Palestine Polytechnic University
Deanship of Graduate Studies and Scientific Research
Master Program of Mathematics

A Study of Graph Theory With Matrix Representation

Submitted by:

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Master of Science Thesis

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Supervisor

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Master thesis submitted and accepted, *Date* :______________.

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2017

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Dedications

This work is dedicated to my family

Maryam

Acknowledgments

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Contents

Abstract

In this thesis, we study the main well known results in graph theory. In particular, we study many formulation and properties of finite simple graphs through their matrix representation such as incidence and adjacency matrix, etc.

In addition, we compute the spectral of adjacency, Laplacian, and antiadjecency matrices of some special graphs.

Furthermore, we compare the largest eigenvalue of antiadjecency matrices that is constructed by some Boolean operations.

Introduction

In the eighteen century the The Königsberg bridge problem is the start of graph theory. The Königsberg bridge originated in Königsberg, formally in Germany, now known as Kaliningrad and it's a part of Russia, the city had seven bridges, which connected two islands with the main land by these bridges, on every Sunday afternoon the citizens in this city always wondered whether was there any way to walk on all bridges once and only once. See figure 1.

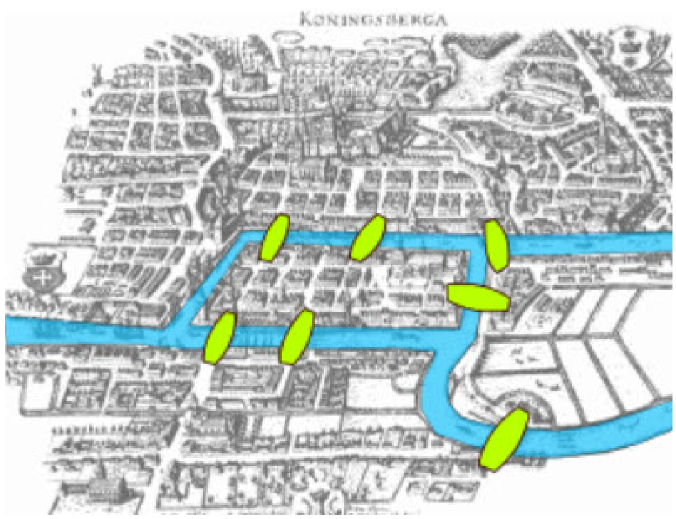


Figure 1: The old town of Königsberg has seven bridges

Since no citizen could achieve this puzzle. In 1736 Euler come out with the solution, first he draw a graph consisting of a node that represented the landmass and lines represented brigades that connected the mass. See figure 2.

After that problem, graph theory developed rapidly by using many techniques and strategies for solving various problems and applications in different fields.
The material of this thesis lies in four chapters, each contains basic definitions, examples and important Theorems.

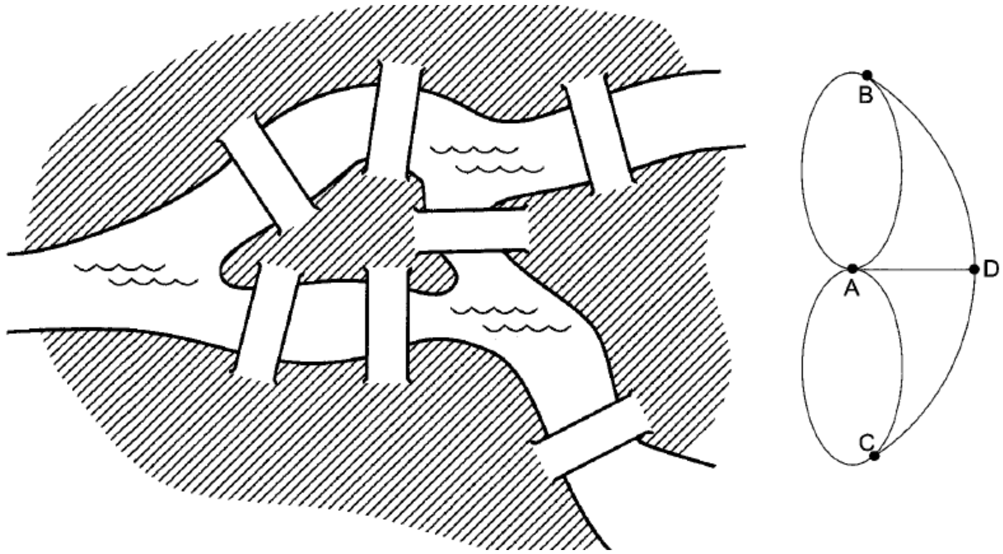


Figure 2

Chapter one: In this chapter we begin with basic definitions and some basic theorems needed in this work. It contains five sections. Section-1 contains the definition of undirected graph and we present many types of graphs and examples. In section-2, we give some unary and binary operations on graphs and cuts of edge and vertex. In section-3, we give the definition of isomorphic graphs. In section-4, we present the directed graph and it's related concepts. In section-5, we display the definition of tree and other related concepts, such as fundamental circuit and fundamental cut.

Chapter two: In this chapter we study the graph by using it's representation incidence matrix and other related matrices . It contains four sections. Section-1, contains the definition of incidence matrix of directed and undirected graph and some properties of this matrix . In section-2, we define the rank and compute the determinant of directed and undirected graphs. In section-3, we display the Moore-Penrose inverse to show a result on connected graphs. In section-4, we define the circuit and cut matrices and other related matrices.

Chapter three: In this chapter we study the graph by using another type of matrix representations such as adjecency matrix, Laplacian and antiadjecncy matrices. It contains four sections. Section-1, contains the definition of adjecency matrix of graphs and some properties of these matrices, we use this matrix to compute the number of walks that have the length k, also we characterize the the isomorphic graphs and finally we find it's determinant of such matrices. Section-2, contains the definition of Laplacian matrix of graphs and some properties of this matrix and it's relation with the incidence matrix. Section-3, contains the definition of antiadjecency matrix of graphs and some related results. In section-4, we study the Boolean operations and their adjacency and antiadjecency matrices.

Chapter four: In this chapter we study the spectral of graphs which is a useful way to study their properties. It consists of four sections. Section-1, contains the characteristic polynomial of adjacency matrix and some basic properties of it's spectral, also we display some type of graphs and their spectral properties. Section-2, contains some basic properties of spectral of laplacian matrix. Section-3, contains some results on comparing the largest eigenvalue of antiadjecency matrices resulted by Boolean operations. Section-4, contains the Page rank application of spectral properties of *link matrix*.

Chapter 1

Fundamentals of graph theory

This chapter is devoted to listing some definitions and results that will be used in succeeding chapters. It is not intended to be an exhaustive study of any topic nor it is a complete list of all of the facts which will be used in later chapters. Instead, it is intended to be a collection of those results which will play important roles in what follows.

1.1 Definitions and Basic concepts

This section describes the graph in mathematics, we give the definition and description of undirected graph and discuss some examples .

An undirected graph G=(V,E) is a collection of vertices (nodes, or points) that are connected by edges (lines). G consists of a set of vertices V=$\{v_1, v_2, ..., v_n\}$ and another set of edge E =$\{e_1, e_2, ..., e_m\}$, a graph with at least two vertices is called a *nontrivial* graph. Each edge e_k in the set E is defined with unorderd pairs (v_i,v_j) of vertices. The vertices v_i and v_j are called the *end vertices* of e_k .

In Figure 1.1 an edge e_3=(v_3,v_2) has v_3 and v_2 as end vertices, e_3 and e_5 share common edge vertex and are called *adjacent*. Two vertices v_i and v_j are adjacent if there exist an edge (v_i,v_j) that connect them. For example in figure 1.1 v_3 and v_4 are adjacent, an edge e_k

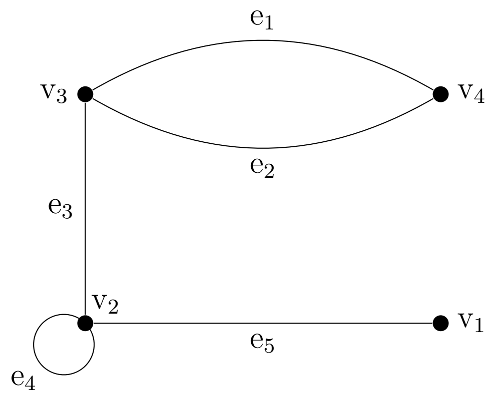


Figure 1.1

is called *self loop* if it has the same vertex as both end vertex. If more than one edge have the same pair of end vertices then these edges are called *parallel* in figure 1.1 for example e_1 and e_2 are parallel.

Definition 1.1.1 *Let G=(V,E) be undirected graph with vertex set V=*$\{v_1, v_2, ..., v_n\}$*, the degree of a vertex* $v \epsilon V$*, is the number of edges for which v is an end vertex, denoted deg(v)or d(v).*

The maximum degree of a graph G, denoted by $\Delta(G)$ and the minimum degree of a graph G denoted by $\delta(G)$ are defined as follows the maximum degree of G is the largest vertex degree of G, and the minimum degree of G is the smallest vertex degree of G, for example in figure 1.1 the maximum degree is 4 of vertex v_2 and the minimum degree is 1 for vertex v_1. A vertex whose degree is 1 is called *pendent* vertex, also a vertex whose degree is zero is called *isolated* vertex. For example in figure 1.1 it doesn't has an isolated vertex but it has v_1 as a pendent vertex.

Theorem 1.1.1 *[23] If G=(V,E) is an undirected graph with V=* $\{v_1, v_2, ..., v_n\}$*and E =*$\{e_1, e_2, ..., e_m\}$*, then*

$$\sum_{i=1}^{n} d(v_i) = 2m$$

Proof

Since the degree of a vertex v of a graph G=(V, E) is the number of edges for which v is an end vertex. Let e=(v_i,v_j) be any edge of G, then e has two end vertices v_i and v_j, when we sum the degrees of vertices, edge e get counted twice (once with end vertex v_i and once with end vertex v_j). Thus the sum of degrees equal twice the number of edges. ■

Corollary 1.1.1 *[23] Every undirected graph with an even number of vertices is of odd degree .*

Proof

Let G=(V,E) be any graph with n vertices and m number of edges. If the vertices $\{v_1, v_2, ..., v_k\}$ have odd degree and the vertices $\{v_{k+1}, v_{k+2}, ..., v_n\}$ have even degree Then by theorem 1.1.1 $\sum_{i=1}^{n} d(v_i) = 2m$, that is

$$\sum_{i=1}^{k} d(v_i) \dot{+} \sum_{i=k+1}^{n} d(v_i) = 2m$$

therefore

$$\sum_{i=1}^{k} d(v_i) + (\text{even number}) = (\text{even number})$$

$$\sum_{i=1}^{k} d(v_i) = (\text{even number})$$

Here each d(v_i) is odd number. So the number of odd degree vertices is always even.

Example 1.1.1 *Consider the undirected graph G(V,E), G has a set of vertices* $\{v_1, v_2, v_3, v_4, v_5\}$

d(v_1)=3, d(v_2)=2, d(v_3)=2, d(v_4)=1 and d(v_5)=0, Since d(v_1)=3 and d(v_4)=1 . Thus the number of vertices that has an odd degree is even number.

Definition 1.1.2 *A graph G(V,E) is said to be regular if all it's vertices have the same degree, if the degree of each vertex of G is k then G is said to be k-regular.*

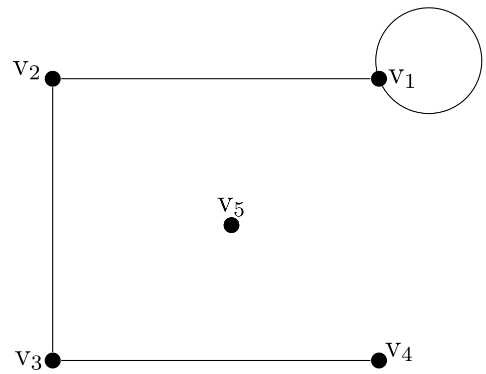


Figure 1.2

A subgraph of a graph G is a graph whose vertices and edges are contained in G. That is if G is a graph with set of vertices V=$\{v_1, v_2, ..., v_n\}$ and a set of edges E =$\{e_1, e_2, ..., e_m\}$ and H is a graph with set of vertices V=$\{v_1, v_2, ..., v_k\}$ such that V(H)$\subset$ V(G) and E(H)$\subset$ E(G) then H is called a *subgrah* of G and G is called *supergraph* of H.

Definition 1.1.3 *If V(H)=V(G) then H is called a spanning subgraph of G otherwise H is called proper subgraph.*

consider the graph G in figure 1.3

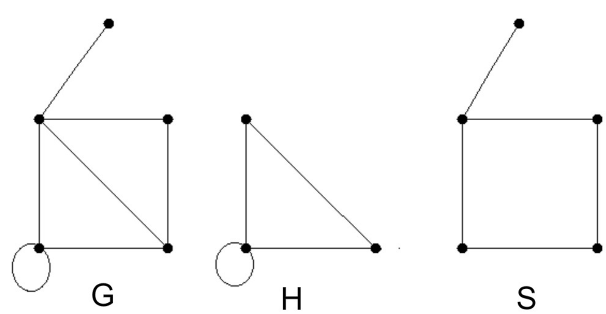


Figure 1.3

G has five vertices and it has a spanning subgraph S, it also has a proper subgraph H.

Definition 1.1.4 *Let v be a vertex in a graph G then the open neighbourhood of v in G is*

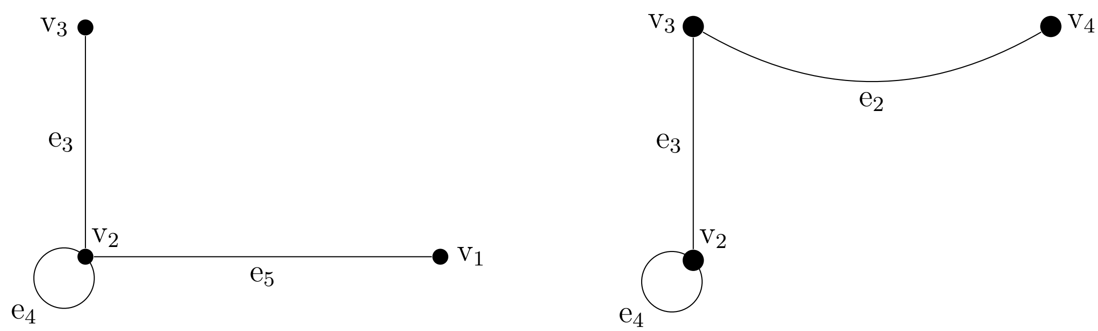
$$N_G(v) = \{u|e = (u, v) \in E(G)\}$$

Also the closed neighbourhood of v is $N_G[v] = N_G(v) \cup \{v\}$

For example, in figure 1.1 N(v_3)=$\{v_4, v_2\}$, and N[v_3]=$\{v_4, v_2, v_3\}$.

Definition 1.1.5 *let G=(V,E) be any undirected graph and let $V_1 \subset V$ be any subset of vertices in G. Then the induced subgraph G[V_1] is the graph whose edges set consists of all edges between the vertices in V.*

For example, the graph G[v_1,v_2,v_3] in figure 1.4a is an induced subgraph of figure 1.1 .



(a) A vertex induced subgraph

(b) An edge induced subgraph

Figure 1.4: A subgraph of a graph G

Definition 1.1.6 *let G=(V,E) be any graph and let $E_1 \subset E$ by any subset of edges of G. Then the induced subgraph G[E_1] is the graph which consists all vertices that are their end points .*

For example, the graph G[e_2,e_3,e_4] in figure 1.4b is an induced edge subgraph of the graph G in figure 1.1.

Definition 1.1.7 *A walk in a graph G is a finite sequence $v_1e_1v_2...v_k$ of graph vertices v_i and graph edges e_i for $1 \leq i \leq k$, where e_i=(v_i,v_{i+1}) .*

The walk v_i-v_k with an initial vertex v_i and a terminal vertex v_k is called v_i-v_k walk, with length is the number of edges.

Note 1.1.1 *If the initial vertex and the terminal vertex are different then v_i-v_k walk is an open walk, otherwise it is a called a closed walk.*

Definition 1.1.8 *A trail is a walk $v_1e_1v_2, ...v_k$ with no repeated edges.*

Definition 1.1.9 *A trail with distinct vertices except possibly the initial and terminal vertices when they are the same is called a path .*

Definition 1.1.10 *A closed path is called a circuit.*

Note 1.1.2 *If a graph G doesn't have a circuit it is called circuitless graph.*

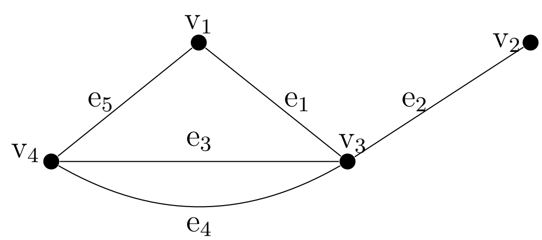


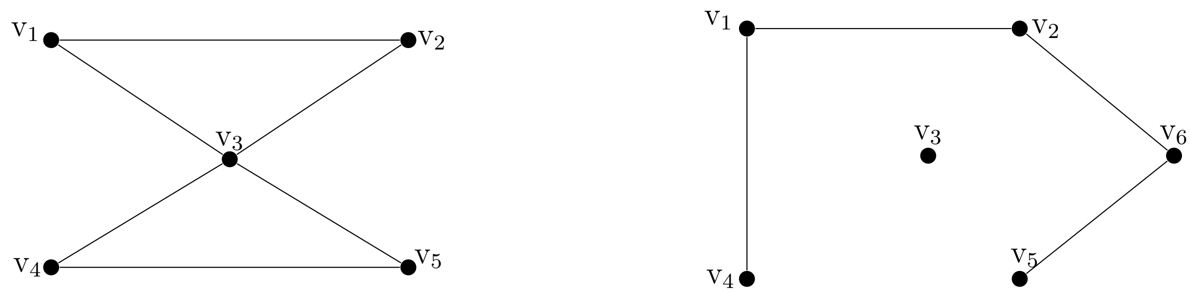
Figure 1.5: A graph for walks

Example 1.1.2 *The graph in figure 1.5 contains many different walks, for example :*

- *The walk $v_1e_1v_3e_2v_2$ is an open walk .*
- *The walk $v_1e_1v_3e_4v_4e_5v_1$ is a closed walk also it is a trail with length 3.*
- *The walk $v_3e_4v_4e_5v_1$ is a path but not a circuit .*
- *The walk $v_3e_4v_4e_5v_1e_1v_3$ is a circuit.*

The connectivity is an important concept in graph theory. Two vertices v_i and v_j are called connected if there is path v_i-v_j.

Definition 1.1.11 *A graph G=(V,E) is connected if any pair of vertices are joined by at least one path, otherwise the graph is called disconnected.*



(a) connected graph

(b) disconnected graph

Figure 1.6

For example, the graph in figure 1.6a is connected whereas the graph in figure 1.6b is disconnected.

Definition 1.1.12 *A simple graph is a graph with no self loops and no parallel edges.*

Definition 1.1.13 *[7] A simple graph in which there exist an edge between every pair of vertices is called a complete graph*

A complete graph with n vertices is denoted by K_n. The graph of the first five complete graphs are given in figure 1.7

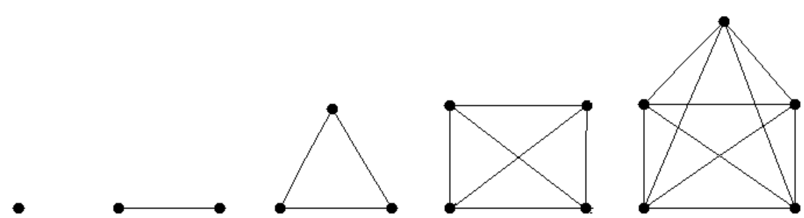


Figure 1.7: Some complete graph

Remark 1.1.1 *1. K_n has exactly $\frac{n(n-1)}{2}$ edges.*

2. A complete graph K_n is a regular graph of degree n-1.

Definition 1.1.14 *let G=(V,E) be any graph, then a complete subgraph of G is called a clique graph .*

For example in figure 1.8 the graph G with set of vertices V=$\{v_1, v_2, v_3, v_4, v_5\}$ have a complete subgraph with set of vertices V=$\{v_1, v_2, v_3\}$, so it's a clique of G.

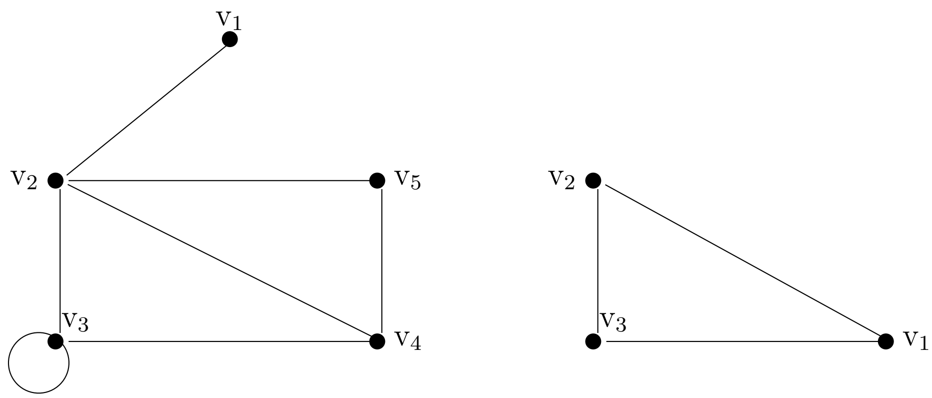


Figure 1.8: A graph with clique subgraph

Definition 1.1.15 *A set V is a partitioned into k nonempty subsets $V_1, V_2, ..., V_k$ if $V_1 \cup\ V_2 \cup\ V_3 ... \cup\ V_k =\ V$ and $V_i \cap\ V_j = \varnothing$, $\forall i \neq j$.*

Definition 1.1.16 *A graph G=(V,E) is said to be bipartite if it's vertex set can be partitioned into different sets V_1 and V_2 with $V_1 \cup V_2 = V$ and $V_1, V_2 \neq \varnothing$ also $V_1 \cap V_2 = \varnothing$ such that every edge in E have an end vertex in V_1 and another one in V_2 .*

Definition 1.1.17 *A complete bipartite graph G=(V,E) is a bipartite graph that contains all possible edges that have one end point in V_1 and the other one in V_2, we denote it by K_{n_1,n_2} where n_1, n_2 are the number of vertices in V_1, V_2 respectively .*

Figure 1.9 display a bipartite graph with a complete bipartite graph

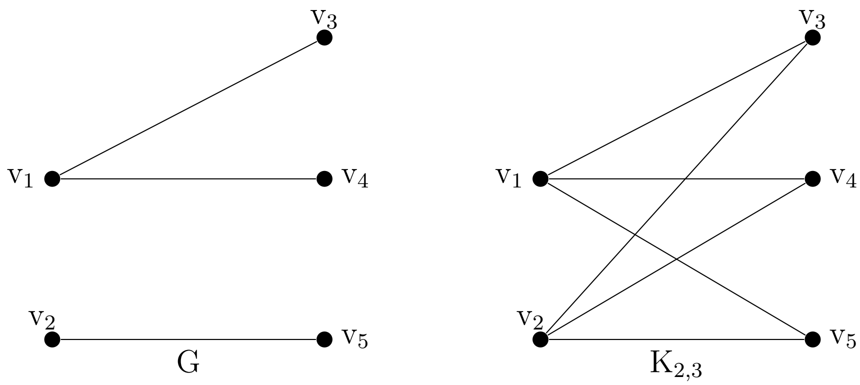


Figure 1.9: A bipartite graph

Definition 1.1.18 *A component of a graph G=(V,E) is the maximal connected subgraph, where maximality condition means that a subgraph* $H \subseteq G$ *is a connected subgraph and for any* $v \in V(G)$, $v \notin V(H)$, $G[V(H) \cup \{v\}]$ *is disconnected*

The only component of a connected graph is the graph itself, moreover the components of a nonconnected graph are all pieces it contains.

Example 1.1.3 *In figure 1.10 the graph G have six vertices which is disconnected graph, it has two components H and S (isolated vertex).*

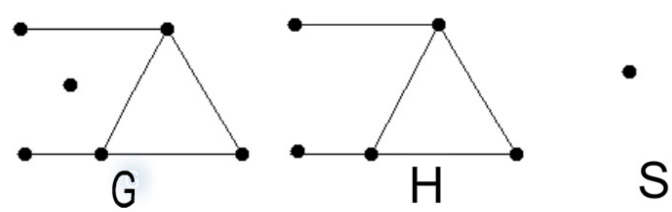


Figure 1.10: A graph with its components

Since $V(H) \cap V(S) = \varnothing$ *and* $V(H) \cup V(S) = V(G)$.

Definition 1.1.19 *The nullity of a graph G=(V,E) with n vertices, m edges and k components is the nonnegative integer* $\mu(G)$*=m-n+k .*

Definition 1.1.20 *The rank of a graph G=(V,E) with n vertices, m edges and k components is the nonnegative integer* $\rho(G)$*=n-k .*

It is easy to see that $\mu(G)$+ $\rho(G)$=m.

In figure 1.11 we have five edges, four vertices and two components then $\rho(G)$=4-2=2, and $\mu(G)$=5-4+2=3.

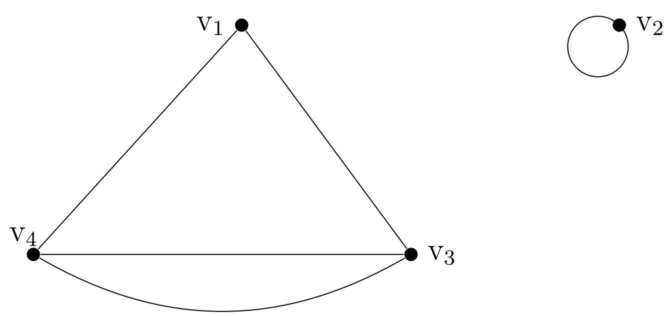


Figure 1.11

Definition 1.1.21 *Let* v_1 *and* v_2 *be two vertices in a graph G, the distance between two vertices is the length of a shortest path from* v_1 *to* v_2 *in G and is denoted* $d(v_1, v_2)$*. If G is disconnected and* v_1 *and* v_2 *in different components we say* $d(v_1, v_2) = \infty$*.*

Definition 1.1.22 *The eccentricity of a vertex v in a graph G=(V,E) is the maximum distance between v and any other vertex in G, denoted by e(v) (i.e* $e(v) = max\ \{d(v, v_j)|v_j \in V(G)\}$*).*

Definition 1.1.23 *Let G=(V,E) be a graph with eccentricity e(v), the diameter of G is the maximum eccentricity of G, denoted by diam(G)=max*$\{e(v)|v \in V(G)\}$*.*

Definition 1.1.24 *Let G=(V,E) is a graph with eccentricity e(v), the radius of G is the minimum eccentricity of G, denoted by rad(G)= min*$\{e(v)|v \in V(G)\}$*.*

Example 1.1.4 *For the graph G in the figure 1.12 with set vertex* $\{v_1, v_2, v_3, v_4, v_5\}$*, we can obtain:*

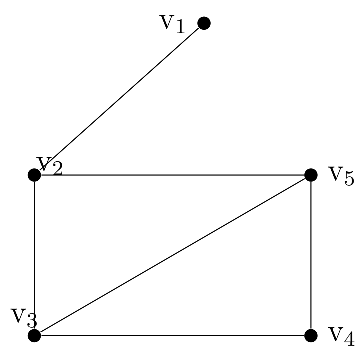


Figure 1.12

- *The distance* $d(v_1, v_4)$*=3 .*
- *The eccentricity e(v) of a vertex in graph G is 3, 2, 2, 3 and 2 respectively .*
- *The diameter of* $G = max\{e(v)|v \in V(G)\}$*=3.*
- *The radius=* $min\{e(v)|v \in V(G)\}$*=2 .*

Theorem 1.1.2 *[27] In any connected graph G*

$$rad(G) \leq diam(G) \leq 2\ rad(G)$$

Proof

By definition diam(G) equal the maximum distance between two vertices in G say u and v, so d(u,v)=diam(G) and say w is a vertex for which e(w)=rad(G). Since rad(G) is distance between two vertices, necessarily rad(G)≤ diam(G). Now

$$\begin{aligned} diam(G) &= d(u,v) \\ &\leq d(u,w) + d(w,v) \\ &= 2\,e(w) \\ &= 2\,rad(G) \end{aligned}$$

■

1.2 Graph operations and cuts

In this section we consider two types of operations on graphs and display the cut idea.

1.2.1 Operations on graphs

We consider some operations on graph. It will produce a new graph from the original by applying operations such as union, intersection, etc. These operations are categorized as unary which creates new graphs from an old one and binary operations create new graphs from two initial graphs.

We have three unary operations on a graph as follows :

1. Let G=(V,E) be a graph and v_i be any vertex in V(G), then the operation that obtained by removing v$_i$ is an induce subgraph of $V - v_i$ is called the *removal of a vertex*.

2. Let G=(V,E) be a graph and $e_i = (v_i, v_j)$ be any edge in E(G), then the operation that obtained by removing e$_i$ an induce subgraph of $E - e_i$ is called the *removal of an edge*.

Example 1.2.1 *let G=(V,E) be a graph with set vertex* $\{v_1, v_2, v_3, v_4\}$ *and set of edges* $\{e_1, e_2, e_3, e_4\}$

the following figure show examples of a graph G with removal of vertex v_5 *also with removal of edge* e_4 *respectively.*

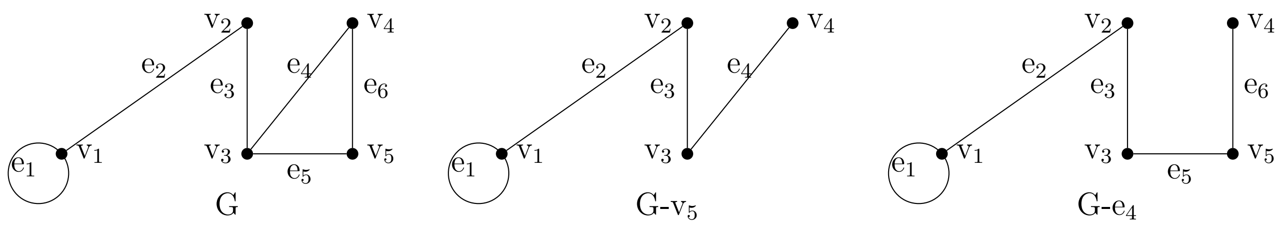


Figure 1.13

3. A complement of a simple graph G=(V,E) is a graph having the same set of vertex but it has a complement edge of G. In multigraph (i.e graph that has a parallel edge) the complement is not defined, also in a graph that is not multigraph but has self loops it's complement is defined by adding a self loop of every vertex that does not have the self loop. One example of complement graph G is given below.

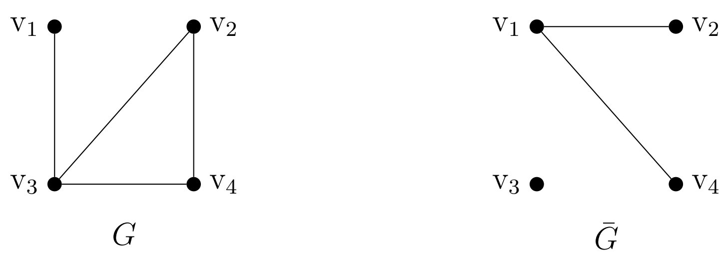


Figure 1.14: A graph with it's complement

Definition 1.2.1 *Two graphs are said to be vertex disjoint if they have no common vertices, and to be edge disjoint if they have no common edges.*

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, then we define three different binary operations on G_1 and G_2 as follow :

1. The union of any two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the new graph G=($V_1 \cup V_2$,$E_1 \cup E_2$). If G_1 and G_2 are disjoint, then we write their union G_1+G_2 as vertex set as V_1+V_2 and the edge set as E_1+E_2 and the edges joining each vertex of V_1 with each vertex V_2, one example of Union of two joint graphs in figure 1.15, also the Union of disjoint graphs in fig 1.16.

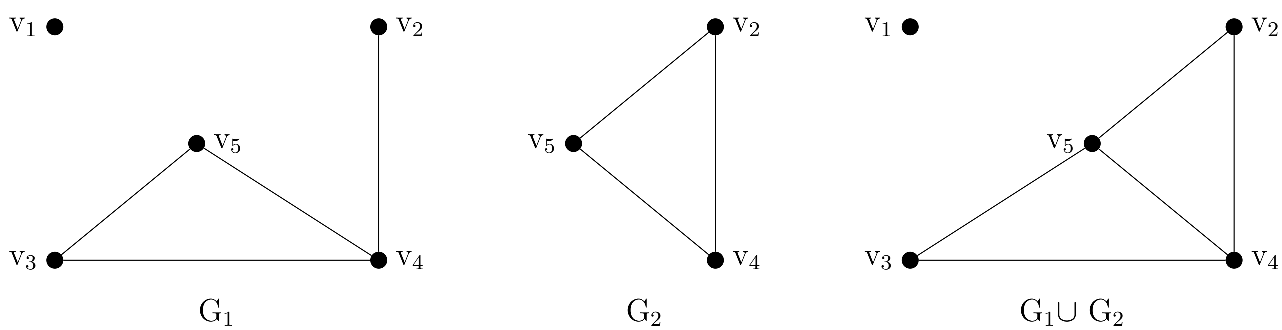


Figure 1.15: A union of joint graph

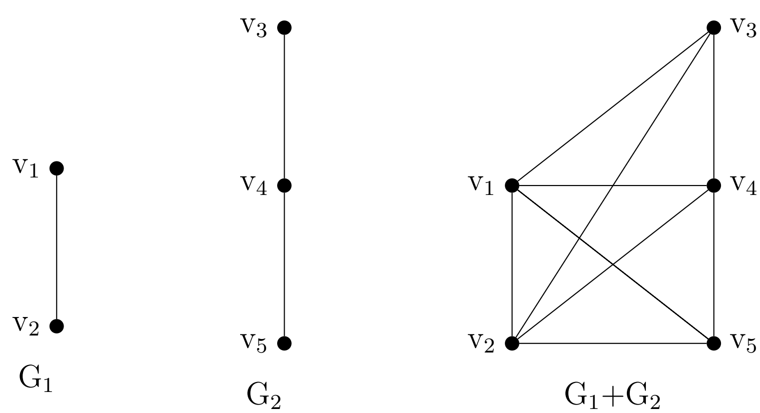


Figure 1.16: A union of disjoint graphs

2. The intersection of any two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a new graph G=($V_1 \cap V_2$, $E_1 \cap E_2$). If G_1 and G_2 are edge disjoint then their intersection is a null graph (i.e a graph with no vertices) , however if the vertex sets are disjoint then their intersection is the empty graph (i.e a graph with no edges). One example of

intersection of two joint graph operation in figure 1.17.

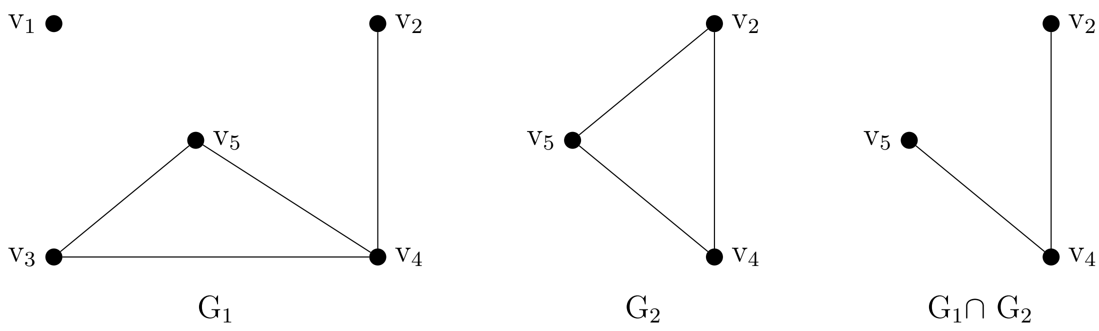


Figure 1.17: A intersection operation of graph

3. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, their ring sum denoted by $G_1 \bigoplus G_2$ is defined by G=(V,E) where the set vertex V=$V_1 \cup V_2$ and the set of edges $E = (E_1 \cup E_2) - (E_1 \cap E_2)$.

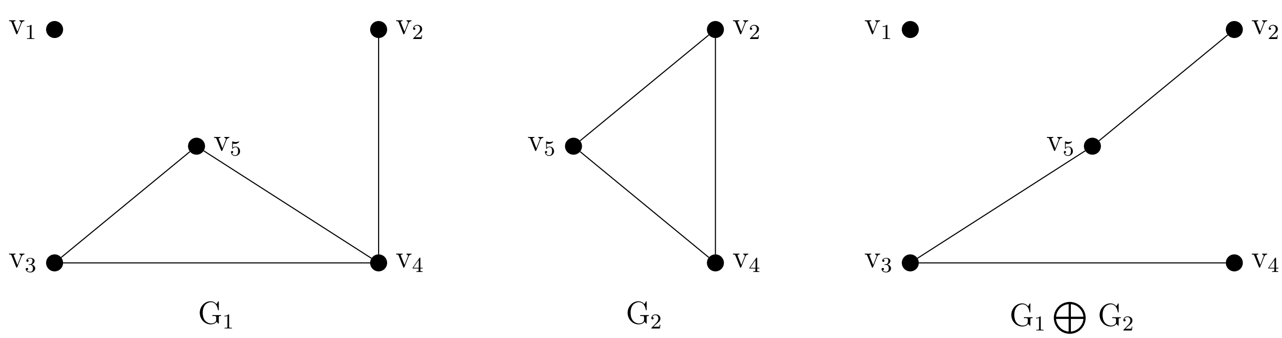


Figure 1.18: A ring sum operation of graph

Although union and intersection and ring sum have been defined for two graphs these can be extended to any finite number of graphs.

1.2.2 Cut on graphs

In this section we define a cut vertex, cut edge and blocks, and display some theorems on cuts.

Definition 1.2.2 *A vertex $v \in V(G)$ is called cut-vertex if G-$v_4$has more connected components than G, i.e (k(G-v)>k(G))*

For example v_4 is a cut vertex in figure 1.19, since G-v_4 has two components an isolated vertex $_2$ and a subgraph with set of vertices $\{v_1, v_3, v_5\}$, v_4 is not the only cut vertex in G, also v_3 is cut vertex.

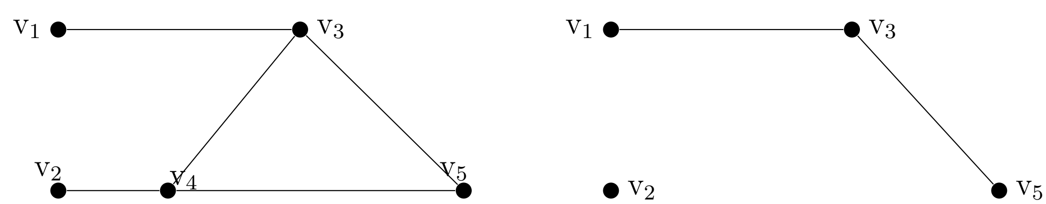


Figure 1.19: A cut vertex example

Definition 1.2.3 *An edge $e \in E(G)$ is called cut edge (bridge) if G-e has more connected components than G, i.e (k(G-e)>k(G))* .

For example e_1 is a bridge in figure 1.20, since G-e_1 has two components. Notice that e_2 is another bridge.

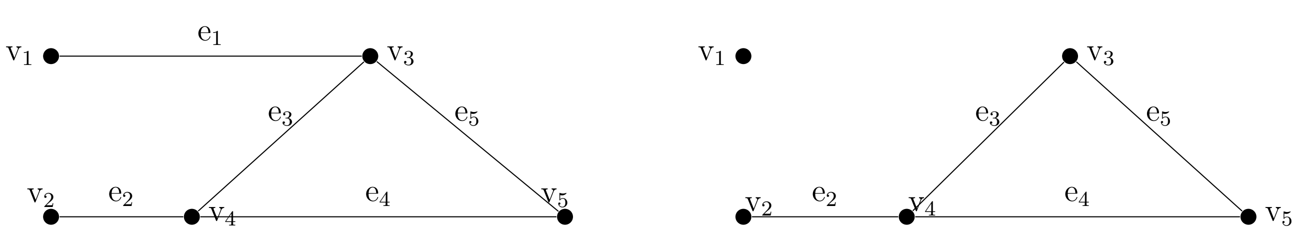


Figure 1.20: A cut edge example

Note 1.2.1 *The only connected graph with a bridge but with no cut vertex is K_2.*

Definition 1.2.4 *A nontrivial connected graph is called non separable if it has no cut vertex. Otherwise it is called separable.*

The graph G in figure 1.19 is separable .

Definition 1.2.5 *A block is a maximal non separable subgraph. In other words a block is a subgraph with as many edges as possible and no cut vertex.*

In figure 1.21 the graph G has two cut vertices, v_3 and v_4 and it has three blocks .

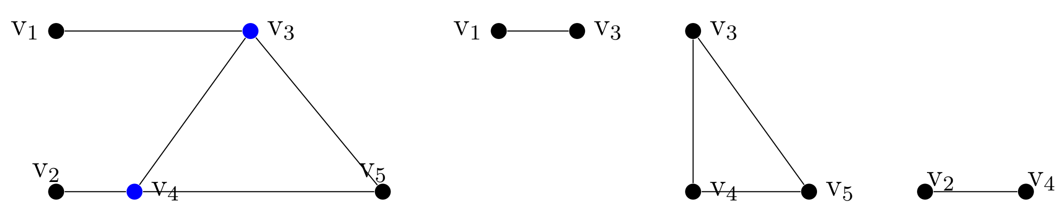


Figure 1.21: A graph and its blocks graph

We have the following two results about cut vertex.

Theorem 1.2.1 *[26] A vertex v is cut vertex of a connected graph G if and only if there exist two vertices u and w distinct from v such that v lies in every u-w path.*

Proof:

Assume G=(V,E) be a connected graph, and let v∈ V(G) be a cut vertex, we will prove that v is on every u-w path of G. So if v is a cut vertex then G-v is disconnected, also it has at least two components say G_1 that contains u and G_2 that contains w, so there does not exist any u-w path in G-v. But G is connected, thus all such u-w paths went through vertex v. Assume there exist u, w $\in$ V(G) and u,w$\neq$ v such that v lies on every u-w path. We will prove that v is cut vertex. Then the vertices u and w are not connected in G-v. Thus the graph G-v is not connected. Hence by definition 1.2.2 v is a cut vertex. ■

Theorem 1.2.2 *[5] In a connected graph G=(V,E) with at least two vertices, G contains at least two vertices that are not cut vertices.*

Proof:

To the contrary, there exist a non trivial connected graph G with at most one non cut vertex. Let u,v be two vertices in G with d(v,u)=diam(G). At least one of u,v is a cut vertex say v, so G-v is disconnected graph, and let w be a vertex in a different component of G-v than u, then every u-w path contains v. Hence

$$d\,(u,w) > d\,(u,v) = dim\,(G)$$

Contradiction. ■

1.3 Isomorphic graphs

Definition 1.3.1 *Two graphs G_1=(V_1,E_1) and G_2=(V_2,E_2) are said to be isomorphic if there exist a bijection function f:$V_1 \rightarrow V_2$, so that for any two vertices u and v the number of edges connecting u to v is the same number of edges connecting f(u) to f(v).*

Remark 1.3.1 *To find an isomorphism f:$V_1 \rightarrow V_2$, we must maintain the adjacency and non adjacency between vertices. In other words if u and v are adjacent in G then f(u) and f(v) are adjacent.*

Example 1.3.1 *let G_1 and G_2 be the two graphs in figure 1.22. To prove that the two graphs are isomorphic we will find a bijection f:$V_1 \rightarrow V_2$. Let*

$f=\begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$, *then f is a bijection such that for every two vertices u and v in G_1 the number of edges connecting u,v is the same number of edges connecting f(u) to f(v).*

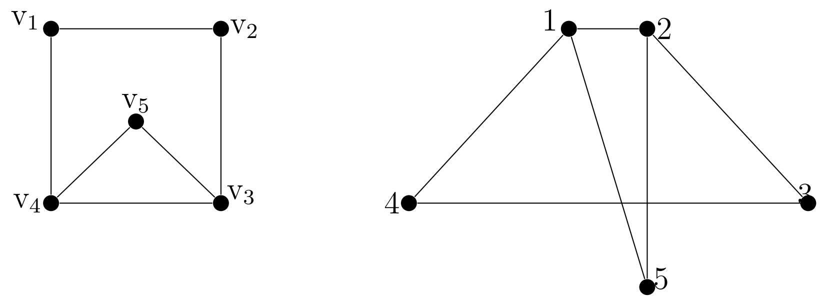


Figure 1.22: The graphs are isomorphic

Note 1.3.1 *If the two graph are isomorphic then they must have*

- *The same number of vertices.*
- *The same number of edges.*
- *The same degree for corresponding vertices.*
- *The same number of connected components .*
- *The same number of loops.*

Using the previous point of common characteristics between isomorphic graphs, it easy to prove the graphs are not isomorphic if any one fails, however we have seen that even if all these characteristics are satisfied but the graphs are not isomorphic. In figure 1.23 we can prove the two graphs satisfies the common characteristics but they are not isomorphic.

The graphs do not have the same number of adjacent vertices so we can't find a bijection that satisfies the condition that every two vertices u and v in G_1 have the number of edges as that of from f(u) to f(v).

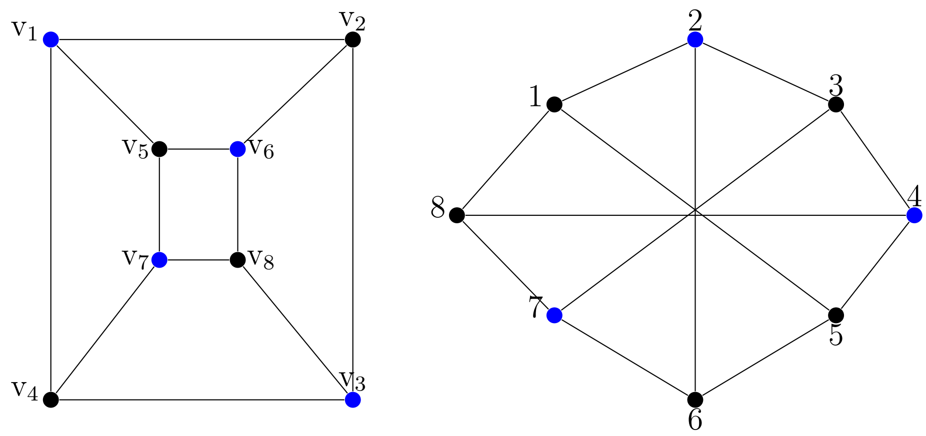


Figure 1.23: The graphs are not isomorphic

Theorem 1.3.1 *If G_1 is not simple graph and G_2 is a simple graph then G_1 and G_2 are not isomorphic.*

Proof:

To the contrary, suppose that there exists a bijection f:$V_1 \rightarrow V_2$ that gives G_1 and G_2 are isomorphic. We have two cases :

First suppose G_1 has a loop at any vertex say v then G_2 must have a loop at f(v). But G_2 is simple graph, a contradiction.

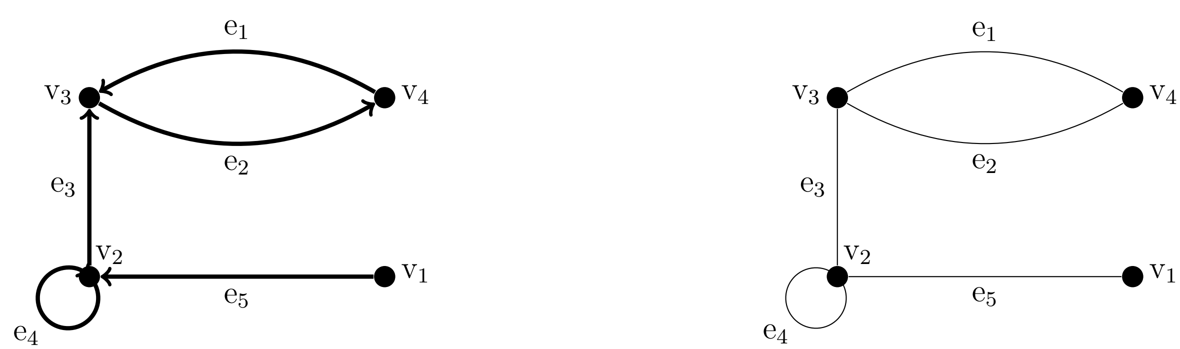
Suppose G_1 has a multiple edge between any two vertices say v and u, then there must exists a multiple edge between f(v) and f(u) in G_2, but G_2 is simple, so it does not have a multiple edge, a contradiction. ■

1.4 Directed graphs

In the last three sections we display several basic results in the theory of undirected graphs. Undirected graphs are not sufficient for representing several situation. For example, the streets map of a city, an electrical network is another example of a physical system whose representation requires a directed graph and there is a lot of other applications. In this section we define a graph with notation of directed edge (arc) and display some definitions and results in the theory of directed graphs.

Definition 1.4.1 *A graph G=(V,E) with a set vertex V=*$\{v_1, v_2, ..., v_n\}$ *and a set of edge (arcs) E=*$\{e_1, e_2, ..., e_m\}$ *is called directed graph or (digraph) if each edge* e_k *is an ordered pair* (v_i,v_j).

Figure 1.24a contains an example of a directed graph. Notice that the arc e_1=(v_4,v_3) have v_4 is an initial vertex and v_3 as a terminal vertex, we can't say e_1 and e_2 parallel because they have a different direction. The vertices u and v in G is called *adjecent vertices* if there exist a path from u to v, so v_2 and v_3 are adjacent vertices, also e_5 and e_3 are adjacent edges.



(a) A directed graph (b) underlying graph

Figure 1.24: Digraph G and its underlying graph

Definition 1.4.2 *let G=(V,E) be a directed graph if replacing every arc in G by an undirected edge then G is said to be underlying graph .*

Figure 1.24 represents a graph G and it's underlying graph.

Definition 1.4.3 *Let G=(V,E) be a graph, if there is a digraph H such that G is underlying graph of H then H is called an orientation of the graph G.*

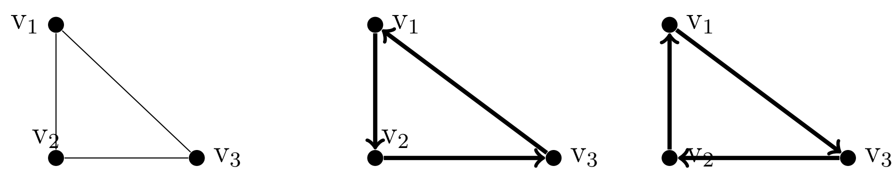


Figure 1.25: The graph G with two different orientations

Definition 1.4.4 *An in-degree of a vertex v in a graph G is the number of arcs whose terminal vertex is v and is denoted by* $deg^-(v)$ *or* $d^-(v)$*, also an out-degree of v is the number of arcs whose initial vertex is v, denoted by* $deg^+(v)$ *or* $d^+(v)$*.*

Definition 1.4.5 *Let G=(V,E) be a graph with a set vertex* $V=\{v_1, v_2, ..., v_n\}$ *and a set of arcs* $E=\{e_1, e_2, ..., e_m\}$*, the degree of a vertex* $v \in V$ *is the sum of out-degree and in-degree of a vertex v.*

For example, the out-degree and in-degree of the four vertices of the digraph shown in figure 1.24a as follows

$$d^-(v_1) = 0 \qquad d^-(v_2) = 2 \qquad d^-(v_3) = 2 \qquad d^-(v_4) = 1$$

$$d^+(v_1) = 1 \qquad d^+(v_2) = 2 \qquad d^+(v_3) = 1 \qquad d^+(v_4) = 1$$

Theorem 1.4.1 *The sum of in-degree of all vertices in a directed graph is equal to the sum of the out-degree of all vertices.*

Definition 1.4.6 *Let v be a vertex in a digraph G then the in-neighborhood of v denoted by* $N_G^-(v)$*, is* $N_G^-(v) = \{u|uv \in E(G)\}$*, and the out-neighborhood is denoted by* $N_G^+(v)$*, is* $N_G^+(v) = \{u|vu \in E(G)\}$ *.*

A digraph H is a subdigraph of a digraph G if V(H)$\subset$ V(G) and E(H)$\subset$ E(G). Figure 1.26 is an example of a subdigraph of graph G in figure 1.24a

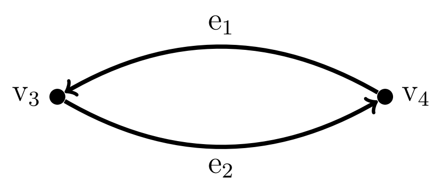


Figure 1.26: A subdigrph of graph G

The walk, path, trail and circuit involving in a directed graph are defined exactly the same way as in the case of undirected graph .
For example, in the digraph shown in figure 1.24a

- The walk $v_1e_5v_2e_3v_3$ is an open walk .
- The walk $v_4e_1v_3e_2v_4$ is a closed walk also it is a trail with length 2 and it is a circuit.

A digraph is connected if the underlying graph is connected, also the component of a digraph G is a subdigraph of G that corresponds to the component of the underlying graph.

Definition 1.4.7 *Let G=(V,E) be a digraph, G is strongly connected if between every two vertices say u and v, there exist a directed u-v path and also a directed v-u path in G.*

Definition 1.4.8 *let G=(V,E) be a digraph, G is said to be weakly connected if the underlying graph is connected but not strongly connected.*

Definition 1.4.9 *A digraph G=(V,E) is quasi-strongly connected if there is at least one vertex say v in G such that there exist a path from v to all the remaining vertices in G.*

Every strongly connected digraphs are is connected, but connected digraphs are not necessarily strongly connected, for example, in figure 1.27a a connected graph G is not strongly connected since there is no path from v_4 to v_2, from $v_4$to v_1 and from v_4 to v_3. Every quasi-strongly connected graph is connected, but a quasi-strongly connected graph is not necessarily strongly connected, for example in figure 1.27b the quasi-strongly connected G is not strongly connected because there is no path from v_3 to v_1.

(a) connected graph (b) quasi-strongly connected

Figure 1.27

Union, intersection, ring sum and other operations involving directed graphs are defined exactly by the same way as in the case of undirected graphs.

1.5 Tree, fundamental circuit and fundamental cut set

Among connected graphs, tree has a simple structure and is the most used in applications. In this section we study tree, cut set and several results associated with them.

1.5.1 Tree

A graph G=(V,E) is said to be *acyclic* if it has no circuit. A *tree* is a connected acyclic graph (forest). A connected subgraph of a tree is a subtree. Consider, for example, all graphs that are shown in figure 1.28, they are all trees.

A *spanning tree* T of a connected graph G is a subtree, that includes all the vertices of

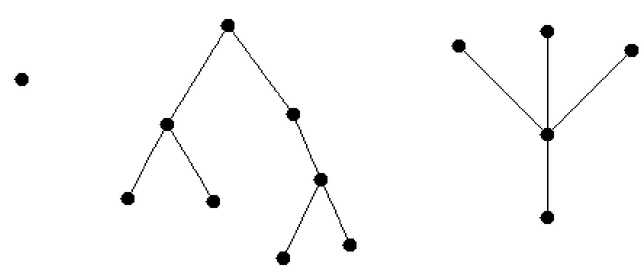


Figure 1.28

that graph and a set of edges that makes it tree. For example the graph H in figure 1.29 is a spanning tree of graph G.

The *cospanning tree* T* of a connected graph G is the subgraph of G having all vertices of

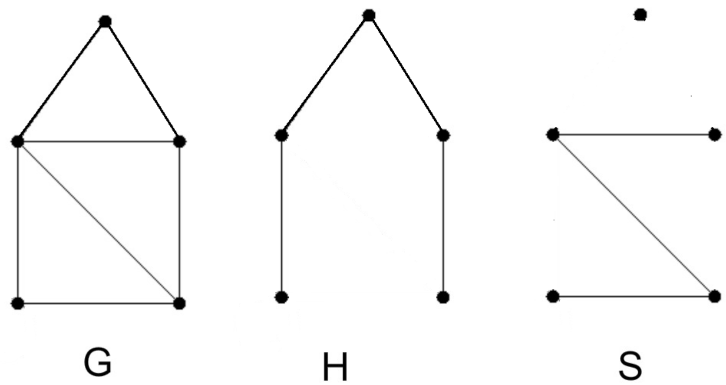


Figure 1.29

G and exactly those edges of G which are not in T. Note that a cospanning tree may not be connected. For example, The graph S in figure 1.29 is a cospanning tree of the graph G .
The edges of a spanning tree T is called branch, and the other edge of corresponding cospanning tree T* is called chord or links.
The following are two characterizations of a tree.

Theorem 1.5.1 *[27] A connected graph is tree if and only if every edge is a bridge.*

Proof:

Let G be a tree graph and let e=(u,w) is an edge in G, then the path u e w is the only path between u and w. So deleting the edge e, the vertices u and w will not be connected, thus in a tree every edge is a bridge.
Conversely, suppose G is a graph in which every edge in it is bridge, so G is connected. Suppose G is not a tree, then there is at least one cycle C in G, let e in this cycle, so e can't be a bridge in G, so there is another path between u and w. This contradicts the hypothesis that every edge is a bridge. Thus G is tree. ■

Theorem 1.5.2 *[2] A connected graph with n vertices is a tree if and only if it has n-1 edges.*

Proof:

Suppose G is a tree with n vertices, we will prove that G has n-1 edges by induction on n. For n=1, the statement is true. Suppose it is true for all m, where $1 < m < n$.

If we delete an edge e=(u,w) from G, so G becomes a disconnected graph with two components $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$, such that there is no common vertex between G_1 and G_2.Thus the number of vertices in G_1 and G_2 is n_1 and n_2 respectively, so both G_1 and G_2 have the number of vertices less than n, the number of edges in both G_1 and G_2 together is $n_1+n_2-2=n-2$ edges. Now if we connect the two graphs G_1 and G_2 by edge e, we get the graph G. So G has (n-2)+1=n-1 edges.

Conversely, let G be a connected graph with n vertices and n -1 edges. Suppose G is not tree, so there is an edge e in G which is not a bridge. If we delete this edge we have a connected subgraph $G_1=(V_1,E_1)$, Continue in this maner until we get a connected subgraph $G_2=(V_2,E_2)$ in which every edge is a bridge. So it's a tree with n vertices and it has n-1 edges, leading to a contradiction.

■

A digraph G=(V,E) is said to have root r if r$\in$ V and there is a directed path that starts in r and ends in each v $\in$V. A digraph is called directed tree if it has a root and it's underlying graph is a tree.

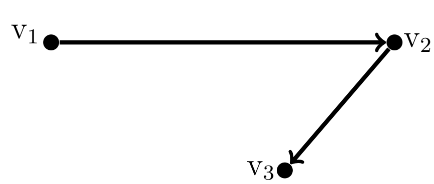


Figure 1.30

For example, the graph in figure 1.30 has a root $r=v_1$.

It is clear that the graph has root r is a quasi strongly connected.

1.5.2 Fundamental circuit and fundamental cut

Let G be a connected graph with n vertices and m edges, and let T be a spanning tree with n-1 branchs denoted by $e_1, \ldots, e_{n-1}$ and let the chords of the corresponding tree T^* denoted by $e_1, \ldots, e_{m-n+1}$. If we add an edge e_i of T^* to T, then T becomes a cyclic that contains exactly one circuit say c_i, where the circuit c_i is called the *fundamental circuit* of the graph G with respect to the chords of the spanning tree T. A fundamental set of circuits of the graph G is the set of all the m-n+1 fundamental circuits $c_1, \ldots, c_{m-n+1}$ of G with respect to the spanning tree T.

For example in figure 1.31 the chords of G is $\{e_3, e_5, e_7\}$, so we have 3 fundamental circuit c_1, c_2, c_3.

We find these fundamental cut sets in figure 1.32 that $\{e_1, e_3, e_6, e_7\}$, $\{e_2, e_3, e_6, e_7\}$, $\{e_4, e_7\}$, $\{e_5, e_6 e_7\}$.

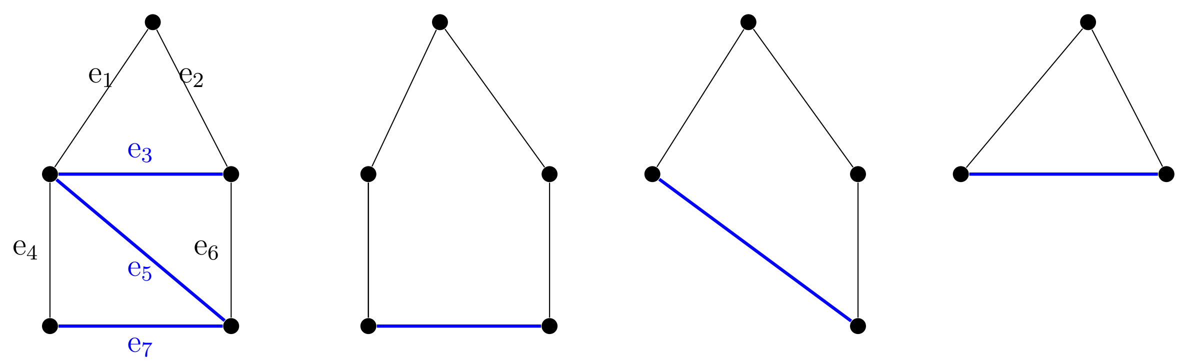


Figure 1.31

In section 1.2.2 we studied the definitions of cuts in graph, now we define the fundamental cut set. Let us consider a graph G and it's spanning tree T in the figure 1.32 . If we remove the branch e_1 from the spanning tree T, it divides the graph into two disjoint sets of vertices $\{v_1, v_3, v_4\}$ and $\{v_2, v_5\}$. Cut set Q of a graph G will contain only one branch e_1 from T, and the remaining edges of Q are chords with respect to T. Such a cut set containing exactly one branch of a spanning tree T is called *fundamental cut set* with respect to T.

We find these fundamental cut sets in figure 1.32, that is $\{e_1, e_3, e_6, e_7\}$, $\{e_2, e_3, e_6, e_7\}$, $\{e_4, e_7\}$ and $\{e_5, e_6, e_7\}$

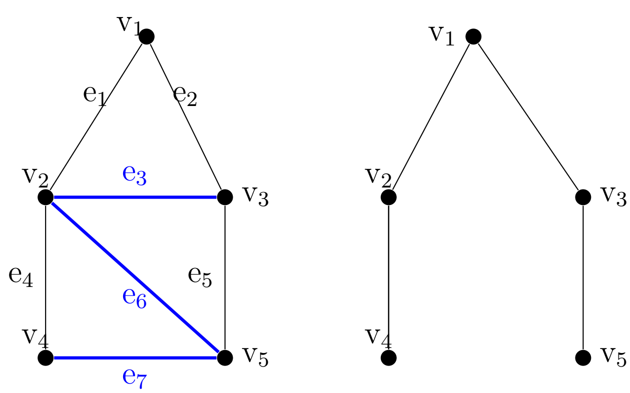


Figure 1.32

Chapter 2

Incidence matrix and other related matrices

In the last chapter we have studied graphs by graphical representation which is comfortable for visual study but this is only possible when the number of edges and vertices small, a matrix representation of a graph to computer is a very useful way to deal with a finite number of graphs. Matrices and graphs have many important applications in a electrical networks analysis and operation research, as well as in other fields. Representing the graph in matrix form lies on the fact that many results of matrix algebra can be applied to study structural proprieties of graph from an algebraic point of view. The graph can have representation in many ways such as incidence matrix, adjacency matrix, etc. In this chapter we will talk about the special type which is the incidence matrix and display other related matrices.

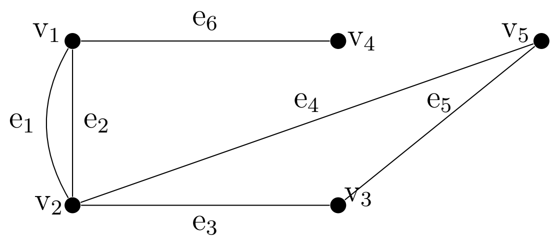
2.1 Incidence matrix of directed and undirected graphs

let G=(V,E) be a graph with set of vertices V=$\{v_1, v_2, ..., v_n\}$ and set of edges E=$\{e_1, e_2, ..., e_m\}$ and no self loop. The incidence matrix is an n$\times$ m matrix A=(a$_{ij}$) defined by

$$
a_{ij}=\begin{cases} 1 & \text{if } v_i \text{ is an end vertex of } e_j \\ 0 & \text{other wise} \end{cases}
$$

For example, the graph G in the figure 2.1a with it's incidence matrix A(G) are as follows

.



(a)

$$
A = \begin{array}{cc} & \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} v_1 \rightarrow \\ v_2 \rightarrow \\ v_3 \rightarrow \\ v_4 \rightarrow \\ v_5 \rightarrow \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}
$$

(b)

It is clear that any entry of A is 0 or 1 and their incidence matrix is also known as binary matrix .

We have the following observations about the incidence matrix:

- Each column of A(G) contains exactly two unit entry .
- Each vertex v_i of V(G) has degree equals the number of unit entries corresponding to v_i .
- A row with single unit entry corresponding to a pendent vertex, also the row with all entries is zero corresponding to an isolated vertex.
- Parallel edge in a graph produce identical columns in it's incidence matrix.
- If a graph G is disconnected and consists of n components, then the incidence matrix A(G) of G can be written in a block diagonal $A(G_i)$ where the $G_{i's}$ are the components

of G, for instance if G has two components G_1 and G_2 then the incidence matrix of G is

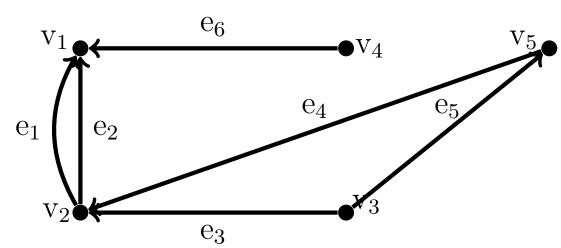
$$A(G)=\begin{bmatrix} A(G_1) & 0 \\ 0 & A(G_2) \end{bmatrix}$$

Where $A(G_1)$ and $A(G_2)$ are the incidence matrices of G_1 and G_2 respectively.

Now we will define an incidence matrix $A=[a_{ij}]$ of a digraph G=(V,E) as follows

$$a_{ij}=\begin{cases} 1 & \text{if } v_i \text{ is the initial vertex of } e_j \\ -1 & \text{if } v_i \text{ is the terminal vertex of } e_j \\ 0 & \text{other wise} \end{cases}$$

A digraph and it's incidence matrix are shown in figure 2.2



(a)

$$A=\begin{bmatrix} -1 & -1 & 0 & 0 & 0 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

Figure 2.2: digraph with it's incidence matrix

As the entries in A(G) are 0,1 or -1 and each column containes 1,-1 exactly once, then the sum of each column is zero. Other proprieties are similar to those for undirected graph .

Definition 2.1.1 *[13] The reduced incidence matrix A_f for a graph G is a submatrix of A(G) obtained by removing one row corresponding to a chosen vertex that is called reference vertex.*

2.2 Rank and determinant

In this section we introduce the rank of incidence matrix of directed and undirected graphs. First we consider the rank and determinant of incidence submatrix of directed graph .

Theorem 2.2.1 *[7] If G is connected digraph with n vertices, then rank A(G)=n-1 .*

Proof:

Let G be a connected graph with n vertices and m edges. let A(G) be an incidence matrix and let x be a vector of left null space of A, that is, $x^T A=0$, then if v_i and v_j are adjacent x_i-x_j =0. Since G is a connected graph, all entries of x must be equal. Thus, the left null space of A is at most one dimensional and the rank(A)≥n-1, also the rows of A are linearly dependent, thus rank(A)≤n-1. Hence rank(A)=n-1. . ■

If G is a disconnected graph the next result displays the rank of G.

Corollary 2.2.1 *[26] If an n vertex graph has k components, then the rank of its all vertex incidence matrix is equal n-k.*

Proof:

Let G be non connected graph with $G_1, \dots G_k$ connected components, so by the proprieties of the incidence matrix we can write A(G) as

$$A(G)=\begin{bmatrix} A(G_1) & 0 & \cdots & 0 \\ 0 & A(G_2) & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & A(G_k) \end{bmatrix}$$

Where G_i is an incidence matrix for i=1, . . ., k, so by theorem 2.2.1, rank of G_i=n_i-1 then

$$\begin{aligned} rankA(G) &= rankA(G_1) + ... + rankA(G_k) \\ &= n_1 - 1 + ... + n_k - 1 \\ &= n - k \end{aligned}$$

. ■

Remark 2.2.1 *This confirms the well known result in linear algebra rank A(G)+ nullity A(G) =m .*

Theorem 2.2.2 *[20] Let G be a directed tree on n vertices then any submatrix of A(G) of order n-1 is nonsingular.*

Proof:

Let B be a submatrix of an incidence matrix A(G) created by the rows 1, 2, $\cdots$, n-1, if we add all rows of B to the last row of it, then this row is equal to the negative of the last row of A(G). Let M be another submatrix of A(G) created by the rows 1, 2, $\cdots$, n-2, n, so det(B)= -det(M), by continuing in this manner if det(B)=0, then det(M)=0, we can show that any square submatrix of order n-1 is singular. In fact, if we can show that any square submatrix of order n-1 is singular, then all of them also are singular, by theorem 2.2.1 the rank of A(G)=n-1, so at least one of these submatrices must be nonsingular. ■

Now we consider the incidence matrix of an undirected graph to find it's rank.

Theorem 2.2.3 *[20] Let G be a connected undirected graph with n vertices and let A be the incidence matrix of G.Then the rank of A is n-1 if G is bipartite and n otherwise.*

Proof:

Let G be a bipartite graph, so G has two disjoint sets of vertices say X and Y. Orient each edge of G from X to Y and let $\bar{A}$ be the incidence matrix of the directed graph G. Consider the columns $j_1, \cdots, j_{n-1}$ corresponding to the spanning tree of G and let B be a submatrix of these columns, then by theorem 2.2.2 any n-1 rows of B are linearly independent and the corresponding rows of incidence matrix of undirected graph G are also linearly independent then rank(A)$\geq$n-1. Let z be a vector defined as follow

$$z_i = \begin{cases} 1 & , \quad i \in X \\ -1 & , \quad i \in Y \end{cases}$$

So z^TA=0, and the rows of A are linearly dependent, thus rank A = n-1.

To show that the rank of A is n for a graph G, suppose x in R^n be a vectors such that x^TA=0, if the vertex v_i and vertex $v_j$are adjacent, then x_i+x_j=0. Since G is connected it follows that $| x_i |= \alpha$, i=1, 2, ..., n for some constant α, suppose G has an odd cycle let it

$v_1, \ldots, v_k$, then if we go around the cycle and use the preceding observation we get $x_i+x_j=0$ if v_i and v_j are adjacent, so we find $\alpha=-\alpha$, hence $\alpha=0$. Thus if G has an odd cycle then the rank of A is n. ■

The next lemma gives a condition when the m columns of G are linearly independent.

Lemma 2.2.1 *[20] Let G be a directed graph on n vertices. Columns $j_1,j_2, \ldots ,j_k$ of A(G) are linearly independent if and only if the corresponding edge of G induce an acyclic graph.*

Proof:

let G be a graph with n vertices and m edges. Consider $j_1,j_2, \ldots ,j_k$ and suppose there is a cycle in the corresponding induced subgraph, without loss of generality. Suppose the columns $j_1,j_2, \ldots ,j_p$ form a cycle. After relabeling the vertices if necessary, so we see that the submatrix of the A(G) formed by $j_1,j_2, \ldots ,j_p$ columns, say M with $n\times p$ is of the form $\begin{bmatrix} B \\ 0 \end{bmatrix}$, where B is a matrix of order p incidence matrix of the cycle formed by the edges $j_1,j_2, \ldots ,j_p$ with column sum zero. Thus B is singular and column $j_1,j_2, \ldots ,j_p$ are linearly dependent. Conversely, suppose the edge $j_1,j_2, \ldots ,j_k$ induce an acyclic graph it's tree, if it has K components then by theorem 2.2.1 the rank of correspond submatrix formed by the columns $j_1,j_2, \ldots ,j_k$ is n-K, therefor the columns $j_1,j_2, \ldots ,j_k$ are linearly independent. ■

Definition 2.2.1 *A square matrix whose determinant is 0, 1,or -1 is called unimodular. A matrix A(G) is totally unimodular if the determinant of every submatrix of A(G) has value 0, 1, or -1.*

Theorem 2.2.4 *[11] Let A(G) be the incidence matrix of a digraph G. Then A(G) is totally unimodular.*

Proof:

let A(G) be an incidence matrix of a digraph G. We must show that any $k\times k$ submatrix of

A(G) has determinant 0, 1 or -1. We'll use induction. For k=1, each entry in A(G) is either 0, 1 or -1. Assume the statements true for k-1. Consider a k$\times$ k submatrix B of A(G), we have three cases :

- Case 1: B has a zero column, so determinant of B is zero.
- Case 2: If each column of B have a 1 and -1, then the determinant of B is zero.
- Case 3: If B has a column with only one non zero entry, which must be +1 or -1. Calculating the determinant of B using this column and using the induction assumption, the determinant must be 0, 1 or -1.

■

Corollary 2.2.2 *If B is a non singular square submatrix of A(G) for a diagraph G, then the determinant of B is ±1.*

We can use the matrix A_f to calculate the number of spanning trees in a diagraph G.

Theorem 2.2.5 *(Binet-Cauchy formula). Let A and B be m×n and n×m matrices respectively. If m≤n and C=AB, then*

$$detC = \sum_{1 \leq j_1 \dots j_m \leq n} A \begin{pmatrix} 1 & \dots & m \\ j_1 & \dots & j_m \end{pmatrix} B \begin{pmatrix} j_1 & \dots & j_m \\ 1 & \dots & m \end{pmatrix}$$

That is, the determinant of the product AB is equal to the sum of the products of all possible minors of order m of A with corresponding minors of B of the same order

Lemma 2.2.2 *[18] If B is a submatrix of order n-1 of A, then B is nonsingular if and only if the edges corresponding to the columns of B determine a spanning subtree of a diagraph G.*

Proof:

If H denotes the spanning subtree of G with n-1 edges corresponds to the columns of B, so B is reduced incidence matrix of order n-1, by theorem 2.2.2 then B is non singular.

Let B is a non singular square submatrix of order n-1 of an incidence matrix. So it has a rank n-1, hence H is connected, so H is tree. . ■

Theorem 2.2.6 *[12](Matrix Tree theorem). Let G be a directed graph, and let $A_f(G)$ be it's reduced incidence matrix. The number of spanning trees is det($A_f(G)$ $A_f^T(G)$)*

proof:

Let G be a graph with A_f be the reduced incidence matrix of G, B and C be submatrices of A_f, the determinant of A_f A_f^T can be written by Binet Cauchy formula as

$$det(A_f.A_f^T) = \sum_{1 \leq j_1 \ldots j_{n-1} \leq m} B \begin{pmatrix} 1 & \ldots & n-1 \\ j_1 & \ldots & j_{n-1} \end{pmatrix} C \begin{pmatrix} j_1 & \ldots & j_{n-1} \\ 1 & \ldots & n-1 \end{pmatrix}$$

$$= \sum det(B).det(B^T)$$

$$= \sum (detB)^2$$

$$= \sum_{\text{B nonsingular}} (detB(G))^2 + \sum_{\text{B singular}} (detB)^2$$

by lemma 2.2.2 and lemma 2.2.2

$$det(A_f.A_f^T) = \sum_{\text{nonsingular B}} 1$$

= number of non singular (n-1)$\times$ (n-1) submatrices B of A_f

= number of spanning trees of G

. ■

2.3 Moore-Penrose inverse

In this section, we study a relation between the incidence matrix and it's generalized inverse of connected diagraph.

Definition 2.3.1 *A generalized inverse of any matrix $A_{n\times m}$ denotes by A^+ , is any matrix of order $m\times n$ that satisfies the condition $AA^+A=A$.*

Clearly, the matrix A^+ is not unique, to make the generalized inverse unique, additional conditions must hold

$AA^+A=A$

$A^+AA^+=A^+$

$(AA^+)^T=AA^+$

$(A^+A)^T=A^+A$

This leads to a unique Moore-Penrose inverse G.[25]

Theorem 2.3.1 *[20] If G is connected digraph with n vertices, then I-$AA^+=\frac{1}{n}J$. Where J is a matrix in which all entries are ones*

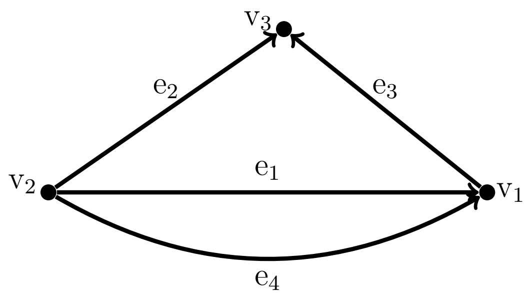
Proof:

Let G be a connected digraph and A be the incidence matrix of G. Let A^+ be a Moore-Penrose inverse of G so $AA^+A=A$, then $(I-AA^+)A=0$, any row of $I-AA^+$ is in the left null space of A that is spanned by the vector 1^T, any row in $I-AA^+$ is a multiple of any other row. Since $I-AA^+$ is symmetric, it follows that all elements of $I-AA^+$ are nonzero constants , since A cannot have right inverse. Now using the fact that $I-AA^+$ is idempotent, so $I-AA^+$ must equal $\frac{1}{n}J$. ■

Remark 2.3.1 *If A(G) is an $n\times m$ incidence matrix with rank r, there exist an $n\times r$ matrix say R and an $r\times m$ matrix say S, both have the same rank r such that A=RS (this is called* *a full rank factorization of A) then*

$$A^{+}{=}S^{+}\ R^{+}{=}\ S^{T}\left(SS^{T}\right)^{-1}\left(R^{T}R\right)^{-1}R^{T}$$

For example, in figure 2.3 the graph G with it's incidence matrix



$$\mathrm{A}{=}\begin{bmatrix} -1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

(a)

Figure 2.3: digraph with incidence matrix

A digraph G has a rank=2, there exist a 3×2 matrix say R and an 2×4 matrix say S, both have the same rank r. Indeed $\mathrm{R}{=}\begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$ and $\mathrm{S}{=}\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

Notice that

$$\mathrm{R}^{+}{=}\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)^{-1}\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}{=}\tfrac{1}{3}\begin{bmatrix} -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

and

$$\mathrm{S}^{+}{=}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\right)^{-1} = \tfrac{1}{5}\begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$$

Finally, $A^{+}=S^{+}\ R^{+}=\frac{1}{15}\begin{bmatrix} -3 & 3 & 0 \\ 1 & 4 & 5 \\ 4 & 1 & -5 \\ -3 & 3 & 0 \end{bmatrix}$

and $I\text{-}AA^{+}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}-\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

2.4 Circuit and cut matrices

In this section we define another type of matrices which is important in many application such as electrical network application [6] [13].

Definition 2.4.1 *let G=(V,E) be a loopless graph with set of vertices V=$\{v_1, v_2, ..., v_n\}$ and set of edge E=$\{e_1, e_2, ..., e_m\}$. We enumerate the circuits c_1, . . . c_l. Then the circuit matrix of undirected graph is defined as*

$$c_{ij}=\begin{cases} 1 & \textit{if } e_i \textit{ is in the } c_j \\ 0 & \textit{other wise} \end{cases}$$

If G is a directed graph

$$c_{ij}=\begin{cases} 1 & \textit{if } c_i \textit{ consists of } e_j \textit{ and they are in the same direction} \\ -1 & \textit{if } c_i \textit{ consists of } e_j \textit{ and they are in the opposite direction} \\ 0 & \textit{other wise} \end{cases}$$

For example, the graph G that is shown in figure 2.3a have three circuits they are $\{e_1, e_3, e_2\}$, $\{e_4, e_3, e_2\}$ and $\{e_4, e_1\}$, then their circuit matrix C(G) is as follows

$$
C = \begin{array}{cc} & \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} c_1 \rightarrow \\ c_2 \rightarrow \\ c_3 \rightarrow \end{array} & \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \end{array}
$$

A *Fundamental circuit matrix* C_f of a graph G with n vertices and m edges is an (m-n+1)×m submatrix of a circuit matrix which all rows corresponds to a set of fundamental circuit, similarly if G is a connected digraph, a fundamental circuit matrix of G is defined as the same of undirected graph but the direction of a fundamental circuit is the same as the direction of the corresponding link in T^*.

The fundamental circuit matrix C_f in a connected graph G with n vertices and m edges can be arranging the rows and columns, arranging the columns such that all the (m-n+1) links correspond to the last column. An arranged matrix C_f has the form

$$C_f = \begin{bmatrix} C^* & I_{m-n+1} \end{bmatrix}$$

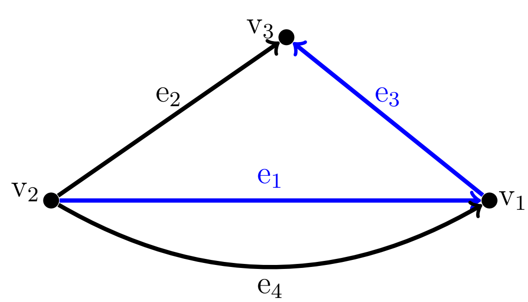


Figure 2.4

For example, the graph G in the figure 2.4 have two fundamental circuits, namely $\{e_2, e_3, e_1\}$ and $\{e_4, e_1\}$ and their fundamental circuit matrix $C_f(G)$ is as follows

$$C_f = \begin{bmatrix} -1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Now we display another type of matrices, called a *cut set matrix* and is defined as follows

$$q_{ij}=\begin{cases} 1 & \text{if } e_i \text{ is in the } I_j \text{ (the cut is interpreted an edge set)} \\ 0 & \text{other wise} \end{cases}$$

If G is a directed graph, then the cut matrix defined as follows

$$q_{ij}=\begin{cases} 1 & \text{if } c_i \text{ consists of } I_j \text{ and they are in the same direction} \\ -1 & \text{if } c_i \text{ consists of } I_j \text{ and they are in the opposite direction} \\ 0 & \text{other wise} \end{cases}$$

For example,

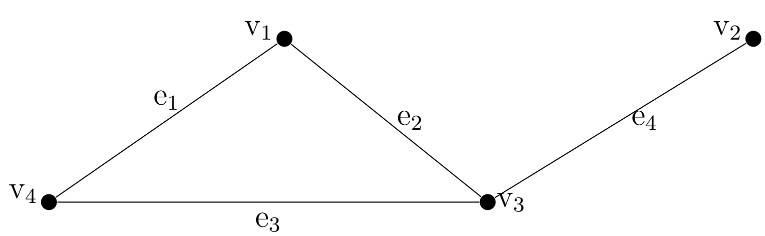


Figure 2.5

The cuts of the undirected graph in figure 2.5 are $I_1=\{e_4\}$, $I_2=\{e_1,e_2\}$, $I_3=\{e_1,e_3\}$, $I_4=\{e_2,e_3\}$, $I_5=\{e_4,e_2,e_3\}$, $I_6=\{e_1,e_2,e_3\}$ and $I_7=\{e_4,e_2,e_1\}$. The cut matrix is

$$Q = \begin{array}{cc} & \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} l_1 \rightarrow \\ l_2 \rightarrow \\ l_3 \rightarrow \\ l_4 \rightarrow \\ l_5 \rightarrow \\ l_6 \rightarrow \\ l_7 \rightarrow \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

For example,

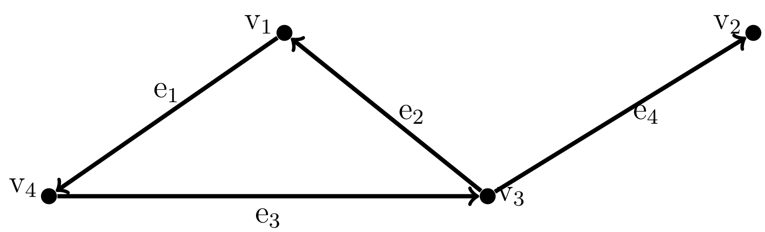


Figure 2.6

The cuts of the digraph in figure 2.6 are I_1=$\{e_4\}$, I_2=$\{e_1, e_2\}$, I_3=$\{e_1, e_3\}$, I_4=$\{e_2, e_3\}$, I_5=$\{e_4, e_2, e_3\}$, I_6=$\{e_1, e_2, e_3\}$ and I_7=$\{e_4, e_2, e_1\}$. The direction of a cut set is the same direction of the first edge in each set, so the cut matrix is

$$Q=\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ 1 & -1 & -1 & 0 \\ -1 & -1 & 0 & 1 \end{bmatrix}$$

A *Fundamental cut matrix* Q_f of a graph G with n vertices and m edges is a submatrix of a cut matrix in which all rows correspond to a set of a fundamental cuts, similarly if G is a connected digraph, a fundamental cut matrix defined as the same as undirected graphs but the direction of a fundamental cut is the same as the direction of the corresponding branch of spanning tree T. If we rearrange the edges of G so that we have the branches first, then the fundamental cut set becomes of the form $Q_f = \begin{bmatrix} I_{n-1} & Q^* \end{bmatrix}$.

Chapter 3

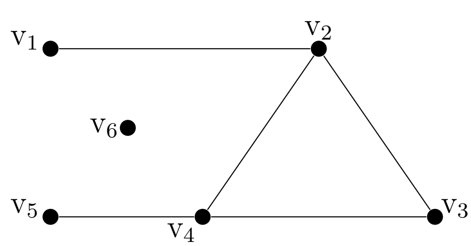
Adjacency matrix and other related matrix representations of graphs

3.1 Adjacency matrix of graphs

let G=(V,E) be a graph with set of vertices V=$\{v_1, v_2, ..., v_n\}$ and set of edges (arcs) E=$\{e_1, e_2, ..., e_m\}$. Then the adjacency matrix is an n× n matrix D=(d_{ij}) where

$$d_{ij}=\begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{other wise} \end{cases}$$

For example, the graph G in figure 3.1a with it's adjacency matrix D(G) are given as follows



(a)

$$D = \begin{array}{cc} & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \begin{array}{c} v_1 \rightarrow \\ v_2 \rightarrow \\ v_3 \rightarrow \\ v_4 \rightarrow \\ v_5 \rightarrow \\ v_6 \rightarrow \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Figure 3.1: A graph with adjacency matrix

The adjacency matrix is not unique, because we can relabel the vertices of the graph which would cause simultaneous permutation of the rows and columns. So for example we could have gotten the following as adjacency matrix of the graph in the figure 3.1a

$$D=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

We have the following observations about the adjacency matrix:

- If a graph has no self loops then the diagonal entries of D(G) are zero .
- A row with single entries corresponding to a pendent vertex, and the row with all entries is zero corresponds to an isolated vertex.
- An adjacency matrix D(G) of a graph G is symmetric if G is an undirected graph .
- Two graph G_1 and G_2 are isomorphic if the adjacency matrix of any one graph can be obtained form the other one by changing some rows or columns .
- If a graph is disconnected and consists of n components the adjacency matrix D(G) of a graph G can be written in a block diagonal $D(G_i)$, so if G has two components G_1 and G_2 the adjacency matrix of G is

  $$D(G)=\begin{bmatrix} D(G_1) & 0 \\ 0 & D(G_2) \end{bmatrix}$$

  Where $D(G_1)$ and $D(G_2)$ are adjacency matrices .

The following result computes the number of walks of length k between any two vertices .

Theorem 3.1.1 *[17] Let G be a graph with vertices v_1,v_2, ..., v_n and adjacency matrix D(G)=$[d_{ij}]$, then the entry d^k_{ij} in row i and column j of D^k(G) is the number of distinct v_i-v_j walks of length k in G .*

Proof:

We will prove the theorem by induction on k

For k=1, we have the matrix D=D^1 and any entry of D can be denoted by

$$d_{ij}=\begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

so there is a v_i-v_j walk of length one in G if v_i and v_j are adjacent. For the induction hypothesis, assume that for a positive integer k, the number of v_i-v_j walks of length k in G is d^k_{ij} . We show that the entry d^{k+1}_{ij} in D^{k+1} gives the number of v_i-v_j walks of length k+1. The entry d^{k+1}_{ij} of a matrix D^{k+1} is represented by

$$D^{k+1}=\begin{bmatrix} d^{k+1}_{11} & d^{k+1}_{12} & . & . & . & d^{k+1}_{1n} \\ d^{k+1}_{21} & d^{k+1}_{22} & . & . & . & d^{k+1}_{2n} \\ . & & & & & \\ . & & & d^{k+1}_{ij} & & \\ . & & & & & \\ d^{k+1}_{n1} & d^{k+1}_{n2} & . & . & . & d^{k+1}_{n3} \end{bmatrix} = \begin{bmatrix} d^{k}_{11} & d^{k}_{12} & . & . & . & d^{k}_{1n} \\ d^{k}_{21} & d^{k}_{22} & . & . & . & d^{k}_{2n} \\ . & & & & & \\ . & & & d^{k}_{im} & & \\ . & & & & & \\ d^{k}_{n1} & d^{k}_{n2} & . & . & . & d^{k}_{n3} \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & . & . & . & d_{1n} \\ d_{21} & d_{22} & . & . & . & d_{2n} \\ . & & & & & \\ . & & & d_{mj} & & \\ . & & & & & \\ d_{n1} & d_{n2} & . & . & . & d_{n3} \end{bmatrix}$$

Thus the entry d^{k+1}_{ij} can be obtained by taking the product of row i of D^k and column j of D.

$$d^{k+1}_{ij} = \sum_{m=1}^{n} d^k_{im} d_{mj} = d^k_{i1}d_{1j} + d^k_{i2}d_{2j} + ... + d^k_{in}d_{nj}$$

then every v_i-v_j walk of length k+1 consists of $v_i - v_m$ walks of length k, and v_m adjecent to v_j, then the total number of v_i-v_j walk of length k+1 in G is d^{k+1}_{ij}.

. ■

Corollary 3.1.1 *[20] Let G be a connected undirected graph with vertices set $\{v_1, v_2, ..., v_n\}$ and let D(G) be an adjacent matrix of G. If v_i and v_j are vertices of G with d(v_i , v_j)=m, then the matrices I,D^1, D^2, ... D^m are linearly independent .*

Proof:

Let v_i and v_j are distinct vertices of a graph G, and let D(G) be an adjacency matrix such that d(v_i , v_j)=m, then the shortest path between v_i and v_j is m so every element d_{ij} in I,D^1, D^2, ... D^{m-1} is zero by theorem 3.1.1, whereas the element d_{ij} of D^m is nonzero .
Hence I, D^1, D^2, ... D^m are linearly independent . ■

In the previous chapter we defined the isomorphism between two graphs by creating a bijection function between them, the next result display an important characterization of isomorphism.

Theorem 3.1.2 *[12] Let G_1=(V_1,E_1) and G_2=(V_2,E_2) with n=$|V_1|$=$|V_2|$, the homomorphism f:$G_1 \to G_2$ is an ismorphism if and only if there exist a matrix P such that D(G_2)=PD(G_1)P^{-1} Where P is a n× n permutation matrix which comes from the identity matrix I_n upon performing row permutations corresponding to f .*

Proof:

suppose G_1 is isomorphic to G_2, then the rows and columns of D(G) are permuted correspondingly. Thus D(G_2)=PD(G_1)P^{-1} , where P is the corresponding row permutation matrix, left multiply by P permutes the rows and right multiplication by P^{-1} permutes the columns .
conversely, let D(G_2)=PD(G_1)P^{-1}, then there exist a mapping f:$G_1 \to G_2$ with

$$e_i = (v_i, v_j) \in E_1 \,(\text{i.e } d_{ij} = 1) \Leftrightarrow d_{f(i)f(j)} = 1 \left(\text{i.e } e_i = \left(v_{f(i)}, v_{f(j)}\right) \in E_2\right)$$

■

Now, we compute the determinant of adjacency matrix.

Definition 3.1.1 *A liner subgraph of a graph G is a subgraph of G whose components are a single edge or cycles.*

Theorem 3.1.3 *[10] let D(G) be adjacency matrix of a simple undirected graph G then* $detD(G) = \sum_H (-1)^{e(H)} 2^{c(H)}$ *where the summation is over spanning linear subgraphs H of G, and e(H) and c(H) denote, respectively,the number of even components and the number of cycles in H.*

Proof:

let G be a graph with V(G)=$\{v_1, ..., v_n\}$ and let D(G)=d_{ij} be an adjacency matrix of G, the determinant of D(G) is defined as

$$\text{det D}=\sum_\pi sgn\,(\pi)\, d_{1\pi(1)}...d_{n\pi(n)}$$

Where π is a permutation on $\{1, 2, ...n\}$, sgn(π) is equal 1 or -1 according to whether π is even or odd permutation and the term $d_{1\pi(1)}...d_{n\pi(n)}$ is zero if and only if the permutation $d_{i\pi(i)}$=0 for some $1\leqslant i \leqslant n$, $d_{i\pi(i)}$=0 if $\pi\,(i)$=i or $\pi\,(i)$=j such that $v_iv_j \notin$ E(G). Other wise $d_{1\pi(1)}...d_{n\pi(n)}$ is nonzero if and only if the permutation π is a product of disjoint cycles of length at least 2. So the term sgn(π)$d_{1\pi(1)}...d_{n\pi(n)}$ = sgn(π) 1 . . .1=sgn(π). Each cycle of length 2 in π corresponds to a single edge, also the cycle of length r>2 in π corresponds to a cycle. Thus, each term in the expansion of detD gives rise to a linear subgraph H of G. For any cycle of any subgraph sgn(π) is 1 or -1 according to whether it is even or odd cycle. Hence sgn(π)=$(-1)^{e(H)}$ where e(H) is the number of even components of H. Moreover any cycle of H has two different orientations. Hence, each undirected cycle of H with length at least 3 yield two distinct even cycles.

■

Example 3.1.1 *Consider the graph G*

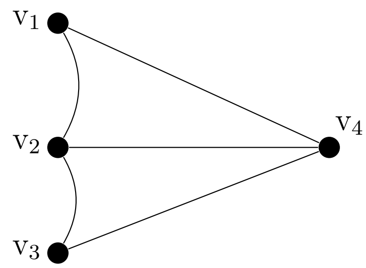
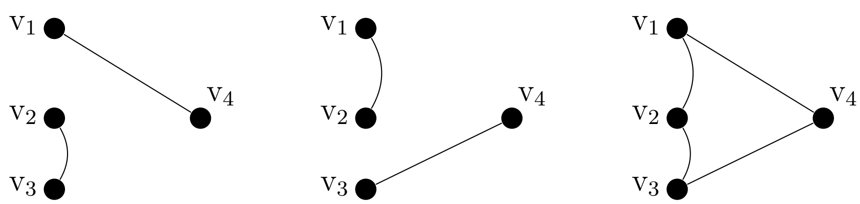


Figure 3.2

There are three spanning liner subgraphs of G, given by H_1, H_2, H_3 respectively



detA $=\sum_H (-1)^{e(H)} 2^{C(H)} = (-1)^2 2^0 + (-1)^2 2^0 + (-1)^1 2^1 = 0$

3.2 Laplacian matrix

The second type of matrix representation of graphs is the Laplacian matrix, denoted by L(G). If G is a graph with V(G)=$\{v_1, v_2, \cdots, v_n\}$ and E(G)=$\{e_1, e_2, \cdots, e_m\}$. The Laplacian matrix is defined as follows:

$$L(G)=\begin{cases} d_i & \text{if i=j} \\ \text{-1} & \text{if } i \text{ and } j \text{ } are \text{ adjacent} \\ 0 & \text{otherwise} \end{cases}$$

Where d_i is the degrees of the i^{th} vertices. This is closely related to adjacency matrix and some time written as L= $\tilde{D} - D$, where D is adjacency matrix, $\tilde{D}$ is the diagonal matrix with degrees of vertices on the diagonal.

For example, the laplacian matrix of the graph in figure 3.2 is

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Remark 3.2.1 *Let A be an incidence matrix of directed graph G, then L=AA^T.*

Some basic properties of the laplacian matrix are summarized below[3]:

1. L(G) is symmetric, positive semidefinite matrix.

2. The off-diagonal entries of L(G) are nonpositive (in fact, they are either 0 or -1).

3. The diagonal entries of L(G) are the vertex degree, also the rows sums and the columns sum are all zero.

4. The rank of L(G) is n-k, where k is the number of connected components of G. In particular if G is connected, then the rank of L(G) is n-1.

Proof:

let G be a graph with V(G)=$\{v_1, \cdots, v_n\}$ and let D(G) be it's adjacency matrix.

1. Recall L(G)= AA^T. Let x be any vector in R^n so we will prove L is a positive semidefinite matrix (i.e x^{T}Lx $\geq$ 0)

$$x^T Lx = x^T AA^T x = \|Ax\|_2 \geq 0$$

For all x.

2. Trivial.

3. Trivial.

4. Recall L(G)= $A(G)A(G)^T$

$$rankL(G) = rankA(G)A(G)^T = rankA(G)$$

So by using corollary 2.2.1, rank A(G)= n-k.

■

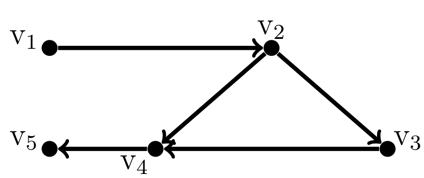
3.3 Antiadjecency Matrix

let G=(V,E) be a graph with set vertices V=$\{v_1, v_2, ..., v_n\}$ and set of edges E=$\{e_1, e_2, ..., e_m\}$. Then the antiadjacency matrix is an n× n matrix B=(b_{ij}) defined by

$$b_{ij}=\begin{cases} 0 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 1 & \text{other wise} \end{cases}$$

In other words we can define the antiadjecency matrix as B= J-D, where J is an n ×n matrix each entry is 1 and D is the adjacentcy matrix .

For example, the graph G in figure 3.3 with it's antiadjacency matrix B(G) is given as follows



(a)

$$B=\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Figure 3.3: A graph with it's antiadjacency matrix

-

Definition 3.3.1 *Hamiltonian path is a path between two vertices of a graph that visits each vertex once and only once.*

For example, the graph G in figure 3.4a has a Hamiltonian path v_2 v_3 v_1 v_4 , also the graph G in figure 3.4b has no Hamiltonian path.

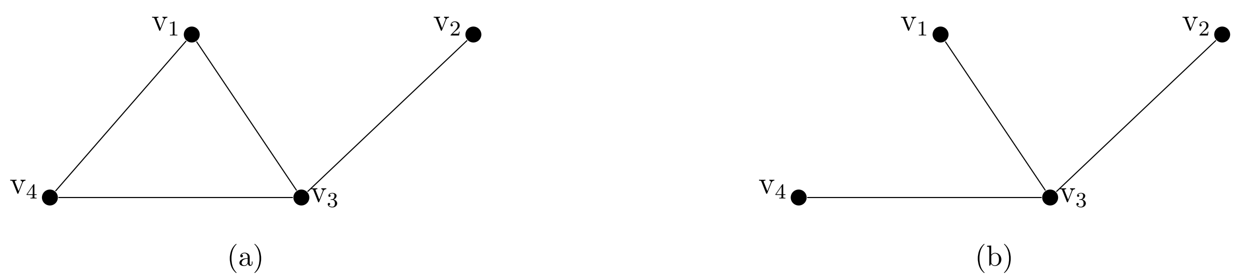


Figure 3.4

The determinant of antiadjecency matrix can be obtained by the next result.

Lemma 3.3.1 *[20] Let G be a directed, acyclic graph with V(G)=$\{v_1, v_2, ..., v_n\}$. Let B be the antiadjacency matrix of G, then detB=1 if G has a Hamiltonian path, and detB=0, otherwise.*

Proof:

Let G be a directed acyclic graph, suppose G has a Hamiltonian path say v_1 e_1 v_2 ... v_n, since G is acyclic, there can't be an edge form v_i to v_j for i≥j, hence b_{ij}=1 if i≥j, so b_{12} =b_{23}= ... =b_{n-1n}=0, then b_{12} in a matrix B equal 0, subtracting the second column from the first one, then all entries of the first column is 0 except b_{11}=1. Expand the determinant along the first column and use that induction on n to complete the proof. then the determinant equals 1.

Conversely, suppose G is a directed graph with no Humiliation path. Since G is the acyclic, so G must have a vertex which is a source(i.e. a vertex of in-degree 1) and without loss of generality let it be v_1. In G-v_1 there is a source say v_2, continuing in this way, let v_i be the source of G-$\{v_1, v_2, ...v_{i-1}\}$, then there is no edge from v_j to v_i for i>j, hence b_{ij}=1 if i≥j, since G has no Humiliation path, so there must exist v_i in v_1, v_2, ... v_{n-1} such that b_{ii+1}=1, let it be b_{12}, so the first two columns are equal, then the determinant equal 0. ■

The next result Computes the number of paths between any number of vertices in a directed graph.

Theorem 3.3.1 *[20] Let G be a directed, acyclic graph with V(G)=$\{v_1, v_2, \cdots, v_n\}$. Let B be the antiadjacency matrix of G, and Let*

$$det(xB+I)=\sum_{i=1}^{n} c_i x^i$$

Then $c_0 = 1$ and c_i equals the number of directed paths of i vertices in G, i=1,2,$\cdots$, n.

Proof:

let B be antiadjacency matrix of acyclic directed graph G.

$$\begin{aligned} det(xB + I) &= \sum_{i=1}^{n} c_i x^i \\ &= c_0 x^0 + c_1 x^1 + \cdots + c_n x^n \end{aligned}$$

we can see that the coefficient of x_i is the principle minor of B of order i. Any principle minor matrix is an antiadjecency matrix by corollary 3.3.1 the determinant of principle minor of order i is equal 1 if and only if the subgraph induced by corresponding vertices contains a Hamiltonian path that can't have another one, otherwise it containes a cycle. Thus the nonsingular minor of order i equals the number of paths in G of i vertices.

■

3.4 Boolean Operations on adjacency and antiadjecency matrices

In this section we will apply some Boolean operations on graph matrices . We introduce the operations on two adjacency or two antiadjacency matrices of graphs and we consider four types of Boolean operations OR, AND, XOR and NXOR.

First we recall the Boolean operations, AND, OR, XOR and NXOR respectively by the tables below

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

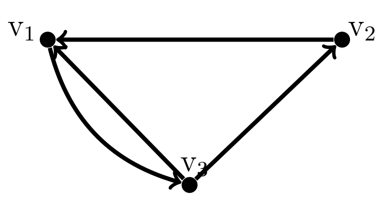
p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

p	q	$p \oplus q$
1	1	0
1	0	1
0	1	1
0	0	0

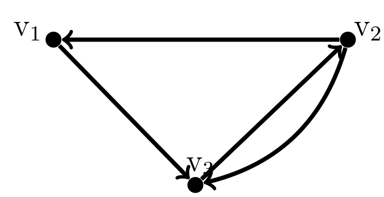
p	q	$p \bar{\oplus} q$
1	1	1
1	0	0
0	1	0
0	0	1

Let G be a graph with $V(G) = \{v_1, \cdots, v_n\}$ and $E(G) = \{e_1, \cdots, e_m\}$, we will define Boolean operations for two adjacency and two antiadjacency matrices of graphs G_1 and G_2 respectively.

Let $B(G_1)$ and $B(G_2)$ be antiadjacency matrices of G_1 and G_2 in figure 3.5 respectively.



(a)



(b)

Figure 3.5

$$B(G_1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B(G_2) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The following result shows what graph results by the OR product of some antiadjacency matrices.

Theorem 3.4.1 *[8] Let $G_1,\ G_2,\ \cdots\ ,\ G_m$ be graphs of n vertices each with $V(G_1) = V(G_2) = \cdots = V(G_m) = \{v_1, v_2, \cdots, v_n\}$. Let $B_1,\ B_2,\ \cdots,\ B_m$ be the antiadjacency matrices of $G_1,\ G_2,\ \cdots\ ,\ G_m$, respectively. Then, the OR product of $B_1,\ B_2,\ \cdots,\ B_m$ is an antiadjacency matrix which represents a graph G with $v_iv_j \in E(G)$ if v_i and v_j are adjacent vertices in all graphs $G_1,\ G_2,\ \cdots\ ,\ G_m$.*

Proof:

let B_1, B_2, $\cdots$, B_m be an antiadjacency matrices of the graphs G_1, G_2, $\cdots$, G_m, and let M be antiadjacency matrix of a graph G such that $M=B_1 \vee B_2 \cdots \vee B_m$, now any two vertices of G are adjacent or not adjacent. Suppose $v_iv_j \in E(M)$, then $M_{ij}=0$, so

$$\begin{aligned} M_{ij} &= b_{1_{ij}} \vee b_{2_{ij}} \cdots \vee b_{m_{ij}} \\ &= 0 \end{aligned}$$

Thus $b_{1_{ij}} \vee b_{2_{ij}} \cdots \vee b_{m_{ij}} = 0$, if v_iv_j are adjacent in all graphs.

Conversely, Suppose $v_iv_j \in E(G_i)$ so $b_{1_{ij}} = b_{2_{ij}} = \cdots = b_{m_{ij}} = 0$ then $M_{ij}=0$ and thus $v_iv_j \in E(G)$. ■

For example, The antiadjacency matrix of $G_1 \vee G_2$ in figure 3.5is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

We can see that $E(G_1 \vee G_2) = E(G_1) \cap E(G_2)$ and thus the graph G is a subgraph of G_1 and G_2.

The following result shows what graph is resulted by the AND product of some antiadjacency matrices.

Theorem 3.4.2 *[8] Let $G_1,\ G_2,\ \cdots\ ,\ G_m$ be graphs of n vertices each. Let $B_1,\ B_2,\ \cdots,\ B_m$ be the antiadjacency matrices of $G_1,\ G_2,\ \cdots\ ,\ G_m$, respectively. Then, the AND product of $B_1,\ B_2,\ \cdots,\ B_m$ is an antiadjacency matrix which represents a graph G with $v_iv_j \notin E(G)$ if*

v_i and v_j are not adjacent vertices in all graph $G_1,\ G_2, \cdots, G_m$.

Proof:

The proof is similar to the proof of theorem 3.4.1 ■

For example, The antiadjacency matrix of $G_1 \wedge G_2$ in figure 3.5 is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We can see that $E\left(G_1 \wedge G_2\right) = E\left(G_1\right) \cup E\left(G_2\right)$ and thus the graph G_i is a subgraph of G.

The following theorem shows what graph is resulted by the XOR product of some antiadjacency matrices.

Theorem 3.4.3 *[1] Let G_1 and G_2 be graphs of n vertices each with $V(G_1) = V\left(G_2\right) = \{v_1, v_2, \cdots, v_n\}$. Let $B_1,\ B_2$ be the antiadjacency matrices of $G_1,\ G_2$ respectively. Then, the XOR product of B_1 and B_2 is an antiadjacency matrix which represents a graph G with $v_i v_j \notin E\left(G\right)$ if and only if v_i is adjacent to v_j in one and only one of the graphs G_1 or G_2.*

Proof:

let B_1 and B_2 be an antiadjacency matrices of graphs G_1 and G_2, and let M be antiadjacency matrix of the graph G such that $M = B_1 \bigoplus B_2$, now any two vertices of G are adjacent or not adjacent. Suppose $v_i v_j \notin E(G)$, then $M_{ij}=1$, whice mean one and only one $v_i v_j$ are adjacent in G_1 and G_2.

Conversely, without loss of generality, suppose v_i and v_j are adjacent in G_1 but not adjacent in G_2, then $M_{ij} = b_{1_{ij}} \bigoplus b_{2_{ij}} = 1$, thus $v_i v_j \notin E(G)$.

■

For example, The antiadjacency matrix of $G_1 \bigoplus G_2$ in figure 3.5 is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

For finite XOR products of graphs the following theorem shown that.

Theorem 3.4.4 *[1] Let $G_1,\ G_2,\ \cdots\ ,\ G_m$ be graphs of n vertices each with $V(G_1) = V(G_2) = \cdots = V(G_m) = \{v_1, v_2, \cdots, v_n\}$. Let $B_1,\ B_2,\ \cdots,\ B_m$ be the antiadjacency matrices of $G_1,\ G_2,\ \cdots\ ,\ G_m$, respectively. Then, the XOR product of $B_1,\ B_2,\ \cdots,\ B_m$ is an antiadjacency matrix which represent a graph G with $v_iv_j \notin E(G)$ if and only if v_i is not adjacent to v_j in an odd number of graphs.*

Proof:

let B_1, B_2, $\cdots$, B_m be an antiadjacency matrices of graphs G_1, G_2, $\cdots$, G_m respectively, and let M be antiadjacency matrix of a graph G such that $M=B_1 \bigoplus B_2 \cdots \bigoplus B_m$. Suppose $v_iv_j \notin E(G)$, then $M_{ij}=1$, since $v_iv_j \notin E(G)$ for odd number of graphs.

Conversely, without loss of generality, suppose v_i and v_j are not adjacent in odd number of graphs, there is an odd number of entries in $B_{k_{ij}}$ equal one, for $k=1, \cdots, m$. Thus $M_{ij}=1$, $v_iv_j \notin E(G)$.

■

The following theorem shows what graph is resulted by the NXOR product of some antiadjacency matrix.

Theorem 3.4.5 *[1] Let $G_1,\ G_2,\ \cdots\ ,\ G_m$ be graphs of n vertices each. Let $B_1,\ B_2,\ \cdots,\ B_m$ be the antiadjacency matrices of $G_1,\ G_2,\ \cdots\ ,\ G_m$, respectively. Then, the NXOR product* *of B_1, B_2, $\cdots$, B_m is an antiadjacency matrix which represents a graph G with $v_iv_j \notin E(G)$ if and only if v_i is not adjacent to v_j in an even number of graphs.*

For example, the antiadjecency matrix of $G_1 \bar{\oplus}\, G_2$ in figure 3.5 is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Remark 3.4.1 *1. Let G_1 and G_2 be two graphs with the same n vertices. Let D_1 and D_2 be their adjacency matrices, and let B_1 and B_2 be their antiadjacency matrices respectively, and let $(B_1 \wedge B_2)$ and $(B_1 \vee B_2)$ are antiadjecency matrices of L_1=$(G_1 \wedge G_2)$ and L_2=$(G_1 \vee G_2)$ respectively, and let $(D_1 \wedge D_2)$ and $(D_1 \vee D_2)$ are adjacency matrices of H_1= $(G_1 \wedge G_2)$ and H_2=$(G_1 \vee G_2)$ respectively, so $H_1 = L_2$ and $H_2 = L_1$.*

2. Let G_1 and G_2 be two graphs with the same n vertices. Let D_1 and D_2 be their adjacency matrices, and let B_1 and B_2 be their antiadjacency matrices respectively, and let $(B_1 \oplus B_2)$ and $(B_1 \bar{\oplus} B_2)$ are antiadjecency matrix of L_1=$(G_1 \oplus G_2)$ and L_2=$(G_1 \bar{\oplus} G_2)$ respectively, and let $(D_1 \oplus D_2)$ and $(D_1 \bar{\oplus} D_2)$ are adjacency matrices of H_1= $(G_1 \oplus G_2)$and H_2=$(G_1 \bar{\oplus} G_2)$ respectively, so $H_1 = L_2$ and $H_2 = L_1$.

Chapter 4

Spectral properties of graphs

One of the most useful ways of studying large graphs is the study of the eigenvalues of the matrices (i.e spectra of matrices). By looking at these eigenvalues it is possible to get information about the graph. In this chapter, we display properties of graph G from what we know about the eigenvalues and compare some graphs that are produced by Boolean operation.

4.1 Spectral of adjacency matrix

In this section we will find the eigenvalues of a graph G by using the adjacency matrix. All graphs considered in this section are undirected graphs.

The characteristic polynomial of a graph G is the determinant of $(\lambda I - D(G))$ where D is adjacency matrix and it's roots are the eigenvalues of a graph.

Theorem 4.1.1 *[24]*

let $x^n + a_1x^{n-1} + ... + a_n$ be the characteristic polynomial of G, then

$$a_i = \sum_H (-1)^{w(H)} 2^{c(H)}$$

Where the summation is over all linear subgraphs H of order i of G, w(H) and c(H) denoted respectively the number of components and the number of cycle components of H.

Proof:

let G be a graph with V(G)= $\{v_1, \cdots, v_n\}$ and let D(G) be it's adjacency matrix, recall $a_i = (-1)^i \sum_H detD$ where H is all induced subgraphs of order i of G. By theorem 3.1.3 detD(G)=$\sum_H (-1)^{e(H)} 2^{c(H)}$, so we can say det H=$\sum_{H_i} (-1)^{e(H_i)} 2^{c(H_i)}$, where H_i is spanning liner subgraph of G and e(H_i) and c(H_i) denotes the number of even components and number of cycles of H_i respectively. The result follows from the fact that i and the number of odd components of H_i have the same parity. ■

The adjacency matrix of a graph G from example 3.1.1 is

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial of this matrix is easily computed by theorem 3.1.3

$$det\,(xI - D) = x^4 + a_1 x^3 + a_2 x^2 + a_3 x^1 + a_4$$

$$a_1 = 0 \qquad a_2 = 5\,(-1)^1\,2^0 = -5$$

$$a_3 = 2\,(-1)^0\,2^1 = 4 \qquad a_4 = 2\,(-1)^2\,2^0 + (-1)^1\,2^1 = 0$$

Now

$$det\,(xI - D) = x^4 - 5x^2 + 4x$$

Recall that if D is a n× n matrix. An eigenvalue of D is the number λ such that det(λI -D)=0 and it's corresponding eigenvector of D is a nonzero vector x such that (λI-D)x=0.

Some basic proprieties of eigenvalues are:

- The eigenvalues are exactly the numbers λ that make the matrix D-λI singular.

- If D is a real symmetric (d_{ij}=d_{ji}) matrix, then all the eigenvalues are real, and there is an orthogonal basis of R^n consisting of eigenvectors .

- The product of all eigenvalues, including the multiple eigenvalues is the determinant of D(G) (i.e $\prod_{i=1}^{n} \lambda_i$=det D(G).

- The sum of all eigenvalues, including the multiple eigenvalues is the trace of D(G) (i.e $\sum_{i=1}^{n} \lambda_i = Tr(D) = \sum_{i=1}^{n} d_{ii}$) .

- The number of nonzero eigenvalues, including the multiple eigenvalues is the rank of D.

- A set of eigenvectors of D, each corresponding to a different eigenvalue of D, is linearly independent .

- The sum of the square of the eigenvalues is $2|E(G)|$ (i.e $\lambda_1^2 + \lambda_2^2 + ... + \lambda_n^2 = 2|E(G)|$).

Now we will give a survey of some of the relationships between the properties of some graphs and the eigenvalues of it's adjacency matrix .

4.1.1 Spectral of complete graph

Recall that for k_n, the adjacency matrix D(G) is given by D= $\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}$

Theorem 4.1.2 *[2] For any positive integer n, the eigenvalues of k_n are n-1 with multiplicity 1 and -1 with multiplicity n-1.*

Proof:

let $D(k_n)=[d_{ij}]$ be adjacency matrix of a graph k_n, so we can write $D(k_n)$ as J_n - I_n, where J_n is an n×n matrix of all entries are ones, I_n is the identity matrix. So J_n is symmetric and has rank 1, hence it has only one nonzero eigenvalue, which is n that equal the trace(J_n)

with multiplicity 1 also it has another eigenvalue 0 with multiplicity n-1, the identity matrix has only one eigenvalue 1 with multiplicity n. Thus the eigenvalues of D(k_n) are n-1 and -1 with multiplicity 1, n-1 respectively.

■

4.1.2 Spectral of the bipartite and complete bipartite graph

Lemma 4.1.1 *[20] Let G be a graph with vertices* $\{v_1, \cdots, v_n\}$ *and let D be the adjacency matrix of G. Let*

$$\phi_\lambda (G) = det (\lambda I - D) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n$$

Be the characteristic polynomial of A. Suppose $c_3 = c_5 = \cdots = c_{2k-1} = 0$ *then G has no odd cycleS of length i,* $3 \leq i \leq 2k - 1$*,. Furthermore the number of (2k+1) cycles in G is* -$\frac{1}{2}c_{2k+1}$

Proof:

Suppose c_3=0, then there are no triangles in G. Thus, any liner subgraph of order 5 must contain a cycle of 5 vertices. By theorem 4.1.1 if c_5=0, then there are no cycles of 5 vertices in G, by continuing in this manner we find that if c_3=c_5 ...=c_{2k-1}=0, then any linear subgraph with 2k+1 vertices must contain a cycle with 2k+1 vertices, hence by theorem 4.1.1 $c_{2k+1} = \sum(-1)^{w(H)}2^{c(H)}$
Where the summation is over all 2k+1 cycles in G. For any 2k+1 cycle say H, w(H)=c(H)=1, so c_{2k+1} is -2 the number of 2k+1 cycles in G.

■

Theorem 4.1.3 *[20] Let G be a graph with vertices* $\{v_1, \cdots, v_n\}$ *and let D be the adjacency* *matrix of G, then the following conditions are equivalent*

1. *G is bipartite*
2. *If*

$$\phi_\lambda (G) = det\,(\lambda I - D) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_n$$

   is the characteristic polynomial of D, then c_{2k+1}=0, k=0,1, ...

3. *The eigenvalues of D are symmetric with respect to the origin, i.e if λ ia an eigenvalue of D with multiplicity k, then -λ is also an eigenvalue of D with multiplicity k.*

Proof:

1$\Rightarrow$ 3 : Let G be bipartite graph, The adjacency matrix (n+m)$\times$ (n+m) of G can be written in the form

D(G)=$\begin{bmatrix} 0 & D \\ D^T & 0 \end{bmatrix}$. Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be a corresponding eigenvalue λ. Now $\begin{bmatrix} 0 & D \\ D^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$

It follows that Dy=λx and D^{T}x=λy. Also D(-y)=-λx and D^{T}x=-λ(-y), therefore, -λ is an eigenvalue and $\begin{bmatrix} x \\ -y \end{bmatrix}$ is it's corresponding eigenvector, hence the eigenvalues of G are symmetric.

It is clear that if we have k linearly independent eigenvectors of λ, the construction will produce k linearly independent eigenvectors for -λ, thus the multiplicity of -λ is also k.

3$\Rightarrow$ 2 : If 3 holds, then we replace λ by -λ then the characteristic polynomial remains the same, this means that the characteristic polynomial is an even function.

2$\Rightarrow$ 1 : Using notation of lemma 4.1.1, if $c_{2k+1} = 0$, k=1,2,... then G has no odd cycles and hence G must be bipartite.

■

Theorem 4.1.4 *[22]. The adjacency matrix eigenvalues of the complete bipartite graph $k_{p,q}$ are zero with multiplicity (p+q-2) and $\pm\sqrt{pq}$.*

Proof:

let $k_{p,q}$ be bipartite graph with bipartition (X,Y) where X have p vertices and Y have q vertices. The adjacency matrix of $k_{p,q}$ is of the form D($k_{p,q}$)= $\begin{bmatrix} 0 & J_{p,q} \\ J_{q,p} & 0 \end{bmatrix}$

Where $J_{r,s}$ is r×s matrix whice all entry is once. So $J_{r,s}$ is symmetric and has a rank 1, hence the rank of D($k_{p,q}$)=rank $J_{p,q}$ + rank $J_{q,p}$ =1+1=2. So D($k_{p,q}$) has two nonzero eigenvalues say λ and -λ, we know that the sum of the squares of the eigenvalues of D($k_{p,q}$) is equal to twice the number of edges (i.e $\lambda^2 + (-\lambda)^2$=2$|E(G)|$), thus λ=$\pm\sqrt{pq}$. The tras of D($k_{p,q}$) is zero, so the other eigenvalue is 0 with multiplicity p+q-2.

■

4.1.3 Spectral of regular graphs

Theorem 4.1.5 *[2] Let G be a k-regular graph of order n, then*

1. *k is an eigenvalue of G.*
2. *If G is connected, every eigenvector corresponding to the eigenvalue k is a multiple of 1 and the multiplicity of k as an eigenvalue of G is one.*
3. *For any eigenvalue λ of G, $|\lambda| \leq k$.*

Proof:

1. Let D be adjacency matrix of graph G, then any rows of D contains k 1's, thus the vector 1 is an eigenvector of corresponding eigenvalue k.

2. Let D be n×n adjacency matrix of a graph G and let $x^T=[x_1, \cdots, x_n]$ be a positive eigenvector of eigenvalue k. Let x_j be the largest entry of x (i.e $x_j=\max\{x_1, \cdots, x_n\}$). The product of j^{th} row of D with the vector x is

   $$D_j x=\sum_{v_i \in N(v_j)} x_i = x_{i1} + x_{i2} + \cdots + x_{ik} = kx_j$$

   Proving the case when $x_{i1} = x_{i2} = \cdots = x_{ik} = x_j$.

   Now let $v_{i1}, v_{i2}, \cdots, v_{ik}$ be neighbors in G. As before, the entries x_p in a vector x that corresponds to these neighbors must all equal to x_j. G is a connected graph, so all vertices of G have neighbors of v_i. Repeating the same argument for all neighbors . Hence $x=x_j[1, 1, \cdots, 1]^T$, and every eigenvector of k is multiple of 1. The space eigenvector x of the eigenvalue λ is one dimensional and the multiplicity of the eigenvalue k of A is 1.

3. Let λ be an eigenvalue of adjacency matrix of a graph G and let y be an eigenvector of the eigenvalue λ. Let y_j be the largest absolute value of the entries of Y. The product of j^{th} row of D with y.

   $$D_j x=\sum_{P=1}^{k} y_{ip} = \lambda y_j$$

   So

   $$|\lambda|\,|y_j| = \left|\sum_{p=1}^{k} y_{ip}\right| \leq \sum_{p=1}^{k} |y_{ip}| \leq k\,|y_j|$$

   Thus $|\lambda| \leq k$

■

4.2 Eigenvalues of Laplacian matrix

The Laplacian matrix of a graph and it's eigenvalues can be used in several areas of mathematical research and have various applications in physical and chemical theories.

In this section we will study the eigenvalues of an undirected graph G by using the Laplacian matrix.

For the graph in figure 3.2 the Laplacian matrix is given by

$$L=\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

With eigenvalues

$$0,\ 2,\ 4,\ 4$$

We can see that all the eigenvalues are real numbers. However there is one special eigenvalue, namely 0. All other eigenvalues are non negative, so in fact we have the following result.

Lemma 4.2.1 *[14] The Laplacian matrix L(G) is singular and positive semidefinite.*

Proof:

Let λ be an eigenvalue of a matrix L(G) with corresponding eigenvector V

$$\begin{aligned} (L(G) - \lambda I)\, V &= 0 \\ \lambda V &= L(G)V \\ \lambda &= V^T L\,(G)\, V \\ \lambda &= V^T Q Q^T V \\ \lambda &= \left(Q^T V\right)^T Q^T V \geq 0 \end{aligned}$$

Furthermore, the summation of each column is zero so L(G) is singular. ■

Remark 4.2.1 *The eigenvector 1 of Laplacian matrix of G is an eigenvector of adjacent matrix if and only if the graph is regular graph.*

Theorem 4.2.1 *[15]*

A graph G has k connected components iff the algebraic multiplicity of 0 in the Laplacian is k.

Proof:

First we show that the multiplicity of the zero eigenvalue is at least the number of connected components. Assume G is a graph with k connected components, so we have V(G_1), V(G_2), …, V(G_k) are the disjoint sets of G. Define k vectors x_1, x_2, …, x_k such that each entry is define as

$$x_j = \begin{cases} \frac{1}{\sqrt{|v_i|}} & \text{if } j \in V(G_i) \\ 1 & \text{other wise} \end{cases}$$

So $\|x_i\| = 1$ for i=1, 2, … k. Also $\langle v_i, v_j \rangle = 0$, since v_i and v_j are disjoint. Thus Lx_i=0, hence there is a set of k orthonormal vectors that are all eigenvectors of L, with eigenvalue 0.

Now, to see that the number of 0 eigenvalues is at most the number of connected components, we know $x^T Lx = \sum_{i<j,(i,j)\in E} (v_i - v_j)^2$ is equal 0 if x is a constant in every connected component. Now we will see there is no way of finding a k+1 vector x that is a zero eigenvector, and orthogonal to x_1, x_2, …, x_k. Notice that x must be nonzero in some coordinate, hence suppose that x is nonzero in the coordinate v_i, otherwise constant on all x_i, in which case x can not be orthogonal to x_i, so there can be no k+1 eigenvector with 0 eigenvalue. ■

4.3 Comparing the largest eigenvalue of matrices resulted by Boolean operation

In this section we will compare the largest eigenvalue of matrices that is generated by Boolean operation.

Let A be an n×n matrix. And let $\lambda_1(A)$ denotes the largest eigenvalue of A. In [19] we can show that if A is a nonnegative matrix, then $\lambda_1(A) > 0$. We need the following result.

Lemma 4.3.1 *[20] If A and B are symmetric n×n matrices, then*

$$\lambda_1(A+B) \leq \lambda_1(A) + \lambda_1(B)$$

Proof:

Recall $\lambda_1(A) = max_{\|x\|=1}\left\{x^T Ax\right\}$.

So,

$$\begin{aligned}\lambda_1(A+B) &= max_{\|x\|=1}\left\{x^T(A+B)x\right\}\\ &\leq max_{\|x\|=1}\left\{x^T Ax\right\} + max_{\|x\|=1}\left\{x^T Bx\right\}\\ &\leq \lambda_1(A) + \lambda_1(B)\end{aligned}$$

■

Theorem 4.3.1 *[19] Let A and B be 0-1 matrices of order n×n. Let λ_1 and μ_1 be the largest eigenvalue of A and B, respectively. If $A \geq B$, then $\lambda_1 \geq \mu_1$*

Lemma 4.3.2 *[8] Let B_1 and B_2 be antiadjacency n×n matrices of G_1 and G_1 respectively. Then,*

$(B_1 \vee B_2) + (B_1 \wedge B_2) = B_1 + B_2$ *and consequently,*

$$\lambda_1((B_1 \vee B_2) + (B_1 \wedge B_2)) = \lambda_1(B_1 + B_2) \leq \lambda_1(B_1 \vee B_2) + \lambda_1(B_1 \wedge B_2)$$

Proof:

Clear, by the definition of $\vee$, $\wedge$ and lemma 4.3.1 ■

Now, we can compare the largest eigenvalues between graphs joind by AND and OR operations on antiadjacency matrices.

Theorem 4.3.2 *[1] Let B_1 and B_2 be antiadjacency matrices of undirected graph with n vertices. Then,*

$\lambda_1(B_1 \wedge B_2) \leq \lambda_1(B_1) + \lambda_1(B_2)$ *and* $\lambda_1(B_1 \vee B_2) \leq \lambda_1(B_1) + \lambda_1(B_2)$.

Proof:

Suppose B=$B_1 \vee B_2$, we know $\lambda_1(B) \geq 0$.

Therefore

$$\lambda_1(B_1 + B_2) - \lambda_1(B) \leq \lambda_1(B_1 + B_2)$$

By Lemma 4.3.1 and Lemma 4.3.2 we conclude

$$\lambda_1(B_1 \wedge B_2) \leq \lambda_1(B_1 + B_2) - \lambda_1(B)$$

$$\leq \lambda_1(B_1 + B_2) \leq \lambda_1(B_1) + \lambda_1(B_2)$$

A similar proof works for the second inequality. ■

Now, we can compare the largest eigenvalues between graphs connected by OR and XOR operations on antiadjacency matrices.

Theorem 4.3.3 *[1] Let B_1 and B_2 be antiadjacency matrices of directed graph. Then,*

$$\lambda_1(B_1 \oplus B_2) \leq \lambda_1(B_1 \vee B_2) \leq \lambda_1(B_1 + B_2)$$

Furthermore, if the graphs are undirected then.

$$\lambda_1(B_1 \oplus B_2) \leq \lambda_1(B_1) + \lambda_1(B_2)$$

Proof:

We know by lemma 4.3.2 $B_1 \vee B_2 \leq B_1 + B_2$.

Also $B_1 \oplus B_2 \leqslant B_1 \vee B_2 \leq B_1 + B_2$

By Lemma 4.3.1 and theorem 4.3.1 we conclude

$$\lambda_1(B_1 \oplus B_2) \leqslant \lambda_1(B_1 \vee B_2)$$

$$\leq \lambda_1(B_1) + \lambda_1(B_2)$$

$\blacksquare$

Now, we can compare the largest eigenvalues between graphs joined by AND and NXOR operations on antiadjacency matrices.

Theorem 4.3.4 *[1] Let B_1 and B_2 be antiadjacency matrices of directed graph. Then,*

$$\lambda_1 (B_1 \wedge B_2) \leq \lambda_1 (B_1 \bar{\oplus} B_2)$$

Proof:

We know $B_1 \bar{\oplus} B_2 \leqslant B_1 \vee B_2 \leq B_1 + B_2$

By Lemma 4.3.1 we conclude $\lambda_1 (B_1 \wedge B_2) \leq \lambda_1 (B_1 \bar{\oplus} B_2)$

$\blacksquare$

4.4 Page Rank Application

In 1998, Brin and Larry Page has presented a page rank algorithm, the aim of this algorithm is to follow some difficulties with content-ranking algorithm of early search engines. Page Rank (score) is a positive real number used to measure the importance of website page by using hyperlinks between pages. A simple graph in figure 4.1 is for website represented by a directed graph. Each page is represented by node v_k. In figure 4.1 we have three Pages or nodes. The links to page one are called *backlinks* for page one. The linking of a page produce a diagraph such as figure 4.1. A page with no outgoing links is called *dangling* node. In figure 4.1 each vertex has at least one outgoing link, so the graph has no outgoing nodes. Also we can see that the graph is strongly-connected.

Let x_k be the number of pages that link to vertex x.

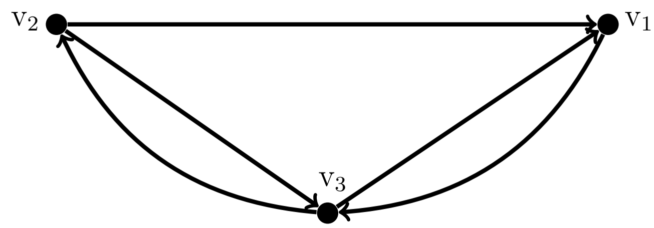


Figure 4.1

To calculate the page rank(score) we find The eigenvectors corresponding to the largest eigenvalue of 1, the link matrix is defined as

$$A = \begin{cases} \frac{1}{n_i} & i \in L_j \\ 0 & o.w \end{cases}$$

Where

L_j is the set of pages that link to page i.

n_i is the number of pages that are linked to page j.

For example, the *link matrix* A of the digraph G in figure 4.1 is

$$A = \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 \end{bmatrix}$$

The eigen values of A is $\lambda_1 = 1, \lambda_2 = -\frac{1}{2}, \lambda_3 = -\frac{1}{2}$

The eigen vector that is associated to λ=1 is $\begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}$

So we can see that page 3 is the most important page in research and the second one is page 2, then page 1.

Definition 4.4.1 *A matrix $A_{n\times n}$ is called column stochastic if it is non negative and the sum of the entries in each column is 1.*

Theorem 4.4.1 *[21] If $A_{n\times n}$ is a stochastic matrix, then its eigenvalues λ_i have the property*

$$|\lambda_i| \leq 1$$

And λ=1 is always an eigenvalue of A.

Proof:

Since $\lambda_1 \leq \|A\|$ and since A is stochastic matrix, then

$$\lambda_1 \leq \|A\| = 1$$

Where $\|.\|$ is the ∞ or 1 matrix norm, depending on whether is row or column stochastic, respectively. On other hand, let x$\in$R^n be a vector whice all entry are once. Because A is stochastic, we either have Ax=x or x^{T}A=x^T($A^T x \neq x$) depending whether A is row or column stochastic respectively. That is, $\lambda = 1$ is always an eigenvalue of A. But this also means that $\lambda_1 \geq 1$, where combined with $\lambda_1 \leq \|A\| = 1$ implies that $\lambda_1 = 1$, therefore, every eigenvalue of A must satisfy $\lambda_1 \leq 1$ ■

There are two challenges to the Page Rank algorithm, namely Nonuniqueness and Dangling nodes.

4.4.1 Nonuniqueness.

The main goal of Page Rank algorithm is to find scores of pages and compare between the pages by these scores. To calculate the Page Rank score we will find an eigen vectors that corresponds to the eigen value 1, this will make our ranking to be unique. Since we have only one eigenvector that represents the eigenvalue 1, which is true if the web is a strongly connected digraph, but if the web is disconnected, then we can have a higher dimensional eigen space of eigenvalue 1. For example the link matrix of graph G in figure 4.2 is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ The eigen values are $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1, \lambda_4 = -1$

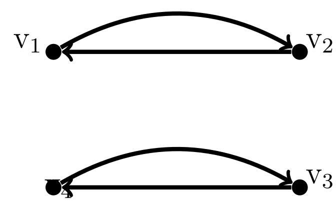


Figure 4.2

Theorem 4.4.2 *[9] Let W be a web with r components* W_1*,* W_2*, . . .,* W_r*. Then the eigenspace of the eigenvalue 1 is at least r-dimensional.*

Proof:

Suppose we label the web by assigning the vertices in W_1 first, then the vertices in W_2, etc., then the link matrix will have a block diagonal from like

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & A_r \end{bmatrix}$$

Where A_1 is the link matrix for the web W_1. If each A_k is column stochastic, so each has an eigenvector v_k with eigenvalue 1, and that can be expanded into a eigenvector w_k for A by letting

$$w_1 = \begin{bmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ w_2 = \begin{bmatrix} 0 \\ v_2 \\ \vdots \\ 0 \end{bmatrix}, \text{ etc.}$$

Each of these eigenvector is linearly independent and part of eigenspace V_1 of eigenvalue.

■

For n pages with multiple subwebs we can generate unambiguous importance score as follows

$$M=(1-m)\,A + mS$$

Where S is an $n \times n$ matrix with all entries are $\frac{1}{n}$ that represent equal probabilities of jumping to any page on web, that means S is a column stochastic matrix. m is a positive number between 0 and 1, the original value of m used by Google was 0.15.

Now we will show that the matrix M has one-dimensional eigenspace corresponding to the eigenvalue 1 when m>1

Theorem 4.4.3 *[4] If M is a positive, column-stochastic matrix, then any eigen vector in $v_1\,(M)$ is of all positive or all negtaive component.*

Proof:

Suppose there exist an eigen vector v of the matrix M.Since M is column-stochastic matrix so for each column

$$\sum_{i=1}^{n} m_{ij} = 1$$

Also, M is positive, we know that

$$|m_{ij}| = m_{ij}$$

Now suppose Mv=v. Therefore, $\|M\|_1 = max_j \sum_{i=1}^{n} |m_{ij}|$, we see that

$$\begin{aligned}
\|v\|_1 &= \|Mv\|_1 \\
&= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} m_{ij} v_j \right| \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |m_{ij}| \, |v_j| \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} m_{ij} \, |v_j| \\
&= \sum_{j=1}^{n} |v_j| = \|v\|_1
\end{aligned}$$

That means that the inequality must be an equality, so

$$\textstyle\sum_{i=1}^{n} \left| \sum_{j=1}^{n} m_{ij} v_j \right| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |m_{ij}| \, |v_j|$$

The equality is true only if m_{ij} $v_j \geq 0$ for each i and j which implies $v_j \geq 0$ or if $m_{ij} v_j \leq 0$ for each i and j which implies $v_j \leq 0$. Furthermore, since

$$v_i = \textstyle\sum_{j=1}^{n} m_{ij} v_j$$

We must have that either all v are zero or all are positive (negative), since v is an eigenvector, it is not the zero vector. ■

Lemma 4.4.1 *[4] Let v and w be linearly independent vectors in R^m, m≥2. Then for some value of s and t that are not both zero, the vector x=sv+tw has both positive and negative components.*

Proof:

Let v and w be linearly independent vectors, so neither v nor w is zero. Let d=$\sum_i v_i$. If d=0, then v must contain componants of differant sign, and taking s=1 and t=0 yields the conclusion.

If d≠0, set s=-$\frac{\sum_i w_i}{d}$, t=1 and x=sv+t. Since v and w are independent x≠0,.However, $\sum_i x_i = 0$, we conclude that x has both positive and negative components. ■

Theorem 4.4.4 *[4] If M is a positive, column-stochastic matrix, then $v_1(M)$ has dimension 1.*

Proof:

Suppose v and w are two linearly independent eigenvectors in the subspsce $\mathrm{v}_1(M)$, we know x=sv+tw $\in \mathrm{v}_1(M)$ for any real numbers s and t that are not both zero, and x have components that are all negative or all positive, by lemma 4.4.1 for some s and t the vector x must contain a mixed sign, that is contradiction.
So we conclude that $\mathrm{v}_1(M)$ can't contain two linearly independent vectors, and so has diminution one.

4.4.2 Dangling nodes

A dangling node problem exists when a vertex in the web with out-degree is zero (i.e with no links). This node produce a column of zeros in a links matrix. That implies that a link matrix is not column stochastic, since some column may sum to zero.
To deal with this problem we have the following theorem Perron–Frobenius.

Theorem 4.4.5 *[9] If A is a matrix with all positive entries, then A contains a real eigenvalue ρ such that :*

1. *For any other eigenvalue λ, we have $|\lambda| < \rho$.*
2. *The eigenspace of ρ is one-dimensional and there ia a unique eigenvector $x=\begin{bmatrix} x_1 & x_2 & \cdots & x_p \end{bmatrix}^T$ with eigenvalue ρ such that $x_i > 0$ for all i and $\sum_{i=1}^{p} x_i = 1$*

This eigenvector is called the Perron vector.

For the proof of the theorem above we refer the reader to [16].

For example, the link matrix of graph G in figure 4.3 is $\begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$. The eigen values are $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = \frac{-1}{3}, \lambda_4 = \frac{1}{3}$

The eigenvector of $\rho = \frac{1}{3}$ is $\begin{bmatrix} \frac{1}{6} \\ \frac{1}{12} \\ \frac{1}{12} \\ \frac{2}{3} \end{bmatrix}$.

So we can see that page 4 is the most important page in research and the second one is page 1, then page 2 and 3.

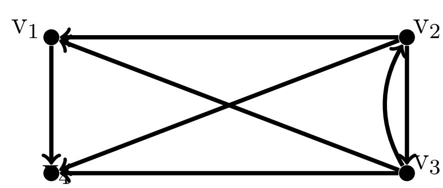


Figure 4.3

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