A) = n - 1$. Therefore,

$$n - 1 = \text{null}(A) \leq M(K_n) \leq Z(K_n) = n - 1.$$

Thus $M(K_n) = n - 1$. □

Proposition 3.2.23. *For* $n \geq 3$, *any cycle* $C_n, M(C_n) = 2 = Z(C_n)$.

Proof. By Observation 2.2.3, $Z(C_n) = 2$. Hence by Proposition 3.1.5, $M(C_n) \leq Z(C_n) = 2$. Hence $M(C_n) = 0$ or 1 or 2. Since C_n is not empty graph or the path graph. Therefore, $M(C_n) = 2$. □

The following theorem includes some families of graphs for which $Z(G) = M(G)$.

Theorem 3.2.24. [1],[13] *For each of the following families of graphs,* $Z(G) = M(G)$.

Graph	$Z(G) = M(G)$
P_n	1
$K_n, n \geq 3$	$n-1$
C_n	2
T_n	n
Q_n	2^{n-1}
$\overline{C_n}, n \geq 5$	$n-3$
$K_s \square P_t$	s
$P_s \square P_t$	$\min\{s,t\}$
$P_s \boxtimes P_t$	$s+t-1$
$C_s \square P_t$	$\min\{s,2t\}$
$K_s \square K_t$	$st-s-t+2$
$C_s \square K_t, s \geq 4$	$2t$
$K_t \circ K_s, t \geq 2$	$st-1$
Petersen	5

3.3 Minimum Rank And Vertex-Sums Of Graphs

This section presents some interesting result on the minimum rank and vertex-sums of graphs. We start by introducing the definition of rank-spread.

Definition 3.3.1. [4],[13] *Let* v *be a vertex in a graph* G*. The* rank-spread *of* G *at* v *is defined as* $r_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G - v)$.

The following proposition states that the minimum rank for the *principal matrix* A_p is equal the same rank of a real symmetric matrix A or less than the rank of matrix A by two which will helpful in the next lemma.

Proposition 3.3.2. [26] *Let* G *be a graph on* n *vertices and let* $A \in S(G)$ *be such that* rank $A = \mathrm{mr}(G) = k$*. Let* $p = \{1, \ldots, n\}$ *and* A_p*, denote the principal matrix obtained by deleting the pth row and column from* A*. Then* rank $A_p = k$ *or* rank $A_p = k-2$ *i.e.,* rank $A_p = k-1$ *is not possible.*

In [4], it was noticed that $0 \leq r_v(G) \leq 2$. In the next lemma we are interested in the matrices A such that satisfy the conditions as follows:

$$A = \begin{bmatrix} a & b^T \\ b & A' \end{bmatrix}, \quad A \in S(G), \quad \text{and} \quad b \in R(A'), \tag{3.15}$$

where $R(A')$ is the *range* of A', i.e., is the span (set of all possible linear combinations) of its column vectors.

Lemma 3.3.3. [4] *Let G be a graph, $v \in V(G)$, and assume $v = 1$. Then*

1. $r_v(G) = 0$ *if and only if* $\min\{\text{rank } A' : A \; satisfies \;\; (3.15)\} = \text{mr}(G - v)$;
2. $r_v(G) = 1$ *if and only if* $\min\{\text{rank } A' : A \; satisfies \;\; (3.15)\} = \text{mr}(G - v) + 1$;
3. $r_v(G) = 2$ *otherwise.*

Proof. 1. Let A satisfy (3.15) with rank $A' = \text{mr}(G - v)$. Then we can choose

$$\tilde{A} := \begin{bmatrix} \alpha & b^T \\ b & A' \end{bmatrix}$$

such that $\tilde{A}$ satisfies (3.15) as well and α is chosen such that $\tilde{A}$ and $0 \oplus A'$ are congruent. Hence by Theorem 1.1.12, we have

$$\text{mr}(G) \leq \text{rank } \tilde{A} = \text{rank } A' = \text{mr}(G - v)$$

Hence

$$\text{mr}(G) \leq \text{mr}(G - v),$$

which implies

$$r_v(G) \leq 0,$$

but

$$r_v(G) \geq 0.$$

Therefore, $r_v(G) = 0$. Conversely, if $r_v(G) = 0$, then for any matrix $A \in S(G)$ such that rank $A = \text{mr}(G) = \text{mr}(G - v)$ will satisfy (3.15)

$$A = \begin{bmatrix} a & c^T \\ c & A' \end{bmatrix}.$$

Since A' is a submatrix of A and

$$\text{mr}(G - v) = \text{rank } A \geq \text{rank } A' \geq \text{mr}(G - v).$$

Therefore,

$$\text{rank } A' = \text{mr}(G - v).$$

2. Let A satisfy (3.15) with rank $A' = \text{mr}(G-v)+1$. With regard to the matrix $\tilde{A}$ defined in the proof of 1., we have

$$\text{mr}(G) \leq \text{rank } \tilde{A} = \text{rank } A' = \text{mr}(G-v)+1$$

Therefore,

$$\text{mr}(G) - \text{mr}(G-v) \leq 1.$$

Hence

$$r_v(G) \leq 1.$$

Since 0 is excluded by 1., we have $r_v(G) = 1$. Conversely, suppose that $r_v(G) = 1$. For any $A \in S(G)$ with rank $A = \text{mr}(G)$, A can be written as

$$A = \begin{bmatrix} a & b^T \\ b & A' \end{bmatrix}.$$

By Proposition 3.3.2, one of the following two cases may occur:

Case 1: rank A = rank A'.

In this case A satisfies (3.15). Moreover, A can be chosen such that A and $0 \oplus A'$ are congruent. Hence

$$\text{mr}(G) = \text{rank } A = \text{rank } A'.$$

Therefore,

$$\begin{aligned} r_v(G) &= \text{mr}(G) - \text{mr}(G-v) \\ 1 &= \text{rank } A' - \text{mr}(G-v) \end{aligned}$$

Hence

$$\text{rank } A' = \text{mr}(G-v)+1.$$

Case 2: rank A = rank $A' + 2$.

In this case A can not satisfy the condition (3.15), so

$$\begin{gathered} \text{mr}(G) = \text{rank } A' + 2 \geq \text{mr}(G-v)+2 \\ r_v(G) \geq 2. \end{gathered}$$

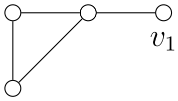
A contradiction. Hence this case is rejected.

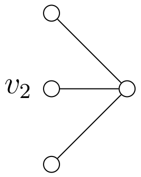
3. Since $r_v(G) \leq 2$, the claim follows from 1. and 2..

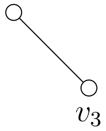
$\square$

For disjoint graphs $G_1, \ldots, G_t$. We select a vertex $v_i \in V(G_i), i = 1, \cdots, t$ and join all G_i's by identifying all v_i's as a unique vertex v. The resulting graph G is called the *vertex-sum of the graphs* $G_1, \ldots, G_t$ *at* v.

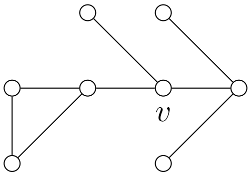
Example 3.3.4. *For the following three graphs, we select* $v_1 \in V(G_1), v_2 \in V(G_2)$, *and* $v_3 \in V(G_3)$, *respectively*







The vertex-sum of G_1, G_2, *and* G_3 *at* v *is*



Theorem 3.3.5. [4] *Let* G *be the vertex-sum of* $G_1, \ldots, G_h$ *at* v. *Then*

$$r_v(G) = \min \left\{ \sum_{i=1}^{h} r_v(G_i), 2 \right\}, \tag{3.16}$$

that is, $\mathrm{mr}(G) = \sum_{i=1}^{h} \mathrm{mr}(G_i - v) + \min \left\{ \sum_{i=1}^{h} r_v(G_i), 2 \right\}$.

Proof. Let $v = 1$ and let $A \in S(G)$. Hence A can be written as follows:

$$A = \begin{bmatrix} a & b^T \\ b & A' \end{bmatrix} = \begin{bmatrix} a & b_1^T & \cdots & b_n^T \\ b_1 & A_1' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_h & 0 & \cdots & A_h' \end{bmatrix}, \tag{3.17}$$

where $A_i' \in S(G_i - v), i = 1, \cdots, h$. It is known that $r_v(G) \leq 2$, in the following we prove

(1) $r_v(G) = 0$ if and only if $\sum_{i=1}^{h} r_v(G_i) = 0$.

Let $r_v(G) = 0$. By Lemma 3.3.3, there exists a matrix A of the form (3.17) such that $b \in R(A')$ and rank $A' = \mathrm{mr}(G-v) = \sum_{i=1}^{h} \mathrm{mr}(G_i - v) = \sum_{i=1}^{h} \mathrm{rank}(A'_i)$. Therefore, $b_i \in R(A'_i)$ and rank $A'_i = \mathrm{mr}(G_i - v), i = 1, 2, \cdots, h$. Hence by applying Lemma 3.3.3, we have $r_v(G_i) = 0, i = 1, 2, \cdots, h$. Thus, $\sum_{i=1}^{h} r_v(G_i) = 0$. Conversely, if $\sum_{i=1}^{h} r_v(G_i) = 0$, we have $r_v(G_i) = 0, i = 1, 2, \cdots, h$. We can find matrices $A_i = \begin{bmatrix} a_i & b_i^T \\ b_i & A'_i \end{bmatrix}$ satisfying (3.15) and rank $A'_i = \mathrm{mr}(G_i - v)$. We can then define a matrix A as in (3.17), where a can be any real number. Clearly $b \in R(A')$ and rank $A' = \sum_{i=1}^{h} \mathrm{rank}\ A'_i = \sum_{i=1}^{h} \mathrm{mr}(G_i - v) = \mathrm{mr}(G - v)$. Therefore, by Lemma 3.3.3, we conclude $r_v(G) = 0$.

(2) $r_v(G) = 1$ if and only if $\sum_{i=1}^{h} r_v(G_i) = 1$.

Let $r_v(G) = 1$. By **(1)**, we then have $\sum_{i=1}^{h} r_v(G_i) \geq 1$. We now prove $\sum_{i=1}^{h} r_v(G_i) \leq 1$. Using Lemma 3.3.3, we can derive a matrix A in the form (3.17) with $b \in R(A')$ and rank $A' = \sum_{i=1}^{h} \mathrm{rank}\ A'_i = \sum_{i=1}^{h} \mathrm{mr}(G_i - v) + 1$. Therefore, there exists $j \in \{1, \cdots, h\}$ such that rank $A'_j = \mathrm{mr}(G_j - v) + 1$ and rank $A'_i = \mathrm{mr}(G_i - v)$ for $i \neq j$. Thus, $\sum_{i=1}^{h} r_v(G_i) \leq 1$. Hence $\sum_{i=1}^{h} r_v(G_i) = 1$. Conversely, if $\sum_{i=1}^{h} r_v(G_i) = 1$, then without loss of generality let $r_v(G_1) = 1$ and $r_v(G_i) = 0, i \geq 2$. By Lemma 3.3.3, we can find matrices

$$A_i := \begin{bmatrix} a_i & b_i^T \\ b_i & A'_i \end{bmatrix}, i = 1, 2, \cdots, h$$

satisfying (3.15) and rank $A'_i = \mathrm{mr}(G_i - v), i = 1, \cdots, h$. Since $b \in R(A'_i)$ we can apply simple Gaussian elimination to the matrix A to get

$$C = \begin{bmatrix} a_1 & b_1^T & 0 & \cdots & 0 \\ b_1 & A'_1 & 0 & \cdots & 0 \\ 0 & 0 & A'_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A'_h \end{bmatrix}.$$

Let $B := \begin{bmatrix} a_1 & b_1^T \\ b_1 & A'_1 \end{bmatrix}$. Note that rank B = rank $A'_1 + 1$ because $r_v(G_1) = 1$. Then by

Theorem 1.1.12, we have

$$\begin{aligned}\operatorname{mr}(A) = \operatorname{rank} A = \operatorname{rank} C &= \operatorname{rank} B + \operatorname{rank} A'_2 + \cdots + \operatorname{rank} A'_h \\ &= \operatorname{rank} A'_1 + 1 + \sum_{i=2}^{h} \operatorname{mr}(G_i - v) \\ &= \operatorname{mr}(G_1 - v) + 1 + \sum_{i=2}^{h} \operatorname{mr}(G_i - v) \\ &= \sum_{i=1}^{h} \operatorname{mr}(G_i - v) + 1 \\ &= \operatorname{mr}(G - v) + 1.\end{aligned}$$

Hence by Lemma 3.3.3, we have $r_v(G) = 1$. Since $r_v(G) \leq 2$, we conclude that $r_v(G) = \min\left\{\sum_{i=1}^{h} r_v(G_i), 2\right\}$. □

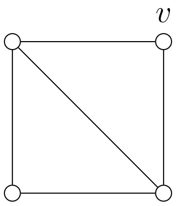
As a simple consequence of the above theorem

$$\operatorname{mr}(G_1) + \operatorname{mr}(G_2) \leq \operatorname{mr}(G) \leq \operatorname{mr}(G_1) + \operatorname{mr}(G_2) + 2,$$

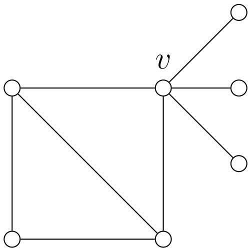
where G is a vertex-sum of graphs G_1 and G_2 at v.

Let G_1 be a graph and select $v \in V(G_1)$. A graph G that is obtained by appending l leaves on a fixed vertex v of G_1. It is easy to note that G is the vertex-sum of graphs $G_1, K_2^{(2)}, K_2^{(3)}, \cdots, K_2^{(l+1)}$.

Example 3.3.6. *Let G_1 be the following graph.*



The graph G obtained by appending three leaves on a vertex v is the following graph.



It is easy to note that the graph G is equivalent to the vertex-sum of $G_1, K_2^{(2)}, K_2^{(3)}$ and $K_2^{(4)}$ at v.

Lemma 3.3.7. [4] *Let G_1 be a graph and consider the graph G obtained by appending l leaves on a vertex v of G_1. Then*

1. *if $l = 1$ and $r_v(G_1) = 0$, then $r_v(G) = 1$ and $\operatorname{mr}(G) = \operatorname{mr}(G_1) + 1$;*
2. *otherwise, $r_v(G) = 2$ and $\operatorname{mr}(G) = \operatorname{mr}(G_1) + 2 - r_v(G_1)$.*

Proof. Let each $G_2, \cdots, G_{l+1}$ denote the graph K_2. For each $i = 2, \cdots, l+1, \operatorname{mr}(G_i) = 1$, while $\operatorname{mr}(G_i - v) = 0$ which implies $r_v(G_i) = 1$. If $l = 1$ and $r_v(G_1) = 0$, then $\sum_{i=1}^{2} r_v(G_i) = r_v(G_1) + r_v(G_2) = 0 + 1 = 1$. Therefore, by Theorem 3.3.5 we have $r_v(G) = 1$ and

$$\operatorname{mr}(G) = \sum_{i=1}^{2} \operatorname{mr}(G_i - v) + r_v(G) = \operatorname{mr}(G_1 - v) + \operatorname{mr}(G_2 - v) + 1 = \operatorname{mr}(G_1) + 1,$$

since $\operatorname{mr}(G_2 - v) = 0$ and $r_v(G_1) = 0$.
On the other hand, if either $l > 1$ or $r_v(G_1) > 0$, then we have by Theorem 3.3.5, $r_v(G) = \min\left\{\sum_{i=1}^{l+1} r_v(G_i), 2\right\} = \min\left\{r_v(G_1) + \underbrace{1 + \cdots + 1}_{l-times}, 2\right\} = 2$, that is,

$$\begin{aligned} r_v(G) &= \operatorname{mr}(G) - \operatorname{mr}(G - v) \\ 2 &= \operatorname{mr}(G) - \sum_{i=1}^{l+1} \operatorname{mr}(G_i - v). \end{aligned}$$

Since $\sum_{i=2}^{l+1} \operatorname{mr}(G_i - v) = 0$ and $\operatorname{mr}(G_1 - v) = \operatorname{mr}(G_1) - r_v(G_1)$ we have

$$\begin{aligned} \operatorname{mr}(G) &= \operatorname{mr}(G_1 - v) + 2 \\ &= \operatorname{mr}(G_1) + 2 - r_v(G_1). \end{aligned}$$

□

3.4 Maximum Nullity Of Trees

In this section, we discuss the maximum nullity of trees. The $M(T) = \Gamma(T) = P(T)$ is proved. In addition, $M(T) = Z(T)$ is shown.

We start with an illustration of a path tree which will be helpful in proving Theorem 3.4.5. A *path tree* is defined as a collection of $P = P(T)$ paths, together with $P - 1$ edges (that connect the paths) such that at least one vertex of each path is a vertex of each of the additional P - 1 edges (and each of the P - 1 edges has vertices of two different paths). So, we realize that due to the minimality of $P(T)$ we have: The two vertices of an extra edge (the edges that connect the paths) cannot be both endpoints of paths in the path tree, as the following figure shown.

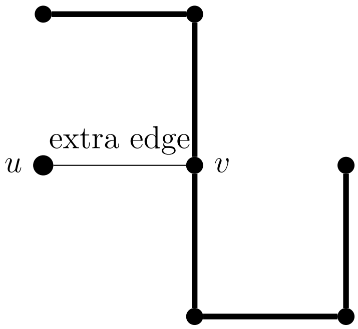


Figure 3.9: The graph T has $P(T) = 2$.

If two extra edges have adjacent vertices, then the other two vertices of these edges cannot both be endpoints of paths in the path tree, as the following figure shows.

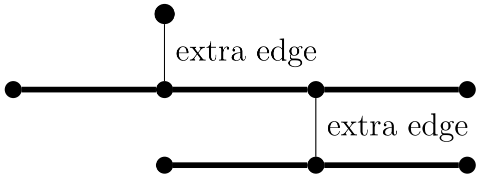


Figure 3.10: The graph T has $P(T) = 3$.

For an illustration of path tree, we present the following example.

Example 3.4.1. *The following figure represents a tree T.*

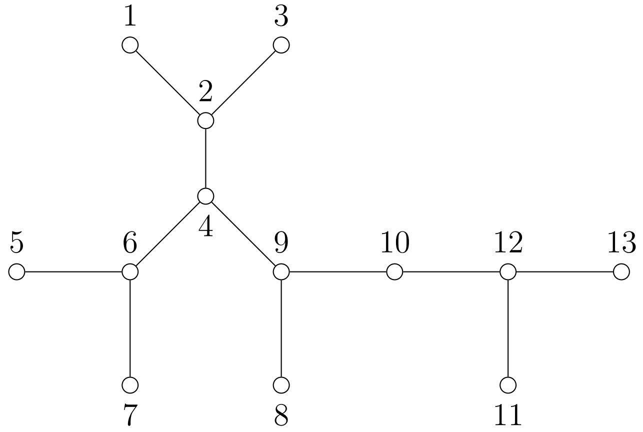
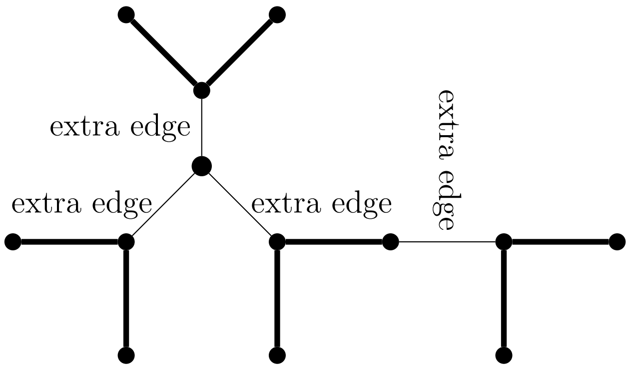
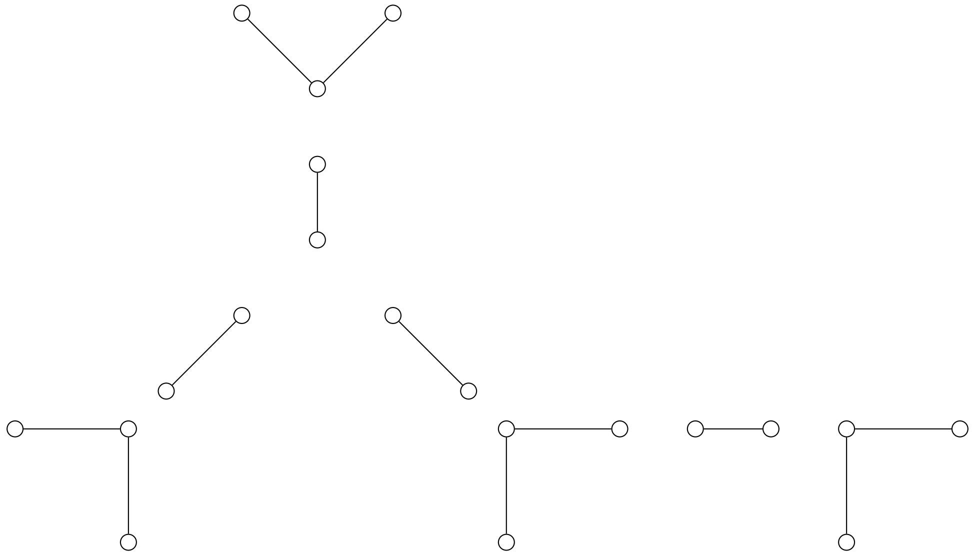


Figure 3.11: The graph T

T has $P(T) = 5$ with 4 extra edge as shown in the following figure.



A path tree of T shown in the following figure.



Delete from T of the vertices from each of the extra edges that is interior to the path tree. In this graph T we deleted $4 = q$ vertices. Hence we have $P(T') = P(T) + q = 5 + 4 = 9$ paths as shown in the following figure. Thus $\Gamma(T) \geq (P(T) + q) - q = (5 + 4) - 4 = 5 = P(T)$.

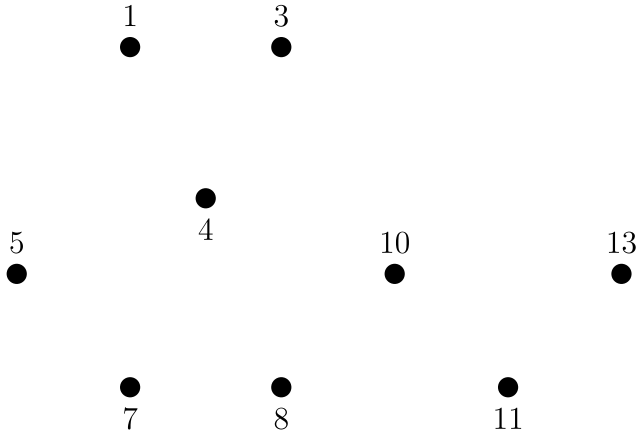


Figure 3.12: The graph T' has $P(T') = 9$.

The following examples illustrate the third inequality in the next theorem's proof.

Example 3.4.2. *The following figure represents a tree T.*

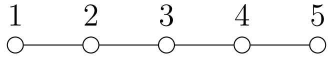


Figure 3.13: The graph T

T has $P(T) = 1$ as shown in the following figure.

Let A an adjacency matrix that associated to the graph T.

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

delete from A the columns whose indices are the first vertex of each path which is column 1 and the rows whose indices are the last vertices of each path which is row 5. Assume B the resulting matrix

$$B := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Note that B is full rank. Hence

$$\begin{aligned} \operatorname{mr}(T) &\geq \operatorname{rank}(A), \\ &\geq \operatorname{rank}(B) = 4 = 5 - 1 = n - P(T). \end{aligned}$$

Example 3.4.3. *The following figure represents a tree T.*

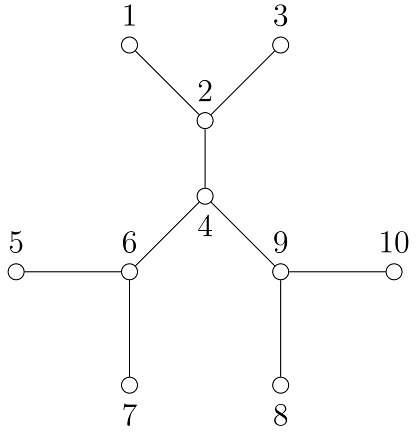
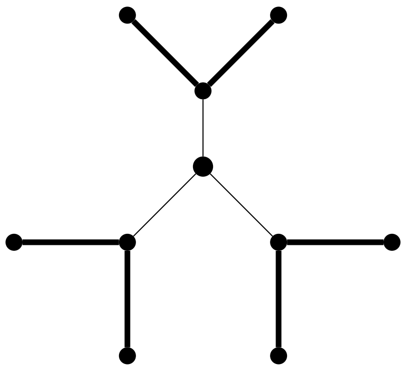


Figure 3.14: The graph T

T has $P(T) = 4$ as shown in the following figure.



The following adjacency matrix A that associated to the graph T.

$$A := \left[\begin{array}{ccc|c|ccc|ccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right],$$

delete from A the columns whose indices are the first vertex of each path which are column 1, column 4, column 5, and column 8 and the rows whose indices are the last *vertices of each path which are row* 3*, row* 4*, row* 7*, and row* 10*. Assume* B *the resulting matrix*

$$B := \left[\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

let

$$W := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, D := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{and } X := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that X *is a direct sum and the submatrices* W, D *and* X *are nonsingular. Thus*

$$\begin{aligned} \operatorname{mr}(T) &\geq \operatorname{rank}(A), \\ &\geq \operatorname{rank}(B) = \operatorname{rank}(W) + \operatorname{rank}(D) + \operatorname{rank}(X) = 2 + 2 + 2 = 6 = 10 - 4 = n - P(T). \end{aligned}$$

Example 3.4.4. *The following figure represents a tree* T'*.*

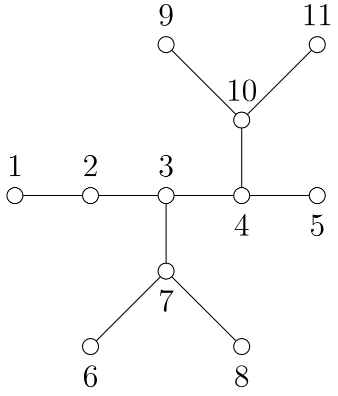
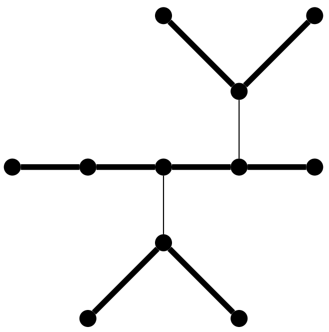


Figure 3.15: The graph T'

T' *has* $P(T) = 3$ *as shown in the following figure.*



Let A an adjacency matrix that associated to the graph T'.

$$A := \left[\begin{array}{ccccc|ccc|ccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right],$$

delete from A the columns whose indices are the first vertex of each path which are column 1, column 6, and column 9 and the rows whose indices are the last vertices of each path which are row 5, row 8, and row 11. Assume B the resulting matrix

$$B := \left[\begin{array}{cccc|cc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}\right].$$

Note that X is not direct sum. So we use elementary row and column operations to *make X a direct summand as the follows*

$$B' := \left[\begin{array}{cccc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

let

$$W := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, D := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{and } X := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that B and B' are congruent. Thus by Theorem 1.1.12, we have $\operatorname{rank}(A) = \operatorname{rank}(B')$. *Hence*

$$\begin{aligned} \operatorname{mr}(T) &\geq \operatorname{rank}(A), \\ &\geq \operatorname{rank}(B) = \operatorname{rank}(B') = \operatorname{rank}(W) + \operatorname{rank}(D) + \operatorname{rank}(X) = 4 + 2 + 2 = 8 = 11 - 3 = n - P(T). \end{aligned}$$

The next theorem indicates the fact that the quantities $M(T), \Gamma(T)$, and $P(T)$ are equal for any tree.

Theorem 3.4.5. [24] *For any tree T we have*

$$M(T) = \Gamma(T) = P(T).$$

Proof. The proof consists of observing the four inequalities:

$$M(T) \geq \Gamma(T) \geq P(T) \geq M(T).$$

If T is a path of at least one vertex, then all inequalities are equals, since $M(T) = \Gamma(T) = P(T) = 1$. Now if T is not a path. To prove the first inequality, suppose that $P^{(1)}, \cdots, P^{(\Gamma(T)+q)}$ be the paths remaining after deletion of an optimal q vertices from T. By Theorem 3.2.3, for any $\mu \in \mathbb{R}$ there exist $A_i \in S(P^{(i)}), i = 1, \cdots, \Gamma(T) + q$ such that $\mu \in \sigma(A_i)$. Define $D := A_1 \oplus A_2 \cdots \oplus A_{\Gamma(T)+q}$ to be an $(n-q)$–by–$(n-q)$ principal submatrix of some $A \in S(T)$. It is know that $\sigma(D) = \sigma(A_1) \cup \cdots \cup \sigma(A_{\Gamma(T)+q})$. Hence μ has multiplicity of $\Gamma(T) + q$. Without loss of generality assume that μ is the smallest,

let $\mu_1 = \cdots = \mu_{\Gamma(T)+q} = \mu$. Hence by Theorem 1.1.15, we have
$\mu_1 \leq \lambda_{1+q} \leq \mu_{1+q}$,
$\mu_2 \leq \lambda_{2+q} \leq \mu_{2+q}$,
$\vdots$
$\mu_{\Gamma(T)} \leq \lambda_{\Gamma(T)+q} \leq \mu_{\Gamma(T)+q}$,
therefore, $\lambda_{1+q} = \lambda_{2+q} = \cdots = \lambda_{\Gamma(T)+q} = \mu$. Thus A has at least $(\Gamma(T) + q) - q = \Gamma(T)$ eigenvalues equal to μ and $\sigma(A - \mu I) = \{\lambda_i - \mu \mid \lambda_i \in \sigma(A)\}$, so we have 0 eigenvalue with multiplicity $\Gamma(T)$. Hence

$$M(T) \geq \Gamma(T).$$

To show the second inequality $\Gamma(T) \geq P(T)$. Let $Q_1, \cdots, Q_{P(T)}$ be the paths remaining when we deleting an optimal q vertices from T. Delete from T one of the vertices of each of an extra edge that is interior to the path tree in this way that no pairs of adjacent vertices in the same path are deleted. If q vertices were deleted, there $P(T) + q$ paths will result. Hence

$$\Gamma(T) \geq P(T) + q - q = P(T).$$

Finally, $P(T) \geq M(T)$ which is equivalent to $\text{mr}(A) \geq n - P(T)$ since $M(T) + \text{mr}(T) = n$. To show this, we prove that for each $A \in S(T), \text{rank } A \geq n - P(T)$. To see this, we delete $P(T)$ rows and another $P(T)$ columns to leave a matrix of full rank, we prove this by induction on the value of $P(T)$. The case $P(T) = 1$, i.e., T a path, we get $A \in S(T)$ full rank matrix by deleting the column which corresponds to one leaf and the row which corresponds to the other leaf. Thus

$$\begin{aligned} \text{rank } A &\geq n - P(T) \\ \text{rank } A &\geq n - 1. \end{aligned}$$

In general, label the paths in a path tree for T whereas label the last path that has one an extra edge to only one other path and number the vertices of T consecutively along each path, beginning with the first. Now, delete from A the columns whose indices are the first vertex of each path and the rows whose indices are the last vertices of each path. Let B the resulting matrix and assume the principal submatrix X of B corresponding to the last path. Now we distinguish between the following two cases:
Case 1: If X is a direct summand of B or empty.
The result holds by the induction hypothesis because X is nonsingular or not present.
Case 2: If X is not a direct summand of B.

Note that X is lower triangular and there is at most one nonzero entry in B outside X in the rows of X and at most one in the columns of X. If both occurred they are not symmetrically placed, because of the way B was chosen. Thus we use elementary row and column operations to eliminate these entries and, perhaps, some below diagonal entries of X, without changing the principal submatrix of B complementary to X. By Theorem 1.1.12, the resulting matrix has the same rank as B and the modified X is a direct summand, so that by applied the induction hypothesis to verify that B has full rank. Thus

$$\begin{aligned} \operatorname{mr}(T) &\geq \operatorname{rank}(A) \geq n - P(T), \\ \operatorname{mr}(T) &\geq n - P(T), \\ n - M(T) &\geq n - P(T), \\ P(T) &\geq M(T). \end{aligned}$$

□

The following proposition shows that the maximum nullity equals the zero forcing number for any tree.

Proposition 3.4.6. [1] *For any tree T, $M(T) = Z(T)$.*

Proof. Let T be a tree on n vertices. Choose a minimum path cover $P(T)$ of T and let Z to be a set of vertices that consists of one endpoints of each path in the minimum path cover. By induction on $P(T)$, we show that Z is a zero forcing set. If $P(T) = 1$, then $T = P_n$ and $M(P_n) = Z(P_n) = 1$. Assume it is true for all trees T' such that $P(T') < P(T)$. Choose a minimum path cover $P(T)$ for T, let Z be defined as above and identify a path P' in the minimum path cover that is joined to the rest of T by only one edge uv not in P', and say $v \in V(P')$. Then by applying the color change rule repeatedly starting at the black endpoint of P' (if v is an endpoint of P' color the other endpoint), all vertices from the black endpoint through v are colored black. Now the path P' is irrelevant to the analysis of the tree $T - V(P')$, so by the induction hypothesis, the black endpoints of the remaining paths are a zero forcing set for $T - V(P')$, and all vertices not in P', including u, can be colored black. Hence the remainder of path P' can also be colored black and Z is a zero forcing set for T. Hence

$$Z(T) \leq 1 + Z(T - V(P')) = 1 + P(T - V(P')) = P(T). \tag{3.18}$$

By Proposition 3.1.5, Theorem 3.4.5, and (3.18) we have

$$M(T) \leq Z(T) \leq P(T) = M(T).$$

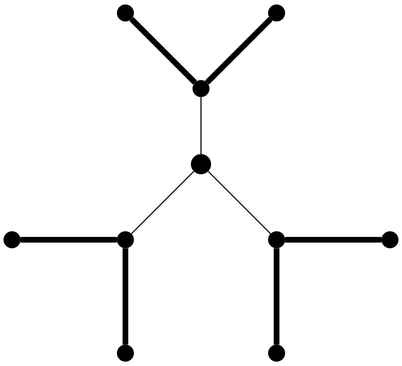
Therefore,

$$M(T) = Z(T) = P(T).$$

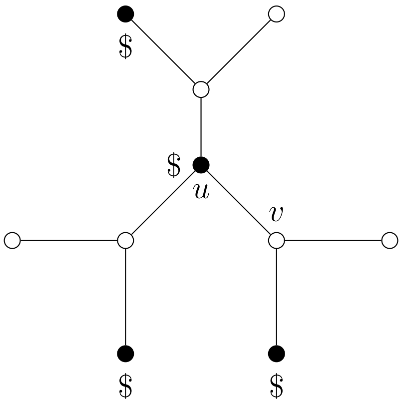
□

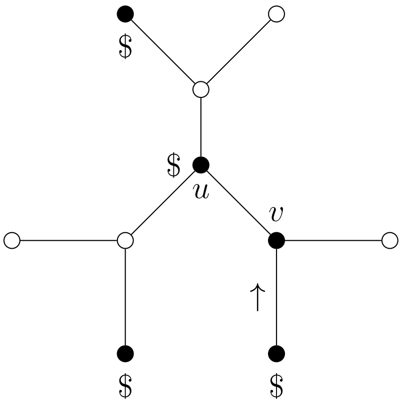
In the following example, we illustrate the proof of Proposition 3.4.6.

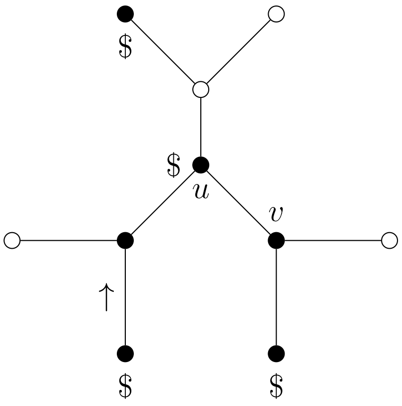
Example 3.4.7. *The following tree T has $P(T) = 4$ as it is shown below.*

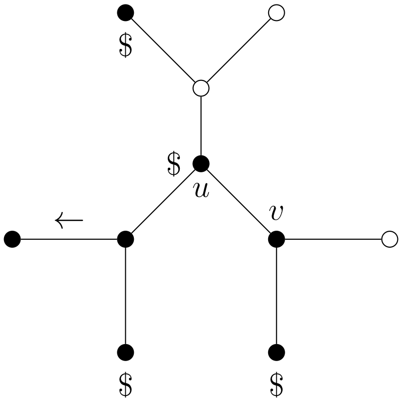


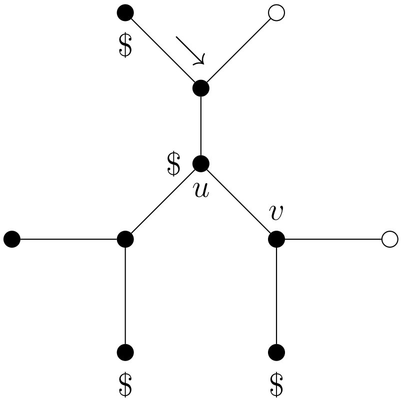
In the following figures we show that $Z(T) \leq 4$.

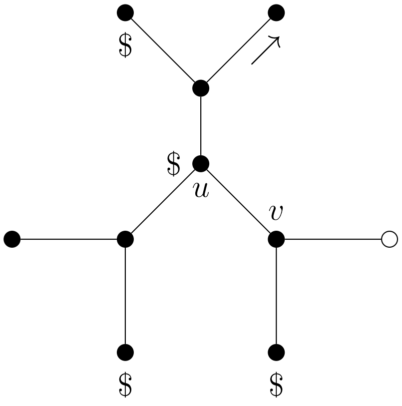


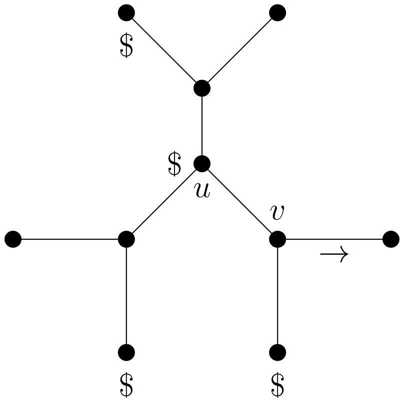












Since we can not do it in three vertices. Therefore, $Z(T) = 4$. *Hence by Proposition 3.4.6,* $M(T) = 4$.

3.5 Maximum Nullity Of Trees With A Fixed q Negative Eigenvalues

In this section, the maximum nullity with a fixed q negative eigenvalues of trees is presented. For any given graph G, the set $S(G)$ can be partitioned into sets $S_q(G), q = 0, 1, \cdots, n$, according to the number of negative eigenvalues, i.e.,

$$S_q(G) := \{A \in S(G) : A \text{ has exactly } q \text{ negative eigenvalues}\}.$$

Clearly $S_0(G)$ consists of all the positive semidefinite matrices in $S(G)$ and $S_q(G), q = 0, 1, \cdots, n$, make up partition of the set $S(G)$. Indeed, $S_q(G) \neq \emptyset$, $\cup_q S_q(G) = S(G)$, and $S_i(G) \cap S_j(G) = \emptyset$ for $i \neq j$. We denote the maximum nullity over $S_q(G)$ as $M_q(G)$. It is easy to note that $M_q(G) \leq M(G)$ for any graph G. The *positive semidefinite maximum nullity* of G is

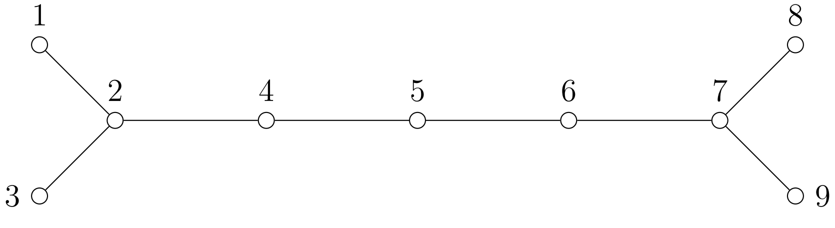
$$M_0(G) := \max\{\text{null}(A) : A \in S(G) \text{ is positive semidefinite}\},$$

equivalently some times written as $M_+(G)$.

Now, let $MD_k(G)$ denote the *maximum disconnection number*, or the largest number of components that result from deleting a collection of k vertices from G. In particular, for a tree T, we have $MD_1(T) = \Delta(T)$. For a given tree T, we define

$$q_0(T) := \min\{q \mid (MD_q(T) - q) = \Gamma(T) = P(T)\}.$$

Example 3.5.1. *For the following tree T.*



It is easy to note that $P(T) = 3$ as it is shown in Figure 3.16. In the following, we determine $q_0(T)$.
$MD_2(T) = 5$ *(remove the vertex 2 and vertex 7) and* $MD_2(T) - 2 = 5 - 2 = 3$,
$MD_3(T) = 6$ *(remove the vertex 2, vertex 5, and vertex 7) and* $MD_3(T) - 3 = 6 - 3 = 3$, *and* $MD_q - q \leq 3$ *for* $q \geq 3$. *Therefore,* $q_0(T) = 2$.

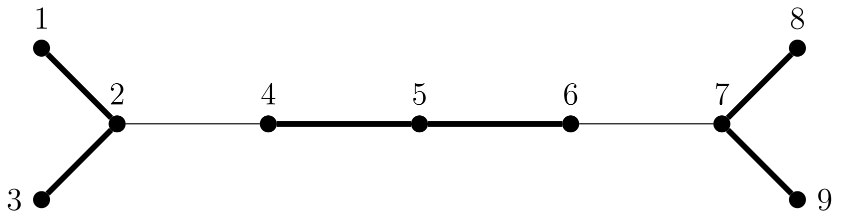


Figure 3.16: The minimum path cover for graph T.

In the following theorem we prove that for any tree, $M_0(T) = 1$.

Theorem 3.5.2. [19] *Let T be a tree on n vertices. Then $M_0(T) = 1$.*

Proof. We use induction on n to show that $M_0(T) \leq 1$. Assume $A \in S_0(T)$ and $\text{null}(A) = M_0(T)$. Now if $n = 1$, we distinguish between the following two cases. $a_{1,1} = 0$. We have $\text{null}(A) = M_0(T) = 1$. Or $a_{1,1} \neq 0$. We have $\text{null}(A) = M_0(T) = 0$.
In either cases we have $M_0(T) \leq 1$. Assume it is true for any tree with $|V(T)| < n$. Let T be a tree on n vertices and let v be a leave vertex in T, without loss of generality let $v = 1$. We distinguish between the following two cases:
Case 1: $a_{1,1} = 0$.
A can written as:

$$A = \left[\begin{array}{cc|ccc} 0 & a_{1,2} & 0 & \cdots & 0 \\ a_{1,2} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ \hline 0 & a_{2,3} & & & \\ \vdots & \vdots & & A_{33} & \\ 0 & a_{2,n} & & & \end{array}\right],$$

let

$$A_{11} := \begin{bmatrix} 0 & a_{12} \\ a_{12} & a_{22} \end{bmatrix}.$$

In this case A_{11} is a principal submatrix of A and $\det(A_{11})$ is negative. Hence A can not be a positive semidefinite. Thus this case is rejected.
Case 2: $a_{1,1} \neq 0$.
A can be written as:

$$= \left[\begin{array}{c|cccc} a_{1,1} & a_{1,2} & 0 & \cdots & 0 \\ \hline a_{1,2} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ 0 & a_{2,3} & & & \\ \vdots & \vdots & & A_{33} & \\ 0 & a_{2,n} & & & \end{array}\right],$$

We apply elementary row operation to make the entry $a_{1,2}$ zero. We multiply the first row by $\frac{-a_{1,2}}{a_{1,1}}$ and add the result to row 2. Then apply the same to column 1 and column 2. Hence we get

$$C := \left[\begin{array}{c|ccc} a_{1,1} & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A_{22} & \\ 0 & & & \end{array}\right],$$

where $A_{22} \in S_0(T')$ and $T' := T - v$ is a tree on $n-1$ vertices. Since C and A are congruent we have by Theorem 1.1.12

$$M_0(T) = \text{null}(A) = \text{null}(C) = \text{null}(a_{1,1}) + \text{null}(A_{22}) = \text{null}(A_{22}).$$

Since $\text{null}(a_{1,1}) = 0$. By induction we have

$$M_0(T) = \text{null}(A_{22}) \leq M_0(T') \leq 1.$$

Hence

$$M_0(T) \leq 1. \tag{3.19}$$

To show that $M_0(T) \geq 1$, let $0 = \lambda_1 < \cdots < \lambda_n$ be distinct real numbers. By Theorem 3.2.3 we can choose $A \in S(T)$ with $\sigma(A) = \{\lambda_1, \cdots, \lambda_n\}$. Thus A is positive semidefinite with $\text{null}(A) = 1$. Hence

$$M_0(T) \geq \text{null}(A) = 1. \tag{3.20}$$

By (3.19) and (3.20) we have

$$M_0(T) = 1.$$

□

Lemma 3.5.3. [9] *If* $A \in S_0(G)$ *and* $M_0(G) = \text{null}(A)$, *then every row (column) in* A *is linearly dependent on the other rows (columns).*

We present the proof of the following lemma as an illustration to the proof of Theorem 3.5.9 below.

Lemma 3.5.4. [2] *Let* T *be a tree on* $n \geq 3$ *vertices with maximum degree* Δ. *Then* $M_1(T) = \Delta - 1$.

Proof. We start proving that $M_1(T) \leq \Delta - 1$. We use induction on n. When $n = 3$, then $T = P_3$ with $\Delta = 2$ and from the fact that $M_q(T) \leq M(T)$ then we have $M_1(T) \leq 1$.

By Theorem 3.2.3 there exist $A \in S(T)$ with spectrum $\lambda_1 < \lambda_2 = 0 < \lambda_3$. Hence $M_1(T) = 1$ so the result holds. Suppose it is true for all trees on m vertices with $m < n$. Suppose on the contrary, that there exists a tree T on n vertices such that $M_1(T) > \Delta - 1$ which is equivalent to $M_1(T) \geq \Delta$. Without loss of generality, we may assume that there exists a matrix $A \in S_1(T)$ such that $\text{null}(A) = \Delta$. Let $v \in V(T)$ be such that $\ell := \deg(v) \geq 2$. Clearly $T - v = \bigcup_{i=1}^{\ell} T_i$ and since $A \in S_1(T)$ then by Theorem 1.1.15, all of $A[T_i], i = 1, \cdots, \ell$ must be positive semidefinite except possibly at most one. Without loss of generality, assume that $A[T_1]$ is positive semidefinite and by permutation similarity, we may assume that A has the following form:

$$A = \begin{bmatrix} A_{11} & a_{12} & 0 \\ a_{12}^T & a_{vv} & a_{23}^T \\ 0 & a_{23} & A_{33} \end{bmatrix},$$

where $A_{11} = A[T_1]$, with $|V(T_1)| = k$ and $a_{12} \in \mathbb{R}^k$ is a vector whose only nonzero entry is a_{kv} since T is a tree. We distinguish the following two cases:

Case 1: A_{11} is nonsingular.

We make the entry a_{kv} to be zero by linear combination from columns A_{11}, i.e., we can always find y such that $y = A_{11}^{-1} \begin{bmatrix} 0 \\ \vdots \\ a_{kv} \end{bmatrix}$. Apply the same to the rows of A_{11} to make a_{vk} zero. Hence D is congruent to A, where

$$D := \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & d_{vv} & a_{23}^T \\ 0 & a_{23} & A_{33} \end{bmatrix} \in S_1(T).$$

Define

$$B := \begin{bmatrix} d_{vv} & a_{23}^T \\ a_{23} & A_{33} \end{bmatrix},$$

and $T' = T - V(T_1)$. Thus by Theorem 1.1.12 and since A_{11} is positive definite we have $B \in S_1(T')$ and

$$\text{null}(A) = \text{null}(D) = \text{null}(B).$$

By the induction hypothesis, $\Delta \leq M_1(T) = \text{null}(B) \leq M_1(T') \leq \Delta(T') - 1 \leq \Delta - 1$, which is a contradiction.

Case 2: A_{11} is singular.

Let $A_{11} \in S_0(T_1)$, since A_{11} is singular and have the graph T_1 we have by Theorem 3.5.2 that $\text{null}(A_{11}) = 1 = M_0(T_1)$. By Lemma 3.5.3, we can make all the entries in the last column and row in A_{11} zeros, then A and A' are congruent, where A' is given as follows:

$$A' := \begin{bmatrix} A_{11}(k) & 0 & 0 & 0 \\ 0 & 0 & a_{kv} & 0 \\ 0 & a_{vk} & a_{vv} & a_{23}^T \\ 0 & 0 & a_{23} & A_{33} \end{bmatrix}.$$

We now use a_{kv} to eliminate all the nonzero entries in a_{23} and we use a_{vk} to eliminate all the nonzero entries in a_{23}^T. Thus C is congruent to A', where

$$C := \begin{bmatrix} A_{11}(k) & 0 & 0 & 0 \\ 0 & 0 & a_{kv} & 0 \\ 0 & a_{vk} & a_{vv} & 0 \\ 0 & 0 & 0 & A_{33} \end{bmatrix}.$$

Since C and A are congruent we have by Theorem 1.1.12, C has exactly one negative eigenvalue, we must have A_{33} to be positive semidefinite with $\deg(v) - 1$ components. Therefore, by Theorem 3.5.2 we have

$$\Delta \leq \text{null}(A) = \text{null}(C) = \text{null}(A_{33}) \leq \deg(v) - 1 \leq \Delta - 1,$$

which is a contradiction. Hence in the two cases,

$$M_1(T) \leq \Delta - 1. \tag{3.21}$$

Now we need to prove $M_1(T) \geq \Delta - 1$. Let $v \in V(T)$ be such that $\deg(v) = \Delta$. Then $T - v$ has Δ components, say $T_1, T_2, \cdots, T_\Delta$. For each $i = 1, 2, \cdots, \Delta$, chose $A_i \in S_0(T_i)$ with nullity 1 which is possibly by Theorem 3.2.3 and define $B := A_2 \oplus A_3 \oplus \cdots \oplus A_\Delta$ and A as follows:

$$A := \begin{bmatrix} A_1 & a_{12} & 0 \\ a_{12}^T & a_{vv} & a_{23}^T \\ 0 & a_{23} & B \end{bmatrix},$$

where $a_{12} \in \mathbb{R}^k$ is a vector whose only nonzero entry is a_{kv} and a_{23} is a vector whose only nonzero entries are those corresponding to the edges between vertex v and its neighbors. Hence $A \in S(T)$. Now by Lemma 3.5.3, we can make the entries in the last column and row in A_1 to be zeros, and repeat the above steps to show that A is congruent to

$$B' := \begin{bmatrix} A_1(k) & 0 & 0 & 0 \\ 0 & 0 & a_{kv} & 0 \\ 0 & a_{kv} & a_{vv} & 0 \\ 0 & 0 & 0 & B \end{bmatrix}.$$

Since B' has exactly one negative eigenvalue (since A_1 is positive semidefinite it follows that the principle submatrix $A_1(k)$ must also be positive semidefinite) with nullity $\Delta - 1$ which implies by Theorem 1.1.12 that $A \in S_1(T)$ with nullity $\Delta - 1$. Therefore,

$$\text{null}(A) = \Delta - 1 \leq M_1(T). \tag{3.22}$$

By (3.21) and (3.22) we have

$$M_1(T) = \Delta - 1.$$

□

Example 3.5.5. *The previous lemma help us to determine $M_1(T)$. For the following tree, we have $M_1(T_1) = \Delta - 1 = 5 - 1 = 4$.*

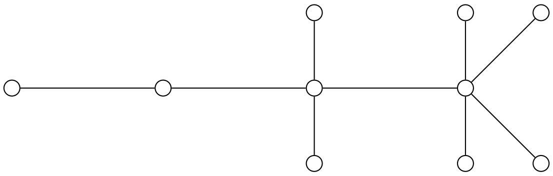


Figure 3.17: The graph T_1 with $\Delta = 5$.

The following theorem presents how we can determine $M_q(T)$ for $q = 0, 1, \cdots, q_0(T)$.

Theorem 3.5.6. [7] *Let T be a tree on n vertices. Then for $q = 0, 1, \cdots, q_0(T)$ the following holds:*

$$M_q(T) = MD_q(T) - q.$$

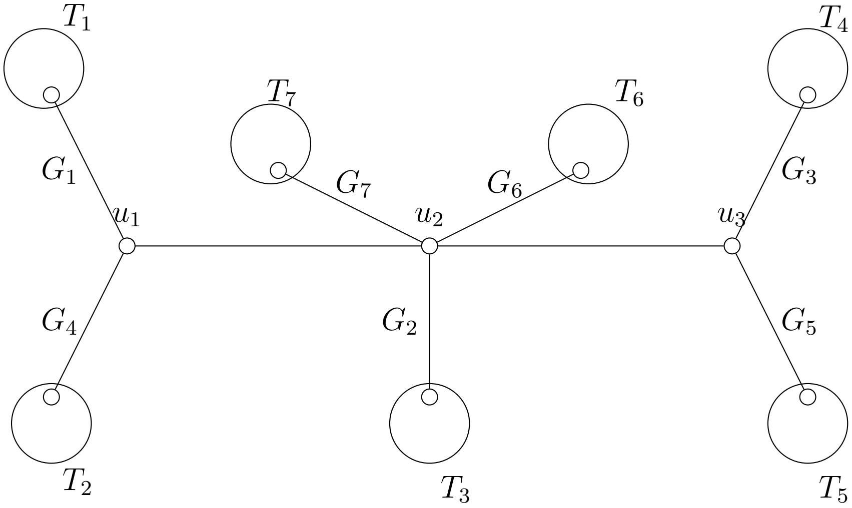
In order to prove Theorem 3.5.9, we need the following lemma.

Lemma 3.5.7. [2] *Let T be a tree on n vertices, $U = \{u_1, u_2, \cdots, u_q\}$ be a set of $q \leq q_0(T)$ vertices whose deletion results in $k := MD_q(T)$ components, $T - U = \cup_{i=1}^{k} T_i$, and $N(U) \cap V(T_i) = V_i, i = 1, \cdots, k$. Then T can be written as*

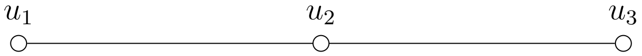
$$T = (\cup_{i=1}^{k} T_i) \cup H \cup (\cup_{i=1}^{k} G_i),$$

where H is the induced subgraph on U and G_i is the induced subgraph of T on the vertices $(U \cap N(V_i)) \cup V_i, i = 1, \cdots, k$ such that $|E(G_1)| = 1$ and $|E(G_i - \{u_1, \cdots, u_{i-1}\})| = 1$, for $i = 2, \cdots, q$.

Example 3.5.8. *As shown in the following tree T. Let $U = \{u_1, u_2, u_3\}$ be a set of* 3 *vertices whose deletion results in $k := MD_3(T) = 7$ components.*



H is induced subgraph on U as shown in the following figure



Clearly $T = (\cup_{i=1}^{7} T_i) \cup H \cup (\cup_{i=1}^{7} G_i)$.

For a given tree T, in the following theorem presents a method how we can build a matrix whose graph is T with a prescribed number of negative eigenvalues with nullity $M_q(T), q = 0, 1, \cdots, q_0(T)$.

Theorem 3.5.9. [2] *Let T be a tree on n vertices. Then for $q = 0, 1, \cdots, q_0(T)$ there exist a matrix $A \in S_q(T)$ such that*

$$\mathrm{null}(A) = M_q(T) = MD_q(T) - q.$$

Proof. For $q = 0, 1$ the result follows by Theorem 3.5.2 and Lemma 3.5.4. In the following assume $2 \leq q \leq q_0(T)$. Let $U = \{u_1, u_2, \cdots, u_q\}$ be a set of q vertices whose deletion results in $k := MD_q(T)$ components and $T - U = \cup_{i=1}^{k} T_i$ and $N(U) \cap V(T_i) = V_i, i = 1, \cdots, k$. Then by Lemma 3.5.7, T can be written as

$$T = (\cup_{i=1}^{k} T_i) \cup H \cup (\cup_{i=1}^{k} G_i),$$

where H and G_i are defined as in Lemma 3.5.7.
By Theorem 3.5.2 there exist $A_i \in S_0(T_i)$ and $\mathrm{null}(A_i) = 1$, for each $i = 1, \ldots, k$.

Without loss of generality, let u_i is adjacent to the first vertex in $T_i, i = 1, \ldots, q$. Define A as follows:

$$A = \left[\begin{array}{c|c|c|c|c|c|c|c}
B & B_1 & B_2 & \cdots & B_q & B_{q+1} & \cdots & B_k \\ \hline
B_1^T & A_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline
B_2^T & 0 & A_2 & 0 & 0 & 0 & \cdots & 0 \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \hline
B_q^T & 0 & 0 & 0 & A_q & 0 & \cdots & 0 \\ \hline
B_{q+1}^T & 0 & 0 & 0 & 0 & A_{q+1} & \cdots & 0 \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
B_k^T & 0 & 0 & 0 & 0 & 0 & \cdots & A_k
\end{array}\right],$$

where $B \in S(H)$ and $B_i[i, \cdots, q|V(T_i)], i = 1, \cdots, q$ has exactly one nonzero entry in the position $(B_i)_{i1}$ as follows. Thus $A \in S(T)$.

$$A = \left[\begin{array}{c|c|c|c|c|c|c|c}
B &
\begin{array}{cccc} (B_1)_{11} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{array} &
\begin{array}{cccc} r_{11} & r_{12} & \cdots & r_{1m} \\ (B_2)_{21} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{array} &
\cdots &
\begin{array}{cccc} w_{11} & w_{12} & \cdots & w_{1m'} \\ w_{21} & w_{22} & \cdots & w_{2m'} \\ w_{31} & w_{32} & \cdots & w_{3m'} \\ \vdots & \vdots & \vdots & \vdots \\ (B_q)_{q1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{array} &
B_{q+1} & \cdots & B_k \\ \hline
\begin{array}{cccccccc} (B_1)_{11} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array} & A_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline
\begin{array}{cccccccc} r_{11} & (B_2)_{21} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ r_{12} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{1m} & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array} & 0 & A_2 & 0 & 0 & 0 & \cdots & 0 \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \hline
\begin{array}{cccccccc} w_{11} & w_{21} & w_{31} & \cdots & (B_q)_{q1} & 0 & \cdots & 0 \\ w_{12} & w_{22} & w_{32} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{1m'} & w_{2m'} & w_{3m'} & \cdots & 0 & 0 & \cdots & 0 \end{array} & 0 & 0 & 0 & A_q & 0 & \cdots & 0 \\ \hline
B_{q+1}^T & 0 & 0 & 0 & 0 & A_{q+1} & \cdots & 0 \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
B_k^T & 0 & 0 & 0 & 0 & 0 & \cdots & A_k
\end{array}\right]$$

where $m = |V(T_2)|$, $m' = |V(T_q)|$ and $r_{ij}, w_{ix} \in \mathbb{R}$. We apply Lemma 3.5.3, on A_i. First we apply it in A_1 to eliminate its entries in its first row and column. Thus in the resulting matrix, the only nonzero entry in its $(q+1)$th column (row) is in position $(1, q+1)((q+1, 1))$, which is $(B_1)_{11}$. Now use $(B_1)_{11}$ to eliminate all the entries in the first row except a_{11}. After permutation similarities, A is congruent to

$$D=\left[\begin{array}{c|c|c|c|c|c|c|c}
\begin{matrix}\mathbf{a_{11}} & 0 & 0 & \cdots & 0\\ 0 & & & & \\ 0 & & B(1) & & \\ \vdots & & & & \\ 0 & & & & \end{matrix} & \begin{matrix}(\mathbf{B_1})_{11} & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{matrix} & B_2 & \cdots & B_q & B_{q+1} & \cdots & B_k \\ \hline
\begin{matrix}(\mathbf{B_1})_{11} & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{matrix} & \begin{matrix}\mathbf{0} & 0 & 0 & \cdots & 0\\ 0 & & & & \\ 0 & & A_1(1) & & \\ \vdots & & & & \\ 0 & & & & \end{matrix} & 0 & 0 & 0 & \cdots & 0 & \\ \hline
B_2^T & 0 & A_2 & 0 & 0 & 0 & \cdots & 0 \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \hline
B_q^T & 0 & 0 & 0 & A_q & 0 & \cdots & 0 \\ \hline
B_{q+1}^T & 0 & 0 & 0 & 0 & A_{q+1} & \cdots & 0 \\ \hline
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \hline
B_k^T & 0 & 0 & 0 & 0 & 0 & \cdots & A_k
\end{array}\right],$$

we are rearranging the matrix as follows

$$D:=\left[\begin{array}{c|c|c|c|c|c|c|c|c}
\begin{matrix}a_{11} & (B_1)_{11}\\ (B_1)_{11} & 0\end{matrix} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline
0 & A_1(1) & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline
0 & 0 & B(1) & B_2(1|\emptyset) & \cdots & B_q(1|\emptyset) & B_{q+1}(1|\emptyset) & \cdots & B_k(1|\emptyset) \\ \hline
0 & 0 & B_2(1|\emptyset)^T & A_2 & 0 & 0 & 0 & \cdots & 0 \\ \hline
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \hline
0 & 0 & B_q(1|\emptyset)^T & 0 & 0 & A_q & 0 & \cdots & 0 \\ \hline
0 & 0 & B_{q+1}(1|\emptyset)^T & 0 & 0 & 0 & A_{q+1} & \cdots & 0 \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
0 & 0 & B_k(1|\emptyset)^T & 0 & 0 & 0 & 0 & \cdots & A_k
\end{array}\right].$$

Now repeat this process sequentially for $j = 2, \cdots, q$ until A is congruent to

$$(\oplus_{i=1}^{q} B_i') \oplus (\oplus_{i=1}^{q} A_i(1)) \oplus (\oplus_{i=q+1}^{k} A_i), \tag{3.23}$$

where

$$B_i' := \begin{bmatrix} a_{ii} & (B_i)_{i1} \\ (B_i)_{i1} & 0 \end{bmatrix}, \quad i = 1. \cdots , q.$$

Since $\det(B_i')$ is negative for $i = 1, \cdots , q$ so each B_i' has exactly one positive and one negative eigenvalue, $A_i(1)$ is positive definite for $i = 1, \cdots , q$ and A_i for $i = q+1, \cdots , k$ have nullity equal to $k - (q+1) + 1 = k - q$. Hence by (3.23), A has exactly q negative eigenvalues and has nullity equal to $k - q$. Thus

$$\mathrm{null}(A) = k - q = MD_q(T) - q = M_q(T)$$

□

Lemma 3.5.10. [2] *Let T be a tree. Then*

$$M_{q_0}(T) > M_{q_0-1}(T).$$

Proof. By the definition of $q_0(T), MD_{q_0}(T) - q_0(T) > MD_{q_0-1}(T) - (q_0(T) - 1)$. Hence by Theorem 3.5.6, we have

$$M_{q_0}(T) > M_{q_0-1}(T).$$

□

Theorem 3.5.11. [2] *Let T be a tree on n vertices with $\Delta(T) \geq 3$. Set $p_0 := MD_{q_0}(T)$ and let $P_{l_1}, P_{l_2}, \cdots , P_{l_{P_0}}$ be the paths that result from the deletion of $q_0(T)$ vertices and let $L := \sum_{i=1}^{p_0} l_i - p_0 = n - (q_0(T) + p_0)$. Then*

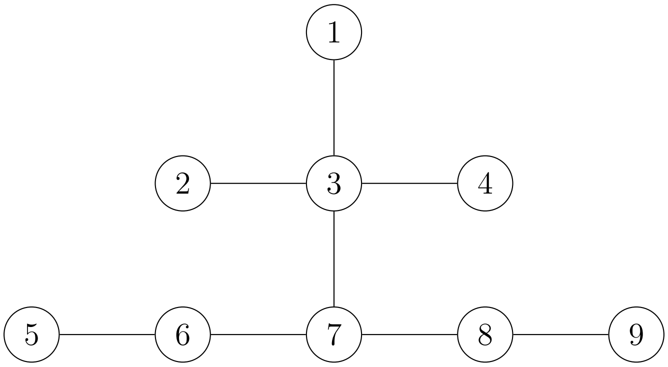
1. $M_0(T) < M_1(T) \leq \cdots \leq M_{q_0-1}(T) < M_{q_0}(T) = M(T)$;
2. $M_{n-k}(T) = k - \xi, \quad k = 1, 2, \cdots , n - q_0(T) - L - 1(= P_0 - 1)$,
   where

$$\xi = \min = \{\eta \mid \eta \in \{0, 1, \cdots , q_0(T)\} \text{ and } \eta + M_\eta(T) \geq k\}$$
$$= \min = \{\eta \mid \eta \in \{0, 1, \cdots , q_0(T)\} \text{ and } MD_\eta(T) \geq k\}; \tag{3.24}$$

3. $M_{n-1}(T) \leq M_{n-2}(T) \leq \cdots \leq M_{q_0+L+1}(T) \leq M_{q_0+L}(T)$;
4. $M_{q_0}(T) = M_{q_0+1}(T) = \cdots = M_{q_0+L}(T) = \Gamma(T) = P(T)$.

We conclude the section by the following example which illustrates the previous theorem.

Example 3.5.12. *For the following tree T,*



we determine $q_0(T)$ as follows:
$MD_0(T) = 1$ *and* $MD_0(T) - 0 = 1 - 0 = 1$.
$MD_1(T) = 4$ *(remove the vertex 3) and* $MD_1(T) - 1 = 4 - 1 = 3$.
$MD_2(T) = 5$ *(remove the vertex 3 and vertex 7) and* $MD_2(T) - 2 = 5 - 2 = 3$.
$MD_3(T) = 6$ *(remove the vertex 3, vertex 6, and vertex 8) and* $MD_3(T) - 3 = 6 - 3 = 3$.
$MD_4(T) = 5$ *(remove the vertex 3,vertex 6, vertex 7, and vertex 8) and* $MD_4(T) - 4 = 5 - 4 = 1$.
Since 3 *is the maximum value we have* $q_0(T) = 1$. *We use Theorem 3.5.6, to determine* $M_q(T)$ *for* $q = 0, 1$. *Hence*
$M_0(T) = 1 - 0 = 1$.
$M_1(T) = 4 - 1 = 3$.
By Theorem 3.5.11,(4), $M_q(T) = 3, q = 2, 3, 4, 5$ *since* $L = 9 - 1 - 4 = 4$. *For* $q = 6, 7, 8$, *we use Theorem 3.5.11,(2), to determine* $M_q(T)$ *we define* $q := 9 - k$ *and depending on* $M_s(T), s = 0, 1$ *to find* ξ *such that* (3.24) *holds as follows:*
$M_8 = M_{9-1} = k - \xi = 1 - 0 = 1$.
$M_7 = M_{9-2} = k - \xi = 2 - 1 = 1$.
$M_6 = M_{9-3} = k - \xi = 3 - 1 = 2$.
Hence we have

q	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
$M_q(T)$	*1*	*3*	*3*	*3*	*3*	*3*	*2*	*1*	*1*

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