Palestine Polytechnic University
Deanship of Graduate Studies and Scientific Research
Master of Mathematics

Matrices Having A Positive Determinant And All Other Minors Nonpositive

Submitted by

Imad Hassona

Supervisor

Dr. Mohammad Adam

Co-Supervisor

Prof. Dr. Jürgen Garloff
University of Konstanz, Germany

Submitted to the Department of Mathematics at Palestine Polytechnic University as a partial fulfilment of the requirement for the master degree.

2022

The undersigned hereby certify that they have read, examined and recommended to the Deanship of Graduate Studies and Scientific Research at Palestine Polytechnic University the approval of a thesis entitled: **Matrices Having A Positive Determinant And All Other Minors Nonpositive**, submitted by **Imad Shokri Mohammad Hassona** in partial fulfillment of the requirements for the degree of Master in Informatics.

Graduate Advisory Committee:

Dr. Mohammad Adam (Supervisor), Palestine Polytechnic University.

Signature:____________________ Date:______________

Prof. Dr. Jürgen Garloff (Co-Supervisor), Palestine Polytechnic University.

Signature:____________________ Date:______________

Dr. Iyad Al-Hribat (Internal committee member), Palestine Polytechnic University.

Signature:____________________ Date:______________

Dr. Mahmoud Shalalfeh (External committee member), Name of University.

Signature:____________________ Date:______________

Thesis Approved

Dr. Nafeth Kamal Nasereddin
Dean of Graduate Studies and Scientific Research
Palestine Polytechnic University

Signature:____________________ Date:______________

DECLARATION

I declare that the Master Thesis entitled **"Matrices Having A Positive Determinant And All Other Minors Nonpositive"** is my original work, and hereby certify that unless stated, all work contained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgement is made in the text.

Imad Shokri Mohammad Hassona

Signature:________________________ Date:________________

STATEMENT OF PERMISSION TO USE

In presenting this thesis in partial fulfillment of the requirements for the master degree in Informatics at Palestine Polytechnic University, I agree that the library shall make it available to borrowers under rules of the library.

Brief quotations from this thesis are allowable without special permission, provided that accurate acknowledgement of the source is made.

Permission for extensive quotation from, reproduction, or publication of this thesis may be granted by my main supervisor, or in his absence, by the Dean of Graduate Studies and Scientific Research when, in the opinion of either, the proposed use of the material is for scholarly purposes.

Any coping or use of the material in this thesis for financial gain shall not be allowed without my written permission.

Imad Shokri Mohammad Hassona

Signature:________________________ Date:________________

Abstract

The class of square real matrices of order n having a positive determinant and all other minors up to order $n-1$ nonpositive are called sign regular matrices with signature $(-1,\ldots,-1,1)$. In this thesis, such matrices are introduced and a characterization of them is presented which provides an easy test for their recognition based on the so-called the Cauchon Algorithm. The value of the entry $(2,2)$ of the matrix resulting upon application of the Cauchon algorithm to such a sign regular matrix plays a fundamental role in our characterization. Therefore, the possible values of the entry $(2,2)$ are explored. Finally, it is shown that all matrices lying between two matrices of this class with respect to the so-called checkerboard ordering are contained in this class, too.

DEDICATION

This thesis is dedicated to:
The sake of Allah, my Creator and my Master,
My great supervisor Dr.Mohammad Adam, who encourage and support me,
My external committee member,
My parents, the reason of what I become today.

ACKNOWLEDGEMENT

First and foremost, all praise and thanks are due to Allah for helping me to complete this work. I could never have finished this thesis without his help and guidance. I would like to express my sincere gratitude to my supervisor, Dr. Mohammad Adam, for the continuous support of my Master and related research,for his patience,motivation,and immense knowledge.His guidance helped me in all the time of research and writing of this thesis. I am also thankful to Dr. Mahmoud Shalalfeh and Dr. Iyad Al-Hribat for serving on my thesis committee and for providing constructive commens which have improved my thesis.

Last but not the least,I would like to express my gratitude and appreciation to my family,my sisters and brothers for supporting me spiritually throughout writing this thesis and my life in general.

Contents

List of Figures

Chapter 1

Preliminaries

In this chapter, we present some notation, definitions, and auxiliary results that will be used throughout this thesis.

1.1 Introduction

Several classes of matrices play an important role in various branches of mathematics and applied sciences [13],[18]. In this thesis, we will consider the class of sign regular matrices which are the matrices whose minors of any fixed order have the same sign or are allowed to vanish [8], [11]. Adm and Garloff in [4] worked on a special type of sign regular matrices, namely matrices of order n having a negative determinant and all their minors up to order $n-1$ nonnegative. We are going to work on the opposite kind of these matrices which are matrices having a positive determinant and all other minors nonpositive.

In the current chapter, we present some notation that help us to introduce some definitions, theorems, and lemmas that will be beneficial in showing our results.

In the next chapter, we present the definition of the sign regular, totally nonnegative and totally nonpositive matrices. This will be followed by an introduction of the Cauchon Algorithm, and some theorems that describe the relationship between the totally nonnegative matrices and the Cauchon matrices. In addition, we introduce the lacunary sequences with respect to a Cauchon matrix or a Cauchon diagram that are very useful to facilitate the proofs of some theorems and help us to determine the rank of matrices. Moreover, we present propositions that investigate the relationship between the determinants of the submatrices of the intermediate matrices of the Cauchon Algorithm, these propositions will play an important role in proving some of our new results.

In Chapter three, we present our new results by applying the Cauchon Algorithm on totally nonpositive matrices and on matrices of order n having positive determinants and all other minors nonpositive under the condition that the entry in the position (n, n) is negative. We introduce some properties of the determinants of submatrices and the sign of the entries of the matrices that we obtain after applying the Cauchon Algorithm. We conclude this thesis by utilizing the obtained results to show the interval property of the nonsingular matrices having a positive determinant and all other minors nonpositive.

1.2 Notation

In this section, we present some notation that will be helpful in introducing many theorems and results in this thesis.

First of all, we denote by $\mathbb{R}^{n,m}$ the set of all real $n-$by$-m$ matrices. For a given positive integer n and for each $l \in \{1, \ldots, n\}$, we define

$$\mathbf{Q}_{l,n} := \left\{ \mathbf{q} = (q_1, \cdots, q_l) \in \mathbb{N}^l : 1 \leq q_1 < q_2 < \cdots < q_l \leq n \right\}.$$

That is $\mathbf{Q}_{l,n}$ denotes the set of all strictly increasing sequences of l integers chosen from $\{1, \ldots, n\}$. Let $\alpha = (\alpha_1, \ldots, \alpha_s, \ldots, \alpha_l) \in \mathbf{Q}_{l,n}$ and $s \in \{1, \ldots, l\}$. Then define $\alpha_{\hat{s}} := (\alpha_1, \ldots, \alpha_{s-1}, \alpha_{s+1}, \ldots, \alpha_l)$, where the 'hat' over s indicates that the entry α_s has to be discarded from the index sequence. Moreover, define α^c to be the complement of α, which its given by $\alpha^c = \{1, \ldots, n\} \backslash \{\alpha_1, \ldots, \alpha_l\}$, where the indices are arranged in increasing order.

The square submatrices of $A = (a_{ij}) \in \mathbb{R}^{n,m}$ are denoted by the following

$$A\left[\mathbf{i} \mid \mathbf{j}\right] = A\left[i_1, i_2, \ldots, i_l | j_1, j_2, \ldots, j_l\right] := (a_{i_k, j_h})_{k=1,h=1}^{l},$$

when $\mathbf{i} = (i_1, \ldots, i_l) \in \mathbf{Q}_{l,n}$, $\mathbf{j} = (j_1, \ldots, j_l) \in \mathbf{Q}_{l,m}$, and $l = 1, \ldots, \min\{n, m\}$.
If $\mathbf{i} = \mathbf{j}$, then they called principal submatrices, denoted by

$$A\left[\mathbf{i} \mid \mathbf{i}\right] = A\left[\mathbf{i}\right] = A\left[i_1, i_2, \ldots, i_l | i_1, i_2, \ldots, i_l\right] := (a_{i_k, i_h})_{k=1,h=1}^{l}.$$

Moreover, the *minors* of $A = (a_{ij}) \in \mathbb{R}^{n,m}$ are the determinants of the square submatrices of A, denoted by

$$\det A\left[\mathbf{i} \mid \mathbf{j}\right] = \det A\left[i_1, i_2, \ldots, i_l | j_1, j_2, \ldots, j_l\right] := \det(a_{i_k, j_h})_{k=1,h=1}^{l}.$$

If $\mathbf{i} = \mathbf{j}$, then they are called *principal minors*. By convention, $\det A(\phi) = 1$.

For a matrix $A \in \mathbb{R}^{n,n}$, the *inverse* of A, denoted by $A^{-1} = \left(a_{ij}^{-1}\right)$, is given by

$$a_{ij}^{-1} = (-1)^{i+j} \frac{\det A\left[1,\ldots,j-1,j+1,\ldots,n|1,\ldots,i-1,i+1,\ldots,n\right]}{\det A}, \tag{1.1}$$

provided that $\det A \neq 0$.

For $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbf{Q}_{l,n}$, a measure for the *gap* in an index sequence α is the *dispersion* of α, denoted by $d(\alpha)$, and defined by $d(\alpha) := \alpha_l - \alpha_1 - l + 1$. If $d(\alpha) = 0$ we call α contiguous. Moreover, for $\beta \in \mathbf{Q}_{l,n}$, if $d(\alpha) = d(\beta) = 0$ we call the submatrix $A[\alpha|\beta]$ contiguous.

For distinct $\mathbf{i}, \mathbf{k} \in \mathbf{Q}_{l,n}$, we say that $\mathbf{i} = (i_1, \ldots, i_l)$ is *greater than* $\mathbf{k} = (k_1, \ldots, k_l)$ *with respect to the lexicographical order*, denoted by $\mathbf{i} \geq \mathbf{k}$, if the first non-zero entry in the sequence $i_1 - k_1, i_2 - k_2, \ldots, i_l - k_l$ is positive or $\mathbf{i} = \mathbf{k}$. Moreover, we say that $\mathbf{i} = (i_1, \ldots, i_l)$ is *greater than* $\mathbf{k} = (k_1, \ldots, k_l)$ *with respect to the colexicographical order*, denoted by $\mathbf{i} \geq_c \mathbf{k}$ if the first non-zero entry in the sequence $i_l - k_l, i_{l-1} - k_{l-1}, \ldots, i_1 - k_1$ is positive or $\mathbf{i} = \mathbf{k}$.

For $p = 1, 2, \ldots, \min\{n, m\}$. *the* p^{th} *compound matrix of an* $n \times m$ *matrix* A, denoted by $A^{[p]}$, is the $\binom{n}{p} \times \binom{m}{p}$ matrix that consists of all $p \times p$ minors arranged in lexicographical order, i.e.,

$$A^{[p]} = \left[\det A\left[\mathbf{i}|\mathbf{j}\right]\right]_{\mathbf{i} \in \mathbf{Q}_{p,n}, \mathbf{j} \in \mathbf{Q}_{p,m}, i \geq j},$$

For the matrix $A = (a_{ij}) \in \mathbb{R}^{n,m}$, the notation $0 < A$ means that all the entries of A are positive, i.e., $a_{ij} > 0$. Moreover, the notation $0 \leq A$ means that all the entries of A are nonnegative, i.e., $a_{ij} \geq 0$, for all $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$.

1.3 Auxiliary Results

In this section, we present some definitions and lemmas that will be helpful in introducing and showing several theorems in this thesis.

In the following, we introduce the definition of matrix rank.

Definition 1.1. *[17] Let A be an $n \times m$ matrix. Then rank A is the size of the largest nonzero minor of A.*

In the following, we present the definition of the right shadow and the left sahdow of the submatrices.

Definition 1.2. *[20] Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$. Then*

(1) The right shadow *of the submatrix*

$$A\left[i+1,\ldots,i+r|j+1,\ldots,j+r\right]$$

is the $(i+r)\times(m-j)$ submatrix

$$A\left[1,\ldots,i+r|j+1,\ldots,m\right].$$

(2) The left shadow *of the submatrix*

$$A\left[i+1,\ldots,i+r|j+1,\ldots,j+r\right]$$

is the $(n-i)\times(j+r)$ submatrix

$$A\left[i+1,\ldots,n|1,\ldots,j+r\right].$$

Lemma 1.1. *[9] Partition $A \in \mathbb{R}^{n,n}, n \geq 3$, as follows:*

$$A = \begin{bmatrix} c & A_{12} & d \\ A_{21} & A_{22} & A_{23} \\ e & A_{32} & f \end{bmatrix},$$

where $A_{22} \in \mathbb{R}^{n-2,n-2}$ and c, d, e, f are scalars. Define the submatrices

$$C := \begin{bmatrix} c & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, D := \begin{bmatrix} A_{12} & d \\ A_{22} & A_{23} \end{bmatrix},$$

$$E := \begin{bmatrix} A_{21} & A_{22} \\ e & A_{32} \end{bmatrix}, F := \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & f \end{bmatrix}.$$

Then

$$\det A_{22} \det A = \det C \det F - \det D \det E.$$

The previous lemma is the key of deducing the following lemma that will be used in showing some theorems.

Lemma 1.2. *[1] Let* $A \in \mathbb{R}^{n,m}, \alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbf{Q}_{l,n}$*, and* $\beta = (\beta_1, \ldots, \beta_{l-1}) \in \mathbf{Q}_{l-1,m-1}$ *with* $0 < d(\beta)$*. Then for all* η *such that* $\beta_{l-1} < \eta \leq m, k \in \{1, \ldots, l\}, s \in \{1, \ldots, h\}$*, and* $\beta_h < t < \beta_{h+1}$ *for some* $h \in \{1, \ldots, l-2\}$ *or* $\beta_{l-1} < t < \eta$ *the following determinantal identity holds:*

$$\det A\left[\alpha_{\hat{\alpha}_k} \mid \beta_{\hat{\beta}_s} \cup \{t\}\right] \det A\left[\alpha \mid \beta \cup \{\eta\}\right] = \det A\left[\alpha_{\hat{\alpha}_k} \mid \beta_{\hat{\beta}_s} \cup \{\eta\}\right] \det A\left[\alpha \mid \beta \cup \{t\}\right] + \det A\left[\alpha_{\hat{\alpha}_k} \mid \beta\right] \det A\left[\alpha \mid \beta_{\hat{\beta}_s} \cup \{t, \eta\}\right].$$

The following is an illustrative example of the previous lemma.

Example 1.1. *Let* $A = \begin{bmatrix} 1 & 0 & -1 & 1 & -1 & 2 \\ 0 & -2 & 3 & 0 & -1 & 1 \\ 0 & 3 & 1 & -1 & 2 & -3 \\ -1 & 2 & 0 & 1 & -3 & 2 \\ -3 & 0 & 2 & -1 & 0 & 1 \\ 2 & -1 & 0 & 3 & 0 & -1 \end{bmatrix}$*. Suppose* $\alpha = (1, 2, 3, 4)$*,* $\beta = (1, 3, 4)$*,* $\eta = 5$*,* $k = 3$*,* $s = 1$*,* $t = 2$*, and* $h = 1$*. By applying Lemma 1.2 we get*

$$\begin{aligned} \det A[1,2,4|2,3,4] \cdot \det A[1,2,3,4,|1,3,4,5] &= \det A[1,2,4|3,4,5] \cdot \det A[1,2,3,4|1,2,3,4] \\ &+ \det A[1,2,4|1,3,4] \cdot \det A[1,2,3,4|2,3,4,5], \\ \begin{vmatrix} 0 & -1 & 1 \\ -2 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & -1 & 1 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 2 \\ -1 & 0 & 1 & -3 \end{vmatrix} &= \begin{vmatrix} -1 & 1 & -1 \\ 3 & 0 & -1 \\ 0 & 1 & -3 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & -1 & 1 \\ 0 & -2 & 3 & 0 \\ 0 & 3 & 1 & -1 \\ -1 & 2 & 0 & 1 \end{vmatrix} \\ &+ \begin{vmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ -1 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 0 & -1 & 1 & -1 \\ -2 & 3 & 0 & -1 \\ 3 & 1 & -1 & 2 \\ 2 & 0 & 1 & -3 \end{vmatrix}, \\ -8 \times -1 &= 5 \times -26 + 6 \times 23 = 8. \end{aligned}$$

The following lemma relates the minors of a given nonsingular matrix to the minors of its inverse. In particular, it plays a fundamental role in proving that for a given $n \times n$

sign regular matrix A with a given signature, i.e., matrices whose minors of any fixed order have a fixed sign or allowed to vanish, the matrix $SA^{-1}S^{-1}$ is a sign regular matrix with the same signature, where $S = \text{diag}\,(1, -1, \ldots, (-1)^{n+1})$.

Lemma 1.3. *[9] Let $A \in \mathbb{R}^{n,n}$ be nonsingular. Then for any nonempty subsets $\alpha, \beta \subseteq \{1, \ldots, n\}$ with $|\alpha| = |\beta|$ the following equality holds*

$$\det A^{-1}[\alpha \mid \beta] = (-1)^s \frac{\det A\left[\beta^c \mid \alpha^c\right]}{\det A},$$

where $s := \sum_{\alpha_i \in \alpha} \alpha_i + \sum_{\beta_i \in \beta} \beta_i$.

We conclude this section by recalling the following famous identity. Assume $B = CD$, where B is an $n \times m$ matrix, C is an $n \times r$ matrix, and D is an $r \times m$ matrix. Then the *Cauchy-Binet* formula [20] states that for each $\mathbf{i} \in \mathbf{Q}_{p,n}$, $\mathbf{j} \in \mathbf{Q}_{p,m}$,

$$\det B[\mathbf{i}|\mathbf{j}] = \sum_{\mathbf{k} \in \mathbf{Q}_{p,r}} \det C[\mathbf{i}|\mathbf{k}] \det D[\mathbf{k}|\mathbf{j}].$$

Chapter 2

Sign Regular Matrices And The Cauchon Algorithm

In this chapter, we present the definition of sign regular matrices and some of its properties. Interesting subclasses of the sign regular matrices are the totally nonnegative and the totally nonpositive matrices. Therefore, we concentrate on them and introduce some necessary and sufficient conditions for a given matrix to be in their classes by using the Cauchon Algorithm.

2.1 Totally Nonnegative Matrices

In this section, we present the definition of sign regular matrices, totally nonnegative matrices, and the Cauchon Algorithm. In addition, we will introduce a procedure that will be helpful in constructing lacunary sequences for the Cauchon matrices which are useful to prove many theorems in our thesis.

In the following, we present the definition of sign regular matrices.

Definition 2.1. *An* $n \times m$ *matrix* A *is called* sign regular of order k*,* $k \leq \min\{m, n\}$*, with signature* $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ *if for each* $j = 1, \ldots, k$*, the sign of all of its minors of order* j *coincides with* ε_j *or vanishes. If* $k = \min\{m, n\}$*, then a sign regular matrix of order* k *is simply called* sign regular*. Moreover,* A *is called* strictly sign regular with signature ε *if it is sign regular with signature* ε *and all submatrices of* A *have nonzero determinants.*

In the following definition, we introduce the totally nonnegative matrices.

Definition 2.2. *Let A be an $n \times m$ matrix. Then A is said to be* totally nonnegative, *denoted by* TN *if A is sign regular of order $k = \min\{n, m\}$ with signature $\varepsilon = (+1, +1, \dots, +1)$, i.e.,*

$$\det A \left[\mathbf{i}|\mathbf{j}\right] = \det A \left[i_1, \dots, i_l | j_1, \dots, j_l\right] \geq 0, \tag{2.1}$$

for all $\mathbf{i} = (i_1, \dots, i_l) \in \mathbf{Q}_{l,n}$, $\mathbf{j} = (j_1, \dots, j_l) \in \mathbf{Q}_{l,m}$, and all $l = 1, \dots, \min\{n, m\}$. If the strict inequalities occur in (2.1), then the matrix A is called totally positive, *denoted by* TP.

Since the compound matrices involve all the minors, Definitions 2.1 and 2.2 can be rewritten by using the compound matrices concept as the following definition states.

Definition 2.3. *Let A be an $n \times m$ matrix. Then it is said to be sign regular matrix if all the nonzero entries in the p^{th}* compouned matrix *have the same sign, it is said to be totally nonnegative if all entries of the p^{th} compound matrix of A are nonnegative, $p = 1, 2, \dots, \min\{n, m\}$.*

Example 2.1. *Let* $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix}$.

To show that A is a sign regular matrix we compute all the p^{th} compound matrices of A for $p = 1, 2, 3$.

$$A^{[1]} = A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{bmatrix}_{\binom{4}{1}\times\binom{3}{1}=4\times 3}, \quad A^{[2]} = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 12 & 16 \\ 3 & 21 & 30 \\ 1 & 7 & 12 \\ 2 & 16 & 30 \\ 1 & 9 & 20 \end{bmatrix}_{\binom{4}{2}\times\binom{3}{2}=6\times 3}, \quad A^{[3]} = \begin{bmatrix} 2 \\ 6 \\ 6 \\ 2 \end{bmatrix}_{\binom{4}{3}\times\binom{3}{3}=4\times 1}.$$

It is clear that all the entries of $A^{[1]}, A^{[2]}$, and $A^{[3]}$ are nonnegative. Therefore, A is a sign regular matrix with signature $(+1, +1, +1)$. Hence A is a TN matrix. Moreover, since all the entries are positive then the matrix A is TP.

In order to be ready to present some lemmas, propositions, and the Cauchon Algorithm, we first need to present the following notation.

Define the set $E^\circ := \{1,\dots,n\} \times \{1,\dots,m\} \backslash \{(1,1)\}, E := E^\circ \cup \{(n+1,2)\}$. Let $(s,t) \in E^\circ$. Then $(s,t)^+ := \min\{(i,j) \in E \mid (s,t) \leq (i,j), (s,t) \neq (i,j)\}$; here the minimum is taken with respect to the lexicographical order.

In the following, we present the definition of the Cauchon diagram, Cauchon matrix and the Cauchon Algorithm.

Algorithm 2.1. *[12] [The Cauchon Algorithm] Let $A \in \mathbb{R}^{n,m}$. As r runs in decreasing order over the set E with respect to the lexicographical order, we define matrices $A^{(r)} = \left(a_{ij}^{(r)}\right) \in \mathbb{R}^{n,m}$ as follows:*

1. Set $A^{(n+1,2)} := A$.

2. For $r = (s,t) \in E^\circ$, define the matrix $A^{(r)} = \left(a_{ij}^{(r)}\right)$ as follows:

(a) If $a_{st}^{\left(r^+\right)} = 0$, then put $A^{(r)} := A^{\left(r^+\right)}$.

(b) If $a_{st}^{(r+)} \neq 0$, then put

$$a_{ij}^{(r)} := \begin{cases} a_{ij}^{\left(r^+\right)} - \frac{a_{it}^{\left(r^+\right)} a_{sj}^{(r+)}}{a_{st}^{\left(r^+\right)}}, & \text{for } i < s \text{ and } j < t, \\ a_{ij}^{\left(r^+\right)}, & \text{otherwise.} \end{cases}$$

3. Set $\tilde{A} := A^{(1,2)}$; $\tilde{A}$ is called the matrix obtained from A (by the Cauchon Algorithm).

Definition 2.4. *An $n \times m$* Cauchon diagram *C is a grid consisting of $n \cdot m$ squares colored black and white, where each black square has the property that either every square to its left (in the same row) or every square above it (in the same column) is black.*

Example 2.2. *(a) The following diagram is a Cauchon diagram, since its satisfying the properity that for each black square either every square to its left or every square above it is black.*

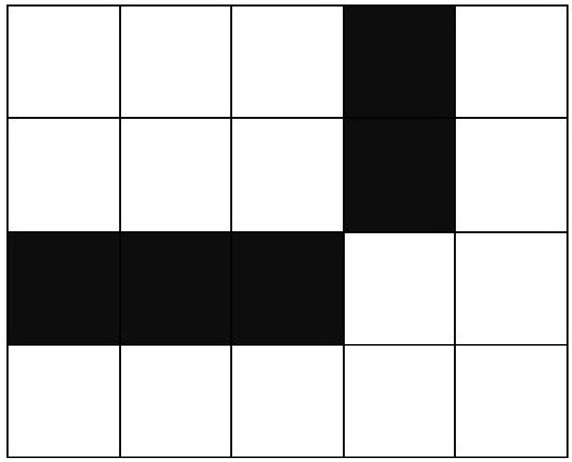

Figure 2.1: An example of a Cauchon diagram

(b) The following diagram is not a Cauchon diagram. Since for the black square in position (3,3) neither all squares above it are black nor all squares to the left of it are black.

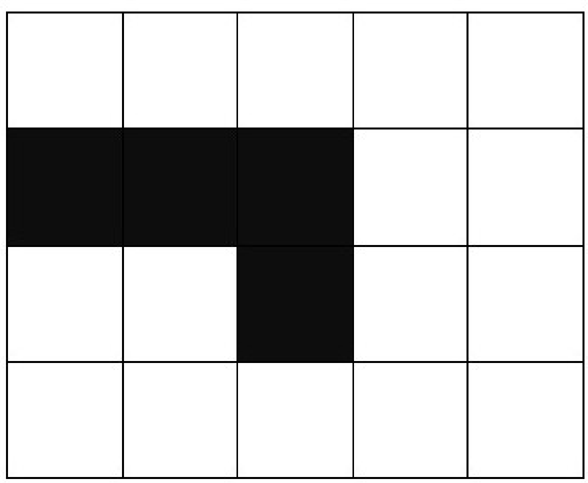

Figure 2.2: An example of a non-Cauchon diagram

We denote by $\mathcal{C}^{n,m}$ the set of the $n \times m$ Cauchon diagrams, when $n = m$ we write $\mathcal{C}^n$.

Definition 2.5. *[16] An $n \times m$ matrix $A = (a_{ij})$ is called* Cauchon matrix *if for each a_{ij} with $a_{ij} = 0$ for some $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, then $a_{sj} = 0$ for all $s = 1, \ldots, i-1$ or $a_{it} = 0$ for $t = 1, \ldots, j-1$.*

By replacing zero entries by black squares and nonzero entries by white squares and vice versa, the Cauchon diagram and Cauchon matrices will be used interchangeable.

Example 2.3. *Let* $A = \begin{bmatrix} 8 & 5 & 4 & 1 \\ 5 & 4 & 4 & 1 \\ 5 & 4 & 4 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. *Then by application of the Cauchon Algorithm on A we obtain*

$$A^{(5,2)} = A, \quad A^{(4,4)} = \begin{bmatrix} 7 & 4 & 3 & 1 \\ 4 & 3 & 3 & 1 \\ 4 & 3 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad A^{(4,3)} = \begin{bmatrix} 4 & 1 & 3 & 1 \\ 1 & 0 & 3 & 1 \\ 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$A^{(4,2)} = \begin{bmatrix} 3 & 1 & 3 & 1 \\ 1 & 0 & 3 & 1 \\ 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = A^{(4,1)}, \quad A^{(3,4)} = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$A^{(3,4)} = A^{(3,3)} = A^{(3,2)} = A^{(3,1)} = A^{(2,4)} = A^{(2,3)} = A^{(2,2)} = A^{(2,1)} = A^{(1,4)} = A^{(1,3)} = A^{(1,2)} = \tilde{A}.$$

Its clear that $\tilde{A}$ *is not a Cauchon matrix since* $\tilde{a}_{32} = 0$ *but* $\tilde{a}_{31} \neq 0$ *and* $\tilde{a}_{12} \neq 0$.

Example 2.4. *Let* $A = \begin{bmatrix} 4 & 4 & 4 \\ 4 & 8 & 8 \\ 4 & 8 & 16 \end{bmatrix}$. *Then by application of the Cauchon Algorithm on A we obtain*

$$A^{(4,2)} = A, \quad A^{(3,3)} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 4 & 8 \\ 8 & 8 & 16 \end{bmatrix}, \quad A^{(3,2)} = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 4 & 8 \\ 8 & 8 & 16 \end{bmatrix} = A^{(3,1)},$$

$$A^{(2,3)} = \begin{bmatrix} 2 & 0 & 4 \\ -2 & 4 & 8 \\ 8 & 8 & 16 \end{bmatrix}, \quad A^{(2,2)} = \begin{bmatrix} 2 & 0 & 4 \\ -2 & 4 & 8 \\ 8 & 8 & 16 \end{bmatrix} = A^{(2,1)} = A^{(1,3)} = A^{(1,2)} = \tilde{A}.$$

It is clear that $\tilde{A}$ *is a Cauchon matrix since it satisfies Definition 2.5.*

The following theorem gives necessary and sufficient conditions for a given matrix to be totally nonnegative or totally positive by using the Cauchon Algorithm.

Theorem 2.1. *[12] Let* $A \in \mathbb{R}^{n,m}$. *Then A is totally nonnegative (totally positive) if and only if* $0 \leq \tilde{A}$ $(0 < \tilde{A})$ *and* $\tilde{A}$ *is a Cauchon matrix.*

By the above theorem, it is easy to show that the above matrices are not TN.

Theorem 2.2. *[12] Let $A \in \mathbb{R}^{n,m}$ be TN. Then A is a Cauchon matix.*

The previous theorem states that one necessary condition for a given matrix A to be TN is to be a Cauchon matrix while the converse is not true as the following example illustrates.

Example 2.5. *The following matrix*

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 0 & 4 \\ -5 & 6 & 9 \end{bmatrix},$$

satisfies the definition of a Cauchon matrix but clear that the matrix A is not TN since the entry $a_{31} = -5$ which is not nonnegative.

In the following, we present the definition of a lacunary sequence that will be used in introducing and proving some theorems.

Definition 2.6. *[16] Let $C \in \mathcal{C}^{n,m}$. We say that a sequence*

$$\gamma := \left((i_k, j_k),\ k = 0, 1, \ldots, t\right),$$

which is strictly increasing in both arguments is a lacunary sequence with respect to C *if the following conditions hold:*

1. $(i_k, j_k) \notin C, k = 1, \ldots, t$.
2. $(i, j) \in C$ *for* $i_t < i \leq n$ *and* $j_t < j \leq m$.
3. *Let* $s \in \{1, \ldots, t-1\}$. *Then* $(i, j) \in C$

   (a) either for all $(i, j), i_s < i < i_{s+1}$ *and* $j_s < j$,
   or for all $(i, j), i_s < i < i_{s+1}$ *and* $j_0 \leq j < j_{s+1}$

   and

   (b) either for all $(i, j), i_s < i$ *and* $j_s < j < j_{s+1}$,
   or for all $(i, j), i < i_{s+1}$, *and* $j_s < j < j_{s+1}$.

We call t the length of γ.

The following procedure will be beneficial to construct a lacunary sequence that will be very useful in proving some theorems.

Procedure 2.1. *[2] Let $A \in \mathbb{R}^{n,m}$ be a Cauchon matrix. Construct the lacunary sequence*

$$\gamma = \left(\left(i_p, j_p \right), \ldots, \left(i_0, j_0 \right) \right),$$

as follows: Put $(i_{-1}, j_{-1}) := (n+1, m+1)$. For $k = 0, 1, \ldots, p$, define

$$M_k := \left\{ (i,j) \mid 1 \leq i < i_{k-1}, 1 \leq j < j_{k-1}, a_{ij} \neq 0 \right\}.$$

If $M_k = \phi$, put $p := k-1$.
Otherwise, put $(i_k, j_k) := \max M_k$, where the maximum is taken with respect to the lexicographical order.

The following is an illustration example for the above procedure.

Example 2.6. *Let* $A = \begin{bmatrix} 1 & 0 & -3 & -5 \\ 4 & 0 & 1 & 2 \\ 1 & 0 & 9 & 3 \\ 3 & 2 & -4 & 6 \end{bmatrix}$. *Then A is a Cauchon matrix and we use Procedure 2.1 to construct a lacunary sequence as follows*
$(i_{-1}, j_{-1}) := (n+1, m+1) = (5,5)$.
For the case that $k = 0$ we get

$$M_0 := \left\{ (i,j) \mid 1 \leq i < 5, 1 \leq j < 5, a_{ij} \neq 0 \right\},$$

i.e., $(i_0, j_0) := \max M_0 = (4,4)$, *since* $a_{4,4} = 6 \neq 0$.

For the case that $k = 1$ we get

$$M_1 := \left\{ (i,j) \mid 1 \leq i < 4, 1 \leq j < 4, a_{ij} \neq 0 \right\},$$

i.e., $(i_1, j_1) := \max M_1 = (3,3)$, *since* $a_{3,3} = 9 \neq 0$.

For the case that $k = 2$ we get

$$M_2 := \left\{ (i,j) \mid 1 \leq i < 3, 1 \leq j < 3, a_{ij} \neq 0 \right\},$$

i.e., $(i_2, j_2) := \max M_2 = (2,1)$, *since* $a_{2,2} = 0$ *and* $a_{2,1} = 4 \neq 0$.

For the case that $k = 3$ we get

$$M_3 := \left\{ (i,j) \mid 1 \leq i < 2, 1 \leq j < 1, a_{ij} \neq 0 \right\},$$

i.e., $(i_3, j_3) := \max M_3 = \max \phi$.

Hence, the lacunary sequence is $\gamma = \big((i_2, j_2), (i_1, j_1), (i_0, j_0)\big) = \big((2,1), (3,3), (4,4)\big)$.

Procedure 2.1 helps to deduce the following theorem that is used to calculate the rank of a given matrix A from $\tilde{A}$, provided that $\tilde{A}$ is Cauchon matrix.

Theorem 2.3. *[2] Let* $A \in \mathbb{R}^{n,m}$ *be such that* $\tilde{A}$ *is a Cauchon matrix. Then* $\operatorname{rank} A = p+1$, *where* p *is the length of the sequence which is obtained by application of Procedure 2.1 to* $\tilde{A}$.

The following is an illustration example for the above theorem.

Example 2.7. *Let* $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 6 & 8 \\ 4 & 8 & 12 \end{bmatrix}$. *Then after applying the Cauchon Algorithm to* A *we get that* $\tilde{A} = \begin{bmatrix} 0.5 & 0 & 4 \\ 0 & 0.6 & 8 \\ 4 & 8 & 12 \end{bmatrix}$ *which is a Cauchon matrix. By application of Procedure 2.1 to* $\tilde{A}$ *we construct the lacunary sequance* $\gamma = \big((1,1), (2,2), (3,3)\big)$, *hence* $p = 2$. *Then by Theorem 2.3 we get that* $\operatorname{rank} A = p + 1 = 2 + 1 = 3$.

In the following theorem, we will present how to find the value of some minors of A by using lacunary sequences with respeect to $\tilde{A}$, provided that $\tilde{A}$ is Cauchon matrix.

Theorem 2.4. *[2] Let* $A \in \mathbb{R}^{n,m}$ *be such that* $\tilde{A} = (\tilde{a}_{ij})$ *is a Cauchon matrix and let* $\gamma = \big((i_k, j_k),\ k = 0, 1, \ldots, p\big)$ *be a lacunary sequence. Then the following representation holds:*

$$\det A\left[i_0, \ldots, i_p \mid j_0, \ldots, j_p\right] = \tilde{a}_{i_0,j_0}\tilde{a}_{i_1,j_1} \cdots \tilde{a}_{i_p,j_p}.$$

Corollary 2.1. *[5] Let* $A \in \mathbb{R}^{n,n}$ *and assume that* $\tilde{A} = (\tilde{a}_{ij})$ *is a Cauchon matrix with* $\tilde{a}_{ii} \neq 0$, $i = 1, \ldots, n$. *Then the following equality holds*

$$\det A = \tilde{a}_{11} \cdots \tilde{a}_{nn}.$$

Example 2.8. *Suppose we have the same matrix* A *that we had in the Example 2.4. Then the lacunary sequnces* $\gamma_1 = \big((2,2), (3,3)\big)$ *and* $\gamma_2 = \big((1,1), (2,2), (3,3)\big)$. *Hence by Theorem 2.4 we obtain that*

$$\det A[2,3|2,3] = \begin{vmatrix} 8 & 8 \\ 8 & 16 \end{vmatrix} = \tilde{a}_{22} \cdot \tilde{a}_{33} = 4 \cdot 16 = 64,$$

$$\det A = \begin{vmatrix} 4 & 4 & 4 \\ 4 & 8 & 8 \\ 4 & 8 & 16 \end{vmatrix} = \tilde{a}_{11} \cdot \tilde{a}_{22} \cdot \tilde{a}_{33} = 2 \cdot 4 \cdot 16 = 128.$$

In the following two propositions and the lemma, we investigate the relationship between the minors of the intermediate matrices of the Cauchon Algorithm. These propositions will play important roles in proving some of the new results in this thesis.

Proposition 2.1. *[12] Let $A = \left(a_{ij}\right) \in \mathbb{R}^{n,m}$ and $r = (s,t) \in E^{\circ}$. Let $a_{st} \neq 0$, $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbf{Q}_{l,n}$ and $\beta = (\beta_1, \ldots, \beta_l) \in \mathbf{Q}_{l,m}$, where $l \leq \min\{n, m\}$ such that $(\alpha_l, \beta_l) = r$. Then*

$$\det A^{(r^+)}[\alpha \mid \beta] = \det A^{(r)}[\alpha_{\hat{s}} \mid \beta_{\hat{t}}] \;.\; a_{st}.$$

The following is an illustration example for the above proposition.

Example 2.9. *Suppose we have the same matrix A that we had in the Example 2.3.*

Let $r = (4,3)$, $\alpha = (1,2,4)$ and $\beta = (1,2,3)$. Then

$$\begin{aligned} \det A^{(4,4)}[1,2,4|1,2,3] &= \begin{vmatrix} 7 & 4 & 3 \\ 4 & 3 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -1, \\ &= \det A^{(4,3)}[1,2|1,2] \cdot a_{43} = \begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix} \cdot 1 = -1. \end{aligned}$$

Proposition 2.2. *[12] Let $A = \left(a_{ij}\right) \in \mathbb{R}^{n,m}$ and $r = (s,t) \in E^{\circ}$.* Let *$\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbf{Q}_{l,n}$, $\beta = (\beta_1, \ldots, \beta_l) \in \mathbf{Q}_{l,m}$, where $l \leq \min\{n, m\}$, and $(\alpha_l, \beta_l) < r$. If $a_{st} = 0$, or if $\alpha_l = s$, or if $t \in \{\beta_1, \ldots, \beta_l\}$, or if $t < \beta_1$, then*

$$\det A^{(r^+)}[\alpha \mid \beta] = \det A^{(r)}[\alpha \mid \beta].$$

The following is an illustration example for the above proposition.

Example 2.10. *Suppose we have the same matrix A that we had in the Example 2.3.*

Let $r = (3,4)$, $\alpha = (1,3)$, and $\beta = (2,3)$. Then

$$\det A^{(4,2)}[1,3|2,3] = \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} = \det A^{(3,4)}[1,3|2,3] = \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 3.$$

Lemma 2.1. *[12] Let* $A = \big(a_{ij}\big) \in \mathbb{R}^{n,m}$*, and let* $r = (s,t) \in E^{\circ}$*. Let* $\delta = \det A[\alpha|\beta]$ *with* $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbf{Q}_{l,n}$*,* $\beta = (\beta_1, \dots, \beta_l) \in \mathbf{Q}_{l,m}$*. Assume that* $a_{st} \neq 0$ *and that* $\alpha_l < s$ *while* $\beta_h < t < \beta_{h+1}$ *for some* $h \in \{1, \dots, l\}$ *(by convention,* $\beta_{l+1} := m+1$*). Then*

$$\delta^{(r^+)} = \delta^{(r)} + \sum_{k=1}^{h} (-1)^{k+h} \delta^{(r)}_{\beta_k \to t} a_{s,\beta_k} a_{st}^{-1}.$$

The following is an illustration example for the above lemma.

Example 2.11. *Suppose we have the same matrix A that we had in the Example 2.3.*

Let $r = (4,2)$*,* $\alpha = (1,2,3)$ *,* $\beta = (1,3,4)$*, and* $t = 2$*. Then* $\delta = \det A[1,2,3,|1,3,4]$ *and* $h = 1$*.*

$$\begin{aligned}
\delta^{(4,3)}[1,2,3|1,3,4] &= \delta^{(4,2)}[1,2,3|1,3,4] + \sum_{k=1}^{1} (-1)^{k+1} \cdot \delta^{(4,2)}[1,2,3|2,3,4] \cdot a_{41} \cdot a_{42}^{-1} \\
\begin{vmatrix} 4 & 3 & 1 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{vmatrix} &= \begin{vmatrix} 3 & 3 & 1 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{vmatrix} = 0.
\end{aligned}$$

2.2 Totally Nonpositive Matrices

In this section, we introduce the definition of totally nonpositive matrices and some important results on this class of matrices.

Definition 2.7. *[9] Let A be an* $n \times m$ *matrix. Then A is said to be* totally nonpositive *(t.n.p.) if A is a sign regular matrix of order* $k = \min\{n, m\}$ *with signature* $\varepsilon = (-1, -1, \dots, -1)$*, i.e.,*

$$\det A\Big[\mathbf{i}|\mathbf{j}\Big] = \det A\Big[i_1, \dots, i_l | j_1, \dots, j_l\Big] \leq 0, \tag{2.2}$$

for all $\mathbf{i} = (i_1, \dots, i_l) \in \mathbf{Q}_{l,n}$ *,* $\mathbf{j} = (j_1, \dots, j_l) \in \mathbf{Q}_{l,m}$*, and all* $l = 1, \dots, k$*. In addition, if A is nonsingular then we denote it by Ns.t.n.p. If the strict inequality occur in (2.2), then the matrix A is called* totally negative *(t.n.).*

In the following definition, we can write Definition 2.7 on *t.n.p* matrices by utilizing the concept of compound matrices as follows.

Definition 2.8. *Let A be an* $n \times m$ *matrix. Then it is said to be t.n.p. if all entries of the* p^{th} *compound matrix of A are nonpositive,* $p = 1, 2, \dots, \min\{n, m\}$*.*

Example 2.12. *Let* $B = \begin{bmatrix} -2 & -5 & -9 & -15 \\ -5 & -5 & -9 & -15 \\ -9 & -9 & -9 & -15 \\ -15 & -15 & -15 & -15, \end{bmatrix}$.

To show that A is a t.n.p matrix we compute all the p^{th} compound matrices of A for $p = 1, 2, 3, 4$.

$$B^{[1]} = B = \begin{bmatrix} -2 & -5 & -9 & -15 \\ -5 & -5 & -9 & -15 \\ -9 & -9 & -9 & -15 \\ -15 & -15 & -15 & -15, \end{bmatrix}_{\binom{4}{1}\times\binom{4}{1}=4\times 4},$$

$$B^{[2]} = \begin{bmatrix} -15 & -27 & -45 & 0 & 0 & 0 \\ -27 & -36 & -105 & 0 & -60 & 0 \\ -45 & -105 & -195 & -60 & -105 & -90 \\ 0 & -36 & -60 & -36 & -60 & 0 \\ 0 & -60 & -105 & -60 & -105 & -90 \\ 0 & 0 & -90 & 0 & -90 & -90 \end{bmatrix}_{\binom{4}{2}\times\binom{4}{2}=6\times 6},$$

$$B^{[3]} = \begin{bmatrix} -105 & -180 & 0 & 0 \\ -180 & -450 & -270 & 0 \\ 0 & -270 & -630 & -360 \\ 0 & 0 & -360 & -360 \end{bmatrix}_{\binom{4}{3}\times\binom{4}{3}=4\times 4}, \quad B^{[4]} = \begin{bmatrix} -1080 \end{bmatrix}_{\binom{4}{4}\times\binom{4}{4}=1\times 1}.$$

It is clear that all the entries of $B^{[1]}, B^{[2]}, , B^{[3]}$, and $B^{[4]}$ are nonpositive. Therefore, B is a sign regular matrix with signature $(-1, -1, -1, -1)$. Hence A is $Ns.t.n.p.$.

The following theorem presents equivalent statements for a nonsingular matrix to be *t.n.p.* using a few number of minors.

Theorem 2.5. *[14] Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ with $2 \le n$ be nonsingular. Then the following three statements are equivalent:*
(a) A is t.n.p.
(b) For any $k \in \{1, \ldots, n-1\}$,

$$a_{11} \le 0, \quad a_{nn} \le 0, \quad a_{n1} < 0, \quad a_{1n} < 0, \tag{2.3}$$

$$\det A[\alpha \mid k+1,\ldots,n] \leq 0, \text{ for all } \alpha \in \mathbf{Q}_{n-k,n}, \tag{2.4}$$

$$\det A[k+1,\ldots,n \mid \beta] \leq 0, \text{ for all } \beta \in \mathbf{Q}_{n-k,n}, \tag{2.5}$$

$$\det A[k,\ldots,n] < 0. \tag{2.6}$$

(c) For any $k \in \{1,\ldots,n-1\}$,

$$a_{11} \leq 0, \quad a_{nn} \leq 0, \quad a_{n1} < 0, \quad a_{1n} < 0, \tag{2.7}$$

$$\det A[\alpha \mid 1,\ldots,k] \leq 0, \text{ for all } \alpha \in \mathbf{Q}_{k,n}, \tag{2.8}$$

$$\det A[1,\ldots,k \mid \beta] \leq 0, \text{ for all } \beta \in \mathbf{Q}_{k,n}, \tag{2.9}$$

$$\det A[1,\ldots,k+1] < 0. \tag{2.10}$$

In the following theorem, two equivalent statements are given to present the relation between a *t.n.p.* matrix and the matrix obtained by the Cauchon Algorithm.

Theorem 2.6. *[5] Let* $A = \left(a_{ij}\right) \in \mathbb{R}^{n,n}$ *have all entries negative except possibly* $a_{11} \leq 0$. *Then the following two properties are equivalent:*
(i) A *is a Ns.t.n.p. matrix.*
(ii) $\tilde{A}$ *is a Cauchon matrix and* $\tilde{A}[1,\ldots,n-1]$ *is a nonnegative matrix with positive diagonal entries.*

The following theorem indicates that the entries of $\tilde{A}$ can be represented as ratios of contiguous minors if A is $Ns.t.n.p.$

Theorem 2.7. *[5] Let* $A = \left(a_{ij}\right) \in \mathbb{R}^{n,n}$ *be Ns.t.n.p. with* $a_{nn} < 0$. *Then the entries* $\tilde{a}_{kj}$ *of the matrix* $\tilde{A}$ *can be represented as* $(k, j = 1,\ldots,n)$

$$\tilde{a}_{kj} = \frac{\det A[k,\ldots,k+p \mid j,\ldots,j+p]}{\det A[k+1,\ldots,i+p \mid j+1,\ldots,j+p]},$$

with a suitable $0 \leq p \leq n-k$, *if* $j \leq k$ *and* $0 \leq p \leq n-j$, *if* $k < j$.

The following three theorems will play an important role in proving our main and new results.

Theorem 2.8. *[7] Let* $A \in \mathbb{R}^{(n,m)}$ *of rank* r, *and* ε *a signature sequence. If*

$$\varepsilon_k \det A[\alpha \mid \beta] \geqslant 0 \quad \text{for } \alpha \in Q_{k,n}, \beta \in Q_{k,m}, k = 1,2,\ldots,\min(n,m),$$

is valid whenever $d(\beta) \leqslant m-r$, *then* A *is sign regular with signature* ε.

Theorem 2.9. *[1] Let* $A \in \mathbb{R}^{n,m}$ *be t.n.p. and let* $\alpha = (i+1,\ldots,i+r), \beta = (j+1,\ldots,j+r)$ *for some* $i \in \{1,\ldots,n\}, j \in \{1,\ldots,m\}$, *and* $2 \leq r < \min\{n,m\} - 1$. *If* $A[\alpha \mid \beta]$ *has rank* $r-1$, *then*

(i) *either the rows* $i+1,\ldots,i+r$ *or the columns* $j+1,\ldots,j+r$ *of* A *are linearly dependent,*
or

(ii) *the right or left shadow of* $A[i+1,\ldots,i+r \mid j+1,\ldots,j+r]$ *has rank* $r-1$.

The following theorem present that the principal minors of $Ns.t.n.p.$ matrices are negative.

Theorem 2.10. *[19] Let* $A=\left(a_{ij}\right)\in\mathbb{R}^{n,n}$ *be* $Ns.t.n.p.$ *with* $a_{11}<0$ *and* $a_{nn}<0$*. Then* $\det A[\alpha]<0$ *for all* $\alpha\in Q_{k,n}, k=1,2,\ldots,n$.

Chapter 3

Main Results

In this chapter, we present our results on the characterization of a special class of sign regular matrices by applying the Cauchon Algorithm. In the first section, we will extend some results to rectangular totally nonpositive matrices. In the second section, we will introduce some characerizations and necessary and sufficent conditions for a given square matrix to be in the class of matrices whose determinants are positive and all other minors are nonpositive. We conclude this thesis by section three wherein we show that the so-called interval property holds for the class of matrices considered in section two.

3.1 Totally Nonpositive Matrices And The Cauchon Algorithm

In this section, we will extend some results on totally nonpositive matrices, which will be helpful in proving and deducing new results on the matrices we are interested in.

Here, we are interested in *totally nonpositive* matrices that do not contain zero rows or zero columns. The following Lemma present that for a given totally nonpositive matrix and the last entry is negative, then all entries in the last row and column are negative or the matrix has a zero row or a zero column.

Lemma 3.1. *Let* $A = (a_{ij}) \in \mathbb{R}^{n,m}$ *be a t.n.p. matrix with* $a_{nm} < 0$*. Then all entries in the last row and column are negative or* A *has a zero row or a zero column.*

Proof. Firstly, assume $a_{im} = 0$ for some $i \in \{1, \ldots, n-1\}$, i.e., the last column has a zero entry. Then for any $j \in \{1, \ldots, m-1\}$ consider

$$\begin{aligned} 0 \geq \det A[i, n|j, m] &= a_{ij}a_{nm} - a_{im}a_{nj} \\ &= a_{ij}a_{nm} \geq 0, \end{aligned}$$

since A is *t.n.p.*, $a_{im} = 0$, because $a_{nm} < 0$, we conclude that $a_{ij} = 0$ for all $j = 1, \ldots m-1$, i.e., A has a zero row. If $a_{nk} = 0$ for some $k \in \{1, \ldots, m-1\}$, i.e., the last row has a zero entry, we proceed similarly to conclude that A has a zero column. □

In the following theorem, we investigate the entries of the intermediate matrices that result by the application of the Cauchon Algorithm to a given *t.n.p.* matrix A. We proceed parallel to the case of a *Ns.t.n.p.* matrix which was derived in [1].

Theorem 3.1. *Let $A = (a_{ij}) \in \mathbb{R}^{n,m}$ be t.n.p. with $a_{nm} < 0$. If we apply the Cauchon Algorithm on A, then the following properties hold*

(i) *All entries of $A^{(n,t)}[1, \ldots, n-1|1, \ldots, m-1]$ are nonnegative for all $t = 2, \ldots, m$.*

(ii) *$A^{(n,t)}[1, \ldots, n-1|1, \ldots, t-1]$ is TN for all $t = 2, \ldots, m$.*

(iii) *$A^{(n,t)}[1, \ldots, n-1|1, \ldots, m-1]$ is TN for all $t = 2, \ldots, m$.*

(iv) *$A^{(n,2)}$ is a Cauchon matrix.*

(v) *For $t = 2, \ldots, m$, $\det A^{(n,t)}[\alpha|\beta] \leq 0$ for all $\alpha \in \mathbf{Q}_{l,n-1}, \beta = (\beta_1, \beta_2, \ldots, \beta_l) \in \mathbf{Q}_{l,m}$ with $\beta_l = m$ and $l = 1, \ldots, \min\{n-1, m\}$.*

Proof. (i) If $t = m$, then set $r = (n, m) \in E^\circ$. By Proposition 2.1, we have

$$\det A[i, n|j, m] = \det A^{(r+)}[i, n|j, m] = \det A^{(r)}[i|j] \cdot a_{nm} = a_{ij}^{(r)} \cdot a_{nm},$$

for $i \in \{1, \ldots, n-1\}$, $j \in \{1, \ldots, m-1\}$. Since A is *t.n.p.*, i.e., $\det A[i, n|j, m] \leq 0$ and $a_{nm} < 0$ it follows that

$$a_{ij}^{(r)} \geq 0 \text{ for all } i = 1, \ldots, n-1 \text{ and } j = 1, \ldots, m-1.$$

This proves the case $t = m$.

Now for the other cases, let $r = (n, t)$. Since the last row index in the underlying submatrices of the following minors equals n, we apply Proposition 2.2 to conclude that

$$\det A^{(r+)}[i, n|j, t] = \cdots = \det A[i, n|j, t] \text{ for } \textit{all } t \leq m-1. \tag{3.1}$$

By Proposition 2.1, we obtain

$$\det A[i,n|j,t] = \det A^{(r+)}[i,n|j,t] = \det A^{(r)}[i|j] \cdot a_{nt} = a_{ij}^{(r)} \cdot a_{nt}.$$

Now by Lemma 3.1, $a_{nt} < 0$ because A has no zero rows or columns and since the right hand side is nonpositive by the total nonpositivity of A, we conclude that $a_{ij}^{(r)} \geq 0$. This completes the proof of (i).

(ii) If $t = m$, then by (i) $A^{(n,m)}[1,\ldots,n-1|1,\ldots,m-1]$ is a nonnegative matrix. Furthermore, it follows from Proposition 2.1 and Proposition 2.2 with $r = (n,m)$ that

$$0 \geq \det A[\alpha_1,\ldots,\alpha_i,n|\beta_1,\ldots,\beta_i,m] = \det A^{(n+1,2)}[\alpha_1,\ldots,\alpha_i,n|\beta_1,\ldots,\beta_i,m] = \det A^{(n,m)}[\alpha_1,\ldots,\alpha_i|\beta_1,\ldots,\beta_i] \cdot a_{nm},$$

for all $(\alpha_1,\ldots,\alpha_i) \in \mathbf{Q}_{i,n-1}$, $(\beta_1,\ldots,\beta_i) \in \mathbf{Q}_{i,m-1}$, and $i = 1,2,\ldots,\min\{n-1,m-1\}$. Since A is *t.n.p.* and $a_{nm} < 0$, we obtain

$$\det A^{(n,m)}[\alpha_1,\ldots,\alpha_i|\beta_1,\ldots,\beta_i] \geq 0.$$

Hence $A^{(n,m)}[1,\ldots,n-1|1,\ldots,m-1]$ *is* TN.

This proves the case $t = m$. For the other cases, let $r = (n,t)$, $t < m$, and $\alpha \in \mathbf{Q}_{i,n-1}$, $\beta \in \mathbf{Q}_{i,t-1}$. Since the last row index in the underlying submatrices of the following minors equals n, we apply Proposition 2.2 so that

$$\det A^{(r+)}[\alpha_1,\ldots,\alpha_i,n|\beta_1,\ldots,\beta_i,t] = \cdots = \det A[\alpha_1,\ldots,\alpha_i,n|\beta_1,\ldots,\beta_i,t] \leq 0. \quad (3.2)$$

Also by Proposition 2.1 with $r = (n,t)$, we get

$$\det A^{(r+)}[\alpha_1,\ldots,\alpha_i,n|\beta_1,\ldots,\beta_i,t] = \det A^{(r)}[\alpha_1,\ldots,\alpha_i|\beta_1,\ldots,\beta_i] \cdot a_{nt}.$$

Now by Lemma 3.1, $a_{nt} < 0$ and since the right hand side is nonpositive by the total nonpositivity of A, we conclude by (3.2) that

$$\det A^{(n,t)}[\alpha_1,\ldots,\alpha_i|\beta_1,\ldots,\beta_i] \geq 0.$$

Hence $A^{(n,t)}[1,\ldots,n-1|1,\ldots,t-1]$ is TN, for all $t = 2,\ldots,m-1$. This the completes proof of (ii).

(iii) We will prove this statement by decreasing primary induction on the iteration number t and secondary induction on the order of the minors l.
For $t = m$, $A^{(n,m)}[1,\ldots,n-1|1,\ldots,m-1]$ is TN by (ii).

Suppose that $A^{(n,t+1)}[1,\ldots,n-1|1,\ldots,m-1]$ is TN, we want to show that $A^{(n,t)}[1,\ldots,n-1|1,\ldots,m-1]$ is TN, i.e., $0 \leq \det A^{(n,t)}[\alpha|\beta]$ for all $\alpha \in \mathbf{Q}_{l,n-1},\ \beta \in \mathbf{Q}_{l,m-1}$.
For the case $l = 1$, all entries of $A^{(n,t)}[1,\ldots,n-1|1,\ldots,m-1]$ are nonnegative for $t = 1,\ldots,m$ by (i).
Now assume the statement is true for all iterations $m-1,\ldots,t+1$ and all minors of order $1,\ldots,l-1$. We want to show the claim for iteration t and minors of order l.

Let $\alpha = (\alpha_1,\ldots,\alpha_l) \in \mathbf{Q}_{l,n-1}$ and $\beta = (\beta_1,\ldots,\beta_l) \in \mathbf{Q}_{l,m-1}$.
If $\beta_l < t$, then the matrix $A^{(n,t)}[\alpha_1,\alpha_2,\ldots,\alpha_l|\beta_1,\beta_2,\ldots,\beta_l]$ is a submatrix of $A^{(n,t)}[1,\ldots,n-1|1,\ldots,t-1]$ which is TN by (ii) and so $\det A^{(n,t)}[\alpha|\beta] \geq 0$.
If $t < \beta_1$ or t is contained in β, then by Proposition 2.2 , we have

$$\det A^{(n,t+1)}[\alpha|\beta] = \det A^{(n,t)}[\alpha|\beta],$$

which implies by the induction hypothesis on t that

$$\det A^{(n,t)}[\alpha|\beta] \geq 0.$$

Hence it remains to consider the case, where there exists h, $1 \leq h \leq l-1$, such that $\beta_h < t < \beta_{h+1}$ which implies $d(\beta) > 0$.

In order to prove the statement in this case we follow [1] and simplify the notation where we proceed by setting

$$[\alpha \mid \beta] := \det A^{(n,t)}[\alpha \mid \beta], \quad [\alpha \mid \beta]^+ := \det A^{(n,t+1)}[\alpha \mid \beta]$$

and for $j \in \{1,\ldots,h\}$, $\beta'_{\hat{j}} := \left(\beta_1,\ldots,\beta_{j-1},\beta_{j+1},\ldots,\beta_{l-1}\right)$, where $\beta'_{\hat{j}}$ has the length of $l-2$.

Now since $d(\beta) > 0$ and $\beta_h < t < \beta_{h+1}$, we use Lemma 1.2 to conclude that

$$\begin{aligned}&\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right] \cdot [\alpha \mid \beta] = \\ &\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{\beta_l\}\right] \cdot \left[\alpha \mid \beta'_{\hat{j}} \cup \left\{\beta_j, t\right\}\right] + \left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \left\{\beta_j\right\}\right] \cdot \left[\alpha \mid \beta'_{\hat{j}} \cup \{t,\beta_l\}\right],\end{aligned} \quad (3.3)$$

$k = 1,\ldots,l$.
It follows from the induction hypothesis on l that the minors $\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right]$,

$\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{\beta_l\}\right]$, and $\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{\beta_j\}\right]$ are nonnegative because they have order $l-1$.

Furthermore, since t is contained in the column set of the submatrices corresponding to the following minors, then by Proposition 2.2 we get that

$$\left[\alpha \mid \beta'_{\hat{j}} \cup \{\beta_j, t\}\right] = \left[\alpha \mid \beta'_{\hat{j}} \cup \{\beta_j, t\}\right]^+,$$

and

$$\left[\alpha \mid \beta'_{\hat{j}} \cup \{t, \beta_l\}\right] = \left[\alpha \mid \beta'_{\hat{j}} \cup \{t, \beta_l\}\right]^+.$$

Hence by induction on t the latter minors are also nonnegative.

Hence all of these equalities together imply that the left-hand side of (3.3) is nonnegative. If $0 < \left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right]$ for some k and j, then $0 \leq [\alpha|\beta]$, as desired. If for all k, j, $\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right] = 0$, then it follows by the Laplace expansion on column β_l that $\left[\alpha \mid \beta'_{\hat{j}} \cup \{\beta_l, t\}\right] = 0$. Then by Lemma 2.1 we have

$$\det A^{(n,t)^+}[\alpha \mid \beta] = \det A^{(n,t)}[\alpha \mid \beta].$$

Hence we obtain by induction on t that $0 \leq \det A^{(n,t)}[\alpha \mid \beta]$, as desired. This completes the induction step for the proof of (iii).

(iv) Since the entries in the last row and last column of A are negative (because they are not changed when running the Cauchon Algorithm). By Theorem 2.1, $A^{(n,2)}[1,\ldots,n-1|1,\ldots,m-1]$ is a Cauchon matrix since by (iii) the latter submatrix is TN. Hence $A^{(n,2)}$ is a Cauchon matrix.

(v) We prove the claim by decreasing primary induction on t and induction on l as in the proof of statement (iii). For $l = 1$, since $\beta_l = m$ we are referring to the entries in the last column which are negative by Lemma 3.1 as $a_{nm} < 0$ and the fact that the entries of the last column and row do not change during the application of the Cauchon Algorithm. If $t = m$, then by Proposition 2.2 we have $\det A^{(n,m)}[\alpha \mid \beta] = \det A[\alpha \mid \beta] \leq 0$ since $\beta_l = m$.

Suppose that the statement is true for all minors of order less than l (secondary induction) and for all steps $t+1, \ldots, m-1$ (primary induction). Let $\alpha \in \mathbf{Q}_{l,n-1}$

and $\beta = (\beta_1, \beta_2, \ldots, \beta_l) \in \mathbf{Q}_{l,m}$ with $\beta_l = m$.

If $t < \beta_1$ or $t = \beta_h$ for some $h = 1, \ldots, l$, then by Proposition 2.2 we have

$$\det A^{(n,t)^+}[\alpha \mid \beta] = \det A^{(n,t)}[\alpha \mid \beta],$$

which implies by the induction hypothesis on t that $\det A^{(n,t)}[\alpha \mid \beta] \leq 0$.
If $\beta_h < t < \beta_{h+1}$ for some $h = 1, \ldots, l-1$, then by Lemma 1.2 we have

$$\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right] \cdot [\alpha \mid \beta] =$$
$$\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{\beta_l\}\right] \cdot \left[\alpha \mid \beta'_{\hat{j}} \cup \left\{\beta_j, t\right\}\right] + \left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \left\{\beta_j\right\}\right] \cdot \left[\alpha \mid \beta'_{\hat{j}} \cup \{t, \beta_l\}\right].$$

The minors $\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right]$, $\left[\alpha \mid \beta'_{\hat{j}} \cup \left\{\beta_j, t\right\}\right]$, $\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \left\{\beta_j\right\}\right]$ are nonnegative by (iii).

$\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{\beta_l\}\right]$ is nonpositive by the induction hypothesis on l, $\left[\alpha \mid \beta'_{\hat{j}} \cup \{t, \beta_l\}\right] = \left[\alpha \mid \beta'_{\hat{j}} \cup \{t, \beta_l\}\right]^+$ by Proposition 2.2, and by the induction hypothesis on t the latter minor is nonpositive. All of these inequalities yield

$$\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right] \cdot [\alpha \mid \beta] \leq 0.$$

If $0 < \left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right]$ for some k and j, then we have $[\alpha \mid \beta] \leq 0$.
If for all k, j, $\left[\alpha_{\hat{k}} \mid \beta'_{\hat{j}} \cup \{t\}\right] = 0$, then proceeding parallel to the last part of (iii) we get

$$\det A^{(n,t+1)}[\alpha \mid \beta] = \det A^{(n,t)}[\alpha \mid \beta].$$

Hence we obtain by induction on t that $0 \leq \det A^{(n,t)}[\alpha \mid \beta]$, as desired.

□

By sequentially repeating the steps of the proof of Theorem 3.1, we obtain the following theorem.

Theorem 3.2. *Let* $A = (a_{ij}) \in \mathbb{R}^{n,m}$ *be t.n.p. with* $a_{nm} < 0$. *Then the following statements hold:*

(i) $A^{(s,t)}[1, \ldots, s-1 \mid 1, \ldots, t-1]$ *is TN for all* $s = 2, \ldots, n$ *and* $t = 2, \ldots, m$.

(ii) $A^{(s,2)}[1, \ldots, s-1 | 1, \ldots, t-1]$ *is* TN *for all* $s = 2, \ldots, n$ *and* $t = 2, \ldots, m$.

(iii) $\tilde{A}[1, \ldots, n-1 | 1, \ldots, m-1]$ *is a nonnegative matrix.*

(iv) $\tilde{A}$ *is a Cauchon matrix.*

3.2 Matrices Having A Positive Determinant and All Other Minors Nonpositive

In this section, we employ the results obtained so far to investigate and characterize the matrices having positive determinants and all other minors nonpositive by using the Cauchon Algorithm.

A square matrix that has a positive determinant and all other minors nonpositive, i.e., a nonsingular sign regular matrix with signature $(-1,-1,\ldots,-1,1)$, is denoted by $t.n.p^+$.

In the following theorem, we present some properties of the entries in the matrix that we obtain after applying the Cauchon Algorithm to a $t.n.p^+$ matrix.

Theorem 3.3. *Let $A=(a_{ij})\in\mathbb{R}^{n,n}$ be $t.n.p^+$ and $a_{nn}<0$. Then application of the Cauchon Algorithm to A results in the following properties:*

(1) $\tilde{a}_{ii}\neq 0$ for $i=3,\ldots,n-1$.

(2) If $\tilde{a}_{2j}=0$ for $j>2$, then $\tilde{a}_{2i}=0$ for all $i=2,\ldots,j$ or $\tilde{a}_{1j}=0$.

(3) If $\tilde{a}_{i2}=0$ for $i>2$, then $\tilde{a}_{j2}=0$ for all $j=2,\ldots,i$ or $\tilde{a}_{i1}=0$.

Proof. Since A is a nonsingular matrix, the rows and columns of A are linearly independent. Moreover, since A is $t.n.p^+$ we conclude that $A[1,\ldots,n|2,\ldots,n]$ and $A[2,\ldots,n|1,\ldots,n]$ are $t.n.p.$. Let $\tilde{A}$ be the matrix obtained by the application of the Cauchon Algorithm to A. Hence by Theorem 3.2

$$\tilde{A}[1,\ldots,n|2,\ldots,n] \text{ and } \tilde{A}[2,\ldots,n|1,\ldots,n]$$

are Cauchon matrices since all entries of the above matrices coincide with the corresponding entries of the matrices obtained by the running the Cauchon Algorithm on $A[1,\ldots,n|2,\ldots,n]$ and $A[2,\ldots,n|1,\ldots,n]$. The reason is that the entries of the first column and first row do not affect the calculation of the entries of $A[1,\ldots,n|2,\ldots,n]$ and the entries of $A[2,\ldots,n|1,\ldots,n]$, respectively.

(1) We prove this statement by decreasing induction on i for $i=n-1,\ldots,3$. Suppose that $\tilde{a}_{jj}>0$ for $j=n-1,\ldots,i+1$ and $\tilde{a}_{ii}=0$. Since $\tilde{A}[1,\ldots,n|2,\ldots,n]$ and $\tilde{A}[2,\ldots,n|1,\ldots,n]$ are Cauchon matrices, we distinguish between the following three cases:

Case (1) $\tilde{a}_{i,s} = 0$ for all $s = 1, \ldots, i-1$, i.e.,

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \ldots & \tilde{a}_{1,i-1} & \tilde{a}_{1i} & \tilde{a}_{1,i+1} & \ldots & \tilde{a}_{1,n} \\ \tilde{a}_{21} & \ldots & \tilde{a}_{2,i-1} & \tilde{a}_{2i} & \tilde{a}_{2,i+1} & \ldots & \tilde{a}_{2,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \tilde{a}_{i,i+1} & \ldots & \tilde{a}_{i,n} \\ \tilde{a}_{i+1,1} & \ldots & \tilde{a}_{i+1,i-1} & \tilde{a}_{i+1,i} & \tilde{a}_{i+1,i+1} & \ldots & \tilde{a}_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n,1} & \ldots & \tilde{a}_{n,i-1} & \tilde{a}_{n,i} & \tilde{a}_{n,i+1} & \ldots & \tilde{a}_{nn} \end{bmatrix}.$$

By applying Procedure 2.1 to $\tilde{A}[i, \ldots, n|1, \ldots, n]$ we construct the lacunary sequence $\gamma = \big((n,n), (n-1, n-1), \ldots, (i+1, i+1)\big)$ for the Cauchon matrix $\tilde{A}[i, \ldots, n|1, \ldots, n]$. By Theorem 2.3 the rank of the matrix $A[i, \ldots, n|1, \ldots, n]$ is equal to $n-(i+1)+1 = n-i$, which is a contradiction to the linear independence of the rows of A.

Case (2) $\tilde{a}_{t,i} = 0$ for all $t = 1, \ldots, i-1$, i.e.,

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \ldots & \tilde{a}_{1,i-1} & 0 & \tilde{a}_{1,i+1} & \ldots & \tilde{a}_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{i-1,1} & \ldots & \tilde{a}_{i-1,i-1} & 0 & \tilde{a}_{i-1,i+1} & \ldots & \tilde{a}_{i-1,n} \\ \tilde{a}_{i1} & \ldots & \tilde{a}_{i,i-1} & 0 & \tilde{a}_{i,i+1} & \ldots & \tilde{a}_{i,n} \\ \tilde{a}_{i+1,1} & \ldots & \tilde{a}_{i+1,i-1} & \tilde{a}_{i+1,i} & \tilde{a}_{i+1,i+1} & \ldots & \tilde{a}_{i+1,i} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n,1} & \ldots & \tilde{a}_{n,i-1} & \tilde{a}_{n,i} & \tilde{a}_{n,i+1} & \ldots & \tilde{a}_{nn} \end{bmatrix}.$$

By applying Procedure 2.1 to $\tilde{A}[1, \ldots, n|i, \ldots, n]$ we obtain the lacunary sequence $\gamma = \big((n,n), (n-1, n-1), \ldots, (i+1, i+1)\big)$ for the Cauchon matrix $\tilde{A}[1, \ldots, n|i, \ldots, n]$. By Theorem 2.3 the rank of the matrix $A[1, \ldots, n|i, \ldots, n]$ is equal to $n-i$, which is a contradiction to the linear independence of the columns of A.

Case (3) $\tilde{a}_{is} = 0$ for all $s = 2, \ldots, i-1$ and $\tilde{a}_{i1} \neq 0$ and $\tilde{a}_{ti} = 0$ for all $t = 2, \ldots i-1$ and $\tilde{a}_{1i} \neq 0$.
Since $\tilde{A}[1, \ldots, n|2, \ldots, n]$ and $\tilde{A}[2, \ldots, n|1, \ldots, n]$ are Cauchon matrices, then $\tilde{A}$ has the

following form:

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1,i-1} & \tilde{a}_{1,i} & \tilde{a}_{1,i+1} & \dots & \tilde{a}_{1,n} \\ \tilde{a}_{21} & 0 & \dots & 0 & 0 & \tilde{a}_{2,i+1} & \dots & \tilde{a}_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{i-1,1} & 0 & \dots & 0 & 0 & \tilde{a}_{i-1,i+1} & \dots & \tilde{a}_{i-1,n} \\ \tilde{a}_{i1} & 0 & \dots & 0 & 0 & \tilde{a}_{i,i+1} & \dots & \tilde{a}_{i,n} \\ \tilde{a}_{i+1,1} & \tilde{a}_{i+1,2} & \dots & \tilde{a}_{i+1,i-1} & \tilde{a}_{i+1,i} & \tilde{a}_{i+1,i+1} & \dots & \tilde{a}_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n,1} & \tilde{a}_{n,2} & \dots & \tilde{a}_{n,i-1} & \tilde{a}_{n,i} & \tilde{a}_{n,i+1} & \dots & \tilde{a}_{nn} \end{bmatrix}.$$

By applying Procedure 2.1 to $\tilde{A}[2,\dots,n|1,\dots,n]$ we construct the lacunary sequence $\gamma = \big((n,n),(n-1,n-1),\dots,(i+1,i+1),(i,1)\big)$ for the Cauchon matrix $\tilde{A}[2,\dots,n|1,\dots,n]$. By Theorem 2.3 the rank of the matrix $A[2,\dots,n|1,\dots,n]$ is equal to $n-i+1$. Since $i \geq 3$ then $n-i+1 \leq n-2$ which is a contradiction to the linear independence of the rows of this matrix.

Hence

$$a_{ii} \neq 0, \text{ for all } i = n-1,\dots,3.$$

(2) Let $\tilde{a}_{2j} = 0$ for $j > 2$. Then since $\tilde{A}[1,\dots,n|2,\dots,n]$ is a Cauchon matrix we get that either $\tilde{a}_{2i} = 0$ for all $i = 2,\dots,j-1$, or $\tilde{a}_{1j} = 0$.

(3) Let $\tilde{a}_{i2} = 0$ for $i > 2$. Then since $\tilde{A}[2,\dots,n|1,\dots,n]$ is a Cauchon matrix we get that either $\tilde{a}_{j2} = 0$ for all $j = 2,\dots,i-1$, or $\tilde{a}_{i1} = 0$.

□

Corollary 3.1. *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be t.n.p$^+$ and $a_{nn} < 0$. Then*

(1) If $\tilde{a}_{ij} = 0$ for $i > 2$ and $i < j$, then $\tilde{a}_{sj} = 0$ for all $s = 1,\dots,i-1$.

(2) If $\tilde{a}_{ij} = 0$ for $j > 2$ and $i > j$, then $\tilde{a}_{it} = 0$ for all $t = 1,\dots,j-1$.

Proof. Since $A = (a_{ij}) \in \mathbb{R}^{n,n}$ is $t.n.p^+$ and $a_{nn} < 0$. Then as in the proof of the above theorem, $\tilde{A}[1,\dots,n|2,\dots,n]$ and $\tilde{A}[2,\dots,n|1,\dots,n]$ are Cauchon matrices.

(1) Suppose that $\tilde{a}_{ij} = 0$ for $i > 2$ and $i < j$. We want to show that $\tilde{a}_{sj} = 0$ for all $s = 1,\dots,i-1$.
Suppose on the contrary that $\tilde{a}_{sj} \neq 0$ for some $s = 1,\dots,i-1$. Since $\tilde{a}_{ij} = 0$, $\tilde{a}_{sj} \neq 0$ for some $s = 1,\dots,i-1$, and $\tilde{A}[1,\dots,n|2,\dots,n]$ is a Cauchon matrix then $\tilde{a}_{ik} = 0$ for all $k = 2,\dots,j-1$. Now since $j > i$ we conclude that $\tilde{a}_{ii} = 0$ for $i > 2$ which provides a contradiction to Theorem 3.3 since $\tilde{a}_{ii} \neq 0$ for all $i \geq 3$.

(2) Suppose that $\tilde{a}_{ij} = 0$ for $j > 2$ and $i > j$. We want to show that $\tilde{a}_{it} = 0$ for all $t = 1, \ldots, j-1$.
Suppose on the contrary that $\tilde{a}_{it} \neq 0$ for some $t = 1, \ldots, j-1$. Now since $\tilde{a}_{ij} = 0$, $\tilde{a}_{it} \neq 0$ for some $t = 1, \ldots, j-1$ and $\tilde{A}[2, \ldots, n|1, \ldots, n]$ is a Cauchon matrix then we get that $\tilde{a}_{lj} = 0$ for all $l = 2, \ldots, i-1$. Now since $i > j$ we conclude that $\tilde{a}_{ii} = 0$ for $i > 2$. This is a contradiction to Theorem 3.3 since $\tilde{a}_{jj} \neq 0$ for all $j \geq 3$. □

In the following theorem, we present some properties for the matrix that we obtain after applying the Cauchon algorithm to a *t.n.p*$^+$ matrix, where the entry $(2, 2)$ in the obtained matrix equals zero.

Theorem 3.4. *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be t.n.p$^+$ with $a_{nn} < 0$, and let $\tilde{A} = (\tilde{a}_{ij})$ be the matrix obtained by the Cauchon Algorithm satisfying $\tilde{a}_{22} = 0$. Then*

(i) $\tilde{a}_{12}, \tilde{a}_{21} \neq 0$.

(ii) $\det A[1, \ldots, n-1|2, \ldots, n], \det A[2, \ldots, n|1, \ldots, n-1] < 0$.

(iii) $\det A[2, \ldots, n-1] < 0$.

(iv) $\tilde{a}_{i,i-1}, \tilde{a}_{i-1,i} > 0$ *for* $i = 3, \ldots, n-1$.

(v) If $\tilde{a}_{2j} = 0$ *for some* $j \in \{4, \ldots, n-1\}$ *then* $\tilde{a}_{1j} = 0$.

(vi) If $\tilde{a}_{i2} = 0$ *for some* $i \in \{4, \ldots, n-1\}$ *then* $\tilde{a}_{i1} = 0$.

Proof. Since A is a *t.n.p*$^+$ matrix, the rows and columns of A are linearly independent and $A[1, \ldots, n|2, \ldots, n]$ and $A[2, \ldots, n|1, \ldots, n]$ are *t.n.p.* with $a_{nn} < 0$. By Theorem 3.2 we get that

$$\tilde{A}[1, \ldots, n|2, \ldots, n] \text{ and } \tilde{A}[2, \ldots, n|1, \ldots, n]$$

are Cauchon matrices.

(i) Suppose by contradiction that $\tilde{a}_{12} = 0$, then $\tilde{A}$ looks like

$$\tilde{A} = \begin{bmatrix} \tilde{a}_{11} & 0 & \tilde{a}_{13} & \ldots & \tilde{a}_{1,n} \\ \tilde{a}_{21} & 0 & \tilde{a}_{23} & \ldots & \tilde{a}_{2,n} \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} & \ldots & \tilde{a}_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n,1} & \tilde{a}_{n,2} & \tilde{a}_{n,3} & \ldots & \tilde{a}_{n,n} \end{bmatrix}.$$

By Theorem 3.3, $\tilde{a}_{ii} > 0$ for $i = 3, \ldots, n-1$. Applying Procedure 2.1 to $\tilde{A}[1, \ldots, n|2, \ldots, n]$, we obtain the lacunary sequence $\gamma = \big((n,n), (n-1, n-1), \ldots, (3,3)\big)$. Hence by Theorem 2.3, the rank of the matrix $A[1, \ldots, n|2, \ldots, n]$ is $n-2$ which is a contradiction to the linear independence of the columns of $A[1, \ldots, n|2, \ldots, n]$. For $\tilde{a}_{21} = 0$, proceed similarly as in the case $\tilde{a}_{12} = 0$ and work on $\tilde{A}\left[2, \ldots, n|1, \ldots, n\right]$.

(ii) Since A is a nonsingular matrix then A^{-1} exist. Set the matrix $C = \big(c_{ij}\big) := SA^{-1}S^{-1}$, where $S = \text{diag}\ (1, -1, \ldots, (-1)^{n+1})$.
Now for $\alpha, \beta \in \mathbf{Q}_{l,n}$, $\det S[\alpha|\alpha] = \det S^{-1}[\alpha|\alpha] = (-1)^{\sum_{\alpha_i \in \alpha} \alpha_i + l}$ and $\det S[\alpha|\beta] = 0$ for $\alpha \neq \beta$. By using the Cauchy-Binet formula we get

$$\begin{aligned}
\det C[\alpha|\beta] &= \det S[\alpha|\alpha] \cdot \det A^{-1}[\alpha|\beta] \cdot \det S^{-1}[\beta|\beta] \\
&= (-1)^{\sum_{\alpha_i \in \alpha} \alpha_i + l} \cdot \det A^{-1}[\alpha|\beta] \cdot (-1)^{\sum_{\beta_i \in \beta} \beta_i + l} \\
&= (-1)^{\sum_{\alpha_i \in \alpha} \alpha_i + \beta_i} \cdot \det A^{-1}[\alpha|\beta].
\end{aligned}$$

By using Lemma 1.3, we get

$$\det C[\alpha|\beta] = \frac{\det A[\beta^c|\alpha^c]}{\det A}. \tag{3.4}$$

Since for $l = 1, \ldots, n-1$, $\det A[\beta^c|\alpha^c] \leq 0$ and $\det A > 0$, we obtain that $\det C[\alpha|\beta] \leq 0$ for all $l = 1, \ldots, n-1$. Moreover, for $\alpha, \beta = (1, \ldots, n)$,

$$\begin{aligned}
\det C &= \det S \det A^{-1} \det S^{-1} \\
&= \det A^{-1} \\
&= \frac{1}{\det A} > 0.
\end{aligned}$$

Hence C is $t.n.p^{+}$.
Now by (3.4), we get that

$$c_{1n} = \frac{\det A[1, \ldots, n-1|2, \ldots, n]}{\det A}. \tag{3.5}$$

We claim that c_{1n} is negative by which we conclude that $\det A[1, \ldots, n-1|2, \ldots, n]$ is negative. Suppose to the contrary that $c_{1n} = 0$. Since C is nonsingular, then the matrix has neither a zero row nor a zero column. Hence we may assume that $c_{1j}, c_{in} \neq 0$ for some $i \in \{2, \ldots, n\}$, $j \in \{1, \ldots, n-1\}$. Hence

$$\begin{aligned}
0 \geq \det C[1, i|j, n] &= c_{1j}c_{in} - c_{1n}c_{ij} \\
&= c_{1j}c_{in},
\end{aligned}$$

but since c_{1j}, c_{in} are negative, the right hand side is positive which is a contradiction. Therefore, $c_{1n} < 0$. Hence by (3.5)

$$\det A[1,\ldots,n-1|2,\ldots,n] < 0.$$

Now for $\det A[2,\ldots,n|1,\ldots,n-1]$ we apply the same steps on c_{n1}, where

$$c_{n1} = \frac{\det A[2,\ldots,n|1,\ldots,n-1]}{\det A}, \tag{3.6}$$

to conclude that

$$\det A[2,\ldots,n|1,\ldots,n-1] < 0.$$

(iii) Since $\tilde{a}_{ii} \neq 0$ for $i = 3,\ldots,n$, we construct the lacunary sequance $\big((2,2),(3,3),\ldots,(n,n)\big)$ with respect to the Cauchon matrix $\tilde{A}[2,\ldots,n|1,\ldots,n]$, and by the signature of A we get that the matrix $A[2,\ldots,n]$ is a *t.n.p.*. Hence by Theorem 3.2 we obtain that $\tilde{A}[2,\ldots,n]$ is a Cauchon matrix. Moreover, by Theorem 2.4 we conclude that

$$\det A[2,\ldots,n] = \tilde{a}_{22}\cdot\tilde{a}_{33}\cdots\tilde{a}_{nn} = 0, \tag{3.7}$$

since $\tilde{a}_{22} = 0$. Application of Lemma 1.1 to the matrix A yields

$$\begin{aligned}\det[2,\ldots,n-1]\det A &= \det A[1,\ldots,n-1]\det A[2,\ldots,n]\\ &- \det A[1,\ldots,n-1|2,\ldots,n]\det A[2,\ldots,n|1,\ldots,n-1].\end{aligned}$$

Since $\det A > 0$, $\det A[1,\ldots,n-1|2,\ldots,n], \det A[2,\ldots,n|1,\ldots,n-1] < 0$ by (ii), and $\det[2,\ldots,n] = 0$ by (3.7), we conclude that

$$\det[2,\ldots,n-1] < 0.$$

(iv) By the signature of A and Theorem 3.3, we get that the matrix $A[i,\ldots,n]$, $i = 3,\ldots,n-1$, are $Ns.t.n.p.$, and $\tilde{a}_{ii} \neq 0$ for $i = 3,\ldots,n-1$.

Since $A[i,\ldots,n-1|i+1,\ldots,n]$ and $A[i+1,\ldots,n|i,\ldots,n-1]$ are principal minors for the $Ns.t.n.p.$ matrices $A[1,\ldots,n-1|2,\ldots,n]$ and $A[2,\ldots,n|1,\ldots,n-1]$ respectively, then by Theorem 2.10 we conclude that

$$\det A[i,\ldots,n-1|i+1,\ldots,n] < 0, \quad \det A[i+1,\ldots,n|i,\ldots,n-1] < 0, \tag{3.8}$$

for $i = 3,\ldots,n-1$.

Application of Lemma 1.1 to $A[2,\ldots,n]$ yields

$$\begin{aligned}\det A[3\ldots,n-1]\det A[2\ldots,n] &= \det A[2\ldots,n-1]\det A[3\ldots,n]\\ &- \det A[2\ldots,n-1|3\ldots,n]\det A[3\ldots,n|2\ldots,n-1].\end{aligned}$$

Since $\det A[2\ldots,n]=0$, we get

$$\det A[2\ldots,n-1]\det A[3\ldots,n]=\det A[2\ldots,n-1|3\ldots,n]\det A[3\ldots,n|2\ldots,n-1].$$

Because $\det A[2\ldots,n-1]\neq 0$ from (ii) and $\tilde{a}_{ii}\neq 0$ for $i=3,\ldots,n$, by Procedure 2.1 we find the lacunary sequence $\big((3,3),\ldots,(n,n)\big)$ and by Theorem 2.4 we obtain $\det A[3\ldots,n]\neq 0$, i.e.,

$$\det A[2\ldots,n-1|3\ldots,n],\ \det A[3\ldots,n|2\ldots,n-1]<0.$$

Hence

$$\det A[i-1,\ldots,n-1|i,\ldots,n],\ \det A[i,\ldots,n|i-1,\ldots,n-1]<0, \tag{3.9}$$

for $i=3,\ldots,n$.

Now to prove that $\tilde{a}_{i,i-1},\tilde{a}_{i-1,i}>0$, we verify the following formulas

$$\tilde{a}_{i,i-1}=\frac{\det A[i,\ldots,n|i-1,\ldots,n-1]}{\det A[i+1,\ldots,n|i,\ldots,n-1]}, \tag{3.10}$$

$$\tilde{a}_{i-1,i}=\frac{\det A[i-1,\ldots,n-1|i,\ldots,n]}{\det A[i,\ldots,n-1|i+1,\ldots,n]}, \tag{3.11}$$

by decreasing induction on i for $i=n-1,\ldots,3$. For $i=n-1$, by Lemma 3.1 we get that $\tilde{a}_{n-1,n}=a_{n-1,n}\neq 0$ and $\tilde{a}_{n,n-1}=a_{n,n-1}\neq 0$. We construct the lacunary sequences $\big((n-2,n-1),(n-1,n)\big)$ and $\big((n-1,n-2),(n,n-1)\big)$ for the matrices $\tilde{A}[n-1,n|n-2,n-1]$ and $\tilde{A}[n-2,n-1|n-1,n]$, respectively. By Theorem 2.4 we conclude that

$$\det A[n-1,n|n-2,n-1]=\tilde{a}_{n-1,n-2}\cdot\tilde{a}_{n,n-1},$$

$$\det A[n-2,n-1|n-1,n]=\tilde{a}_{n-2,n-1}\cdot\tilde{a}_{n-1,n},$$

or equivalently,

$$\tilde{a}_{n-1,n-2}=\frac{\det A[n-1,n|n-2,n-1]}{\det A[n|n-1]},$$

$$\tilde{a}_{n-2,n-1}=\frac{\det A[n-2,n-1|n-1,n]}{\det A[n-1|n]}.$$

Since by (3.9) $\det A[n-1,n|n-2,n-1],\det A[n-2,n-1|n-1,n]<0$, we obtain $\tilde{a}_{n-1,n-2},\tilde{a}_{n-2,n-1}>0$.

Suppose that $\tilde{a}_{i,i-1},\tilde{a}_{i-1,i}>0$, for $i=n-1,\ldots,k+1$, $k\geq 4$. Now, for $i=k$, $\tilde{a}_{k+1,k},\tilde{a}_{k+2,k+1},\ldots,\tilde{a}_{n-1,n-2}\neq 0$ and $\tilde{a}_{k,k+1},\tilde{a}_{k+1,k+2},\ldots,\tilde{a}_{n-2,n-1}\neq 0$ by induction hypothesis. So we find the lacunary sequences $\big((k,k-1),(k+1,k),\ldots,(n,n-1)\big)$ and

$\big((k-1,k),(k,k+1),\dots,(n-1,n)\big)$ for the matrices $\tilde{A}[k,\dots,n|k-1,\dots,n-1]$ and $\tilde{A}[k-1,\dots,n-1|k,\dots,,n]$, respectively, and by Theorem 2.4 we conclude that

$$\det A[k,\dots,n|k-1,\dots,n-1] = \tilde{a}_{k,k-1}\cdot\tilde{a}_{k+1,k}\cdots\tilde{a}_{n,n-1},$$

$$\det A[k-1,\dots,n-1|k,\dots,,n] = \tilde{a}_{k-1,k}\cdot\tilde{a}_{k,k+1}\cdots\tilde{a}_{n-1,n}.$$

Moreover, it follows by the induction hypothesis that

$$\det A[k+1,\dots,n|k,\dots,n-1] = \tilde{a}_{k+1,k}\cdots\tilde{a}_{n,n-1},$$

$$\det A[k,\dots,n-1|k+1,\dots,n] = \tilde{a}_{k,k+1}\cdots\tilde{a}_{n-1,n},$$

which yields

$$\tilde{a}_{k,k-1} = \frac{\det A[k,\dots,n|k-1,\dots,n-1]}{\det A[k+1,\dots,n|k,\dots,n-1]},$$
$$\tilde{a}_{k-1,k} = \frac{\det A[k-1,\dots,n-1|k,\dots,n]}{\det A[k,\dots,n-1|k+1,\dots,n]},$$

where $\det A[k,\dots,n|k-1,\dots,n-1]$, $\det A[k+1,\dots,n|k,\dots,n-1]$, $\det A[k-1,\dots,n-1|k,\dots,,n]$, $\det A[k,\dots,n-1|k+1,\dots,n]$ are negative by (3.8) and (3.9). Thus $\tilde{a}_{k,k-1},\tilde{a}_{k-1,k} > 0$. Hence by induction hypothesis we get

$$\tilde{a}_{i,i-1},\tilde{a}_{i-1,i} > 0, \text{ for } i = 3,\dots,n-1.$$

(v) By the signature of A, the matrix $A[1,\dots,n|2,\dots,n]$ is a $t.n.p.$ matrix with $a_{nn} < 0$. By Theorem 3.2, the matrix $\tilde{A}[1,\dots,n|2,\dots,n]$ is a Cauchon matrix. Now, since $\tilde{a}_{2j} = 0$ for some $j \in \{4,\dots,n-1\}$ and $\tilde{A}[1,\dots,n|2,\dots,n]$ is a Cauchon matrix, by Definition 2.5 we obtain that either $\tilde{a}_{2k} = 0$ for all $k = 2,\dots,j-1$ or $\tilde{a}_{1j} = 0$. Since $\tilde{a}_{23} \neq 0$ by (iv) which excludes $\tilde{a}_{2k} = 0$ for all $k = 2,\dots,j-1$, we get that $\tilde{a}_{1j} = 0$.

(vi) By the signature of A, the matrix $A[2,\dots,n|1,\dots,n]$ is a $t.n.p.$ matrix with $a_{nn} < 0$. By Theorem 3.2, the matrix $\tilde{A}[2,\dots,n|1,\dots,n]$ is a Cauchon matrix. Now, since $\tilde{a}_{i2} = 0$ for some $i \in \{4,\dots,n-1\}$ and $\tilde{A}[2,\dots,n|1,\dots,n]$ is a Cauchon matrix, by Definition 2.5 we obtain that either $\tilde{a}_{k2} = 0$ for all $k = 2,\dots,i-1$ or $\tilde{a}_{i1} = 0$. Since $\tilde{a}_{32} \neq 0$ by (iv) which exclude $\tilde{a}_{k2} = 0$ for all $k = 2,\dots,i-1$, we get that $\tilde{a}_{i1} = 0$.

□

Remark 3.1. *In Theorem 3.4, we did not rely on the value of $\tilde{a}_{22}$ in the proof of (ii), and therefore (ii) is valid at any value of $\tilde{a}_{22} \geq 0$.*

The next theorem presents necessary and sufficient conditions for a given matrix to be $t.n.p^+$ under certain conditions.

Theorem 3.5. *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ have all entries negative,* $\det A[1,\dots,n-1] \le 0$, *and $\tilde{A} = (\tilde{a}_{ij})$ be the matrix obtained by the Cauchon Algorithm satisfying $\tilde{a}_{22} > 0$. Then A is t.n.p$^+$ if and only if the following statements hold*

(i) $\tilde{A}$ is a Cauchon matrix.

(ii) $\tilde{a}_{in}, \tilde{a}_{ni} < 0$ for all $i = 1,\dots,n$.

(iii) $\tilde{a}_{ii} > 0$ for all $i = 3,\dots,n-1$.

(iv) $\tilde{a}_{i,i-1}$, $\tilde{a}_{i-1,i} > 0$ for $i = 2,\dots,n-1$.

(v) $\tilde{a}_{11} < 0$.

(vi) $\tilde{a}_{ij} \ge 0$ for $i = 1,\dots,n-1$, $j = 2,\dots,n-1$, and $i = 2,\dots,n-1$, $j = 1$.

Proof. To prove the necessity, let A be *t.n.p*$^+$ with $a_{nn} < 0$ and $\tilde{a}_{22} > 0$.

(i) Since A is *t.n.p*$^+$ with $a_{nn} < 0$, then as in the proof of Theorem 3.3, $\tilde{A}[1,\dots,n|2,\dots,n]$ and $\tilde{A}[2,\dots,n|1,\dots,n]$ are Cauchon matrices. By Corollary 3.1 if $\tilde{a}_{ij} = 0$ for some $i > 2$ and $i < j$, then $\tilde{a}_{sj} = 0$ for all $s = 1,\dots,i-1$ and if $\tilde{a}_{ij} = 0$ for $j > 2$ or $i > j$, then $\tilde{a}_{it} = 0$ for all $t = 1,\dots,j-1$. If $\tilde{a}_{2j} = 0$ or $\tilde{a}_{i2} = 0$ for some $j > 2$ or $i > 2$, then by Theorem 3.3 and since $\tilde{a}_{22} > 0$, we conclude that $\tilde{a}_{1j} = 0$ or $\tilde{a}_{i1} = 0$. Hence $\tilde{A}$ is a Cauchon matrix.

(ii) It is clear that $\tilde{a}_{in} = a_{in}$, $\tilde{a}_{nj} = a_{nj}$ for all $i,j = 1,\dots,n$ since the entries in the last row and column do not change when applying the Cauchon Algorithm to the matrix A. Moreover, since A is *t.n.p*$^+$, then $a_{in}, a_{nj} \le 0$ for all $i,j = 1,\dots,n-1$. By Lemma 3.1 and since $\tilde{A}$ is a Cauchon matrix and A is nonsingular, then all entries in the last row and column are negative.

(iii) Since A is *t.n.p*$^+$ and $a_{nn} < 0$, then by Theorem 3.3 we get that $\tilde{a}_{ii} \ne 0$ for all $i = 3,\dots,n-1$. To prove the positivity of these entries, we will proceed using decreasing induction on $i = n-1,\dots,3$.
For $i = n-1$, since $\tilde{A}$ is a Cauchon matrix, then $\big((n-1,n-1),(n,n)\big)$ is a lacunary sequence for $\tilde{A}$. Moreover, since $\tilde{a}_{ii} \ne 0$ for $i = n-1, n$, then by Theorem 2.4 we have

$$0 \ge \det A\,[n-1,n] = \tilde{a}_{n-1,n-1} \cdot \tilde{a}_{nn} \ne 0,$$

since $\tilde{a}_{nn} = a_{nn} < 0$, we get that $\tilde{a}_{n-1,n-1} > 0$.

Suppose that $\tilde{a}_{n-1,n-1},\dots,\tilde{a}_{i+1,i+1}$ are positive. Since $\tilde{A}$ is a Cauchon matrix, then $\big((i,i),(i+1,i+1),\dots,(n,n)\big)$ is a lacunary sequence for $\tilde{A}$. Since $\tilde{a}_{ii} \ne 0$ for all $i = n,\dots,3$, then by Theorem 2.4 we get that

$$0 \geq \det A\left[i,\ldots,n\right] = \tilde{a}_{ii} \cdot \tilde{a}_{i+1,i+1} \cdots \tilde{a}_{nn} \neq 0.$$

Since $\tilde{a}_{nn} = a_{nn} < 0$ and $\tilde{a}_{n-1,n-1},\ldots,\tilde{a}_{i+1,i+1}$ are positive by the induction hypothesis, we conclude that $\tilde{a}_{ii} > 0$. This completes proof of statement (iii).

(iv) By Theorem 3.4 and Remark 3.1,

$$\det A[1,\ldots,n-1|2,\ldots,n],\ \det A[2,\ldots,n|1,\ldots,n-1] < 0$$

By proceeding as in proof of Theorem 3.4, we conclude that

$$\tilde{a}_{i,i-1}, \tilde{a}_{i-1,i} > 0 \text{ for } i = 2,\ldots,n-1.$$

(v) Since $\tilde{A}$ is a Cauchon matrix, $\tilde{a}_{22} > 0$, and $\tilde{a}_{ii} \neq 0$ for $i = 3,\ldots,n$, then $\big((1,1),\ldots,(n,n)\big)$ is a lacunary sequence for $\tilde{A}$. Since A is nonsingular with $\det A > 0$, then by Theorem 2.4

$$0 < \det A = \tilde{a}_{11} \cdot \tilde{a}_{22} \cdots \tilde{a}_{nn}.$$

Since $\tilde{a}_{nn} = a_{nn} < 0$, $\tilde{a}_{22},\ldots,\tilde{a}_{n-1,n-1}$ are positive, then we conclude that $\tilde{a}_{11} < 0$.

(vi) Since A is $t.n.p^{+}$, then $A\big[1,\ldots,n|2,\ldots,n\big]$ is $t.n.p.$ and by Theorem 3.2 we conclude that the matrix $\tilde{A}\big[1,\ldots,n-1|2,\ldots,n-1\big]$ is a nonnegative matrix, i.e.,

$$\tilde{a}_{ij} \geq 0, \text{ for } \quad \text{all } i = 1,\ldots,n-1,\ j = 2,\ldots,n-1. \tag{3.12}$$

Now by the same steps on the matrix $A\big[2,\ldots,n|1,\ldots,n\big]$ we get by Theorem 3.2 that the matrix $\tilde{A}\big[2,\ldots,n-1|1,\ldots,n-1\big]$ is nonnegative, i.e.,

$$\tilde{a}_{ij} \geq 0,\ \textit{for all}\, i = 2,\ldots,n-1,\ j = 1,\ldots,n-1. \tag{3.13}$$

Hence by (3.12) and (3.13) we get that

$$\tilde{a}_{ij} \geq 0,$$

for all $i = 1,\ldots,n-1$, $j = 2,\ldots,n-1$, and for all $i = 2,\ldots,n-1$, $j = 1$, which completes the proof of (v) and the first direction.

For the converse direction, we aim at showing that the matrix A is $t.n.p^+$. We will prove that

$$\det A[\alpha|\beta] \leq 0,$$

for all $\alpha, \beta \in \mathbf{Q}_{k,n}$, $k = 1, \ldots, n-1$ and $\det A > 0$.
To prove the latter inequality, since $\tilde{A}$ is a Cauchon matrix, $a_{nn} < 0$, $\tilde{a}_{22} > 0$, and by (iii) and (v) we have that $\tilde{a}_{ii} \neq 0$ for all $i = 1, \ldots, n$. Since $\big((1,1), \ldots, (n,n)\big)$ is a lacunary sequence for $\tilde{A}$, we conclude by Corollary 2.1 that

$$\det A[1, \ldots, n] = \det A = \tilde{a}_{11} \cdot \tilde{a}_{22} \cdots \tilde{a}_{n-1,n-1} \cdot \tilde{a}_{nn}.$$

Hence $\det A > 0$.

In the same manner we obtain that $\det A[k, \ldots, n] < 0$ for $k = 2, \ldots, n-1$. Moreover, by following the proof of Theorem 3.9 in [1], we have

$$\det A[\alpha|\beta] \leq 0,$$

for all $\alpha = (\alpha_1, \ldots, \alpha_l), \beta = (\beta_1, \ldots, \beta_l) \in \mathbf{Q}_{l,n}$ with $\alpha_l = n$ or $\beta_l = n$. Hence $A[2, \ldots, n]$ is $Ns.t.n.p.$ by Theorem 2.5 and the fact that $\tilde{A}[2, \ldots, n] = A[2, \ldots, n]$.

Furthermore, in the same manner and by (iv), we have

$$\det A[i, \ldots, n|i-1, \ldots, n-1] < 0,$$

$$\det A[i-1, \ldots, n-1|i, \ldots, n] < 0,$$

for $i = 2, \ldots, n$, and for $k = 2, \ldots, n$,

$$\det A[k, \ldots, n|\beta] \leq 0, \text{ for all } \beta \in \mathbf{Q}_{n-k+1,n-1},$$

$$\det A[\alpha|k, \ldots, n] \leq 0, \text{ for all } \alpha \in \mathbf{Q}_{n-k+1,n-1}.$$

In addition, since $A[2, \ldots, n]$ is $Ns.t.n.p.$, we obtain for $k = 3, \ldots, n$, that

$$\det A[\alpha|k-1, \ldots, n-1] \leq 0, \text{ for all } \alpha \in \mathbf{Q}_{n-k+1,\{2,\ldots,n\}},$$

$$\det A[k-1, \ldots, n-1|\beta] \leq 0, \text{ for all } \beta \in \mathbf{Q}_{n-k+1,\{2,\ldots,n\}}.$$

Hence by Theorem 2.5, $A[2, \ldots, n|1, \ldots, n-1]$ and $A[1, \ldots, n-1|2, \ldots, n]$ are $Ns.t.n.p.$

In order to complete the proof, by employing Theorem 2.8, it is sufficient to show that

$$\det A[\alpha|\beta] \leq 0,$$

for $\alpha = (s+1,\dots,s+l)\,,\beta = (\beta_1,\dots,\beta_l) \in \mathbf{Q}_{l,n}$ with $s = 0,1,\dots,n-1-l$, $l = 1,\dots,n-1$ and $\beta_l < n$. For $l = 1$, it is done by assumption. If $s+1 \geq 2$ and $\beta_1 \geq 2$, then $\det A[\alpha|\beta] \leq 0$ since $A[\alpha|\beta]$ will be a submatrix in $A[2,\dots,n]$. If $s+1 \geq 2$ and $\beta_1 = 1$, then $A[\alpha|\beta]$ will be a submatrix in $A[2,\dots,n|1,\dots,n-1]$ in this case $\det A[\alpha|\beta] \leq 0$ since $A[2,\dots,n|1,\dots,n-1]$ is a $Ns.t.n.p.$ If $s+1 = 1$ and $\beta_1 \geq 2$, then $\det A[\alpha|\beta] \leq 0$ since $A[\alpha|\beta]$ is a submatrix in the $Ns.t.n.p.$ matrix $A[1,\dots,n-1|2,\dots,n]$. In the following we will only consider the case $s+1 = 1$ and $\beta_1 = 1$.

By Lemma 1.2 and properties of determinants, we have

$$\begin{aligned}
&\det A[s+2,\dots,s+l|\beta_{\hat{\beta}_1}] \det A[s+1,\dots,s+l,t_1|\beta \cup \{t_2\}] \\
&= \det A[s+2,\dots,s+l,t_1|\beta_{\hat{\beta}_1} \cup \{t_2\}] \det A[s+1,\dots,s+l|\beta] \\
&- \det A[s+2,\dots,s+l,t_1|\beta] \det A[s+1,\dots,s+l|\beta_{\hat{\beta}_1} \cup \{t_2\}],
\end{aligned}$$

for all $t_1 > s+l$ and $t_2 \in \{1,\dots,n\}\backslash\beta$.
After some arrangements of the latter equality, we have

$$\begin{aligned}
&\det A[s+1,\dots,s+l|\beta] \det A[s+2,\dots,s+l,t_1|\beta_{\hat{\beta}_1} \cup \{t_2\}] \\
&= \det A[s+2,\dots,s+l|\beta_{\hat{\beta}_1}] \det A[s+1,\dots,s+l,t_1|\beta \cup \{t_2\}] \\
&+ \det A[s+2,\dots,s+l,t_1|\beta] \det A[s+1,\dots,s+l|\beta_{\hat{\beta}_1} \cup \{t_2\}]. \qquad (3.14)
\end{aligned}$$

The minors $\det A[s+1,\dots,s+l,t_1|\beta_{\hat{\beta}_1} \cup \{t_2\}]$, $\det A[s+2,\dots,s+l|\beta_{\hat{\beta}_1}]$, $\det A[s+2,\dots,s+l,t_1|\beta]$, and $\det A[s+1,\dots,s+l|\beta_{\hat{\beta}_1} \cup \{t_2\}]$ are nonpositive since the corresponding submatrices lie in $A[2,\dots,n]$ or $A[1,\dots,n-1|2,\dots,n]$ or $A[2,\dots,n|1,\dots,n-1]$ which are $t.n.p.$ matrices. In the following we first consider the case $l < n-1$.

For $t_1 = n$ or $t_2 = n$, $\det A[s+1,\dots,s+l,t_1|\beta \cup \{t_2\}] \leq 0$.
If for $t_1 = n$ or $t_2 = n$, and $\det A[s+2,\dots,s+l,t_1|\beta_{\hat{\beta}_1} \cup \{t_2\}] < 0$, then we conclude that $\det A[s+1,\dots,s+l|\beta] \leq 0$, as desired. Otherwise, $\det A[s+2,\dots,s+l,t_1|\beta_{\hat{\beta}_1} \cup \{t_2\}] = 0$ for $t_1 = n$ and $t_2 \in \{1,\dots,n\}\backslash\beta$ or $t_2 = n$ and $t_1 > s+l$.
If $\det A[s+2,\dots,s+l|\beta_{\hat{\beta}_1}] < 0$, then together with $\det A[s+2,\dots,s+l,t_1|\beta_{\hat{\beta}_1} \cup \{n\}] = 0$ for $t_1 > s+l$, we conclude that $A[2,\dots,n]$ is singular which is a contradiction.
Hence in the following we assume that $\det A[s+2,\dots,s+l|\beta_{\hat{\beta}_1}] = 0$. By (3.14) $\det A[s+2,\dots,s+l,t_1|\beta] \det A[s+1,\dots,s+l|\beta_{\hat{\beta}_1} \cup \{t_2\}] = 0$ for all $t_1 > s+l$ and $t_2 \in \{1,\dots,n\}\backslash\beta$.

If the rows of $A[s+2,\dots,s+l|\beta]$ or the columns of $A[s+1,\dots,s+l|\beta_{\hat{\beta}_1}]$ are linearly dependent, then $\det A[s+1,\dots,s+l|\beta] = 0$, as desired.

Hence rows of $A[s+2,\ldots,s+l|\beta]$ and columns of $A[s+1,\ldots,s+l|\beta_{\hat{\beta}_1}]$ are linearly independent. Moreover, $\det A[s+2,\ldots,s+l,t_1|\beta]=0$ for all $t_1>s+l$ or $\det A[s+1,\ldots,s+l|\beta_{\hat{\beta}_1}\cup\{t_2\}]=0$ for $t_2\in\{1,\ldots,n\}\backslash\beta$, since otherwise by (3.14) we have a nonzero quantity equals a zero quantity.
Whence by $\det A[s+2,\ldots,s+l,t_1|\beta]=0$ for all $t_1>s+l$ and linear independence of the rows of $A[s+2,\ldots,s+l|\beta]$ we conclude that rank $A[2,\ldots,n|\beta]\le l-1$ which is a contradiction with the nonsingularity of $A[2,\ldots,n]$. This completes the proof of the theorem.

Now for case $l=n-1$, then $\det A[s+1,\ldots,s+l|\beta]=\det A[1,\ldots,n-1]$ which is nonpositive by assumption. □

The condition that $A[1,\ldots,n-1]\le 0$ in Theorem 3.5, is necessary to conclude that a given matrix is $t.n.p^+$ as the following example shows.

Example 3.1. *Let* $A=\begin{bmatrix}-6&-10&-2\\-4&-7&-1\\-5&-8&-1\end{bmatrix}$. *The application of the Cauchon Algorithm yields*

$$A^{(4,2)}=A,\quad A^{(3,3)}=\begin{bmatrix}4&6&-2\\1&1&-1\\-5&-8&-1\end{bmatrix},$$

$$A^{(3,2)}=\begin{bmatrix}\frac{2}{8}&6&-2\\\frac{3}{8}&1&-1\\-5&-8&-1\end{bmatrix}=A^{(3,1)},\quad A^{(3,2)}=\begin{bmatrix}-\frac{1}{2}&4&-2\\\frac{3}{8}&1&-1\\-5&-8&-1\end{bmatrix},$$

$$A^{(2,2)}=\begin{bmatrix}-2&4&-2\\\frac{3}{8}&1&-1\\-5&-8&-1\end{bmatrix}=A^{(2,1)}=A^{(1,3)}=A^{(1,2)}=\tilde{A}.$$

It is clear that $\tilde{A}$ satisfies all the conditions listed in Theorem 3.5, but the matrix A is not $t.n.p^+$ since $\det A[1,2]=\begin{vmatrix}-6&-10\\-4&-7\end{vmatrix}=2>0$.

3.3 Matrix Intervals

In this section, we turn to study the intervals of matrices of the class $t.n.p^+$.

Let $A = \left(a_{ij}\right), B = \left(b_{ij}\right) \in \mathbb{R}^{n,n}$. Then $A \le B$ is defined by $\le B - A$, this called usual entry-wize partial ordering.

The *checkerboard partial ordering* is defined as follows. Let $S := \operatorname{diag}(1, -1, \ldots, (-1)^{n+1})$ and $A^* := SAS$. Then we define

$$A \le^* B :\Leftrightarrow A^* \le B^*,$$

i.e.,

$$A \le^* B :\Leftrightarrow (-1)^{i+j} a_{ij} \le (-1)^{i+j} b_{ij}, \quad i, j = 1, \ldots, n.$$

We consider matrix intervals with respect to this partial ordering, i.e., for $A, B \in \mathbb{R}^{n,n}$ with $A \le^* B$ let

$$[A, B] := \left\{ Z \in \mathbb{R}^{n,n} \mid A \le^* Z \le^* B \right\}.$$

The matrices A and B are called the *corner matrices*. By $\mathbb{I}\left(\mathbb{R}^{n,n}\right)$ we denote the set of all matrix intervals of order n with respect to the checkerboard partial ordering. In [1],[3],[5],[6], [10], the matrix intervals of $NsTN$ and $Ns.t.n.p.$ have been studied.

Lemma 3.2. *[15] Let $A, B, Z \in \mathbb{R}^{n,n}$ and let A and B be nonsingular with $0 \le A^{-1}, B^{-1}$. If $A \le Z \le B$, then Z is nonsingular and $B^{-1} \le Z^{-1} \le A^{-1}$.*

Theorem 3.6. *[5] Let $[A, B] \in \mathbb{I}\left(\mathbb{R}^{n,n}\right)$ with $Z \in [A, B]$. If A and B are Ns.t.n.p. with $b_{nn} < 0$, then $\tilde{A} \le^* \tilde{B}$, $\tilde{Z} \in [\tilde{A}, \tilde{B}]$, and Z is $Ns.t.n.p.$*

Theorem 3.7. *Let $[A, B] \in \mathbb{I}\left(\mathbb{R}^{n,n}\right)$ with $Z \in [A, B]$. If A and B are $t.n.p^+$ matrices with negative entries, then Z is $t.n.p^+$.*

Proof. First of all, since all entries of $A = \left(a_{ij}\right)$,$B = \left(b_{ij}\right)$ are negative and $Z = \left(z_{ij}\right) \in [A, B]$, i.e.,

$$(-1)^{i+j} a_{ij} \le (-1)^{i+j} z_{ij} \le (-1)^{i+j} b_{ij}, \quad i, j = 1, \ldots, n,$$

then all the entries of Z are negative, too.

Now using (1.1) for $A^{-1} = \left(a_{ij}^{-1}\right)$ of A to get

$$a_{ij}^{-1} = (-1)^{i+j} \frac{\det A\left[1, \ldots, j-1, j+1, \ldots, n | 1, \ldots, i-1, i+1, \ldots, n\right]}{\det A}, \qquad (3.15)$$

where $\det A\left[1,\ldots,j-1,j+1,\ldots,n|1,\ldots,i-1,i+1,\ldots,n\right]\leq 0$ and $\det A>0$. Hence if $i+j$ is even, then $a^{-1}_{ij}\leq 0$ and if $i+j$ is odd, then $a^{-1}_{ij}\geq 0$. Thus $-SA^{-1}S\geq 0$. Proceeding in the same manner we obtain $-SB^{-1}S\geq 0$.

It is easy to see that $-SAS\geq -SZS\geq -SBS$ since $A\leq^{*} Z\leq^{*} B$ and by $-SA^{-1}S\geq 0$, $-SB^{-1}S\geq 0$, we obtain by Lemma 3.2 that $-SZS$ is nonsingular and so Z is nonsingular. Moreover,

$$-SA^{-1}S\leq -SZ^{-1}S\leq -SB^{-1}S. \tag{3.16}$$

By Lemma 1.1, we get that

$$\begin{aligned}\det A[2,\ldots,n-1]\det A &= \det A[1,\ldots,n-1]\det A[2,\ldots,n]\\ &-\det A[1,\ldots,n-1|2,\ldots,n]\det A[2,\ldots,n|1,\ldots,n-1].\end{aligned} \tag{3.17}$$

Then we have two cases that depend on the value of $\det A[2,\ldots,n-1]$.

Case (1) $\det A[2,\ldots,n-1]=0$.
By (3.17) we get

$$\det A[1,\ldots,n-1]\det A[2,\ldots,n]=\det A[1,\ldots,n-1|2,\ldots,n]\det A[2,\ldots,n|1,\ldots,n-1].$$

Since $\det A[1,\ldots,n-1|2,\ldots,n]\det A[2,\ldots,n|1,\ldots,n-1]\neq 0$ by Remark 3.1 we get that $A[1,\ldots,n-1]$ and $A[2,\ldots,n]$ are nonsingular, and since A is $t.n.p^{+}$ we get that $A[1,\ldots,n-1]$ and $A[2,\ldots,n]$ are $Ns.t.n.p.$ By (3.16), we also have that $\det B[1,\ldots,n-1]$ and $\det B[2,\ldots,n]$ are negative since $\det A[1,\ldots,n-1], A[2,\ldots,n]<0$. Hence $B[1,\ldots,n-1]$ and $B[2,\ldots,n]$ are $Ns.t.n.p$, too.

Now, since $A\leq^{*} Z\leq^{*} B$,

$$A[1,\ldots,n-1]\leq^{*} Z[1,\ldots,n-1]\leq^{*} B[1,\ldots,n-1],$$

it follows from Theorem 3.6, that $Z[1,\ldots,n-1]$ is $Ns.t.n.p.$ because $A[1,\ldots,n-1]$ and $B[1,\ldots,n-1]$ are $Ns.t.n.p$ with $b_{n-1,n-1}<0$. Moreover, in the same manner we get that $Z[2,\ldots,n]$, $Z[1,\ldots,n-1|2,\ldots,n]$, and $Z[2,\ldots,n|1,\ldots,n-1]$ are $Ns.t.n.p.$ since $A[2,\ldots,n]$, $B[2,\ldots,n]$, $A[1,\ldots,n-1|2,\ldots,n]$, $B[1,\ldots,n-1|2,\ldots,n]$, $A[2,\ldots,n|1,\ldots,n-1]$, $B[2,\ldots,n|1,\ldots,n-1]$ are $Ns.t.n.p.$ (the latter follows by Remark 3.1) and

$$A[2,\ldots,n]\leq^{*} Z[2,\ldots,n]\leq^{*} B[2,\ldots,n],$$

$$B[1,\ldots,n-1|2,\ldots,n]\leq^{*} Z[1,\ldots,n-1|2,\ldots,n]\leq^{*} A[1,\ldots,n-1|2,\ldots,n],$$

$$B[2,\ldots,n|1,\ldots,n-1]\leq^{*} Z[2,\ldots,n|1,\ldots,n-1]\leq^{*} A[2,\ldots,n|1,\ldots,n-1].$$

Case (2) $\det A[2,\dots,n-1] \neq 0$.

If $A[1,\dots,n-1]$ $\big(or\ A[2,\dots,n]\big)$ is $Ns.t.n.p.$, then we proceed as in Case(1) to conclude that $Z[1,\dots,n-1]$ $\big(or\ Z[2,\dots,n]\big)$ is $Ns.t.n.p.$ In the following, we consider the case that $A[1,\dots,n-1]$ and $A[2,\dots,n]$ are singular matrices. Define the matrix $A_{\varepsilon_{ij}}$ to be the matrix obtained from A by adding a small real positive number ε to the entry a_{ij}, i.e., $A_{\varepsilon_{ij}} := A + \varepsilon E_{ij}$, where $E_{ij} = \big(e_{ij}\big)$ is the standard basis matrix where the only nonzero entry is $e_{ij} = 1$.

In the following, assume $\det A[1,\dots,n-1] = 0$ and put

$$A_{\varepsilon_{11}}[1,\dots,n-1] = \begin{bmatrix} a_{11}+\varepsilon & a_{12} & \dots & a_{1,n-1} \\ a_{21} & a_{22} & \dots & a_{2,n-1} \\ \vdots & \vdots & \dots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1}, \end{bmatrix}.$$

By application of Laplace expansion to $A_{\varepsilon_{11}}[1,\dots,n-1]$ along its first row, we get

$$\begin{aligned} \det A_{\varepsilon_{11}}[1,\dots,n-1] &= (a_{11}+\varepsilon)\det A_{\varepsilon_{11}}[2,\dots,n-1] - a_{12}\det A_{\varepsilon_{11}}[2,\dots,n-1|1,3,\dots,n-1] \\ &+ \dots + (-1)^{1+n-1} a_{1,n-1}\det A_{\varepsilon_{11}}[2,\dots,n-1|1,\dots,n-2] \\ &= \varepsilon \det A[2,\dots,n-1] + a_{11}\det A[2,\dots,n-1] \\ &- a_{12}\det A[2,\dots,n-1|1,3,\dots,n-1] \\ &+ \dots + (-1)^{1+n-1} a_{1,n-1}\det A[2,\dots,n-1|1,\dots,n-2] \\ &= \varepsilon \det A[2,\dots,n-1] + \det A[1,\dots,n-1]. \end{aligned}$$

The rigth hand side is negative, because $\varepsilon > 0$, $\det A[2,\dots,n-1] < 0$, and $\det A[1,\dots,n-1] = 0$, and we conclude that $A_{\varepsilon_{11}}[1,\dots,n-1]$ is nonsingular. Now, to show that $\det A_{\varepsilon_{11}}[1,\dots,n-1]$ is $t.n.p.$, we firstly note that for $\mathbf{i} = (i_1,\dots,i_l), \mathbf{j} = (j_1,\dots,j_l) \in \mathbf{Q}_{l,n-1}$, for $i_1 \neq 1$ or $j_1 \neq 1$

$$\det A_{\varepsilon_{11}}[i_1,\dots,i_l|j_1,\dots,j_l] = \det A[i_1,\dots,i_l|j_1,\dots,j_l] \leq 0.$$

Moreover, for $i_1 = 1$ and $j_1 = 1$, then by Laplace expansion along the first row we get that

$$\det A_{\varepsilon_{11}}[1,i_2,\dots,i_l|1,j_2\dots,j_l] = \det A[1,i_2,\dots,i_l|1,\dots,j_l] + \varepsilon \det A[i_2,\dots,i_l|j_2\dots,j_l].$$

Since $\det A[1,i_2,\dots,i_l|1,j_2,\dots,j_l], \det A[i_2,\dots,i_l|j_2\dots,j_l] \leq 0$ and $\varepsilon > 0$ and therefore

$$\det A_{\varepsilon_{11}}[1,i_2,\dots,i_l|1,j_2\dots,j_l] \leq 0.$$

Hence $A_{\varepsilon_{11}}[1,\ldots,n-1]$ is $Ns.t.n.p.$ In the same way we can conclude that $B_{\varepsilon_{11}}[1,\ldots,n-1]$ is a $Ns.t.n.p$ matrix.

Since $A \leq^* Z \leq^* B$, we get that

$$A_{\varepsilon_{11}}[1,\ldots,n-1] \leq^* Z_{\varepsilon_{11}}[1,\ldots,n-1] \leq^* B_{\varepsilon_{11}}[1,\ldots,n-1].$$

By Theorem 3.6, since $A_{\varepsilon_{11}}[1,\ldots,n-1]$ and $B_{\varepsilon_{11}}[1,\ldots,n-1]$ are $N.s.t.n.p$ matrices with $b_{n-1,n-1} < 0$, we conclude that $Z_{\varepsilon_{11}}[1,\ldots,n-1]$ is a $Ns.t.n.p$ matrix. Since this is true for all sufficiently small $\varepsilon_{11} > 0$, by letting $\varepsilon_{11} \to 0$, we get that $Z[1,\ldots,n-1]$ is a $t.n.p$ matrix.

Now, we continue by considering the matrix $Z[2,\ldots,n]$. Since $A \leq^* Z \leq^* B$ we have

$$A_{\varepsilon_{n-1,n-1}}[2,\ldots,n] \leq^* Z_{\varepsilon_{n-1,n-1}}[2,\ldots,n] \leq^* B_{\varepsilon_{n-1,n-1}}[2,\ldots,n],$$

for all sufficiently small positive real numbers $\varepsilon_{n-1,n-1}$

By proceeding parallel to the case $\det A[1,\ldots,n-1] = 0$, we conclude that $Z[2,\ldots,n]$ is a $t.n.p.$ matrix. In the same way as in Case (1), $Z[1,\ldots,n-1|2,\ldots,n]$ and $Z[2,\ldots,n|1,\ldots,n-1]$ are $Ns.t.n.p.$ In the following, assume that $\det Z[2,\ldots,n] < 0$. Then by Theorem 2.6, $\tilde{z}_{ii} > 0$ for $i = 2,\ldots,n-1$.

Now we will prove that the matrix $\tilde{Z}$ is a Cauchon matrix. Since $Z[2,\ldots,n]$ is $t.n.p.$, by Theorem 3.2, and since $\tilde{Z}[2,\ldots,n] = \widetilde{Z[2,\ldots,n]}$, we get that $\tilde{Z}[2,\ldots,n]$ is a Cauchon matix. Now, to complete the proof, we investigate the entries in the first row and the first column in $\tilde{Z}$. We will do it for the first column, and in the same way we proceed for the first row.

Claim: $\tilde{z}_{i1} \geq 0$ for $i = 2,\ldots,n-1$ and $\tilde{z}_{i1} = 0$ if $\tilde{z}_{i2} = 0$.
We proceed by decreasing induction on i.
For $i = n-1$, we consider the submatrix $\tilde{Z}[n-1,n|1,\ldots,n]$. It is easy to show that this matrix is a Cauchon matrix. Now, since all the entries in the last row are negative, we can construct a lacunary sequence $\gamma = \big((n-1,1),(n,2)\big)$ for the matrix $\tilde{Z}[n-1,n|1,\ldots,n]$ since $\tilde{z}_{n2} = z_{n2} < 0$. Hence by Theorem 2.4, we get

$$\det Z[n-1,n|1,2] = \tilde{z}_{n-1,1} \cdot z_{n2},$$

i.e.,

$$\tilde{z}_{n-1,1} = \frac{\det Z[n-1,n|1,2]}{z_{n2}}.$$

The numerator and denominator of the last expression are nonpositive since the underlying submatrices lie in $Z[2,\ldots,n|1,\ldots,n-1]$, hence $\tilde{z}_{n-1,1} \geq 0$.

If $\tilde{z}_{n-1,2} = 0$, the sequence $\big((n-1,2),(n,3)\big)$ is lacunary for the matrix $\tilde{Z}[n-1,n|1,\ldots,n]$, whence

$$\det Z[n-1,n|2,3] = \tilde{z}_{n-1,2} \cdot \tilde{z}_{n3} = 0.$$

Thus, the matrix $Z[n-1,n|2,3]$ is of rank 1. By Theorem 2.9, either the rows $n-1$ and n or the columns 2 and 3 are linearly dependent in $Z[2,\ldots,n|1,\ldots,n-1]$ which is a contradiction to the nonsingularity of this matrix, or the right shadow of $Z[n-1,n|2,3]$ which is $Z[2,\ldots,n|2,\ldots,n-1]$ has rank 1 which is a contradiction to the nonsingularity of the principal submatrices of $Z[2,\ldots,n]$ since the latter matrix is $Ns.t.n.p.$ and by Theorem 2.10, its principal minors are negative. Hence the only option that is left is that the left shadow of $Z[n-1,n|2,3]$ which is $Z[n-1,n|1,2,3]$ has rank 1. The sequence $\big((n-1,1),(n,2)\big)$ is lacuanary for the matrix $\tilde{Z}[n-1,n|1,2,\ldots,n]$ and by Theorem 2.4 we get that

$$0 = \det Z[n-1,n|1,2] = \tilde{z}_{n-1,1} \cdot \tilde{z}_{n2}.$$

Since $\tilde{z}_{n2} = z_{n2} < 0$, we get that $\tilde{z}_{n-1,1} = 0$.

As the induction hypothesis, suppose that $\tilde{z}_{i1} \geq 0$ and $\tilde{z}_{i1} = 0$ if $\tilde{z}_{i2} = 0$ for $i = k+1,\ldots,n-1$. Then the matrix $\tilde{Z}[k,\ldots,n|1,\ldots,n]$ is a Cauchon matrix. Define

$$(i_s, j_s) := \min\{(i,j) : i_{s-1} < i \leq n, j_{s-1} < j \leq n, \tilde{z}_{ij} \neq 0\},$$

where the minimum is taken with respect to the colexicographical order with $(i_0, j_0) = (k,1)$, $s = 1,\ldots,r$. Assume that the sequence that is produced by this procedure is $\big((k,1),(i_1,j_1),\ldots,(i_r,j_r)\big)$. By the construction, it is easy to show that this sequence is a lacunary sequence for the Cauchon matrix $\tilde{Z}[k,\ldots,n|1,\ldots,n]$, $i_r = n$ and $j_r \leq n-1$ since $\tilde{z}_{ii} > 0$ for $2,\ldots,n-1$ and $z_{ni} < 0$ for $i = 1,\ldots,n$. Hence by Theorem 2.4, we have

$$\tilde{z}_{k1} = \frac{\det Z[k,i_1,\ldots,i_r|1,j_1,\ldots,j_r]}{\det Z[i_1,\ldots,i_r|j_1,\ldots,j_r]},$$

where the underlying submatrices in the numenator and domenator lie in $Z[2,\ldots,n|1,\ldots,n-1]$. Hence $\tilde{z}_{k,1} \geq 0$, since the latter submatrix is $Ns.t.n.p.$

If $\tilde{z}_{k2} = 0$, then by the above procedure construct a lacunary sequence starting from $(k,2)$ and call the resulting sequence $\big((k,2),(\alpha_1,\beta_1),\ldots,(\alpha_s,\beta_s)\big)$. Then by Theorem 2.4, we get

$$\tilde{z}_{k2} = \frac{\det Z[k,\alpha_1,\ldots,\alpha_s|2,\beta_1,\ldots,\beta_s]}{\det Z[\alpha_1,\ldots,\alpha_s|\beta_1,\ldots,\beta_s]} = 0,$$

where $\alpha_s = n$ and $\beta_s \leq n-1$. By Theorem 2.7, the above ratio can be written as a ratio of two contiguous minors as they lie in $\tilde{Z}[2,\dots,n|1,\dots,n-1]$. Hence we obtain

$$\tilde{z}_{k,2} = \frac{\det Z[k,k+1,\dots,k+s|2,3,\dots,s+2]}{\det Z[k+1,\dots,k+s|3,\dots,s+2]} = 0,$$

which implies that $Z[k,k+1,\dots,k+s|2,3,\dots,s+2]$ has rank s.

By Theorem 2.9 and using the same arguments as above, the left shadow of $Z[k,\dots,k+s|2,\dots,s+2]$ has rank s. By construction and since $\tilde{z}_{k2} = 0$, it is easy to show that the sequence $\big((k,1),(\alpha_1,\beta_1),\dots,(\alpha_s,\beta_s)\big)$ is lacunary for $\tilde{Z}[k,\dots,n|1,\dots,n]$.
Hence by Theorem 2.7, we get that

$$\tilde{z}_{k,1} \quad = \quad \frac{\det Z[k,\alpha_1,\dots,\alpha_s|1,\beta_1,\dots,\beta_s]}{\det Z[\alpha_1,\dots,\alpha_s|\beta_1,\dots,\beta_s]} = 0$$

since $Z[k,\alpha_1,\dots,\alpha_s|1,\beta_1,\dots,\beta_s]$ lies in the left shadow of $Z[k,\dots,k+s|2,\dots,s+2]$, we obtain $\tilde{z}_{k1} = 0$.

Hence $\tilde{Z}$ is a Cauchon matrix with $\tilde{z}_{ii} > 0$ for $i = 2,\dots,n-1$, and $\tilde{z}_{nn} < 0$. Furthermore, $\big((1,1),\dots,(n,n)\big)$ is a lacunary sequence for the Cauchon matrix $\tilde{Z}$. By Theorem 2.4, we get

$$\tilde{z}_{11} = \frac{\det Z}{\det Z[2,\dots,n]} < 0.$$

Since $Z[2,\dots,n]$ is $t.n.p$ and $\tilde{Z}[2,\dots,n] = Z[\widetilde{2,\dots},n]$ we conclude by Theorem 3.2 that $\tilde{Z}[1,\dots,n-1|2,\dots,n-1]$ is nonnegative and $\tilde{Z}_{i1} \geq 0$ for $i = 2,\dots,n-1$. Moreover, since $Z[1,\dots,n-1|2,\dots,n]$ and $Z[2,\dots,n|1,\dots,n-1]$ are $Ns.t.n.p.$ as in proof of (iv) in Theorem 3.5 we get that $\tilde{z}_{i,i-1},\ \tilde{z}_{i-1,i} > 0$. Hence by Theorem 3.5, Z is $t.n.p^+$.

For $\det Z[2,\dots,n] = 0$, we work with $Z_{\varepsilon_{22}}$ for sufficiently small ε_{22}. It is easy to show that $Z_{\varepsilon_{22}}$ is $t.n.p^+$ by applying the above arguments on $Z_{\varepsilon_{22}}$. As $\varepsilon_{22} \to 0$, we arrive at Z is $t.n.p^+$. This completes the proof. $\square$

Bibliography

[1] Adm, M. (2016). *Perturbation and Intervals of Totally Nonnegative Matrices and Related Properties of Sign Regular Matrices*, University of Konstanz, Germany (Doctoral dissertation).

[2] Adm, M., Al Muhtaseb, K., Ghani, A. A., Fallat, S., and Garloff, J. (2018). Further applications of the Cauchon Algorithm to rank determination and bidiagonal factorization. *Linear algebra Appl.*, 545, $240-255$.

[3] Adm, M., Al Muhtaseb, K., Ghani,A A., and Garloff, J. (2020). Relaxing the nonsingularity assumption for intervals of totally nonnegative matrices. *Electron. J. Linear Algebra*, 36, $106-123$.

[4] Adm, M., and Garloff, J. (2021). Characterization, perturbation, and interval property of certain sign regular matrices. *Linear Algebra Appl.*, 612, 146-161.

[5] Adm, M., and Garloff, J. (2016). Intervals of special sign regular matrices. *Linear and Multilinear Algebra*, 64(7), $1424-1444$

[6] Adm, M., and Garloff, J. (2013). Intervals of totally nonnegative matrices. *Linear Algebra Appl.*, 439(12), $3796-3806$.

[7] Ando T. (1987). Totally positive matrices . *Linear Algebra Appl.*, 90,165-219.

[8] Cortés, V., and Peña, J. M. (2008). Decompositions of strictly sign regular matrices. Linear algebra appl., $429(5-6), 1071-1081$.

[9] Fallat, S. M., and Johnson, C. R. (2011). *Totally Nonnegative Matrices*, Princeton Ser. Appl. Math. . Princeton University Press, Princeton and Oxford.

[10] Garloff, J., Al-Saafin, D., and Adm, M. (2021). *Further Matrix Classes Possessing the Interval Property.* Konstanzer Schriften in Mathematik.

[11] Gasca M. Micchelli C. A. (Eds).(1996). *Total Positivity and its Applications* ($vol.$359). Springer Science and Business Media.

[12] Goodearl, K. R., Launois, S., and Lenagan, T. H. (2011). Totally nonnegative cells and matrix Poisson varieties. *Adv. Math.*, $226(1), 779 - 826$.

[13] Holtz O., and Tyaglov M. (2012). Structured matrices, continued fractions, and root localization of polynomials. *SIAM Review*, 54(3), $421 - 509$.

[14] Huang R. and Chu D. (2010). Total nonpositivity of nonsingular matrices. *Linear Algebra Appl.*, $432, 2931–2941$.

[15] Kuttler, J. R. (1971). A fourth-order finite-difference approximation for the fixed membrane eigenproblem. *Mathematics of Computation*, 25(114), $237 - 256$.

[16] Launois S., Lenagan T.H. (2014). Efficient recognition of totally nonnegative matrix cells, *Found. Comput. Math.* $14, 371 - 387$.

[17] Meyer, C. D. (2000). *Matrix Analysis and Applied Linear Algebra*. Library of congress cataloging in publication data . SIAM.

[18] Peña J M , editor. (1999). *Shape Preserving Representations in Computer-aided Geometric Design*. Nova Science Publishers, Inc., New York.

[19] Peña J M . (2003). On nonsingular sign regular matrices. *Linear Algebra Appl.*, 359, 91-100.

[20] Pinkus A. (2010). *Totally Positive Matrices*, Cambridge Tracts in Mathematics 181, Cambridge Univ, Press, Cambridge, UK.