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Bounds for the Eigenvalues of Matrix Polynomials

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DEDICATION

This thesis is dedicated to:
The sake of Allah, my Creator and my Master,
My great supervisors Dr.Mohammad Adam and Dr. Ali Zein, who encourage and support me,
My external committee member,
My parents, the reason of what I become today.

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Abstract

Developing bounds for the eigenvalues of matrix polynomials is an interesting problem which has a lot of applications. In this thesis, several known bounds for the eigenvalues of matrix polynomials are presented. In addition, we derive new bounds for the eigenvalues of matrix polynomials with commuting coefficients. These bounds are based on norms, numerical radius, and spectral radius of the coefficient matrices. Various tools are used in the derivations, such as Frobenius companion matrix, the numerical radius inequalities, and matrix norms. In general, it is not possible to compare the sharpness of these bounds analytically. Therefore, we compare our new bounds with each other and with other known bounds numerically through a set of examples.

Contents

Chapter 1

Introduction

Polynomial root finding is one of the most important subjects in scientific computation. In particular, polynomial eigenvalue problems (PEP) consist of finding a nonzero complex eigenvector $\mathbf{x}$ and a complex eigenvalue λ such that $P(\lambda)\mathbf{x} = 0$, where P is a matrix polynomial. Polynomial eigenvalue problems have important applications in applied mathematics and can be found in many field of engineering such as vibration analysis, structural and fluid mechanics, and acoustics, see, e.g, [12, 31]. In addition, information about the location of the eigenvalues by using iterative methods [40, 41] is useful in the computation of pseudospectra [42, 43].

In general, it is not easy, and in many cases it is hard, to compute the eigenvalues of a matrix polynomial. But it is relatively easy to obtain bounds for them. The polynomial eigenvalue problem has recently received much attention due to its importance and applications. Several bounds of eigenvalues of matrix polynomials have been derived based on various inequalities obtained in many researches.

Fujii and Kubo [9] used the norm of the coefficients of a matrix polynomial to provide bounds for the eigenvalues. Upper and lower bounds for the absolute value of the eigenvalues of a matrix polynomial were derived by Higham and Tisseur [10], based on norm and numerical radius inequalities. In [6, 14] new bounds were established by applying several numerical radius inequalities to the Frobenius companion matrices. Burqan et al. [15] employed several numerical radius inequalities to the square Frobenius companion matrices to provide new bounds. Bounds for the eigenvalues of matrix polynomials with commuting coefficients can be found in [4, 11, 16]. Melman [13] showed how ℓ-ifications, namely, lower order matrix polynomials with the same eigenvalues as a given matrix polynomial, can be used to produce eigenvalue bounds. Le et al. [44]

established upper and lower bounds for the eigenvalues of matrix polynomials using the norm of their coefficients. Cameron [8] proved that the eigenvalues of any matrix polynomial, with unitary coefficients, lie inside the open annulus $\frac{1}{2} < |\lambda| < 2$. Hadimani and Jayaraman [18] showed that under certain assumptions, matrix polynomials with either doubly stochastic matrix coefficients or Schur stable matrix coefficients also have eigenvalues within similar annular regions.

In this thesis we focus on studying bounds of the eigenvalues of matrix polynomials. Also, we propose new upper bounds for the eigenvalues of these polynomials. Frobenius companion matrices are employed to present several bounds on eigenvalues.

In Chapter two, basic definitions and results in matrix theory are introduced. This includes some definitions, properties, and examples of matrix norms. The chapter also presents important numerical radius inequalities and relationships between the spectral radius, the numerical radius, and the spectral norm.

In Chapter three, the discussion focuses on Frobenius companion matrix of matrix polynomials and relationships between its spectrum and those of the corresponding matrix polynomials. The chapter also illustrates how ℓ-ifications of matrix polynomial can be employed to establish bounds for the eigenvalues. Bounds for the eigenvalues are obtained in different ways; some bounds derived using matrix norm and by applying numerical radius inequalities to Frobenius companion matrices. Other bounds are proved using square Frobenius companion matrices. Moreover, bounds for eigenvalues of matrix polynomials with commuting coefficients are also introduced. Also, the generalized Cauchy and generalized Pellet theorems are discussed.

In Chapter four, special cases of matrix polynomials are addressed, including lower and upper bounds for eigenvalues of matrix polynomials whose coefficients are unitary, doubly stochastic, and Schur stable.

In Chapter five, new bounds for the eigenvalues of monic matrix polynomials with commuting coefficients are introduced. By the fact that similar matrices have the same spectral radius, numerical radius inequalities are applied to various decompositions and partitions to derive these new bounds.

Chapter 2

Preliminaries

In this chapter, we introduce some results in matrix theory that will be helpful in the following chapters. We present the definitions of matrix norm and numerical radius with basic properties and inequalities.

2.1 Matrix norm

In this section, we present definitions of matrix norms and some examples of matrix norms. Most of the material in this section can be found in Horn and Johnson [1].

Definition 2.1. *A function* $\|.\| : \mathbb{M}_n(\mathbb{C}) \to \mathbb{R}$ *is called a* matrix norm *if, for all* $A, B \in \mathbb{M}_n(\mathbb{C})$ *it satisfies the following axioms:*

1. $\|A\| \geq 0$ *and* $\|A\| = 0$ *iff* $A = 0$.
2. $\|\alpha A\| = |\alpha|\|A\|$ *for all* $\alpha \in \mathbb{C}$.
3. $\|A + B\| \leq \|A\| + \|B\|$.
4. $\|AB\| \leq \|A\|\|B\|$.

A matrix norm is also known as a ring norm. The first three axioms of a matrix norm are identical to the axioms for a vector norm. A norm on matrices that does not satisfy condition 4 for every A and B is a vector norm on matrices. There are many interesting examples of matrix norms. Here, important matrix norms are mentioned which will be used later.

Definition 2.2. ***(Induced norm)***
Let $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C}),\ \mathbf{x} \in \mathbb{C}^n$. *The* matrix norm induced by the vector norm $\|.\|$ *on* $\mathbb{C}^n$ *is defined by*

$$\|A\| = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|.$$

For $\mathbf{x} \in \mathbb{C}^n$ and $p = 1, 2, \ldots, \infty$, the *p-vector norm* on $\mathbb{C}^n$ is

$$\|\mathbf{x}\|_p := \left(\sum_{j=1}^{n} |x_j|^p \right)^{\dfrac{1}{p}}.$$

Using this vector norms we get the following induced matrix norms.

Example 2.1. ***(p-matrix norm)***
Let $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C}),\ \mathbf{x} \in \mathbb{C}^n$, *and* $\|.\|_p$ *is the p-vector norm on* $\mathbb{C}^n$. *Then the* p-matrix norm *of* A *is defined by*

$$\|A\|_p = \max_{\|\mathbf{x}\|_p=1} \|A\mathbf{x}\|_p.$$

Example 2.2. ***(The maximum column sum norm)***
The maximum column sum norm *for* $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ *is defined by*

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|.$$

Example 2.3. ***(The maximum row sum norm)***
The maximum row sum norm *for* $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ *is defined by*

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

The set of all eigenvalues of $A \in \mathbb{M}_n(\mathbb{C})$ is called the *spectrum* of A and is denoted by $\sigma(A)$. Also, the *spectral radius* of A is $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$. The *Hermitian adjoint* of a matrix $B = [b_{ij}] \in \mathbb{M}_{n,m}(\mathbb{C})$ is denoted by B^* and defined by $B^* = \bar{\mathrm{B}}^T$, in which $\bar{\mathrm{B}}$ is the entrywise conjugate.

One of the most important and widely used norms is the spectral norm (2–norm) which is defined as follows.

Example 2.4. ***(Spectral norm)***
Let $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ *and* $\rho(A)$ *be the spectral radius of* A. *Then the* spectral norm $\|.\|_2$ *for* A *is defined by*

$$\|A\|_2 = \sqrt{\rho(A^* A)}.$$

Note that the maximum column sum norm, maximum row sum norm, and spectral norm are special cases of p-matrix norm with $p = 1, 2, \infty$, respectively [1].

Example 2.5. ***(Frobenius norm, Schur norm, or Hilbert–Schmidt norm)*** *Let $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$. Then the* Frobenius *or* Hilbert–Schmidt norm *is defined as*

$$\|A\|_F = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} = \big(tr(AA^*)\big)^{\frac{1}{2}},$$

where $tr(A)$ is the trace of a square matrix A.

The *singular values* of a matrix $A \in \mathbb{M}_n(\mathbb{C})$ are the square roots of the eigenvalues of the matrix AA^* or A^*A, and they are denoted and ordered as $\mathbf{s}_1 \geq \mathbf{s}_2 \geq \ldots \geq \mathbf{s}_n \geq 0$. So, we can define the spectral norm and Frobenius norm by using the singular values as follows.

$$\|A\|_2 = \mathbf{s}_1, \text{ the largest singular values of A}$$

$$\text{and } \|A\|_F = \left(\sum_{i=1}^{n} \mathbf{s}_i^2 \right)^{\frac{1}{2}}.$$

Definition 2.3. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then*

(1) A is called normal *if $AA^* = A^*A$.*

(2) A is called Hermitian *if $A = A^*$.*

(3) A is called unitary *if $AA^* = A^*A = I$.*

A matrix norm $\|.\|$ on $\mathbb{M}_n(\mathbb{C})$ is called unitary invariant if $\|UAV\| = \|A\|$ for all $A, U, V \in \mathbb{M}_n(\mathbb{C})$ and U, V are unitary matrices. The matrix norms $\|.\|_2$ and $\|.\|_F$ are unitary invariant [1].

Theorem 2.1. *Let $\|.\|$ be any matrix norm on $\mathbb{M}_n(\mathbb{C})$, $A \in \mathbb{M}_n(\mathbb{C})$, $\rho(A)$ be the spectral radius of A and let λ be an eigenvalue of A. Then*

$$|\lambda| \leq \rho(A) \leq \|A\|.$$

Note that if $A \in \mathbb{M}_n(\mathbb{C})$ is normal. Then $\rho(A) = \|A\|_2$.

Theorem 2.2. ***(The Spectral Mapping Theorem)***
Let $A \in \mathbb{M}_n(\mathbb{C})$*. Then for every complex polynomial* p*,* $\sigma(p(A)) = p(\sigma(A))$*.*

In the following, we will denote the spectral norm by $\|.\|$ instead of $\|.\|_2$.

2.2 Numerical radius

In this section, we start by presenting the definition of numerical radius and then introducing the inequalities related to numerical radius and spectral norm. Numerical radius inequalities are useful in geting bounds to the eigenvalues of matrix polynomials by applying them to the *Frobenius companion matrices.*

Definition 2.4. *[1] Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T,\ \mathbf{y} = (y_1, y_2, \ldots, y_n)^T \in \mathbb{C}^n$*. The* Euclidean inner product *of* $\mathbf{x}$ *and* $\mathbf{y}$ *is defined by*

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i \bar{y}_i.$$

Now, we present the definition of numerical radius. Numerical radius is one of the most important concepts in matrix theory. It is also useful in studying and deriving bounds for the eigenvalues of matrix polynomials.

Definition 2.5. ***(Numerical radius)*** *[2]*
Let $A \in \mathbb{M}_n(\mathbb{C})$*. Then the* numerical radius *of* A *is given by*

$$w(A) = \sup_{\|\mathbf{x}\|=1} |(A\mathbf{x}, \mathbf{x})|.$$

Numerical radius is a vector norm on matrices but it is not a matrix norm. It is satisfy the first three axioms of a matrix norm but it is not sub-multiplicative. Since the numerical radius is a vector norm we get the next Theorem.

Theorem 2.3. *[2] Let* $A, B \in \mathbb{M}_n(\mathbb{C})$*. Then* $w(A+B) \leq w(A) + w(B)$*.*

In the next theorem, important inequality is introduced. This inequality connects the spectral radius, the numerical radius, and the spectral norm.

Theorem 2.4 (e.g., [19])**.** *Let* $A \in \mathbb{M}_n(\mathbb{C})$*. Then*

$$\rho(A) \leq w(A) \leq \|A\|,$$

with equality if A *is normal.*

Theorem 2.5 (e.g., [2]). *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then*

$$\frac{\|A\|}{2} \leq w(A) \leq \|A\|.$$

The first inequality becomes an equality if $A^2 = 0$.

This inequality provides both a lower and an upper bound for the numerical radius in terms of the spectral norm. Also, it helps in computing the numerical radius when $A^2 = 0$. In the following Theorem, we present the power inequality for the numerical radius.

Theorem 2.6 (e.g., [2]). *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then $w(A^k) \leq w^k(A)$, where $k \in \mathbb{N}$.*

The coming theorem provides results that concerning on the numerical radius and spectral properties for 2×2 block matrices.

Theorem 2.7. *(e.g., [3, Lemma 2]) Let $A \in \mathbb{M}_k(\mathbb{C})$, $B \in \mathbb{M}_{l,k}(\mathbb{C})$, $C \in \mathbb{M}_{k,l}(\mathbb{C})$, and $D \in \mathbb{M}_l(\mathbb{C})$. Then the following statements hold:*

(a) $w\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) = \max\{w(A), w(D)\}.$

(b) $w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & C \\ B & 0 \end{bmatrix}\right).$

(c) If $l = k$, then we have
$w\left(\begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix}\right) = w(B).$

Computing the spectral radius is easier than computing the numerical radius, so the coming theorem helps in computing the numerical radius of a matrix with nonnegative entries through the spectral radius.

Theorem 2.8 (e.g., [20]). *Let $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ be such that $a_{i,j} \geq 0$ for all $i, j = 1, 2, \ldots, n$. Then*

$$w(A) = \frac{1}{2}\rho\left([a_{ij} + a_{ji}]\right).$$

In the following chapters we will deal with block matrices, therefore the next theorem contributes in estimating the spectral radius, the numerical radius, and the spectral norm of a block matrix depending on its matrix entries.

Theorem 2.9. *[5, Theorem 1.1] Let* $A = [A_{ij}]$ *be an* $m \times m$ *block matrix, where* $A_{ij} \in \mathbb{M}_{n_i,n_j}(\mathbb{C})$ *with* $\sum_{i=1}^{m} n_i = m$ *and let* $B := [\|A_{ij}\|]$. *Then*

(1) $w(A) \leq w(B)$.

(2) $\|A\| \leq \|B\|$.

(3) $\rho(A) \leq \rho(B)$.

Theorem 2.10. *[3, Theorem 1] Let* $A = [A_{ij}]$ *be an* $m \times m$ *block matrix, where* $A_{ij} \in \mathbb{M}_{n_i,n_j}(\mathbb{C})$ *with* $\sum_{i=1}^{m} n_i = m$. *Then*

$$w(A) \leq w([c_{ij}]),$$

where

$$c_{ij} = \begin{cases} w(A_{ij}) & if\ i = j, \\ \|A_{ij}\| & if\ i \neq j. \end{cases}$$

Theorem 2.10 gives an upper bound for the numerical radius of a block matrix A by considering the numerical radius and norms of its individual blocks A_{ij}. The importance of this Theorem lies in simplifying the estimation of bounds for the numerical radius of large block matrices by breaking them down into smaller blocks.

The following corollary follows by employing the above theorem and Theorem 2.8.

Corollary 2.1. *[3, Corollary 1] Let* $A \in \mathbb{M}_k(\mathbb{C})$, $B \in \mathbb{M}_{k,l}(\mathbb{C})$, $C \in \mathbb{M}_{l,k}(\mathbb{C})$, $D \in \mathbb{M}_l(\mathbb{C})$, *and let* $T = \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$. *Then*

$$\begin{aligned} w(T) &\leq w\left(\begin{bmatrix} w(A) & \|B\| \\ \|D\| & w(D) \end{bmatrix}\right) \\ &= \frac{1}{2}\left(w(A) + w(D) + \sqrt{(w(A) - w(D))^2 + (\|B\| + \|D\|)^2}\right). \end{aligned}$$

The next theorem provide other upper bound for the numerical radius of a block matrix A. It is similar to Theorem 2.10 with difference in the construction of the entries that are not on the diagonal.

Theorem 2.11. *[3, Theorem 2] Let $A = [A_{ij}]$ be an $m \times m$ block matrix, where $A_{ij} \in \mathbb{M}_{n_i,n_j}(\mathbb{C})$ matrix $\sum_{i=1}^{m} n_i = m$. Then*

$$w(A) \leq w([a'_{ij}]),$$

where

$$a'_{ij} = \begin{cases} w(A_{ij}) & if\ i = j, \\ w\left(\begin{bmatrix} 0 & A_{ij} \\ A_{ji} & 0 \end{bmatrix}\right) & if\ i \neq j. \end{cases}$$

The coming corollary follows by using the above theorem and Theorem 2.8.

Corollary 2.2. *[3, Corollary 2] Let $A \in \mathbb{M}_k(\mathbb{C})$, $B \in \mathbb{M}_{k,l}(\mathbb{C})$, $C \in \mathbb{M}_{l,k}(\mathbb{C})$, $D \in \mathbb{M}_l(\mathbb{C})$, and let $T = \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$ and $T_0 = \left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)$. then*

$$\begin{aligned} w(T) &\leq w\left(\begin{bmatrix} w(A) & w(T_0) \\ w(T_0) & w(D) \end{bmatrix}\right) \\ &= \frac{1}{2}\left(w(A) + w(D) + \sqrt{(w(A) - w(D))^2 + 4w^2(T_0)}\right). \end{aligned}$$

The next theorem provides the eigenvalues for a special case of tridiagonal matrix.

Theorem 2.12 (e.g. [21])**.** *Let $T_n \in \mathbb{M}_n(\mathbb{C})$ be tridiagonal matrix given by*

$$T_n = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & & 0 \\ 0 & \frac{1}{2} & & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ & & & & 0 & \frac{1}{2} \\ 0 & 0 & \cdots & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Then the eigenvalues of T_n are given by

$$\lambda_j = \cos\left(\frac{\pi j}{n+1}\right) \ for\ \ j = 1, 2, \ldots, n.$$

Not that the matrix T_n is symmetric and so it is normal. Therefor, by Theorem 2.4,

$$w(T_n) = \|T_n\| = \rho(T_n) = \cos\left(\frac{\pi}{n+1}\right).$$

In the following, we introduce the definition of *Kronecker product* of two matrices.

Definition 2.6. ***(Kronecker product)****[25]*
The Kronecker product *of the matrix $A \in \mathbb{M}_{p,q}(\mathbb{C})$ with the matrix $B \in \mathbb{M}_{r,s}(\mathbb{C})$ is defined as*

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1q}B \\ \vdots & & \vdots \\ a_{p1}B & \cdots & a_{pq}B \end{bmatrix}.$$

Theorem 2.13. *[7] If $A, B \in \mathbb{M}_n(\mathbb{C})$, and if $\sigma(A), \sigma(B)$ denote their respective spectra. Then the spectrum of $A \otimes B$ is the set of products $\sigma(A) \bullet \sigma(B) = \{\lambda \cdot \mu : \lambda \in \sigma(A),\ \mu \in \sigma(B)\}$.*

The subsequent theorem is a generalization of Theorem 2.12 to the tridiagonal block matrix.

Theorem 2.14. *(e.g. [6, Lemma 6]) Let T be the $m \times m$ block matrix given by*

$$T = \begin{bmatrix} 0 & \frac{1}{2}I & 0 & \cdots & 0 \\ \frac{1}{2}I & 0 & \frac{1}{2}I & \cdots & 0 \\ \vdots & \vdots\ddots & \ddots & \ddots & \vdots \\ & & & & \frac{1}{2}I \\ 0 & 0 & \cdots & \frac{1}{2}I & 0 \end{bmatrix},$$

where I is the identity matrix in $\mathbb{M}_n(\mathbb{C})$. Then the eigenvalues of T are

$$\lambda_j = \cos\left(\frac{\pi j}{m+1}\right) \text{ for } j = 1, 2, \ldots, m,$$

with each eigenvalue has multiplicity n.

Proof. Note that $T = T_m \otimes I$, where T_m is given in Theorem 2.12 and $I \in \mathbb{M}_n(\mathbb{C})$. Then by Theorem 2.13 and Theorem 2.12 we have

$$\begin{aligned}
\sigma(T) &= \sigma(T_m) \bullet \sigma(I) \\
&= \left\{\cos\frac{\pi}{m+1}, \cos\frac{2\pi}{m+1}, \ldots, \cos\frac{m\pi}{m+1}\right\} \bullet \{1, 1, \ldots 1\} \\
&= \left\{\cos\frac{\pi}{m+1}, \ldots, \cos\frac{\pi}{m+1}, \ldots, \cos\frac{m\pi}{m+1}, \ldots, \cos\frac{m\pi}{m+1}\right\}.
\end{aligned}$$

Therefor, the eigenvalues of T are $\lambda_j = \cos\left(\frac{\pi j}{m+1}\right)$ for $j = 1, 2, \ldots, m$, with each eigenvalue has multiplicity n.

□

In the following, the numerical radius of a new block matrix is given. This matrix often appears when deriving bounds for the eigenvalues of matrix polynomials by using the Frobenius companion matrices. Therefore, we need to calculate the numerical radius for it as shown.

Theorem 2.15. *(e.g. [6, Lemma 7]) Let L be the $m \times m$ block matrix given by*

$$L = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ I & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}.$$

Then

$$w(L) = \cos\left(\frac{\pi}{m+1}\right).$$

Proof. It follow from Theorem 2.8 and Theorem 2.14.

□

The next theorem gives an upper bound for the numerical radius of block matrix whose entries are zeros expect for the first row. This bound depends on the numerical radius and norms of its matrix entries.

Theorem 2.16. *[22, Theorem 2.5] Let $Q_i \in \mathbb{M}_n(\mathbb{C})$ for $i = 1, 2, \ldots, m$. Then*

$$w\left(\begin{bmatrix} Q_1 & Q_2 & \cdots & Q_m \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left(w(Q_1) + \sqrt{w^2(Q_1) + \sum_{i=2}^{m} \|Q_i\|^2}\right).$$

Based on Theorem 2.16 and Theorem 2.3, the coming corollary provides an upper bound for the numerical radius of any block matrix.

Corollary 2.3. *[22, Corollary 2.6] Let $A = [A_{ij}]$ be an $n \times n$ block matrix with $A_{ij} \in \mathbb{M}_{n_i,n_j}(\mathbb{C})$. Then*

$$w(A) \leq \frac{1}{2}\sum_{i=1}^{n}\left(w(A_{ii}) + \sqrt{w^2(A_{ii}) + \sum_{\substack{j=1 \\ i \neq j}}^{n} \|A_{ij}\|^2}\right).$$

The following theorem enables us to calculate the numerical radius of a matrix whose entries are zeros expect for the first row. It is similar to Theorem 2.16 but it is not a special case for it.

Theorem 2.17. *[9] Let $N = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_n \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$, where $b_i \in \mathbb{C}$ for $i = 1, 2, \ldots, n$. Then*

$$w(N) = \frac{1}{2}\left(|b_1| + \sqrt{\sum_{j=1}^{n} |b_j|^2}\right).$$

The next two theorems provide other upper bounds for the numerical radius.

Theorem 2.18. *[14, Lemma 4.1] Let $Q_{1i}, Q_{1i} \in \mathbb{M}_n(\mathbb{C})$ for $i = 1, 2, \ldots, m$. Then*

$$w\left(\begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1m} \\ Q_{21} & 0 & \cdots & 0 \\ \\ \vdots & \vdots & & \vdots \\ Q_{m1} & 0 & \cdots & 0 \end{bmatrix}\right) \leq \frac{1}{2}\left(w(Q_{11}) + \sqrt{w^2(Q_{11}) + 4\sum_{i=2}^{m} w^2\left(\begin{bmatrix} 0 & Q_{1i} \\ Q_{i1} & 0 \end{bmatrix}\right)}\right).$$

For a given matrix $A \in \mathbb{M}_n(\mathbb{C})$, A^*A is positive semidefinite and $|A| = (A^*A)^{\frac{1}{2}}$ is defined.

Theorem 2.19. *[23] Let $A \in \mathbb{M}_n(\mathbb{C})$. Then*

$$w(A) \leq \frac{1}{2}\| |A| + |A^*| \|.$$

Later we will see that one of ways to obtain new bounds of eigenvalues of matrix polynomials is to write the Frobenius companion matrix as sum of other matrices. The following two theorems provide upper bounds for the numerical radius for the sum of two matrices and they are useful for proving some theorems in the following chapter.

Theorem 2.20. *[24, Lemma 2.9] Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then*

$$w(A+B) \leq \sqrt{w^2(A) + w^2(B) + \|A\|\|B\| + w(B^*A)}.$$

Theorem 2.21. *[4, Lemma 2.6] Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then*

$$w(A+B) \leq \sqrt{\max\{\|A\|^2, \|B\|^2\} + \|A\|\|B\| + 2w(A^*B)}.$$

Remark 2.1. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the spectral radius of A is*

$$\rho(A) = \frac{1}{2}(a + d + \sqrt{(a-d)^2 + 4bc}).$$

The subsequent theorem presents an upper bound for the numerical radius of 3×3 block matrix.

Theorem 2.22. *[22, Theorem 2.11] Let $T = [T_{ij}]$ be an 3×3 block matrix with $T_{ij} \in \mathbb{M}_{n_i,n_j}(\mathbb{C})$. Then*

$$w(T) \leq \max\{t_{11}, t_{23}\} + \max\{t_{22}, t_{13}\} + \max\{t_{33}, t_{12}\},$$

where $t_{ij} = w\left(\begin{bmatrix} 0 & T_{ij} \\ T_{ji} & 0. \end{bmatrix}\right)$ for $i, j = 1, 2, 3$ and $t_{ii} = w(T_{ii})$, for $i = 1, 2, 3$.

In the next theorem, a generalization of the third part of Theorem 2.9 is introduced in the case of commuting entries. This inequality help in establishing bounds for the eigenvalues of matrix polynomials with commuting coefficients.

Theorem 2.23. *[11, Theorem 2.2] Let $A = [A_{ij}]$ be an $m \times m$ block matrix be such that the entries $A_{ij} \in \mathbb{M}_n(\mathbb{C})$ are commuting for $i, j = 1, 2, \ldots, m$. Then*

$$\rho(A) \leq \rho([\rho(A_{ij})]).$$

Chapter 3

Bounds for the eigenvalues of matrix polynomials

In this chapter, we start by introducing the Frobenius companion matrices and the strong ℓ-ifications of matrix polynomials and some of their properties. Also, we present several bounds for the eigenvalues of matrix polynomials. In addition, the proof of generalized Cauchy and generalized Pellet theorems are provided at the end of this chapter.

3.1 Frobenius companion matrix and ℓ-ifications

Frobenius companion matrix is one of the most popular matrices used for computing the eigenvalues of matrix polynomials. In this section, we introduce the Frobenius companion matrix of the matrix polynomials (see, e.g., [9, 26]). In addition we address the *strong ℓ-ifications* of square complex matrix polynomials. In [27] and [28] ℓ-ifications of a matrix polynomials are derived and in [13] ℓ-ifications for a square matrix polynomials are re-derived in simpler way.

In the follwoing definition, we define the square complex matrix polynomial and its eigenvalues as will as its eigenvectors.

Definition 3.1. *[8] An* $n \times n$ matrix polynomial of degree m *is a mapping* $P : \mathbb{C} \to \mathbb{M}_n(\mathbb{C})$ *defined by*

$$P(z) = A_{m+1}z^m + A_m z^{m-1} + \cdots + A_2 z + A_1,$$

where $A_i \in \mathbb{M}_n(\mathbb{C})$ *and* $A_{m+1} \neq 0$. *We say that* $\lambda \in \mathbb{C}$ *is an* eigenvalue *of* $P(z)$ *if* $\det P(\lambda) = 0$. *A nonzero vector* $\mathbf{x} \in \mathbb{C}^n$ *is called* eigenvector *of* $P(z)$ *corresponding to* λ

if $P(\lambda)\mathbf{x} = 0$. *We denote the set of all eigenvalues of* $P(z)$ *by* $\sigma(P)$, *and call this set the* spectrum *of* $P(z)$.

We say ∞ is an eigenvalue of $P(z)$, if the *reverse matrix polynomial*

$$revP(z) = z^m P(1/z) = A_1 z^m + A_2 z^{m-1} + \cdots + A_m z + A_{m+1}$$

has zero as an eigenvalue. We say that $P(z)$ is *regular matrix polynomial* if $\det P(\lambda) \not\equiv 0$. $P(z)$ has mn eigenvalues (counting multiplicities) and possible infinite ones. It is well known (see, e.g., [13]) that if A_{m+1} is singular then some eigenvalues of $P(z)$ are infinite, and if A_1 is singular then the zero is an eigenvalue of $P(z)$.

In the ensuing example, we consider the case in which A_{m+1} is singular.

Example 3.1. *Consider the quadratic matrix polynomial*

$$P(z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} z^2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It clear that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *is singular and* $\det P(z) = (z^2 + z + 1)(z + 1)$. *Thus,* $\lambda_1 = -1$, $\lambda_2 = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$, *and* $\lambda_3 = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$ *are eigenvalues of* $P(z)$.
Now,

$$revP(z) = z^2 P(1/z) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z^2,$$

and $\det(revP(z)) = (z^2 + z + 1)(z^2 + z)$. *Since zero is an eigenvalue of* $revP(z)$, *then* ∞ *is an eigenvalue of* $P(z)$.

In the following example, we consider the case in which A_1 is singular.

Example 3.2. *Let*

$$P(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z^2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then $\det P(z) = z^4 + 2z^3 + 3z^2 + 2z$ *and so zero is an eigenvalue of* $P(z)$.

In the next theorem, Frobenius companion matrix is presented, and it is showed that it has the same eigenvalues as $P(z)$.

Theorem 3.1. *(e.g. [9, Lemma 7]) Consider the monic matrix polynomial* $P(z) = Iz^m + A_m z^{m-1} + \cdots + A_2 z + A_1$, *of degree* $m \geq 2$, *with* I *be the identity matrix in* $\mathbb{M}_n(\mathbb{C})$. *Then the* Frobenius companion matrix *of* $P(z)$ *is the* $mn \times mn$ *matrix given by*

$$C(P) = \begin{bmatrix} -A_m & -A_{m-1} & -A_{m-2} & \cdots & -A_2 & -A_1 \\ I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix},$$

and λ *is eigenvalue of* $P(z)$ *if and only if* $\lambda \in \sigma(C(P))$.

Proof. Let $\lambda \in \sigma(C(P))$. Then there exist a nonzero vector $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)^T \in \mathbb{C}^{mn}$, where $\mathbf{x}_i \in \mathbb{C}^n$ for $i = 1, 2, \ldots, m$, such that $C(P)\mathbf{x} = \lambda\mathbf{x}$. Hence we have

$$\begin{aligned} \sum_{i=1}^{m} -A_{m-i+1}\mathbf{x}_i &= \lambda\mathbf{x}_1 \\ \mathbf{x}_1 &= \lambda\mathbf{x}_2 \\ \mathbf{x}_2 &= \lambda\mathbf{x}_3 \\ &\vdots \\ \mathbf{x}_{m-1} &= \lambda\mathbf{x}_m. \end{aligned} \tag{3.1}$$

By (3.1), we have $\mathbf{x}_i = \lambda^{m-i}\mathbf{x}_m$, hence $\mathbf{x}_m \neq 0$. And we have

$$\sum_{i=1}^{m} -A_{m-i+1}\mathbf{x}_i = \sum_{i=1}^{m} -A_{m-i+1}\lambda^{m-i}\mathbf{x}_m = \lambda^m\mathbf{x}_m,$$

so $P(\lambda)\mathbf{x}_m = 0$. Therefor $\lambda \in \sigma(P)$.

Conversely, if $\lambda \in \sigma(P)$, then $\det P(\lambda) = 0$, so there exists a nonzero vector $\mathbf{x}_0 \in \mathbb{C}^n$ such that $P(\lambda)\mathbf{x}_0 = 0$. Now, take $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m)^T \in \mathbb{C}^{mn}$, where $\mathbf{x}_i = \lambda^{m-i}\mathbf{x}_0$. Since $P(\lambda)\mathbf{x}_0 = 0$, we get

$$\begin{aligned} \lambda^m\mathbf{x}_0 + \sum_{i=1}^{m} A_{m-i+1}\lambda^{m-i}\mathbf{x}_0 &= 0, \\ \sum_{i=1}^{m} -A_{m-i+1}\lambda^{m-i}\mathbf{x}_0 &= \lambda^m\mathbf{x}_0. \end{aligned}$$

Hence

$$C(P)\mathbf{x} = \begin{bmatrix} \sum_{i=1}^{m} -A_{m-i+1}\lambda^{m-i}\mathbf{x}_0 \\ \lambda^{m-1}\mathbf{x}_0 \\ \vdots \\ \lambda\mathbf{x}_0 \end{bmatrix} = \begin{bmatrix} \lambda^{m}\mathbf{x}_0 \\ \lambda^{m-1}\mathbf{x}_0 \\ \vdots \\ \lambda\mathbf{x}_0 \end{bmatrix} = \begin{bmatrix} \lambda\mathbf{x}_1 \\ \lambda\mathbf{x}_2 \\ \vdots \\ \lambda\mathbf{x}_m \end{bmatrix} = \lambda\mathbf{x}.$$

Therefore, $\lambda \in \sigma(C(P))$.

□

Now, consider the matrix polynomial $P(z) = \sum_{j=0}^{m} A_{j+1}z^j$. We aim to define a so-called strong ℓ-ification for P. An ℓ-ification of P is a matrix polynomial with lower order of P that has the same eigenvalues and also it preserves the eigenstructure of P. Most of the following material can be found in [13].

Definition 3.2. *Let m be divisible by a positive integer k and $\ell = \dfrac{m}{k}$. Then the* strong ℓ-ification *of P is given by $\sum_{j=0}^{\ell} WC_jWz^j$, where W is $kn \times kn$ block exchange matrix $J_k \otimes I_n$, and the $k \times k$ block matrices $C_j \in \mathbb{C}^{kn\times kn}$ are defined by*

$$C_0 = \begin{bmatrix} A_{(k-1)\ell+1} & A_{(k-2)\ell+1} & \cdots & A_{\ell+1} & A_1 \\ -I & 0 & & & \\ & -I & \ddots & & \\ & & \ddots & 0 & \\ & & & -I & 0 \end{bmatrix},$$

$$C_j = \begin{bmatrix} A_{j+1+(k-1)\ell} & A_{j+1+(k-2)\ell} & \cdots & A_{j+1+\ell} & A_{j+1} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad j = 1, \ldots, \ell-1,$$

$$\text{and } C_\ell = \begin{bmatrix} A_{m+1} & & & & \\ & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \end{bmatrix}.$$

The matrix J_k *is the* $k \times k$ exchange matrix *that defined as*

$$J_k = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}.$$

Not that $W^2 = I_{kn}$, also $\sum_{j=0}^{\ell} WC_jWz^j$ and $\sum_{j=0}^{\ell} C_jz^j$ have identical eigenvalues and eigenstructure because they are similar. So it is convenient to work with $\sum_{j=0}^{\ell} C_jz^j$ because it is easier.

The coming lemma will help us to prove the next theorem.

Lemma 3.1. *For a positive integer* $m \geq 3$, *let* $M_j \in \mathbb{C}^{n\times n}$ *for* $j = 1, \ldots .m$ *and* $N_j \in \mathbb{C}^{n\times n}$ *for* $j = 1, \ldots .m-1$. *Then,*

$$\det\left(\begin{bmatrix} M_1 & M_2 & M_3 & \cdots & M_m \\ -I & N_1 & & & \\ & -I & N_2 & & \\ & & \ddots & \ddots & \\ & & & -I & N_{m-1} \end{bmatrix}\right) = \det\left(M_1 \prod_{j=1}^{m-1} N_j + M_2 \prod_{j=2}^{m-1} N_j + \cdots + M_{m-1}N_{m-1} + M_m\right),$$

where the matrix multiplications are from the right with increasing index j.

In the subsequent theorem, we show that the matrix polynomial and its ℓ-ification have identical eigenvalues.

Theorem 3.2. *Let* $P(z) = \sum_{j=0}^{m} A_{j+1}z^j$ *and* $\ell = \dfrac{m}{k}$ *if* m *is divisible by a positive integer* k, *and let* $Q(z) = \sum_{j=0}^{\ell} C_jz^j$, *with the matrices* C_j *defined in Definition 3.2. Then the eigenvalues of* P *and* Q *coincide.*

Proof. For $\ell = \dfrac{m}{k}$, we rewrite P as follows:

$$
\begin{aligned}
P(z) &= A_{m+1}z^m + A_m z^{m-1} + \cdots + A_2 z + A_1 \\
&= z^{(k-1)\frac{m}{k}}\left(A_{m+1}z^{\frac{m}{k}} + A_m z^{\frac{m}{k}-1} + \cdots + A_{(k-1)\frac{m}{k}+1}\right) \\
&\quad + z^{(k-2)\frac{m}{k}}\left(A_{(k-1)\frac{m}{k}} z^{\frac{m}{k}-1} + A_{(k-1)\frac{m}{k}-1} z^{\frac{m}{k}-2} + \cdots + A_{(k-2)\frac{m}{k}+1}\right) + \cdots \\
&\quad + z^{(k-j)\frac{m}{k}}\left(A_{(k-(j-1))\frac{m}{k}} z^{\frac{m}{k}-1} + A_{(k-(j-1))\frac{m}{k}-1} z^{\frac{m}{k}-2} + \cdots + A_{(k-j)\frac{m}{k}+1}\right) + \cdots \\
&\quad + z^{\frac{m}{k}}\left(A_{2\frac{m}{k}} z^{\frac{m}{k}-1} + A_{2\frac{m}{k}-1} z^{\frac{m}{k}-2} + \cdots + A_{\frac{m}{k}+1}\right) + \cdots \\
&\quad + A_{\frac{m}{k}} z^{\frac{m}{k}-1} + A_{\frac{m}{k}-1} z^{\frac{m}{k}-2} + \cdots + A_1 \\
&= \sum_{j=0}^{\ell}\left(A_{j+1+(k-1)\ell} z^j\right) z^{(k-1)\ell} + \sum_{i=2}^{k}\left(\sum_{j=0}^{\ell-1} A_{j+1+(k-i)\ell} z^j\right) z^{(k-i)\ell}.
\end{aligned}
$$

By defining $M_i(z)$ as follows:

$$
M_1(z) := \sum_{j=0}^{\ell} A_{j+1+(k-1)\ell} z^j \quad \text{and} \quad M_i(z) := \sum_{j=0}^{\ell-1} A_{j+1+(k-i)\ell} z^j \quad \text{for } i = 2, \ldots, k,
$$

we get

$$
P(z) = \sum_{i=1}^{k} M_i(z) z^{(k-i)\ell}.
$$

Now, applying Lemma 3.1 with $M_i = M_i(z)$ for $i = 1, \ldots, k$ and $N_i = Iz^\ell$ yields

$$
\det(P(z)) = \det\left(\sum_{i=1}^{k} M_i(z) z^{(k-i)\ell}\right) = \det\left(\begin{bmatrix} M_1(z) & M_2(z) & M_3(z) & \cdots & M_k(z) \\ -I & Iz^\ell & & & \\ & -I & Iz^\ell & & \\ & & \ddots & \ddots & \\ & & & -I & Iz^\ell \end{bmatrix}\right)
$$

$$
= \det\left(\begin{bmatrix} \sum_{j=0}^{\ell} A_{j+1+(k-1)\ell} z^j & \sum_{j=0}^{\ell-1} A_{j+1+(k-2)\ell} z^j & \sum_{j=0}^{\ell-1} A_{j+1+(k-3)\ell} z^j & \cdots & \sum_{j=0}^{\ell-1} A_{j+1} z^j \\ -I & Iz^\ell & & & \\ & -I & Iz^\ell & & \\ & & \ddots & \ddots & \\ & & & -I & Iz^\ell \end{bmatrix}\right).
$$

Thus,

$$\det(P(z)) = \det\left(\sum_{j=0}^{\ell} C_j z^j\right) = \det(Q(z)). \tag{3.2}$$

So, the finite eigenvalues of P and Q coincide. It remains to prove this theorem in the case that the infinite eigenvalues exists. By (3.2) we have

$$\begin{aligned}\det\left(revP(z)\right) &= \det\left(z^m P(1/z)\right) = z^{mn} \det\left(P(1/z)\right) \\ &= z^{\ell(kn)} \det\left(Q(1/z)\right) = \det\left(z^{\ell} Q(1/z)\right) \\ &= \det\left(revQ(z)\right),\end{aligned}$$

this implies that the infinite eigenvalue of P and Q also coincide.

□

Not that, when $k = m$ and $P(z)$ is monic, i.e., $A_{m+1} = I$, then Q becomes $Iz - C(P)$, where $C(P)$ is the Frobenius companion matrix of P.

In the coming example, we construct $Q(z)$ for a matrix polynomial of degree 6.

Example 3.3. *Consider the matrix polynomial*

$$p(z) = A_7 z^6 + A_6 z^5 + A_5 z^4 + A_4 z^3 + A_3 z^2 + A_2 z + A_1.$$

Taking $k = 3$, then $\ell = 2$, thus $P(z)$ has the same eigenvalues as of

$$Q(z) = \begin{bmatrix} A_7 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} z^2 + \begin{bmatrix} A_6 & A_4 & A_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} A_5 & A_3 & A_1 \\ -I & 0 & 0 \\ 0 & -I & 0 \end{bmatrix}.$$

3.2 Bounds for the eigenvalues of matrix polynomials

In this section, bounds for the absolute value of eigenvalues of a matrix polynomials are presented. We use matrix analysis methods and several numerical radius inequalities to the Frobenius companion matrices of monic matrix polynomials to get these bounds.

As we proved in Theorem 3.1, λ is an eigenvalue of $P(z)$ if and only if $\lambda \in \sigma(C(P))$. So if λ is an eigenvalue of $P(z)$, then by Theorem 3.1 and Theorem 2.4, we have

$$|\lambda| \le \rho(C(P)) \le w(C(P)) \le \|(C(P))\|.$$

In 1973, Fujii and Kubo [9] used the norms of the coefficient matrices of a matrix polynomial to establish some classical bounds for the eigenvalues of matrix polynomials as stated in the next two theorems.

Theorem 3.3. *[9, Theorem 8] If λ is any eigenvalue of the monic matrix polynomial $P(z)$, then*

$$|\lambda| \le \max\{1, \|A_1\| + \cdots + \|A_m\|\}.$$

Proof. For a block matrix $A = [A_{ij}]$, where $A_{ij} \in \mathbb{M}_n(\mathbb{C})$ for $i, j = 1, 2, \ldots, m$ define

$$\|A\|_b := \max_{1\le i\le m} \sum_{j=1}^{m} \|A_{ij}\|.$$

Note that $\|.\|_b$ is a matrix norm since axioms 1, 2, and 3 are trivially satisfied, and for the fourth axiom, let $B = [B_{ij}]$, where $B_{ij} \in \mathbb{M}_n(\mathbb{C})$, then it follows that

$$\begin{aligned}
\|AB\|_b &= \max_{1\le i\le m} \sum_{j=1}^{m} \|(AB)_{ij}\| \\
&= \max_{1\le i\le m} \sum_{j=1}^{m} \left\| \sum_{k=1}^{m} A_{ik}B_{kj} \right\| \\
&\le \max_{1\le i\le m} \sum_{j=1}^{m} \sum_{k=1}^{m} \|A_{ik}\|\|B_{kj}\| \\
&= \max_{1\le i\le m} \sum_{k=1}^{m} \|A_{ik}\| \sum_{j=1}^{m} \|B_{kj}\| \\
&\le \left(\max_{1\le k\le m} \sum_{j=1}^{m} \|B_{kj}\| \right) \left(\max_{1\le i\le m} \sum_{k=1}^{m} \|A_{ik}\| \right) \\
&= \|A\|_b\|B\|_b.
\end{aligned}$$

Hence, $\|.\|_b$ is a matrix norm. By Theorem 2.1 we have $|\lambda| \le \|C(P)\|_b$, and since $\|C(P)\|_b = \max\{1, \|A_1\| + \cdots + \|A_m\|\}$ we get the result. □

This result is simple and easy to use. But often it does not give good results especially when the spectral norms of the matrices A_i are large. By the above theorem, we can get the following result.

Theorem 3.4. *[9, Theorem 9] If λ is any eigenvalue of the monic matrix polynomial $P(z)$, then*

$$|\lambda| \leq \|A_1\| + \|A_1 - A_2\| + \cdots + \|A_{m-1} - A_m\| + \|A_m - I\|.$$

Proof. Consider the matrix polynomial $Q(z) := (I - Iz)P(z)$. By expanding the $Q(z)$ we get

$$Q(z) = -Iz^{m+1} + (I - A_m)z^m + (A_m - A_{m-1})z^{m-1} + \cdots + (A_2 - A_1)z + A_1.$$

Now, $\det Q(z) = \det(I - Iz) \det P(z)$. Hence if $\lambda \in \sigma(P)$, then $\lambda \in \sigma(Q)$. Also we have

$$\begin{aligned} 1 &= \|-A_1 + A_1 - A_2 + A_2 - \cdots - A_m + A_m - I\| \\ &\leq \|A_1\| + \|A_1 - A_2\| + \cdots + \|A_{m-1} - A_m\| + \|A_m - I\|. \end{aligned}$$

Applying Theorem 3.3 to $Q(z)$, we get the result.

□

For matrix polynomial $P(z) = A_{m+1}z^m + A_m z^{m-1} + \cdots + A_2 z + A_1$, if A_{m+1} is nonsingular, we can introduce new matrix polynomial associated with $P(z)$:

$$\begin{aligned} P_U &= Iz^m + A_{m+1}^{-1}A_m z^{m-1} + \cdots + A_{m+1}^{-1}A_2 z + A_{m+1}^{-1}A_1 \\ &= Iz^m + U_m z^{m-1} + \cdots + U_2 z + U_1, \end{aligned}$$

where $U_i := A_{m+1}^{-1}A_i$, for $i = 1, 2, \ldots, m$. Therefore, $P(z)$ and $P_U(z)$ have the same eigenvalues. The Frobenius companion matrix of $P_U(z)$ is

$$C(P_U) = \begin{bmatrix} -U_m & -U_{m-1} & -U_{m-2} & \cdots & -U_2 & -U_1 \\ I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix}.$$

Using the companion matrix $C(P_U)$, Higham and Tisseur [10] presented new bounds for eigenvalues of P based on norm and numerical radius inequalities. Here, we mention some of these bounds.

Theorem 3.5. *[10, Lemma 2.2] Every eigenvalue λ of P satisfies*

$$|\lambda| \leq 1 + \sum_{i=1}^{m} \|U_i\|_p, \quad 1 \leq p \leq \infty.$$

Proof. We can write $C(P_U)$ as:

$$C(P_U) = K_1 + K_2 + \cdots + K_m + L,$$

where

$$K_m = \begin{bmatrix} -U_m & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \; K_{m-1} = \begin{bmatrix} 0 & -U_{m-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \ldots,$$

$$K_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -U_1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \; L = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ I & 0 & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & I & 0 \end{bmatrix}.$$

Now, for $i = 1, 2, \ldots, m$, $\|K_i\|_p \leq \|U_i\|_p$ and $\|L\|_p \leq 1$. Thus,

$$\begin{aligned} |\lambda| &\leq \|C(P_U)\|_p \\ &= \|K_1 + K_2 + \cdots + K_m + L\|_p \\ &\leq \|K_1\|_p + \|K_2\|_p + \cdots + \|K_m\|_p + \|L\|_p \\ &\leq 1 + \sum_{i=1}^{m} \|U_i\|_p. \end{aligned}$$

□

The last bound depends on the p-matrix norm $\|.\|_p$, where $1 \leq p \leq \infty$, but the p-matrix norm is more important and easier to calculate when $p = 1, 2, \infty$. So, in the coming theorem, we obtain bounds based on $\|.\|_p$, where $p = 1, \infty$, and also based on the sperctral norm.

Theorem 3.6. *[10, Lemma 2.3] Every eigenvalue λ of P satisfies*

$$\begin{aligned} |\lambda| &\leq \max\left(\|U_1\|_1, 1 + \max_{2 \leq i \leq m} \|U_i\|_1 \right), \\ |\lambda| &\leq \max\left(1, \|U\|_\infty\right), \\ |\lambda| &\leq \|I + UU^*\|^{\frac{1}{2}}, \end{aligned}$$

where $U := [U_1, U_2, \ldots, U_m]$.

Proof. The first two bounds are follows from the facts $|\lambda| \leq \|C(P_U)\|_1$ and $|\lambda| \leq \|C(P_U)\|_\infty$. These facts follows by Theorem 2.1. For the last bound, let

$$R := \begin{bmatrix} -U_m & -U_{m-1} & \cdots & U_1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \ L := \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}.$$

Then $C(P_U) = R + L$ and $R^*L = L^*R = 0$.

Now

$$\begin{aligned} |\lambda|^2 &\leq \|C(P_U)\|^2 \\ &= \|C^*(P_U)C(P_U)\| \\ &= \|R^*R + L^*L\| \\ &\leq \|I + R^*R\| \\ &= \|I + RR^*\| \\ &= \|I + UU^*\| \end{aligned}$$

□

Corollary 3.1. *[10, Corollary 2.4] Every eigenvalue* λ *of* P *satisfies*

$$\begin{aligned} |\lambda| &\leq 1 + \max_{1 \leq i \leq m} \|U_i\|_1, \\ |\lambda| &\leq \max\left(1, \sum_{i=1}^{m} \|U_i\|_\infty\right), \\ |\lambda| &\leq \left(1 + \sum_{i=1}^{m} \|U_i\|^2\right)^{\dfrac{1}{2}}. \end{aligned}$$

In the last bounds we use the norms of $C(P)$ to get them. Based on numerical radius estimations of the companion matrix, we can get many bounds for the eigenvalues of matrix polynomials, as presented in the subsequent theorems.

Theorem 3.7. *[6, Theorem 1] If λ is any eigenvalue of the monic matrix polynomial $P(z)$, then*

$$|\lambda| \leq w(C(P)) \leq \frac{1}{2}\left(w(A_m) + \cos\frac{\pi}{m} + \sqrt{\left(w(A_m) - \cos\frac{\pi}{m}\right)^2 + 4\sum_{i=1}^{m-1} a_i^2}\right),$$

where $a_i := \dfrac{\|A_i\|}{2}$, $i = 1, 2, \ldots, m-2$ *and* $a_{m-1} := w\left(\begin{bmatrix} 0 & -A_{m-1} \\ I & 0 \end{bmatrix}\right)$.

Proof. By applying Theorem 2.4 and Theorem 2.11 to $C(P)$, we have

$$|\lambda| \leq w(C(P)) \leq w\left(\begin{bmatrix} w(A_m) & w\left(T_{-A_{m-1},I}\right) & w\left(T_{-A_{m-2},0}\right) & \cdots & w\left(T_{-A_2,0}\right) & w\left(T_{-A_1,0}\right) \\ w\left(T_{I,-A_{m-1}}\right) & w\left(T_{0,0}\right) & w\left(T_{0,I}\right) & \cdots & & w\left(T_{0,0}\right) \\ w\left(T_{0,-A_{m-2}}\right) & w\left(T_{I,0}\right) & w\left(T_{0,0}\right) & \cdots & \ddots & w\left(T_{0,0}\right) \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ w\left(T_{0,-A_2}\right) & & \ddots & & w\left(T_{0,0}\right) & w\left(T_{0,I}\right) \\ w\left(T_{0,-A_1}\right) & w\left(T_{0,0}\right) & \cdots & & w\left(T_{I,0}\right) & w\left(T_{0,0}\right) \end{bmatrix}\right),$$

where $T_{X,Y} = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ for any two matrices $X, Y \in \mathbb{M}_n(\mathbb{C})$. Now, by Theorem 2.5 and Theorem 2.7, we have for any matrix $A \in \mathbb{M}_n(\mathbb{C})$

$$w\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}\right) = \frac{\|A\|}{2},$$

since $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}^2 = 0$.

Let $a_i = \dfrac{\|A_i\|}{2}$, $i = 1, 2, \ldots, m-2$, $a_{m-1} = w\left(\begin{bmatrix} 0 & -A_{m-1} \\ I & 0 \end{bmatrix}\right)$ and $v = [a_{m-1}, a_{m-2}, \ldots, a_1]$.

Then we have

$$w(C(P)) \leq \left(\begin{bmatrix} w(A_m) & a_{m-1} & a_{m-2} & a_{m-3} & \cdots & a_1 \\ a_{m-1} & 0 & 1/2 & 0 & \cdots & 0 \\ a_{m-2} & 1/2 & 0 & 1/2 & \cdots & 0 \\ & & \vdots\ddots & \ddots & \ddots & \\ \vdots & & & \ddots & \ddots & 1/2 \\ a_1 & 0 & 0 & \cdots & 1/2 & 0 \end{bmatrix}\right)$$

$$= \quad w\left(\begin{bmatrix} w(A_m) & v \\ v^T & T_{m-1} \end{bmatrix}\right) \quad \text{(where } T_{m-1} \text{ is the matrix defined in Theorem 2.12),}$$

$$\leq \quad w\left(\begin{bmatrix} w(A_m) & \|v\| \\ \|v\| & w(T_{m-1}) \end{bmatrix}\right) \quad \text{(by Corollary 2.1),}$$

$$= \quad w\left(\begin{bmatrix} w(A_m) & \left(\sum_{i=1}^{m-1} a_i^2\right)^{\frac{1}{2}} \\ \left(\sum_{i=1}^{m-1} a_i^2\right)^{\frac{1}{2}} & \cos\dfrac{\pi}{m} \end{bmatrix}\right)$$

$$= \quad \frac{1}{2}\left(w(A_m) + \cos\frac{\pi}{m} + \sqrt{\left(w(A_m) - \cos\frac{\pi}{m}\right)^2 + 4\sum_{i=1}^{m-1} a_i^2}\right).$$

□

This upper bound is more complicated than the previous bounds and more difficult to compute, but in most numerical examples it gives more accurate results. Now, by using other properties of numerical radius we present the following result.

Theorem 3.8. *[6, Theorem 2] If λ is any eigenvalue of the monic matrix polynomial $P(z)$, then*

$$|\lambda| \leq w(C(P)) \leq \frac{1}{2}\left(w(A_m) + \sqrt{w^2(A_m) + \sum_{i=1}^{m-1} \|A_i\|^2}\right) + \cos\frac{\pi}{m+1}.$$

Proof. Let $A := \begin{bmatrix} -A_m & -A_{m-1} & \cdots & -A_1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ and L is the matrix defined in Theorem 2.15. It is clear that $C(P) = A + L$, using Theorem 2.16 and Theorem 2.15, we get

$$\begin{aligned} |\lambda| \leq w(C(P)) &= w(A+L) \\ &\leq w(A) + w(L) \quad \text{(triangle inequality)} \\ &\leq \frac{1}{2}\left(w(A_m) + \sqrt{w^2(A_m) + \sum_{i=1}^{m-1} \|A_i\|^2}\right) + \cos\frac{\pi}{m+1}. \end{aligned}$$

□

By using the same proof method in Theorem 3.8 with slight difference in choosing A and L, we obtain the following result for estimation of the numerical radius of $C(P)$.

Theorem 3.9. *[6, Theorem 3] If λ is any eigenvalue of the monic matrix polynomial $P(z)$, then*

$$|\lambda| \leq w(C(P)) \leq \frac{1}{2}\left(w(A_m) + \sqrt{w^2(A_m) + \sum_{i=2}^{m-1} \|A_i\|^2 + \|A_1 - I\|^2}\right) + 1.$$

Theorem 3.10. *[6, Theorem 4] If λ is any eigenvalue of the monic matrix polynomial $P(z)$, then*

$$|\lambda| \leq w(C(P)) \leq \frac{1}{2}\left(w(A_m) + \sqrt{w^2(A_m) + \sum_{i=2}^{m-1} \|A_i\|^2}\right) + \max(1, \frac{\|A_1\| + 1}{2}).$$

In the following result we notice a great similarity with Theorem 3.10 and both provide bounds for the eigenvalues of the monic matrix polynomials. Also, it can be considered as an improvement to Theorem 3.10 under specific conditions.

Theorem 3.11. *[6, Theorem 5] If λ is any eigenvalue of the monic matrix polynomial $P(z)$, then*

$$|\lambda| \leq w(C(P)) \leq \sqrt{\frac{1}{4}\left(w(A_m) + \sqrt{w^2(A_m) + \sum_{i=1}^{m-1} \|A_i\|^2}\right)^2 + \cos^2\frac{\pi}{m+1} + (\sum_{i=1}^{m} \|A_i\|^2)^{\frac{1}{2}}}.$$

Proof. Let A and L be the two matrices defined in Theorem 3.8. Then $C(P) = A + L$. It clear that $\|L\| = 1$ and $L^*A = 0$. Now, by Theorem 2.9 we have

$$\|A\| \leq \left\| \begin{bmatrix} \|A_m\| & \|A_{m-1}\| & \cdots & \|A_1\| \\ 0 & 0 & \cdots & 0 \\ & & & \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\| = (\sum_{i=1}^{m} \|A_i\|^2)^{\frac{1}{2}}.$$

By using triangle inequality and Theorems 2.20, 2.16, and 2.15, we get

$$
\begin{aligned}
|\lambda| \leq w(C(P)) &\leq w(A+L) \\
&\leq \sqrt{\frac{1}{4}\left(w(A_m)+\sqrt{w^2(A_m)+\sum_{i=1}^{m-1}\|A_i\|^2}\right)^2+\cos^2\frac{\pi}{m+1}+(\sum_{i=1}^{m}\|A_i\|^2)^{\frac{1}{2}}}.
\end{aligned}
$$

□

In the next example we take a matrix polynomial of degree 3 and we give a comparison between the bounds of eigenvalues that are obtained in Theorem 3.8, Theorem 3.9, Theorem 3.10, and Theorem 3.11.

Example 3.4. *Consider the monic matrix polynomial* $P(z) = Iz^3 + A_3z^2 + A_2z + A_1$*, where*

$$
A_1 = \begin{bmatrix} 6 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

Then in the following table, we compute the bounds obtained in Theorems 3.8, 3.9, 3.10, and 3.11.

Table 3.1: Comparison of several upper bounds

Theorem	Upper bound
Theorem 3.8	6.092271588321052
Theorem 3.9	5.924428900898052
Theorem 3.10	7.500000000000000
Theorem 3.11	6.345890766020875

This example documents that the bound in Theorem 3.9 is better than the other bounds in Theorems 3.8, 3.10, and 3.11. Also, the eigenvalues of $P(z)$ are

$$
\sigma(P) = \{0.7784 \pm 2.412i,\ 0.4239 \pm 2.1305i,\ 0.4239 \pm 2.1305i,\ -1.5568,\ -0.8477,\ -0.8477\}.
$$

We note that Theorems 3.7, 3.8, 3.9, 3.10, and 3.11 based on the norms of the coefficients A_i and the numerical radius of A_m. Thus, one of the negatives of these results is that it is not easy to compute the numerical radius of A_m. Therefor in Example 3.4 we took Hermitian coefficients. Other similar bound introduce in the next theorem.

Theorem 3.12. *[14, Theorem 5.2] If λ is any eigenvalue of the monic matrix polynomial $P(z)$, then*

$$|\lambda| \le w(C(P)) \le \frac{1}{2}\left(w(A_m) + \sqrt{w^2(A_m) + 4w^2\left(\begin{bmatrix} 0 & A_{m-1} \\ I & 0 \end{bmatrix}\right) + \sum_{i=1}^{m-2} \|A_i\|^2} \right) + \cos\frac{\pi}{m}.$$

Proof. Let $A := \begin{bmatrix} -A_m & -A_{m-1} & \cdots & -A_1 \\ I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ and $R := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}$.

It is clear that $C(P) = A + R$, by Theorem 2.5 $w\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) = \frac{\|A\|}{2}$ for every matrix $A \in \mathbb{M}_n(\mathbb{C})$ because $\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}^2 = 0$, together with Theorem 2.18, we obtain

$$\begin{aligned} |\lambda| \le w(C(P)) &= w(A+R) \\ &\le w(A) + w(R) \\ &\le \frac{1}{2}\left(w(A_m) + \sqrt{w^2(A_m) + 4w^2\left(\begin{bmatrix} 0 & A_{m-1} \\ I & 0 \end{bmatrix}\right) + \sum_{i=1}^{m-2} \|A_i\|^2} \right) + \cos\frac{\pi}{m}. \end{aligned}$$

□

To obtain new bounds for the eigenvalue of the monic matrix polynomial $P(z)$, we apply numerical radius inequalities to the square of Frobenius companion matrices of $P(z)$.

By direct multiplication of matrices we can compute $C^2(P)$ as follows

$$C^2(P) = \begin{bmatrix} B_m & B_{m-1} & \cdots & B_3 & B_2 & B_1 \\ -A_m & -A_{m-1} & \cdots & -A_3 & -A_2 & -A_1 \\ I & 0 & \cdots & 0 & 0 & 0 \\ 0 & I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & I & 0 & 0 \end{bmatrix},$$

where $B_j := A_m A_j - A_{j-1}, j = 1, \dots, m$, with $A_0 := 0$.

By estimating the numerical radius of $C^2(P)$, we obtain the following theorem, which is then used to give bounds for the eigenvalues of $P(z)$.

Theorem 3.13. *[15, Theorem 1] Let $P(z)$ be a monic matrix polynomial. An upper bound of the numerical radius of $C^2(P)$ can be stated as follows:*

$$\begin{aligned} w(C^2(P)) \ &\leq\ \frac{1}{4}(w(B_m) + w(A_{m-1}) + \gamma) + \cos^2\left(\frac{\pi}{mn-1}\right) + \frac{1}{2} \\ &+\frac{1}{2}\sqrt{\frac{1}{4}(w(B_m) + w(A_{m-1}) + \gamma)^2 + \frac{1}{2}\left(\alpha + \eta + \sqrt{(\alpha-\eta)^2 + 4\beta^2}\right)} \\ &+\frac{1}{2}\sqrt{\left(2\cos^2\left(\frac{\pi}{mn-1}\right) + 1\right)^2 + \frac{1}{2}\left(\alpha + \eta + \sqrt{(\alpha-\eta)^2 + 4\beta^2}\right)}, \end{aligned}$$

where

$$\begin{aligned} \alpha &= w^2(T_{I,B_{m-2}}) + \frac{1}{4}\sum_{i=1}^{m-3}\|B_i\|^2, \\ \eta &= w^2(T_{I,A_{m-3}}) + \frac{1}{4}\|A_{m-2}\|^2 + \frac{1}{4}\sum_{i=1}^{m-4}\|A_i\|^2, \\ \beta &= w(T_{B_{m-2},I})\frac{\|A_{m-2}\|}{2} + w(T_{A_{m-3},I})\frac{\|B_{m-3}\|}{2} + \frac{1}{4}\sum_{i=1}^{m-4}(\|A_i\|\|B_i\|), \\ \gamma &= \sqrt{(w(B_m) - w(A_{m-1}))^2 + 4w^2(T_{B_{m-1},-A_m})}. \end{aligned}$$

Proof. For any two matrices $X, Y \in \mathbb{M}_n(\mathbb{C})$, let $T_{X,Y} = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$. Application of Theorem 2.11 on $C^2(P)$, resulted in $w(C^2(P)) \leq w(S)$, where S is the following matrix

$$\begin{bmatrix} w(B_m) & w(T_{B_{m-1},-A_m}) & w(T_{B_{m-2},I}) & w(T_{B_{m-3},0}) & w(T_{B_{m-4},0}) & \cdots & w(T_{B_3,0}) & w(T_{B_2,0}) & w(T_{B_1,0}) \\ w(T_{-A_m,B_{m-1}}) & w(A_{m-1}) & w(T_{-A_{m-2},0}) & w(T_{-A_{m-3},I}) & w(T_{-A_{m-4},0}) & \cdots & w(T_{-A_3,0}) & w(T_{-A_2,0}) & w(T_{-A_1,0}) \\ w(T_{I,B_{m-2}}) & w(T_{0,-A_{m-2}}) & w(0) & w(T_{0,0}) & w(T_{0,I}) & \cdots & w(T_{0,0}) & w(T_{0,0}) & w(T_{0,0}) \\ w(T_{0,B_{m-3}}) & w(T_{I,-A_{m-3}}) & w(T_{0,0}) & w(0) & w(T_{0,0}) & \cdots & w(T_{0,0}) & w(T_{0,0}) & w(T_{0,0}) \\ w(T_{0,B_{m-4}}) & w(T_{0,-A_{m-4}}) & w(T_{I,0}) & w(T_{0,0}) & w(0) & \cdots & w(T_{0,0}) & w(T_{0,0}) & w(T_{0,0}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w(T_{0,B_3}) & w(T_{0,-A_3}) & w(T_{0,0}) & w(T_{0,0}) & w(T_{0,0}) & \cdots & w(0) & w(T_{0,0}) & w(T_{0,I}) \\ w(T_{0,B_2}) & w(T_{0,-A_2}) & w(T_{0,0}) & w(T_{0,0}) & w(T_{0,0}) & \cdots & w(T_{0,0}) & w(0) & w(T_{0,0}) \\ w(T_{0,B_1}) & w(T_{0,-A_1}) & w(T_{0,0}) & w(T_{0,0}) & w(T_{0,0}) & \cdots & w(T_{I,0}) & w(T_{0,0}) & w(0) \end{bmatrix}.$$

By Theorem 2.5 and Theorem 2.7, we have

$$w\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}\right) = \frac{\|A\|}{2},$$

for every matrix $A \in \mathbb{M}_n(\mathbb{C})$. Hence we can write S as

$$S = \begin{bmatrix} w(B_m) & w(T_{B_{m-1},-A_m}) & w(T_{B_{m-2},I}) & b_{m-3} & b_{m-4} & \cdots & b_3 & b_2 & b_1 \\ w(T_{-A_m,B_{m-1}}) & w(A_{m-1}) & \frac{\|A_{m-2}\|}{2} & w(T_{-A_{m-3},I}) & a_{m-4} & \cdots & a_3 & a_2 & a_1 \\ w(T_{B_{m-2},I}) & \frac{\|A_{m-2}\|}{2} & 0 & 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ b_{m-3} & w(T_{-A_{m-3},I}) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b_{m-4} & a_{m-4} & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ b_3 & a_3 & 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 & \frac{1}{2} \\ b_2 & a_2 & 0 & 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ b_1 & a_1 & 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & 0 \end{bmatrix},$$

where $b_i := \dfrac{\|B_i\|}{2}$, $i = 1, 2, \ldots, m-3$, and $a_i := \dfrac{\|A_i\|}{2}$, $i = 1, 2, \ldots, m-4$.
Let us take the Partition of S as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

where

$$\begin{aligned} S_{11} &= \begin{bmatrix} w(B_m) & w(T_{B_{m-1},-A_m}) \\ w(T_{-A_m,B_{m-1}}) & w(A_{m-1}) \end{bmatrix}, \\ S_{12} &= \begin{bmatrix} w(T_{B_{m-2},I}) & b_{m-3} & b_{m-4} & \cdots & b_3 & b_2 & b_1 \\ \frac{\|A_{m-2}\|}{2} & w(T_{-A_{m-3},I}) & a_{m-4} & \cdots & a_3 & a_2 & a_1 \end{bmatrix}, \\ S_{21} &= \begin{bmatrix} w(T_{B_{m-2},I}) & b_{m-3} & b_{m-4} & \cdots & b_3 & b_2 & b_1 \\ \frac{\|A_{m-2}\|}{2} & w(T_{-A_{m-3},I}) & a_{m-4} & \cdots & a_3 & a_2 & a_1 \end{bmatrix}^T, \\ S_{22} &= \begin{bmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & 0 & 0 & \cdots & \frac{1}{2} & \cdots & 0 \\ 0 & \frac{1}{2} & \ddots & \ddots \cdots & \vdots & \vdots \ddots & \vdots \\ 0 & 0 & \frac{1}{2} & \ddots \cdots & 0 & 0 & \frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & 0 \end{bmatrix}_{(mn-2)\times(mn-2)}. \end{aligned}$$

Now, applying Corollary 2.3 gives

$$
\begin{aligned}
w(C^2(P)) \le w(S) \ &\le\ \frac{1}{2}\left(w(S_{11}) + \sqrt{w^2(S_{11}) + \|S_{12}\|^2}\right) \\
&\quad + \frac{1}{2}\left(w(S_{22}) + \sqrt{w^2(S_{22}) + \|S_{21}\|^2}\right). \qquad (3.3)
\end{aligned}
$$

Hence it remains to estimate $\|S_{12}\|, \|S_{21}\|, w(S_{11})$ and $w(S_{22})$. We start by estimating $\|S_{12}\|$ and $\|S_{21}\|$.

$$
\begin{aligned}
\|S_{12}\| &= \sqrt{\rho\left(S_{12}S_{12}^*\right)} \\
&= \sqrt{\rho\left(\begin{bmatrix} \alpha & \beta \\ \beta & \eta \end{bmatrix}\right)} \\
&= \sqrt{\frac{1}{2}\left(\alpha + \eta + \sqrt{(\alpha - \eta)^2 + 4\beta^2}\right)}, \qquad \text{(by Remark 2.1)}, \qquad (3.4)
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha &= w^2(T_{I,B_{m-2}}) + \frac{1}{4}\sum_{i=1}^{m-3}\|B_i\|^2, \\
\eta &= w^2(T_{I,-A_{m-3}}) + \frac{1}{4}\|A_{m-2}\|^2 + \frac{1}{4}\sum_{i=1}^{m-4}\|A_i\|^2, \\
\beta &= w(T_{B_{m-2},I})\frac{\|A_{m-2}\|}{2} + w(T_{-A_{m-3},I})\frac{\|B_{m-3}\|}{2} + \frac{1}{4}\sum_{i=1}^{m-4}(\|A_i\|\|B_i\|).
\end{aligned}
$$

Since $S_{21} = S_{12}^T$, then $\|S_{12}\| = \|S_{21}\|$. Now, for $w(S_{11})$, the matrix S_{11} is Hermitian which implies that S_{11} is normal. So by Theorem 2.4 we have

$$
\begin{aligned}
w(S_{11}) &= \rho(S_{11}) \\
&= \frac{1}{2}(w(B_m) + w(A_{m-1}) + \gamma), \qquad (3.5)
\end{aligned}
$$

where

$$
\gamma = \sqrt{(w(B_m) - w(A_{m-1}))^2 + 4w^2(T_{B_{m-1},-A_m})}.
$$

Finally, to estimate $w(S_{22})$ we write S_{22} as

$$
S_{22} = 2\left(T_{mn-2}^2 - \operatorname{diag}\left(\frac{1}{4}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{4}\right)\right),
$$

where T_{mn-2} is the matrix defined in Theorem 2.12. Using the triangle inequality and the fact $w(T_{mn-2}^2) = w^2(T_{mn-2})$ (since T_{mn-1} is normal), we get

$$\begin{aligned} w(S_{22}) &\leq 2\left(w(T_{mn-2}^2) + w\left(\operatorname{diag}\left(\frac{1}{4}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{4}\right)\right)\right) \\ &\leq 2w^2(T_{mn-2}) + 1 \\ &= 2\cos^2\left(\frac{\pi}{mn-1}\right) + 1. \end{aligned} \tag{3.6}$$

Now, by substituting (3.4),(3.5), and (3.6) in (3.3) we get the result.

□

Now, if $\lambda \in \sigma(P)$, then $\lambda \in \sigma(C(P))$ and by Spectral Mapping Theorem we have $\lambda^2 \in \sigma(C^2(P))$ which implies

$$|\lambda|^2 \leq \rho(C^2(P)) \leq w(C^2(P)). \tag{3.7}$$

Thus, by Theorem 3.13 and (3.7) we obtain a bound on the eigenvalues of $P(z)$.

In the next, we are going to employ the fact that similar matrices have the same eigenvalues. So, we can obtain a new bound for the eigenvalues of a monic matrix polynomials using a similar matrix to $C^2(P)$.

Now, consider the $mn \times mn$ invertible matrix

$$B = \begin{bmatrix} I & I & I & \cdots & I \\ 0 & I & I & \cdots & I \\ 0 & 0 & I & \ddots & I \\ \vdots & \vdots & 0 & \ddots & I \\ 0 & 0 & 0 & \cdots & I \end{bmatrix},$$

and the inverse of B is

$$B^{-1} = \begin{bmatrix} I & -I & 0 & \cdots & 0 \\ 0 & I & -I & \cdots & 0 \\ 0 & 0 & I & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & -I \\ 0 & 0 & 0 & \cdots & I \end{bmatrix}.$$

By direct multiplication of matrices we can compute the $mn \times mn$ matrix $BC^2(P)B^{-1}$ as follows

$$\begin{bmatrix} B_m - A_m + I & B_{m-1} - B_m + A_m - A_{m-1} & B_{m-2} - B_{m-1} + A_{m-1} - A_{m-2} & \cdots & B_2 - B_3 + A_3 - A_2 & B_1 - B_2 + A_2 - A_1 \\ I - A_m & A_m - A_{m-1} & A_{m-1} - A_{m-2} & \cdots & A_3 - A_2 - I & A_2 - A_1 \\ I & 0 & 0 & \cdots & -I & 0 \\ 0 & I & 0 & \cdots & -I & 0 \\ 0 & 0 & \ddots & \cdots & -I & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -I & 0 \end{bmatrix}.$$

From similarity of $C^2(P)$ and $BC^2(P)B^{-1}$, we have

$$|\lambda|^2 \leq \rho(BC^2(P)B^{-1}) \leq w(BC^2(P)B^{-1})$$

where λ is an eigenvalue of $P(z)$. In the following theorem, we introduce a new bound for the eigenvalue of P(z) by estimate the numerical radius of $BC^2(P)B^{-1}$.

Theorem 3.14. *[15, Theorem 2] Let $P(z)$ be a monic matrix polynomial. An upper bound of the numerical radius of $BC^2(P)B^{-1}$ can be stated as follows:*

$$\begin{aligned} |\lambda|^2 &\leq w(BC^2(P)B^{-1}) \\ &\leq \frac{1}{2}\Bigg(\xi + 2\cos^2\left(\frac{\pi}{mn-3}\right) + 1 + \frac{2+\sqrt{5}}{4} + \sqrt{\xi^2 + \left(\frac{1}{2}\left(\alpha + \eta + \sqrt{(\alpha-\eta)^2 + 4\beta^2}\right)\right)^2 + \tau^2} \\ &+ \sqrt{\left(2\cos^2\left(\frac{\pi}{mn-3}\right) + 1\right)^2 + \left(\frac{1}{2}\left(\alpha + \eta + \sqrt{(\alpha-\eta)^2 + 4\beta^2}\right)\right)^2 + \mu} \\ &+ \sqrt{\left(\frac{2+\sqrt{5}}{4}\right)^2 + \tau^2 + \mu}\Bigg), \end{aligned}$$

where

$$\begin{aligned} \xi &= \frac{1}{2}\Bigg(w(B_m - A_m + I) + w(A_m - A_{m-1}) \\ &+ \sqrt{\left(w(B_m - A_m + I) - w(A_m - A_{m-1})\right)^2 + 4w^2\left(T_{B_{m-1} - B_m + A_m - A_{m-1}, I - A_m}\right)}\Bigg), \\ \alpha &= w^2\left(T_{B_{m-2} - B_{m-1} + A_{m-1} - A_{m-2}, I}\right) + \frac{1}{4}\sum_{j=6}^{m} \|B_{j-3} - B_{j-2} + A_{j-2} - A_{j-3}\|^2, \\ \eta &= \frac{1}{4}\|A_{m-1} - A_{m-2}\|^2 + w^2\left(T_{A_{m-2} - A_{m-3}, I}\right) + \frac{1}{4}\sum_{j=7}^{m} \|A_{j-3} - A_{j-4}\|^2, \end{aligned}$$

$$\begin{aligned}
\beta &= \frac{1}{2} w\left(T_{B_{m-2}-B_{m-1}+A_{m-1}-A_{m-2},I}\right)\|A_{m-1}-A_{m-2}\| \\
&+ \frac{1}{2}\|B_{m-3}-B_{m-2}+A_{m-2}-A_{m-3}\| w\left(T_{A_{m-2}-A_{m-3},I}\right) \\
&+ \frac{1}{4}\sum_{j=7}^{m}\|B_{j-4}-B_{j-3}+A_{j-3}-A_{j-4}\|\|A_{j-3}-A_{j-4}\|, \\
\tau &= \frac{1}{2}\Big(\frac{\|B_2-B_3+A_3-A_2\|^2}{4}+\frac{\|A_3-A_2-I\|^2}{4}-\frac{\|B_1-B_2+A_2-A_1\|^2}{4}-\frac{\|A_2-A_1\|^2}{4}\Big) \\
&\Bigg(\Big(\frac{\|B_2-B_3+A_3-A_2\|^2}{4}+\frac{\|A_3-A_2-I\|^2}{4}-\frac{\|B_1-B_2+A_2-A_1\|^2}{4}-\frac{\|A_2-A_1\|^2}{4}\Big)^2 \\
&+\frac{1}{4}\left(\|B_2-B_3+A_3-A_2\|\|B_1-B_2+A_2-A_1\|+\|A_3-A_2-I\|\|A_2-A_1\|\right)^2\Bigg)^{\frac{1}{2}}.
\end{aligned}$$

3.3 Location for the eigenvalues of matrix polynomials with commuting coefficients

In this section, we aim to introduce bounds for the eigenvalues of monic matrix polynomials with commuting coefficients. Kittaneh [11] proved the inequality

$$\rho(A) \leq \rho([\rho(A_{ij})]),$$

for commuting entries which stated in Theorem 2.23. Also he employed Theorem 2.23 to obtain better bounds, as illustrated in the following.

Consider the monic matrix polynomial $P(z) = Iz^m + A_m z^{m-1} + \cdots + A_2 z + A_1$ of degree $m \geq 2$, with A_i are commuting matrices for $i = 1, 2, \ldots, m$. Then by applying Theorem 2.23 on $C(P)$ we get $\rho(C(P)) \leq \rho(\tilde{\mathrm{C}}(P))$, where

$$\tilde{\mathrm{C}}(P) = \begin{bmatrix} \rho(A_m) & \rho(A_{m-1}) & \ldots & \rho(A_2) & \rho(A_1) \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}. \tag{3.8}$$

Hence if λ is an eigenvalue of $P(z)$, then

$$|\lambda| \leq \rho\left(\tilde{\mathrm{C}}(P)\right) \leq w\left(\tilde{\mathrm{C}}(P)\right) \leq \left\|\tilde{\mathrm{C}}(P)\right\|. \tag{3.9}$$

Therefore, we can use the inequalities (3.9) to find bounds for the eigenvalues of $P(z)$. We note that the matrix $\tilde{\mathrm{C}}(P)$ is analogous to the companion matrix of complex polynomials. Thus, the ideas that were used to estimate bounds to the zeros of complex polynomials can be used to estimate the eigenvalues of monic matrix polynomials with commuting coefficients. Several bounds for the zeros of complex polynomials have been given in [9, 32, 33, 34, 35]. Based on Theorem 2 in [33], we have

$$\begin{aligned} |\lambda| &\leq w\left(\tilde{\mathrm{C}}(P)\right) \\ &\leq \frac{1}{2}\left(\rho(A_m) + \cos\frac{\pi}{m} + \sqrt{\left(\rho(A_m) - \cos\frac{\pi}{m}\right)^2 + \left(\rho(A_{m-1}) + 1\right)^2 + \sum_{i=1}^{m-2}\rho^2(A_i)}\right). \end{aligned}$$

Now, consider the $m \times m$ invertible matrix

$$Q = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \ddots & 0 \\ 0 & 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

and the inverse of Q is

$$Q^{-1} = \begin{bmatrix} 1 & -1 & 1 & -1 & \cdots & (-1)^{m-1} \\ 0 & 1 & -1 & 1 & \cdots & (-1)^{m-2} \\ 0 & 0 & 1 & -1 & \cdots & (-1)^{m-3} \\ 0 & 0 & 0 & 1 & \ddots & (-1)^{m-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Let $H := Q\tilde{\text{C}}(P)Q^{-1}$. Then $H = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \cdots & \beta_m \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}$.
where $\beta_i = (-1)^{i+1} + \sum_{k=0}^{i-1}(-1)^{k+i+1}\rho(A_{m-k}),\ i = 1, 2, \ldots, m.$

$\tilde{\text{C}}(P)$ and H have the same eigenvalues since they are similar. Thus, by applying numerical radius inequalities to the various decompositions and partitions of H new bounds for the eigenvalues of the monic matrix polynomial $P(z)$ with commuting coefficients are produced.

Theorem 3.15. *[4, Theorem 2.1] If λ is any eigenvalue of $P(z)$, then*

$$|\lambda| \leq \frac{1}{2}\left(\cos\frac{\pi}{m} + \sqrt{\cos^2\frac{\pi}{m} + \sum_{i=3}^{m}\beta_i^2 + (1 + |\beta_2|)^2}\right) + \max\{1, |\beta_1|\}.$$

Proof. At first, we decompose H as $H = S + R$, where

$$S = \begin{bmatrix} 0 & \beta_2 & \beta_3 & \beta_4 & \cdots & \beta_m \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, R = \begin{bmatrix} \beta_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}.$$

Then, we have

$$\begin{aligned} |\lambda| &\leq \rho(C(P)) \\ &\leq \rho(\tilde{\text{C}}(\text{P})) \\ &= \rho(H) \\ &\leq w(H) \\ &= w(S + R) \\ &\leq w(S) + w(R). \end{aligned} \tag{3.10}$$

Now,

$$w(R) = \max\{1, |\beta_1|\}. \tag{3.11}$$

To estimate $w(S)$, we apply Theorem 2.11 on S which gives

$$\begin{aligned}
w(S) \ &\leq \begin{bmatrix} 0 & w\left(\begin{bmatrix} 0 & |\beta_2| \\ 1 & 0 \end{bmatrix}\right) & \frac{1}{2}|\beta_3| & \frac{1}{2}|\beta_4| & \cdots & \frac{1}{2}|\beta_m| \\ w\left(\begin{bmatrix} 0 & |\beta_2| \\ 1 & 0 \end{bmatrix}\right) & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2}|\beta_3| & \frac{1}{2} & 0 & \frac{1}{2} & \ddots & 0 \\ \frac{1}{2}|\beta_4| & 0 & \frac{1}{2} & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \frac{1}{2} \\ \frac{1}{2}|\beta_m| & 0 & \cdots & 0 & \frac{1}{2} & 0 \end{bmatrix} \\
&= \ w\left(\begin{bmatrix} 0 & u \\ u^* & T_{m-1} \end{bmatrix}\right) \ \text{(where } T_{m-1} \text{ is the matrix defined in Theorem 2.12),} \\
&\leq \ w\left(\begin{bmatrix} 0 & \|u\| \\ \|u\| & \|T_{m-1}\| \end{bmatrix}\right) \ \text{(by Theorem 2.9),}
\end{aligned}$$

where $u = \left[w\left(\begin{bmatrix} 0 & |\beta_2| \\ 1 & 0 \end{bmatrix}\right), \frac{1}{2}|\beta_3|, \frac{1}{2}|\beta_4|, \cdots, \frac{1}{2}|\beta_m| \right]$.

Now, $\|u\| = \sqrt{\dfrac{1}{4}\sum_{i=3}^{m} \beta_i^2 + w^2\left(\begin{bmatrix} 0 & |\beta_2| \\ 1 & 0 \end{bmatrix}\right)}$, and using Theorem 2.8 we have

$$w\left(\begin{bmatrix} 0 & |\beta_2| \\ 1 & 0 \end{bmatrix}\right) = \frac{1}{2}\rho\left(\begin{bmatrix} 0 & 1+|\beta_2| \\ 1+|\beta_2| & 0 \end{bmatrix}\right) = \frac{1}{2}(1+|\beta_2|).$$

Also, $\|T_{m-1}\| = \cos\dfrac{\pi}{m}$.

Thus,

$$\begin{aligned} w(S) &\leq w\left(\begin{bmatrix} 0 & \frac{1}{2}\sqrt{\sum_{i=3}^{m}\beta_i^2+(1+|\beta_2|)^2} \\ \frac{1}{2}\sqrt{\sum_{i=3}^{m}\beta_i^2+(1+|\beta_2|)^2} & \cos\frac{\pi}{m} \end{bmatrix}\right) \\ &= \rho\left(\begin{bmatrix} 0 & \frac{1}{2}\sqrt{\sum_{i=3}^{m}\beta_i^2+(1+|\beta_2|)^2} \\ \frac{1}{2}\sqrt{\sum_{i=3}^{m}\beta_i^2+(1+|\beta_2|)^2} & \cos\frac{\pi}{m} \end{bmatrix}\right) \text{ (since it is normal)}, \\ &= \frac{1}{2}\left(\cos\frac{\pi}{m}+\sqrt{\cos^2\frac{\pi}{m}+\sum_{i=3}^{m}\beta_i^2+(1+|\beta_2|)^2}\right) \text{ (by Remark 2.1).} \qquad (3.12) \end{aligned}$$

□

By substituting (3.11) and (3.12) in (3.10), we get

$$|\lambda| \leq \frac{1}{2}\left(\cos\frac{\pi}{m}+\sqrt{\cos^2\frac{\pi}{m}+\sum_{i=3}^{m}\beta_i^2+(1+|\beta_2|)^2}\right)+\max\{1,|\beta_1|\}.$$

In the next result, Theorems 2.15, 2.17, and 2.20 are applied to other decomposition of H to get a new bound for the eigenvalues of monic matrix polynomials with commuting coefficients.

Theorem 3.16. *[4, Theorem 2.2] If λ is any eigenvalue of $P(z)$, then*

$$|\lambda| \leq \sqrt{\frac{1}{4}\sum_{i=2}^{m}\beta_i^2+\cos^2\frac{\pi}{m+1}+\sqrt{\sum_{i=2}^{m}\beta_i^2}}+\max\{1,|\beta_1|\}.$$

Proof. Write H as $H = K + L + M$, where

$$K = \begin{bmatrix} 0 & \beta_2 & \beta_3 & \cdots & \beta_{m-1} & \beta_m \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

and

$$M = \begin{bmatrix} \beta_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}.$$

Now, by Theorem 2.20

$$\begin{aligned} |\lambda| &\leq w(H) \\ &= w(K+L+M) \\ &\leq w(K+L)+w(M) \\ &\leq \sqrt{w^2(K)+w^2(L)+\|K\|\|L\|+w(L^*K)}.+\max\{1,|\beta_1|\}. \end{aligned}$$

$w(L) = \cos\dfrac{\pi}{m+1}$, $\|L\| = 1$, $L^*K = 0$, using Theorem 2.17 we have

$$w(K) = \frac{1}{2}\left(\sqrt{\sum_{i=2}^{m} \beta_i^2}\right)$$

.

For $\|K\|$, since $K^2 = 0$ we have $\|K\| = 2w(K) = \sqrt{\sum_{i=2}^{m} \beta_i^2}$.

Thus,

$$|\lambda| \leq \sqrt{\frac{1}{4}\sum_{i=2}^{m} \beta_i^2 + \cos^2\frac{\pi}{m+1} + \sqrt{\sum_{i=2}^{m} \beta_i^2}} + \max\{1,|\beta_1|\}.$$

□

In the coming bound, we use the same decomposition of H and the same method of proof as in Theorem 3.16, but we use Theorem 2.21 instead of applying Theorem 2.20.

Theorem 3.17. *[4] If λ is any eigenvalue of $P(z)$, then*

$$|\lambda| \leq \sqrt{\max\left\{\sum_{i=2}^{m} \beta_i^2, 1\right\} + \sqrt{\sum_{i=2}^{m} \beta_i^2}} + \max\{1,|\beta_1|\}.$$

This result provides bound for the absolute value of the eigenvalues, which is expressed in terms of the β_i. In the following result, a similar bound established, but with a slightly different structure in the inequality.

Theorem 3.18. *[4, Theorem 2.4] If λ is any eigenvalue of $P(z)$, then*

$$|\lambda| \leq \frac{1}{2}\sqrt{\sum_{i=3}^{m} \beta_i^2 + (1 + |\beta_2|)^2 + \cos\frac{\pi}{m}} + \max\{1, |\beta_1|\}.$$

Proof. By using triangle inequality to the decomposition of H, where $H = K_1 + L_1 + M$ with

$$K_1 = \begin{bmatrix} 0 & \beta_2 & \beta_3 & \cdots & \beta_{m-1} & \beta_m \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, L_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

and

$$M = \begin{bmatrix} \beta_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix},$$

we get

$$\begin{aligned} |\lambda| &\leq w(H) \\ &= w(K_1 + L_1 + M) \\ &\leq w(K_1) + w(L_1) + w(M). \end{aligned}$$

To complete the proof, we need to estimate $w(K_1), w(L_1)$ and $w(M)$. It clear that $w(M) = \max\{1, |\beta_1|\}$, and $w(L_1) = \cos\frac{\pi}{m}$.

Now, for $w(K_1)$

$$
\begin{aligned}
w(K_1) &\leq \begin{bmatrix} 0 & w\left(\begin{bmatrix} 0 & |\beta_2| \\ 1 & 0 \end{bmatrix}\right) & \frac{1}{2}|\beta_3| & \frac{1}{2}|\beta_4| & \cdots & \frac{1}{2}|\beta_m| \\ w\left(\begin{bmatrix} 0 & |\beta_2| \\ 1 & 0 \end{bmatrix}\right) & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{2}|\beta_3| & 0 & 0 & 0 & \ddots & 0 \\ \frac{1}{2}|\beta_4| & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \frac{1}{2}|\beta_m| & 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \\
&= w\left(\begin{bmatrix} 0 & u \\ u^* & 0 \end{bmatrix}\right) \\
&\leq w\left(\begin{bmatrix} 0 & \|u\| \\ \|u\| & 0 \end{bmatrix}\right) \\
&= \|u\| \\
&= \frac{1}{2}\sqrt{\sum_{i=3}^{m} \beta_i^2 + (1+|\beta_2|)^2},
\end{aligned}
$$

where $u = \left[w\left(\begin{bmatrix} 0 & |\beta_2| \\ 1 & 0 \end{bmatrix}\right), \frac{1}{2}|\beta_3|, \frac{1}{2}|\beta_4|, \cdots, \frac{1}{2}|\beta_m| \right]$.

Therefor,

$$
|\lambda| \leq \frac{1}{2}\sqrt{\sum_{i=3}^{m} \beta_i^2 + (1+|\beta_2|)^2} + \cos\frac{\pi}{m} + \max\{1, |\beta_1|\}.
$$

□

In the next theorem, other bound derived once more using the same decomposition of H as presented in Theorem 3.16. This bound obtained by using triangle inequality and other numerical radius properties.

Theorem 3.19. *[4, Theorem 2.5] If λ is any eigenvalue of $P(z)$, then*

$$
|\lambda| \leq \frac{1}{2}\sqrt{\sum_{i=2}^{m} \beta_i^2} + \cos\frac{\pi}{m+1} + \max\{1, |\beta_1|\}.
$$

Proof. Let K, L, M be the matrices defined in Theorem 3.16. Then $H = K + L + M$.

Thus,

$$\begin{aligned} |\lambda| &\leq w(H) \\ &= w(K + L + M) \\ &\leq w(K) + w(L) + w(M) \\ &= \frac{1}{2}\sqrt{\sum_{i=2}^{m} \beta_i^2} + \cos\frac{\pi}{m+1} + \max\{1, |\beta_1|\}. \end{aligned}$$

□

In the subsequent theorem, we write H as $H = K_2 + M + U$, where

$$K_2 = \begin{bmatrix} 0 & \beta_2 & \beta_3 & \cdots & \beta_{m-1} & \beta_m - 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, M = \begin{bmatrix} \beta_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix},$$

and

$$U = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

It is easy to show that $U^*U = UU^* = I$. Hence U is unitary.

Theorem 3.20. *[4, Theorem 2.6] If λ is any eigenvalue of $P(z)$, then*

$$|\lambda| \leq \frac{1}{2}\sqrt{\sum_{i=2}^{m-1} \beta_i^2 + |1 - \beta_m|^2} + \max\{1, |\beta_1|\} + 1.$$

Proof. By applying the triangle inequality to the new decomposition of H, we get

$$\begin{aligned} |\lambda| &\leq w(H) \\ &= w(K_2 + M + U) \\ &\leq w(K_2) + w(M) + w(U). \end{aligned}$$

Now, using Theorem 2.17 we have $w(K_2) = \frac{1}{2}\sqrt{\sum_{i=2}^{m-1} \beta_i^2 + |1 - \beta_m|^2}$, and also we know that $w(M) = \max\{1, |\beta_1|\}$. And since U is unitary, then $w(U) = 1$.
Thus,

$$|\lambda| \leq \frac{1}{2}\sqrt{\sum_{i=2}^{m-1} \beta_i^2 + |1 - \beta_m|^2} + \max\{1, |\beta_1|\} + 1.$$

□

This result is a combination of a quadratic and linear forms. This showing that the eigenvalues are influenced not just by individual coefficients but also by their combined influence. The following bound follows by employing Theorems 2.3 and 2.17 to other different decomposition of H.

Theorem 3.21. *[4, Theorem 2.7] If λ is any eigenvalue of $P(z)$, then*

$$|\lambda| \leq \frac{1}{2}\left(|\beta_1| + \sqrt{\sum_{i=1}^{m-1} \beta_i^2 + 1}\right) + \frac{1}{2}\left(1 + \sqrt{1 + \beta_m^2}\right) + 1.$$

In the next result, Theorems 2.9, 2.17, and 2.19 are used to obtain a new bound for the eigenvalues.

Theorem 3.22. *[4, Theorem 2.8] If λ is any eigenvalue of $P(z)$, then*

$$|\lambda| \leq \max\left\{1, \frac{1}{2}\left(|\beta_1| + \sqrt{\sum_{i=1}^{m-1} \beta_i^2}\right)\right\} + \max\left\{1, \frac{1 + |\beta_m|}{2}\right\}.$$

Now, consider the partition of H as $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where

$$A = \begin{bmatrix} \beta_1 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_2 & \beta_3 & \cdots & \beta_m \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}.$$

Using this partition of H, a new bound of eigenvalues of $P(z)$ is derived, as stated in the following theorem.

Theorem 3.23. *[4, Theorem 2.9] If λ is any eigenvalue of $P(z)$, then*

$$|\lambda| \leq \frac{1}{2}\left(|\beta_1| + \sqrt{2} + \sqrt{(|\beta_1| - \sqrt{2})^2 + 4\sqrt{\sum_{i=2}^{m} \beta_i^2}} \right).$$

Proof. By the partition of $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and Theorem 2.9, we have

$$\begin{aligned} |\lambda| &\leq \rho(H) \\ &= \rho\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \\ &\leq \rho\left(\begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix}\right) \\ &= \frac{1}{2}\left(\|A\| + \|D\| + \sqrt{(\|A\| - \|D\|)^2 + 4\|B\|\|C\|} \right). \end{aligned}$$

Now, since $\|A\| = |\beta_1|, \|B\| = \sqrt{\sum_{i=2}^{m} \beta_i^2}, \|C\| = 1$, and

$$DD^* = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2 \end{bmatrix}.$$

So, $\|D\| = \sqrt{\rho(DD^*)} = \sqrt{2}$.
Thus,

$$|\lambda| \leq \frac{1}{2}\left(|\beta_1| + \sqrt{2} + \sqrt{(|\beta_1| - \sqrt{2})^2 + 4\sqrt{\sum_{i=2}^{m} \beta_i^2}}\right).$$

□

We note that all Theorems 3.15–3.23 based on the β_i which depends on the spectral radius of A_i. Therefore, it easy to compute these results for many matrix polynomials. Through many numerical examples, we observed that Theorem 3.23 often gives better result than the rest.

In the coming example, we take the same matrix polynomial in Example 3.4 to compare all bounds in Theorems 3.15–3.23.

Example 3.5. *Consider the monic matrix polynomial* $P(z) = Iz^3 + A_3z^2 + A_2z + A_1$, *where*

$$A_1 = \begin{bmatrix} 6 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the following table provides bounds for eigenvalues of $P(z)$.

Table 3.2: Comparison of several upper bounds

Theorem	Upper bound
Theorem 3.15	5.288873605350877
Theorem 3.16	5.755604389124888
Theorem 3.17	9.100356356720601
Theorem 3.18	5.531128874149273
Theorem 3.19	5.514993334118501
Theorem 3.20	5.354101966249683
Theorem 3.21	7.193846301110436
Theorem 3.22	6.081138830084188
Theorem 3.23	3.974536333077138

Therefor, the bound in Theorem 3.23 gives $|\lambda| \leq 3.974536333077138$, for any $\lambda \in \sigma(P)$. For this matrix polynomial we note that the upper bound provided by Theorem

3.23 is better than all upper bounds mentioned in Table 3.1 and Table 3.2.

In the previous theorems in this section, we use the matrix $H = Q\tilde{\mathrm{C}}(P)Q^{-1}$ to derive new upper bounds to the eigenvalues of $P(z)$. Here, other invertible matrix, say Q_1 is presented and then we will define the matrix $H_1 = Q_1\tilde{\mathrm{C}}(P)Q_1^{-1}$ which is similar to the matrix $\tilde{\mathrm{C}}(P)$ to provide other bounds.

Now, Consider the invertible $m \times m$ matrix

$$Q_1 = \begin{bmatrix} 1 & -1 & -1 & -1 & \cdots & -1 \\ 0 & 1 & -1 & -1 & \cdots & -1 \\ 0 & 0 & 1 & -1 & \ddots & -1 \\ 0 & 0 & 0 & 1 & \ddots & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & -1 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

and the inverse of Q_1 is

$$Q_1^{-1} = \begin{bmatrix} 1 & 1 & 2 & 4 & \cdots & (2)^{m-2} \\ 0 & 1 & 1 & 2 & \cdots & (2)^{m-3} \\ 0 & 0 & 1 & 1 & \cdots & (2)^{m-4} \\ 0 & 0 & 0 & 1 & \ddots & (2)^{m-5} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Define H_1 as $H_1 := Q_1\tilde{\mathrm{C}}(P)Q_1^{-1}$. Then $H_1 = \begin{bmatrix} \delta_1 & \delta_2 & \delta_3 & \delta_4 & \cdots & \delta_m \\ 1 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{bmatrix}$.

where

$$\begin{aligned}
\delta_1 &= -1 + \rho(A_m), \\
\delta_2 &= \delta_1 + (-1 + \rho(A_{m-1})), \\
\delta_3 &= \delta_1 + \delta_2 + (-1 + \rho(A_{m-2})), \\
\delta_4 &= \delta_1 + \delta_2 + \delta_3 + (-1 + \rho(A_{m-3})), \\
&\vdots \\
\delta_m &= \delta_1 + \delta_2 + \ldots + \delta_{m-1} + \rho(A_1).
\end{aligned}$$

Now, we write H_1 as $H_1 = K + U + M$, where

$$K = \begin{bmatrix} \delta_1 & \delta_2 & \delta_3 & \delta_4 & \cdots & \delta_m - 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, U = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

and

$$M = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

So, using this decomposition of H_1, the next result is obtained.

Theorem 3.24. *[16, Theorem 2.1] If λ is any eigenvalue of $P(z)$, then*

$$|\lambda| \leq \frac{1}{2}\left(|\delta_1| + \sqrt{\sum_{i=1}^{m-1} \delta_i^2 + (\delta_m - 1)^2}\right) + 1 + \frac{1}{2}\left(1 + \sqrt{m-1}\right).$$

Proof. We have

$$\begin{aligned}
|\lambda| &\leq \rho(C(P)) \\
&\leq \rho(\tilde{\mathrm{C}}(P)) \\
&= \rho(H_1) \\
&\leq w(H_1) \\
&= w(K+U+M) \\
&\leq w(K)+w(U)+w(M).
\end{aligned}$$

Now, we use the fact that $w(U) = \rho(U) = 1$, since U is a unitary matrix. By Theorem 2.17 we have

$$w(K) = \frac{1}{2}\left(|\delta_1| + \sqrt{\sum_{i=1}^{m-1} \delta_i^2 + (\delta_m - 1)^2}\right).$$

Also,

$$\begin{aligned}
w(M) &= w\left(\begin{bmatrix} 0 & u \\ 0 & 1 \end{bmatrix}\right) \quad \text{(where } u = [0\ 1\ 1\ \cdots\ 1]^T\text{)}, \\
&\leq w\left(\begin{bmatrix} 0 & \|u\| \\ 0 & 1 \end{bmatrix}\right) \\
&= w\left(\begin{bmatrix} 0 & \sqrt{m-2} \\ 0 & 1 \end{bmatrix}\right) \\
&= \frac{1}{2}\rho\left(\begin{bmatrix} 0 & \sqrt{m-2} \\ \sqrt{m-2} & 2 \end{bmatrix}\right) \quad \text{(by Theorem 2.8)}, \\
&= \frac{1}{2}\left(1+\sqrt{m-1}\right).
\end{aligned}$$

Thus,

$$|\lambda| \leq \frac{1}{2}\left(|\delta_1| + \sqrt{\sum_{i=1}^{m-1} \delta_i^2 + (\delta_m - 1)^2}\right) + 1 + \frac{1}{2}\left(1+\sqrt{m-1}\right).$$

□

The above theorem gives an upper bound for the eigenvalues in terms of δ_i and the degree of the matrix polynomial $P(z)$. To introduce other bounds for the eigenvalues of $P(z)$ that also depends on δ_i and m, we present a new partition of H_1 as follows:

$$H_1 = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix},$$

where

$$\begin{aligned} A_{11} &= \begin{bmatrix} \delta_1 \end{bmatrix}, \ A_{12} = \begin{bmatrix} \delta_2 & \delta_3 & \cdots & \delta_{m-1} \end{bmatrix}, \ A_{13} = \begin{bmatrix} \delta_m \end{bmatrix}, \\ A_{21} &= \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ A_{22} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}, \ A_{23} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \\ A_{31} &= \begin{bmatrix} 0 \end{bmatrix}, \ A_{32} = \begin{bmatrix} 0 & 0 & \cdots 1 & \end{bmatrix}, \ \text{and} \ A_{33} = \begin{bmatrix} 1 \end{bmatrix}. \end{aligned}$$

Using this partition of H_1, the following result is obtained.

Theorem 3.25. *[16, Theorem 2.2] If λ is any eigenvalue of $P(z)$, then*

$$|\lambda| \leq \max\left\{|\delta_1|, \frac{1}{2}\left(1+\sqrt{m-2}\right)\right\} + \max\left\{\cos\frac{\pi}{m-1}, \frac{|\delta_m|}{2}\right\} + \max\left\{1, \frac{1}{2}\left(1+\sqrt{\sum_{i=2}^{m-1}|\delta_i|^2}\right)\right\}.$$

Proof. By the previous partition of H_1, we have

$$\begin{aligned} |\lambda| &\leq w(H_1) \\ &= w\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}\right) \\ &\leq w\left(\begin{bmatrix} w(A_{11}) & \|A_{12}\| & \|A_{13}\| \\ \|A_{21}\| & w(A_{22}) & \|A_{23}\| \\ \|A_{31}\| & \|A_{32}\| & w(A_{33}) \end{bmatrix}\right) \quad \text{(by Theorem 2.10).} \end{aligned}$$

Now,
$w(A_{11}) = |\delta_1|, \ \ w(A_{22}) = \cos\dfrac{\pi}{m-1}, \ \ w(A_{33}) = 1, \ \ \|A_{12}\| = \sqrt{\sum_{i=2}^{m-1}|\delta_i|^2},$
$\|A_{13}\| = |\delta_m|, \ \|A_{21}\| = 1, \ \ \|A_{23}\| = \sqrt{m-2}, \ \ \|A_{31}\| = 0, \text{ and } \|A_{32}\| = 1.$
So, by applying Theorem 2.22, we get

$$\begin{aligned} |\lambda| \ &\leq \ \max\left\{|\delta_1|, w\left(\begin{bmatrix} 0 & \sqrt{m-2} \\ 1 & 0 \end{bmatrix}\right)\right\} + \max\left\{\cos\frac{\pi}{m-1}, w\left(\begin{bmatrix} 0 & |\delta_m| \\ 0 & 0 \end{bmatrix}\right)\right\} \\ &\quad + \max\{1, w\left(\begin{bmatrix} 0 & \sqrt{\sum_{i=2}^{m-1}|\delta_i|^2} \\ 1 & 0 \end{bmatrix}\right)\}. \end{aligned}$$

Thus,

$$|\lambda| \leq \max\left\{|\delta_1|, \frac{1}{2}\left(1+\sqrt{m-2}\right)\right\} + \max\left\{\cos\frac{\pi}{m-1}, \frac{|\delta_m|}{2}\right\} + \max\left\{1, \frac{1}{2}\left(1+\sqrt{\sum_{i=2}^{m-1}|\delta_i|^2}\right)\right\}.$$

□

In the subsequent two theorems, the previous partition of H_1 is used to give other bounds.

Theorem 3.26. *[16, Theorem 2.3] If λ is any eigenvalue of $P(z)$, then*

$$\begin{aligned} |\lambda| \ &\leq \ \max\left\{|\delta_1|, \cos\frac{\pi}{m-1}, 1\right\} \\ &\quad + \frac{1}{2}\left(\sqrt{\sum_{i=2}^{m}\delta_i^2} + \sqrt{\frac{m-1+\sqrt{m^2-6m+13}}{2}} + 1\right). \end{aligned}$$

Theorem 3.27. *[16, Theorem 2.4] If λ is any eigenvalue of $P(z)$, then*

$$\begin{aligned} |\lambda| \ &\leq \ \max\left\{|\delta_1|, \cos\frac{\pi}{m-1}, 1\right\} \\ &\quad + \frac{1}{2}\max\left\{\left(1+|\delta_m|+\sqrt{\sum_{i=2}^{m-1}\delta_i^2}\right), \left(1+\sqrt{\sum_{i=2}^{m-1}\delta_i^2}+\sqrt{m-2}\right), \left(1+|\delta_m|+\sqrt{m-2}\right)\right\}. \end{aligned}$$

Now, Theorem 2.11, Theorem 2.10, and Corollary 2.3 are used with a new partition of H_1 to prove the following theorem.

Theorem 3.28. *[16, Theorem 2.5] If λ is any eigenvalue of $P(z)$, then*

$$\begin{aligned}|\lambda| &\leq \frac{1}{2}\left(|\delta_1| + \cos\frac{\pi}{m-1} + 1\right)\\ &+\frac{1}{4}\left(\sqrt{4\delta_1^2 + (1+|\delta_2|)^2 + \sum_{i=3}^{m}\delta_i^2 + \sqrt{\delta_m^2 + m + 5}}\right)\\ &+\frac{1}{4}\left(\sqrt{4\cos^2\frac{\pi}{m-1} + (1+|\delta_2|)^2 + \sum_{i=3}^{m-1}\delta_i^2 + m + 1}\right).\end{aligned}$$

Proof. Using Theorem 2.4 and Theorem 2.11, we have

$$\begin{aligned}|\lambda| &\leq w(H_1)\\ &\leq w\left(\begin{bmatrix} |\delta_1| & \frac{1+|\delta_2|}{2} & \frac{|\delta_3|}{2} & \frac{|\delta_4|}{2} & \cdots & \frac{|\delta_{m-1}|}{2} & \frac{|\delta_m|}{2} \\ \frac{1+|\delta_2|}{2} & 0 & \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{2} \\ \frac{|\delta_3|}{2} & \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & \frac{1}{2} \\ \frac{|\delta_4|}{2} & 0 & \frac{1}{2} & 0 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \frac{1}{2} & \frac{1}{2} \\ \frac{|\delta_{m-1}|}{2} & 0 & 0 & \cdots & \frac{1}{2} & 0 & 1 \\ \frac{|\delta_m|}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & 1 & 1 \end{bmatrix}\right).\end{aligned}$$

Now , consider the partition $H_2 = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$, where

$$\begin{aligned} M_{11} &= \begin{bmatrix} |\delta_1| \end{bmatrix},\ M_{12} = \begin{bmatrix} \frac{1+|\delta_2|}{2} & \frac{|\delta_3|}{2} & \frac{|\delta_4|}{2} & \cdots & \frac{|\delta_{m-1}|}{2} \end{bmatrix},\ M_{13} = M_{31} = \begin{bmatrix} \frac{|\delta_m|}{2} \end{bmatrix},\\ M_{21} &= \begin{bmatrix} \frac{1+|\delta_2|}{2} \\ \frac{|\delta_3|}{2} \\ \frac{|\delta_4|}{2} \\ \cdots \\ \frac{|\delta_{m-1}|}{2} \end{bmatrix},\ M_{22} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \frac{1}{2} \\ 0 & 0 & \cdots & \frac{1}{2} & 0 \end{bmatrix},\ M_{23} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \vdots \\ \frac{1}{2} \\ 1 \end{bmatrix},\\ M_{32} &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & 1 \end{bmatrix},\ \text{and}\ M_{33} = \begin{bmatrix} 1 \end{bmatrix}.\end{aligned}$$

Thus, by Theorem 2.10 we obtain

$$\begin{aligned}
|\lambda| &\le w\left(\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}\right) \\
&\le w\left(\begin{bmatrix} w(M_{11}) & \|M_{12}\| & \|M_{13}\| \\ \|M_{21}\| & w(M_{22}) & \|M_{23}\| \\ \|M_{31}\| & \|M_{32}\| & w(M_{33}) \end{bmatrix}\right) \\
&= w\left(\begin{bmatrix} |\delta_1| & \frac{1}{2}\sqrt{(1+|\delta_2|)^2 + \sum_{i=3}^{m-1}\delta_i^2} & \frac{|\delta_m|}{2} \\ \frac{1}{2}\sqrt{(1+|\delta_2|)^2 + \sum_{i=3}^{m-1}\delta_i^2} & \cos\frac{\pi}{m-1} & \frac{\sqrt{m+1}}{2} \\ \frac{|\delta_m|}{2} & \frac{\sqrt{m+1}}{2} & 1 \end{bmatrix}\right) \\
&\le \frac{1}{2}\left(|\delta_1| + \cos\frac{\pi}{m-1} + 1\right) \\
&\quad + \frac{1}{4}\left(\sqrt{4\delta_1^2 + (1+|\delta_2|)^2 + \sum_{i=3}^{m}\delta_i^2 + \sqrt{\delta_m^2 + m + 5}}\right) \\
&\quad + \frac{1}{4}\left(\sqrt{4\cos^2\frac{\pi}{m-1} + (1+|\delta_2|)^2 + \sum_{i=3}^{m-1}\delta_i^2 + m + 1}\right) \quad \text{(by Corollary 2.3).}
\end{aligned}$$

□

Now, once again we take the matrix polynomial given in Example 3.4 and compare the upper bounds of eigenvalues obtained in Theorem 3.24, Theorem 3.25, Theorem 3.26, Theorem 3.27, and Theorem 3.28.

Example 3.6. *Let $P(z) = Iz^3 + A_3z^2 + A_2z + A_1$, where*

$$A_1 = \begin{bmatrix} 6 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we have the following table.

Table 3.3: Comparison of several upper bounds

Theorem	Upper bound
Theorem 3.24	7.830582164166344
Theorem 3.25	7.999999999999998
Theorem 3.26	7.797276724936021
Theorem 3.27	7.999999999999998
Theorem 3.28	7.634801217463689

If λ is any eigenvalue of $P(z)$, then the bound in Theorem 3.28 give $|\lambda| \leq 7.634801217463689$, but that in Theorem 3.23 gives better estimates than all estimates mentioned in Table 3.1, Table 3.2, and Table 3.3.

In [16], the bounds in Theorems 3.15–3.28 are compared by the the following matrix polynomial.

$$Q(z) = Iz^3 + \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} z^2 + \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} z + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The bound in Theorem 3.22 gives the best estimation for the eigenvalues of $Q(z)$ comparing to Theorems 3.15–3.28 which is $|\lambda| \leq 2.707$. We note that in our example the bound in Theorem 3.23 provides the best estimation and in the other example Theorem 3.22 provides the best estimation comparing to Theorems 3.15–3.28. In addition, the order of theorems according to preference change. Therefore, we conclude that in general it is not possible to compare the sharpness of these bounds of eigenvalues of matrix polynomials. But we can only compare them through some special cases by numerical examples as we did in Example 3.4, Example 3.5, and Example 3.6.

3.4 Cauchy and Pellet theorems for matrix polynomials

In 1829, Cauchy presented a simple but classical result [36, Theorem 27.1], which states that all zeros of the polynomial

$$p(z) = a_{m+1}z^m + a_m z^{m-1} + \cdots + a_2 z + a_1$$

with complex coefficients and $a_{m+1} \neq 0$, lie in the disk $|z| \leq c$, where c is the unique positive root of

$$|a_{m+1}|z^m - |a_m|z^{m-1} - \cdots - |a_2|z - |a_1| = 0.$$

In [10, 17, 37], a generalization of Cauchy's classical result was derived. Before stating it, we introduce the generalized Rouché theorem for matrices which help to prove the generalized Cauchy theorem.

Theorem 3.29. *[17] (**Generalized Rouché Theorem for matrices**) Let $A, B : \Omega \to \mathbb{M}_n(\mathbb{C})$ be analytic matrix-valued functions, where Ω is an open connected subset of $\mathbb{C}$ and assume that $A(z)$ is nonsingular for all z on the simple closed curve $\Gamma \subseteq \Omega$. If $\|A(z)^{-1}B(z)\| < 1$ for all $z \in \Gamma$, then $\det(A+B)$ and $\det(A)$ have the same number of zeros inside Γ, counting multiplicities.*

Theorem 3.30. *[10, 17, 37](**Generalized Cauchy Theorem**) All eigenvalues of the matrix polynomial*

$$P(z) = A_{m+1}z^m + A_m z^{m-1} + \cdots + A_2 z + A_1,$$

where $A_i \in \mathbb{M}_n(\mathbb{C})$, for $i = 1, 2, \ldots, m+1$, lie in $|z| \le R$ when A_{m+1} is nonsingular, and lie in $|z| \ge r$ when A_1 is nonsingular, where R and r are the unique positive roots of

$$u(x) = \|A_{m+1}^{-1}\|^{-1}x^m - \|A_m\|x^{m-1} - \cdots - \|A_2\|x - \|A_1\| = 0$$

and

$$l(x) = \|A_{m+1}\|x^m + \|A_m\|x^{m-1} + \cdots + \|A_2\|x - \|A_1^{-1}\|^{-1} = 0,$$

respectively.

Proof. At first, we note that the polynomial $l(x)$ has one sign change, then by the Descartes' rule of signs, $l(x)$ has unique real positive root, say r. Also, $l(0) = -\|A_1^{-1}\|^{-1} < 0$ and $l(r) = 0$, which implies that $l(x) < 0$ for all $0 \le x < r$. Since $l(x) < 0$ for $0 \le x < r$, we have,

$$\|A_{m+1}\|x^m + \|A_m\|x^{m-1} + \cdots + \|A_2\|x < \|A_1^{-1}\|^{-1},$$

so that

$$\left(\|A_1^{-1}\|\right)\left(\|A_{m+1}\|x^m + \|A_m\|x^{m-1} + \cdots + \|A_2\|x\right) < 1, \quad \text{for all } 0 \le x < r.$$

Thus, for all $|z| < r$ we have

$$\begin{aligned}
&\|(A_1^{-1})(A_{m+1}z^m + A_m z^{m-1} + \cdots + A_2 z)\| \\
&\le \|(A_1^{-1})\| \, \|(A_{m+1}z^m + A_m z^{m-1} + \cdots + A_2 z)\| \\
&\le \left(\|A_1^{-1}\|\right)\left(\|A_{m+1}\||z|^m + \|A_m\||z|^{m-1} + \cdots + \|A_2\||z|\right) < 1.
\end{aligned}$$

Now, define the simple closed curve Γ as $\Gamma = (r-\epsilon)e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Then for all $\epsilon > 0$ and all $z \in \Gamma$

$$\|(A_1^{-1})(A_{m+1}z^m + A_m z^{m-1} + \cdots + A_2 z)\| < 1.$$

So, by Theorem 3.29, $P(z)$ and A_1 have the same number of eigenvalues of moduli less than r, but since A_1 is nonsingular then A_1 has no eigenvalues of moduli less than r. Therefore, all eigenvalues of $P(z)$ lie in $|z| \geq r$. The proof of upper bound is similar.

□

Pellet's Theorem for scalar polynomial was obtained in ([38], [39, Theorem 28,1]) which says that for the polynomial

$$p(z) = a_{m+1}z^m + a_m z^{m-1} + \cdots + a_2 z + a_1$$

with complex coefficients, $m \geq 2$, and $a_1 a_{k+1} \neq 0$ for some k with $1 \leq k \leq m-1$. If the polynomial

$$|a_{m+1}|x^m + |a_m|x^{m-1} + \cdots + |a_{k+2}|x^{k+1} - |a_{k+1}|x^k + |a_k|x^{k-1} + \cdots + |a_2|x + |a_1|$$

have two distinct positive roots t_1 and t_2 with $t_1 < t_2$. Then p has exactly k zeros in the disk $|z| \leq t_1$ and no zeros in the annulus $t_1 < |z| < t_2$. Note that Cauchy's result can be considered as a special limit case of Pellet's Theorem [17].

Similar to Cauchy's result, a generalization of Pellet's Theorem was obtained in [37, 17]. Also generalized Cauchy Theorem can be considered as a limit case of generalized Pellet's Theorem for $k = 0, m$. In the following theorem we give the generalized Pellet Theorem.

Theorem 3.31. *[17](**Generalized Pellet Theorem**) Let*

$$P(z) = A_{m+1}z^m + A_m z^{m-1} + \cdots + A_2 z + A_1$$

be a regular matrix polynomial with $m \geq 2$, $A_i \in \mathbb{M}_n(\mathbb{C})$, for $i = 1, 2, \ldots, m+1$, and $A_1 \neq 0$. Let A_{k+1} be invertible for some k with $1 \leq k \leq m-1$ and let the polynomial

$$\begin{aligned} f_k(x) &= \|A_{m+1}\|x^m + \|A_m\|x^{m-1} + \cdots + \|A_{k+2}\|x^{k+1} - \|A_{k+1}^{-1}\|^{-1}x^k \\ &\quad + \|A_k\|x^{k-1} + \cdots + \|A_2\|x + \|A_1\| \end{aligned}$$

have two distinct positive roots x_1 and x_2 with $x_1 < x_2$. Then $\det(P(z))$ has exactly kn zeros in the disk $|z| \leq x_1$ and no zeros in the annulus $x_1 < |z| < x_2$.

Proof. $f_k(x)$ has two sign changes, so by the Descartes' rule of signs, $f_k(x)$ has either two or no positive roots. Assume that $f_k(x)$ has two positive roots x_1 and x_2 with $x_1 < x_2$. Since $f_k(0) = \|A_1\| > 0$ and $f_k(x_1) = f_k(x_2) = 0$ we have that $f_k(x) < 0$ for all $x_1 < x < x_2$, which implies that

$$\left(\|A_{k+1}^{-1}\|x^{-k}\right)\left(\sum_{\substack{i=0\\ i\neq k}}^{m}\|A_{i+1}\|x^{i}\right) < 1, \text{ for all } x_1 < x < x_2.$$

Now, for all $x_1 < |z| < x_2$ we have

$$\begin{aligned}\left\|\left(A_{k+1}z^{k}\right)^{-1}\left(\sum_{\substack{i=0\\ i\neq k}}^{m}A_{i+1}z^{i}\right)\right\| &\leq \left(\|A_{k+1}^{-1}\|\;|z|^{-k}\right)\left(\sum_{\substack{i=0\\ i\neq k}}^{m}\|A_{i+1}\|\;|z|^{i}\right)\\ &< 1.\end{aligned}$$

Define the simple closed curves $\Gamma_1 = (x_1+\epsilon)e^{i\theta}$ and $\Gamma_2 = (x_2-\epsilon)e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Then for all $\epsilon > 0$ and all $z \in \Gamma_1$ and all $z \in \Gamma_2$ we have

$$\left\|\left(A_{k+1}z^{k}\right)^{-1}\left(\sum_{\substack{i=0\\ i\neq k}}^{m}A_{i+1}z^{i}\right)\right\| < 1.$$

It clear that the zero is eigenvalue of $A_{k+1}z^k$ with multiplicity kn, then by Theorem 3.29 $P(z)$ has kn eigenvalues lie in $|z| \leq x_1$. Also by Theorem 3.29 $P(z)$ has kn eigenvalues lie in $|z| < x_2$. Therefore, $P(z)$ has exactly kn eigenvalues in the disk $|z| \leq x_1$ and no eigenvalues in the annulus $x_1 < |z| < x_2$.

□

Chapter 4

Location of the eigenvalues of special matrix polynomials

In this chapter, we will deal with matrix polynomials with certain conditions in its coefficients. We present the definitions of *doubly stochastic* matrices and *Schur stable* matrices. In addition, we study the upper and lower bounds for the matrix polynomials with unitary, doubly stochastic, and Schur stable coefficients.

In [8] it was shown that the eigenvalues of matrix polynomials with unitary coefficients lie inside the annulus $\frac{1}{2} < |\lambda| < 2$, and in [18, 29] the same result was displayed for matrix polynomials with doubly stochastic coefficients. Also in [18] some results for the matrix polynomials with Schur stable coefficients were proved.

Now, consider the family of all matrix polynomials with unitary coefficients

$$\mathcal{U} = \left\{ \sum_{i=0}^{m} A_{i+1} z^i : A_i \text{ are } n \times n \text{ unitary matrices}, n, m \in \mathbb{N} \right\},$$

and let

$$\sigma_{\mathcal{U}} = \{\lambda : \lambda \in \sigma(P) \text{ where } P \in \mathcal{U}\} \text{ and } |\sigma_{\mathcal{U}}| = \{|\lambda| : \lambda \in \sigma_{\mathcal{U}}\}.$$

Based on Rouché Theorem, Generalized Cauchy Theorem and Intermediate Value Theorem upper and lower bounds on $|\sigma_{\mathcal{U}}|$ are provide in the coming theorem.

Theorem 4.1. *[8, Theorem 3.2] Let $P(z) \in \mathcal{U}$. Then for any $\lambda \in \mathbb{C}$, such that λ is an eigenvalue of $P(z)$, it follows that*

$$\frac{1}{2} < |\lambda| < 2.$$

Proof. We want to show that 2 is an upper bound for $|\sigma_{\mathcal{U}}|$. Its is known that for any unitary matrix U, $\|U\| = 1$. Since unitary matrices are nonsingular, by this fact and application of the Generalized Cauchy Theorem on $P(z)$ we get that for any eigenvalue of $P(z)$, $|\lambda| \leq R$, where R is the unique positive root of the function $u : \mathbb{R} \to \mathbb{R}$ defined by

$$u(x) = x^m - x^{m-1} - \cdots - 1.$$

Since $2^m > 2^{m-1} + \cdots + 2^0$, it follows that $u(2) > 0$, also we have $u(1) < 0$. Then by the Intermediate Value Theorem $u(x)$ has a root in the interval $(1, 2)$, but $u(x)$ has unique positive root R. Therefore, $R \in (1, 2)$. This implies that the modulus of any eigenvalue of $P(z)$ is bounded above by 2.

We now show that $\frac{1}{2}$ is a lower bound for $|\sigma_{\mathcal{U}}|$. Similarly, by Generalized Cauchy Theorem, the modulus of any eigenvalue of $P(z)$ is bounded below by r, where r is the unique positive root of the function $l : \mathbb{R} \to \mathbb{R}$ defined by

$$l(x) = x^m + \cdots + x - 1.$$

Now,

$$1 = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i > \sum_{i=1}^{m} \left(\frac{1}{2}\right)^i.$$

Therefor, $l(\frac{1}{2}) < 0$ and $l(1) > 0$. Then by intermediate value theorem $r \in (\frac{1}{2}, 1)$. This implies that $|\lambda| > \frac{1}{2}$, where λ is any eigenvalue of $P(z)$.

□

Note that, Theorem 4.1 provides upper and lower bounds on the set $|\sigma_{\mathcal{U}}|$. But we cannot be sure that these bounds are the optimal. In fact, in the next theorem it is shown that they are the least upper bound and greatest lower bound of the set $|\sigma_{\mathcal{U}}|$.

Theorem 4.2. *[8, Theorem 3.3]* $\sup |\sigma_{\mathcal{U}}| = 2$ *and* $\inf |\sigma_{\mathcal{U}}| = \dfrac{1}{2}$.

Proof. By Theorem 4.1, $\frac{1}{2}$ is a lower bound for $|\sigma_{\mathcal{U}}|$. We want to show that there is no number in the interval $(\frac{1}{2}, 2)$ can be a lower bound to the set $|\sigma_{\mathcal{U}}|$. Let $r \in (\frac{1}{2}, 1)$. Then $\sum_{i=1}^{\infty} r^i > 1$, which implies that there exist $t \in \mathbb{N}$ such that $\sum_{i=1}^{t} r^i > 1$. Consider $P(z) \in \mathcal{U}$ defined as

$$P(z) = z^t I + z^{t-1} I + \cdots + zI - I.$$

Therefore $\det P(z) = (z^t + \cdots + z - 1)^n$. Then by Descartes' rule of signs $P(z)$ has a real positive eigenvalue and its the root of the polynomial

$$l(x) = x^t + \cdots + x - 1.$$

Now, $l(r) > 0$ and $l(\frac{1}{2}) < 0$. So, by intermediate value theorem there exist $\lambda \in (\frac{1}{2}, r)$ such that $l(\lambda) = 0$. Therefore, we find a positive eigenvalue of $P(z)$ less than r. Thus, r is not a lower bound of $|\sigma_{\mathcal{U}}|$.

Again, by Theorem 4.1, 2 is a upper bound for $|\sigma_{\mathcal{U}}|$. We now show that there is no number in the interval $(\frac{1}{2}, 2)$ can be an upper bound to the set $|\sigma_{\mathcal{U}}|$. Let $R \in (1, 2)$. Now, define the matrix polynomial $P(z) \in \mathcal{U}$ as

$$P(z) = z^m I - (z^{m-1} I + \cdots + zI + I).$$

Therefore, $\det P(z) = (z^m - (z^{m-1} + \cdots + z + 1))^n$. So, $P(z)$ has a real positive eigenvalue which is the root of the polynomial

$$u(x) = x^m - (x^{m-1} + \cdots + 1).$$

Assume that R is an upper bound for $|\sigma_{\mathcal{U}}|$, then $R^m \geq R^{m-1} + \cdots + 1$. Otherwise, if $R^m < R^{m-1} + \cdots + 1$, then $u(R) = R^m - (R^{m-1} + \cdots + 1) < 0$ and also we have $u(2) > 0$. Once again, by Intermediate Value Theorem there exist $\lambda \in (R, 2)$ such that $u(\lambda) = 0$, which is contradicts our assumption that R is an upper bound for $|\sigma_{\mathcal{U}}|$.
Thus,

$$R^m \geq R^{m-1} + \cdots + 1 = \frac{1 - R^m}{1 - R},$$

since $(1 - R) < 0$, then $(1 - R) \leq \dfrac{1 - R^m}{R^m}$, so we have

$$(2 - R) \leq \frac{1}{R^m}. \tag{4.1}$$

The inequality (4.1) must be true for all $m \in \mathbb{N}$, as $m \to \infty$, we have $(2 - R) \leq 0$, which implies that $R \geq 2$. This is a contradiction. Therefor, R is not an upper bound for $|\sigma_{\mathcal{U}}|$. □

Example 4.1. *Consider the matrix polynomial*

$$P(z) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} z^2 + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then

$$\sigma(P) = \left\{ \frac{-1}{2} \pm \frac{\sqrt{5}}{2}, \frac{1}{2} \pm \frac{\sqrt{5}}{2} \right\}.$$

We note that the coefficients of $P(z)$ are unitary, so the modulus of eigenvalues of $P(z)$ lie in $\frac{1}{2} < |\lambda| < 2$.

In the following, we present the definition of doubly stochastic matrix.

Definition 4.1. *[18] A square matrix of nonnegative real numbers is called* doubly stochastic *matrix if each row and column sums are 1.*

Now, consider the set $\mathcal{D} = \left\{ \sum_{i=0}^{m} A_{i+1} z^i : A_i \text{ are } n\times n \text{ doubly stochastic matrices and } A_{m+1}, A_1 \text{ are } n \times n \text{ permutation matrices } \ n, m \in \mathbb{N} \right\}$, and let

$$\sigma_{\mathcal{D}} = \{\lambda : \lambda \in \sigma(P) \text{ where } P \in \mathcal{D}\} \text{ and } |\sigma_{\mathcal{D}}| = \{|\lambda| : \lambda \in \sigma_{\mathcal{D}}\}.$$

If B is doubly stochastic matrix, then $\|B\| = 1$ [1]. The question is whether the eigenvalues of matrix polynomial with doubly stochastic coefficients lie in the region $\frac{1}{2} < |\lambda| < 2$. The answer to this question is yes, they lie in the same region as in the unitary case. And the result summarized in the subsequent two theorems, also, the proof of the this two theorems is the same as in the unitary case.

Theorem 4.3. *([29], [18, Theorem 2.1]) Let $P(z) \in \mathcal{D}$. Then for any $\lambda \in \mathbb{C}$, such that λ is an eigenvalue of $P(z)$, it follows that*

$$\frac{1}{2} < |\lambda| < 2.$$

Theorem 4.4. *([29], [18, Theorem 2.2])* $\sup |\sigma_{\mathcal{D}}| = 2$ *and* $\inf |\sigma_{\mathcal{D}}| = \frac{1}{2}$.

Remark 4.1. *If A_{m+1} or A_1 or both are doubly stochastic matrices, but not a permutation matrix, then the eigenvalues may not necessarily lie in the region* $\frac{1}{2} < |\lambda| < 2$ *as illustrated in the next example.*

Example 4.2. *Consider the matrix polynomial*

$$P(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z^2 + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} \frac{1}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{1}{7} \end{bmatrix},$$

then

$$\sigma(P) = \left\{ \frac{-1}{2} \pm \frac{3\sqrt{21}}{14},\ \frac{1}{2} \pm i\frac{\sqrt{3}}{2} \right\}.$$

The eigenvalue $\frac{-1}{2} + \frac{3\sqrt{21}}{14} = 0.48198$ *is less than* $\frac{1}{2}$.

Now, we address the matrix polynomials with Schur stable coefficients. Schur stable matrices are defined in the ensuing definition.

Definition 4.2. *[18] A square matrix A is said to be* Schur stable *if all its eigenvalues are located in the open unit disk. That is, the eigenvalues of A has modulus less than 1.*

More generally, for some $r > 0$. Let

$$M_r = \left\{ A : A \text{ is a square matrix with } |\lambda| < r,\ \forall \lambda \in \sigma(A) \right\}.$$

Now, consider the family of all monic matrix polynomials with commuting matrix coefficients whose eigenvalues are of modulus less than r

$$\mathcal{S}_r = \left\{ Iz^m + \sum_{i=0}^{m-1} A_{i+1} z^i : A_i \in M_r \text{ are } n \times n \text{ commuting matrices, } n, m \in \mathbb{N} \right\},$$

and let

$$\sigma_{\mathcal{S}_r} = \{\lambda : \lambda \in \sigma(P) \text{ where } P \in \mathcal{S}_r\} \text{ and } |\sigma_{\mathcal{S}_r}| = \{|\lambda| : \lambda \in \sigma_{\mathcal{S}_r}\}.$$

The coming two lemmas will be used to prove the first theorem for matrix polynomials with Schur stable coefficients.

Lemma 4.1. *(e.g., [1, Theorem 2.3.3]) Let $\mathcal{F} \subseteq \mathbb{M}_n(\mathbb{C})$ be a nonempty commuting family. Then there is a unitary $U \in \mathbb{M}_n(\mathbb{C})$ such that U^*AU is upper triangular for every $A \in \mathcal{F}$.*

Lemma 4.2. *[30] Let $p(z) = a_{m+1}z^m + \cdots + a_2 z + a_1$ be a complex polynomial of degree m. If $\lambda \in \mathbb{C}$ is a root of $p(z)$, then*

$$|\lambda| \leq 1 + \max\left\{\frac{|a_1|}{|a_{m+1}|}, \frac{|a_2|}{|a_{m+1}|}, \ldots, \frac{|a_m|}{|a_{m+1}|}\right\}.$$

Theorem 4.5. *[18, Theorem 2.6] Let $P(z) = Iz^m + A_m z^{m-1} + \cdots + A_2 z + A_1 \in \mathcal{S}_r$. If $\lambda \in \mathbb{C}$ is an eigenvalue of $P(z)$, then $|\lambda| < r + 1$.*

Proof. For $i = 1, 2, \ldots, m$, the coefficients A_i's are commute with each other, therefore by Lemma 4.1 there exists a unitary matrix U such that $U^*A_iU = T_i$, where $T_i \in \mathbb{M}_n(\mathbb{C})$ are upper triangular matrices, for all $i = 1, 2, \ldots, m$. Let $t_{11}^{(i)}, t_{22}^{(i)}, \ldots, t_{nn}^{(i)}$ are the eigenvalues of A_i. Since $A_i \in M_r$ and similar to T_i, then for $k = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, m$ we have $\left|t_{kk}^{(i)}\right| < r$. Also, we write T_i as

$$T_i = \begin{bmatrix} t_{11}^{(i)} & t_{12}^{(i)} & \cdots & t_{1n}^{(i)} \\ 0 & t_{22}^{(i)} & \cdots & t_{2n}^{(i)} \\ \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & t_{nn}^{(i)} \end{bmatrix}.$$

Now, define $Q(z)$ as

$$\begin{aligned} Q(z) &= U^*P(z)U \\ &= Iz^m + T_m z^{m-1} + \cdots + T_2 z + T_1. \end{aligned}$$

We note that all coefficients of $Q(z)$ are upper triangular matrices. Hence,

$$\begin{aligned} \det(P(z)) &= \det(Q(z)) \\ &= \prod_{k=1}^{n}(z^m + t_{kk}^{(m)} z^{m-1} + \cdots + t_{kk}^{(2)} z + t_{kk}^{(1)}). \end{aligned}$$

Therefore, the eigenvalues of $P(z)$ are the zeros of the polynomials

$$z^m + t_{kk}^{(m)} z^{m-1} + \cdots + t_{kk}^{(2)} z + t_{kk}^{(1)},$$

for $k = 1, 2, \ldots, n$. Thus, by using Lemma 4.2 we get that for any $\lambda \in \sigma(P)$

$$\begin{aligned} |\lambda| &\leq 1 + \max\left\{|t_{kk}^{(m)}|, \ldots, |t_{kk}^{(2)}|, |t_{kk}^{(1)}|\right\} \\ &< r + 1. \end{aligned}$$

□

The next example is taken from [31]. This is one of the examples that appears in applications.

Example 4.3. *[18, 31] Let $Q(z) = Mz^2 + Cz + K$, where*

$$M = I,\ C = 10T,\ K = 5T,\ T = \begin{bmatrix} 3 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 3 \end{bmatrix}.$$

The quadratic eigenvalue problem $Q(z)x = 0$ arising from a linearly damped mass-spring system. It clear that $\|T\|_\infty = 5$. So $\|C\|_\infty = 10\|T\|_\infty = 50$ and $\|K\|_\infty = 25$. By the fact that $\rho(A) \leq \|A\|_\infty$ for any square matrix A we get that the eigenvalues of M,C and K have modules less than or equal 50. Therefore, the eigenvalues of M,C, and K lie inside the disc of radius $50 + \epsilon$ for any $\epsilon > 0$. This mean $Q(z) \in \mathcal{S}_{50+\epsilon}$. Thus by Theorem 4.5 $|\lambda| < 50 + \epsilon + 1$, for any $\lambda \in \sigma(Q)$. Since $\epsilon > 0$ arbitrary, then $|\lambda| \leq 51$.

Since the eigenvalues of Schur stable matrices has modulus less than 1, then following corollary follows directly from Theorem 4.5.

Corollary 4.1. *[18, Corollary 2.7] Let $P(z) = Iz^m + A_m z^{m-1} + \cdots + A_2 z + A_1$, where A_i's are commuting Schur stable matrices. If $\lambda \in \mathbb{C}$ is an eigenvalue of $P(z)$, then*

$$|\lambda| < 2.$$

Theorem 4.5 provides bounds for $|\sigma_{\mathcal{S}_r}|$. The subsequent theorem confirms that this bounds are optimal.

Theorem 4.6. *[18, Theorem 2.8]* $\sup |\sigma_{\mathcal{S}_r}| = r + 1$ *and* $\inf |\sigma_{\mathcal{S}_r}| = 0$.

Chapter 5

New bounds

In this chapter, we derive new upper bounds for the eigenvalues of monic matrix polynomials with commuting coefficients by employing several numerical radius inequalities to the Frobenius companion matrices of these polynomials. Related results can be found in [15, 45]

Consider the monic matrix polynomial $P(z) = Iz^m + A_m z^{m-1} + \cdots + A_2 z + A_1$ of degree $m \geq 2$, with A_i are commuting matrices for $i = 1, 2, \ldots, m$. As mentioned in Section 3.3, if λ is an eigenvalue of $P(z)$, then

$$|\lambda| \leq \rho\left(\tilde{\mathrm{C}}(P)\right) \leq w\left(\tilde{\mathrm{C}}(P)\right) \leq \left\|\tilde{\mathrm{C}}(P)\right\|,$$

where

$$\tilde{\mathrm{C}}(P) = \begin{bmatrix} \rho(A_m) & \rho(A_{m-1}) & \ldots & \rho(A_2) & \rho(A_1) \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}.$$

Now, consider the $m \times m$ invertible matrix

$$D =: \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \ddots & 1 \\ \vdots & \vdots & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

so the inverse of D is

$$D^{-1} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & -1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Let $V := D\tilde{\mathrm{C}}(P)D^{-1}$, then

$$V = \begin{bmatrix} \rho(A_m)+1 & \rho(A_{m-1})-\rho(A_m) & \rho(A_{m-2})-\rho(A_{m-1}) & \cdots & \rho(A_2)-\rho(A_3) & \rho(A_1)-\rho(A_2)-1 \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 1 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}.$$

The matrices V and $\tilde{\mathrm{C}}(P)$ are similar, so they have the same eigenvalues. Therefore, by using numerical radius inequalities on V we get our new bounds. In the following theorem, we state our first estimate for the eigenvalues of $P(z)$.

Theorem 5.1. *If λ is any eigenvalue of $P(z)$, then*

$$\begin{aligned} |\lambda| \;\leq\; & \frac{1}{2}\left(\sqrt{\sum_{i=2}^{m-1}\big(\rho(A_i)-\rho(A_{i+1})\big)^2 + \big(\rho(A_1)-\rho(A_2)-1\big)^2}\right) \\ & + \frac{1}{2}\sqrt{m-2} + \cos\frac{\pi}{m+1} + \rho(A_m) + 1. \end{aligned}$$

Proof. To prove this theorem we write V as $V = M_1 + M_2 + M_3 + L$ where

$$M_1 = \begin{bmatrix} 0 & \rho(A_{m-1})-\rho(A_m) & \cdots & \rho(A_2)-\rho(A_3) & \rho(A_1)-\rho(A_2)-1 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \ M_3 = \begin{bmatrix} \rho(A_m)+1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}, \ L = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Now,

$$\begin{aligned} |\lambda| &\leq w(V) \\ &= w(M_1 + M_2 + M_3 + L) \\ &\leq w(M_1) + w(M_2) + w(M_3) + w(L). \end{aligned}$$

It remains to estimate $w(M_1)$, $w(M_2)$, $w(M_3)$, $w(L)$. Using Theorem 2.17 we have

$$w(M_1) = \frac{1}{2}\left(\sqrt{\sum_{i=2}^{m-1}\big(\rho(A_i) - \rho(A_{i+1})\big)^2 + \big(\rho(A_1) - \rho(A_2) - 1\big)^2}\right).$$

For $w(M_2)$ we have

$$\begin{aligned} w(M_2) &= w\left(\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}\right) \quad (\text{where } u = [0 \ -1 \ -1 \ \cdots \ -1]^T), \\ &\leq w\left(\begin{bmatrix} 0 & \|u\| \\ 0 & 0 \end{bmatrix}\right) \quad (\text{by Theorem 2.9 (1)}), \\ &= w\left(\begin{bmatrix} 0 & \sqrt{m-2} \\ 0 & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}\rho\left(\begin{bmatrix} 0 & \sqrt{m-2} \\ \sqrt{m-2} & 0 \end{bmatrix}\right) \quad (\text{by Theorem 2.8}), \\ &= \frac{1}{2}\left(\sqrt{m-2}\right) \quad (\text{by Remark 2.1}). \end{aligned}$$

Now, since M_3 is normal, we conclude that

$$w(M_3) = \rho(M_3) = \max\{\rho(A_m) + 1, 1\}.$$

Also, since $\rho(A_m) \geq 0$, then $w(M_3) = \rho(A_m) + 1$. Finally, by Theorem 2.15 we have $w(L) = \cos\frac{\pi}{m+1}$.

Thus,

$$
\begin{aligned}
|\lambda| \;\leq\; & \frac{1}{2}\left(\sqrt{\sum_{i=2}^{m-1}\big(\rho(A_i)-\rho(A_{i+1})\big)^2 + \big(\rho(A_1)-\rho(A_2)-1\big)^2}\right) \\
& + \frac{1}{2}\sqrt{m-2} + \cos\frac{\pi}{m+1} + \rho(A_m) + 1.
\end{aligned}
$$

□

Theorem 5.2. *If λ is any eigenvalue of $P(z)$, then*

$$
\begin{aligned}
|\lambda| \;\leq\; & \frac{1}{4}\left(\rho(A_m)+1+\sqrt{\big(\rho(A_m)+1\big)^2+\big(|\rho(A_{m-1})-\rho(A_m)|+1\big)^2}\right) \\
& + \frac{1}{4}\left(\cos\frac{\pi}{m-2}+1+\sqrt{\left(\cos\frac{\pi}{m-2}-1\right)^2+(1+\sqrt{m-3})^2}\right) \\
& + \frac{1}{2}\left(\frac{1}{4}\left(\rho(A_m)+1+\sqrt{\big(\rho(A_m)+1\big)^2+\big(|\rho(A_{m-1})-\rho(A_m)|+1\big)^2}\right)^2\right. \\
& \left. + \frac{1}{2}\big(a+b^2+1\big)+\sqrt{(a+b^2-1)^2+4b^2}\right)^{\frac{1}{2}} \\
& + \frac{1}{2}\sqrt{\frac{1}{4}\left(\cos\frac{\pi}{m-2}+1+\sqrt{\left(\cos\frac{\pi}{m-2}-1\right)^2+(1+\sqrt{m-3})^2}\right)^2+1},
\end{aligned}
$$

where $a := \sum_{i=2}^{m-2}\big(\rho(A_i)-\rho(A_{i+1})\big)^2$ and $b := \rho(A_1)-\rho(A_2)-1$.

Proof. Consider the partition of V as follows:

$$V = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix},$$

where

$$K_{11} = \begin{bmatrix} \rho(A_m)+1 & \rho(A_{m-1})-\rho(A_m) \\ 1 & 0 \end{bmatrix},$$

$$K_{12} = \begin{bmatrix} \rho(A_{m-2})-\rho(A_{m-1}) & \cdots & \rho(A_2)-\rho(A_3) & \rho(A_1)-\rho(A_2)-1 \\ 0 & \cdots & 0 & -1 \end{bmatrix},$$

$$K_{21} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \quad K_{22} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 1 & & & -1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}.$$

By applying Corollary 2.3 we get

$$\begin{aligned} w(V) &\leq \frac{1}{2}\left(w(K_{11}) + \sqrt{w^2(K_{11}) + \|K_{12}\|^2}\right) \\ &\quad + \frac{1}{2}\left(w(K_{22}) + \sqrt{w^2(K_{22}) + \|K_{21}\|^2}\right). \end{aligned} \tag{5.1}$$

To complete the proof, we need to estimate $w(K_{11}), w(K_{22}), \|K_{12}\|, \|K_{21}\|$.
Now,

$$\begin{aligned} w(K_{11}) &\leq w\left(\begin{bmatrix} \rho(A_m)+1 & |\rho(A_{m-1})-\rho(A_m)| \\ 1 & 0 \end{bmatrix}\right) \quad \text{(by Theorem 2.9 (1))}, \\ &= \frac{1}{2}\rho\left(\begin{bmatrix} 2\rho(A_m)+2 & |\rho(A_{m-1})-\rho(A_m)|+1 \\ |\rho(A_{m-1})-\rho(A_m)|+1 & 0 \end{bmatrix}\right) \\ &= \frac{1}{2}\left(\rho(A_m)+1+\sqrt{\big(\rho(A_m)+1\big)^2 + \big(|\rho(A_{m-1})-\rho(A_m)|+1\big)^2}\right). \end{aligned} \tag{5.2}$$

Now, we want to estimate $\|K_{12}\|$.

$$K_{12}K_{12}^* = \begin{bmatrix} \sum_{i=2}^{m-2}\big(\rho(A_i)-\rho(A_{i+1})\big)^2 + \big(\rho(A_1)-\rho(A_2)-1\big)^2 & \rho(A_2)-\rho(A_1)+1 \\ \rho(A_2)-\rho(A_1)+1 & 1 \end{bmatrix},$$

so,

$$\begin{aligned} \|K_{12}\| &= \sqrt{\rho(K_{12}K_{12}^*)} \\ &= \sqrt{\frac{1}{2}\left((a+b^2+1)+\sqrt{(a+b^2-1)^2+4b^2}\right)} \quad \text{(by Remark 2.1)}, \end{aligned} \tag{5.3}$$

where $a = \sum_{i=2}^{m-2}\big(\rho(A_i)-\rho(A_{i+1})\big)^2$ and $b = \rho(A_1)-\rho(A_2)-1$.

Now, for $w(K_{22})$, Let $u = \begin{bmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$.

Thus,

$$\begin{aligned}
w(K_{22}) &= w\left(\begin{bmatrix} L_{m-3} & u \\ v & -1 \end{bmatrix}\right) \\
&\leq w\left(\begin{bmatrix} w(L_{m-3}) & \|u\| \\ \|v\| & 1 \end{bmatrix}\right) \quad \text{(Corollary 2.1)}, \\
&= \frac{1}{2}\left(w(L_{m-3}) + 1 + \sqrt{(w(L_{m-3}) - 1)^2 + (\|u\| + \|v\|)^2}\right) \\
&= \frac{1}{2}\left(\cos\frac{\pi}{m-2} + 1 + \sqrt{\left(\cos\frac{\pi}{m-2} - 1\right)^2 + (1 + \sqrt{m-3})^2}\right). \quad (5.4)
\end{aligned}$$

Note that from Theorem 2.15, $w(L_{m-3}) = \cos\frac{\pi}{m-2}$.
Finaly,

$$\|K_{21}\| = 1. \quad (5.5)$$

Hencs, by substitute (5.2), (5.3), (5.4), and (5.5) in (5.1) we have

$$\begin{aligned}
|\lambda| \leq\ & \frac{1}{4}\left(\rho(A_m) + 1 + \sqrt{\left(\rho(A_m) + 1\right)^2 + \left(|\rho(A_{m-1}) - \rho(A_m)| + 1\right)^2}\right) \\
& + \frac{1}{4}\left(\cos\frac{\pi}{m-2} + 1 + \sqrt{\left(\cos\frac{\pi}{m-2} - 1\right)^2 + (1 + \sqrt{m-3})^2}\right) \\
& + \frac{1}{2}\left(\frac{1}{4}\left(\rho(A_m) + 1 + \sqrt{\left(\rho(A_m) + 1\right)^2 + \left(|\rho(A_{m-1}) - \rho(A_m)| + 1\right)^2}\right)^2\right. \\
& \left. + \frac{1}{2}\left(a + b^2 + 1\right) + \sqrt{(a + b^2 - 1)^2 + 4b^2}\right)^{\frac{1}{2}} \\
& + \frac{1}{2}\sqrt{\frac{1}{4}\left(\cos\frac{\pi}{m-2} + 1 + \sqrt{\left(\cos\frac{\pi}{m-2} - 1\right)^2 + (1 + \sqrt{m-3})^2}\right)^2 + 1},
\end{aligned}$$

where $a = \sum_{i=2}^{m-2}\left(\rho(A_i) - \rho(A_{i+1})\right)^2$ and $b = \rho(A_1) - \rho(A_2) - 1$.

□

In the next Theorem, we use the same partition of V which is used in Theorem 5.2. But we use Corollary 2.1 instead of Corollary 2.3.

Theorem 5.3. *If λ is any eigenvalue of $P(z)$, then*

$$|\lambda| \leq \frac{1}{2}\left(\alpha + \beta + \sqrt{(\alpha-\beta)^2 + (\gamma+1)^2}\right),$$

where

$$\begin{aligned}
\alpha &= \frac{1}{2}\left(\rho(A_m) + 1 + \sqrt{\left(\rho(A_m)+1\right)^2 + \left(|\rho(A_{m-1}) - \rho(A_m)| + 1\right)^2}\right),\\
\beta &= \frac{1}{2}\left(\cos\frac{\pi}{m-2} + 1 + \sqrt{\left(\cos\frac{\pi}{m-2} - 1\right)^2 + (1+\sqrt{m-3})^2}\right),\\
\gamma &= \sqrt{\frac{1}{2}\left((a+b^2+1) + \sqrt{(a+b^2-1)^2 + 4b^2}\right)},
\end{aligned}$$

and $a = \sum_{i=2}^{m-2}\left(\rho(A_i) - \rho(A_{i+1})\right)^2$, $b = \rho(A_1) - \rho(A_2) - 1$.

Proof. Consider the partition $V = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$, where K_{11}, K_{12}, K_{21}, K_{22} are defined in Theorem 5.2. Then application of Corollary 2.1 to gives

$$\begin{aligned}
w(V) &\leq w\left(\begin{bmatrix} w(K_{11}) & \|K_{12}\| \\ \|K_{21}\| & w(K_{22}) \end{bmatrix}\right)\\
&= \frac{1}{2}\left(w(K_{11}) + w(K_{22}) + \sqrt{(w(K_{11}) - w(K_{22}))^2 + (\|K_{12}\| + \|K_{21}\|)^2}\right),
\end{aligned}$$

where $w(K_{11})$, $w(K_{22})$, $\|K_{12}\|$, $\|K_{21}\|$ estimates are computed in Theorem 5.2.
By take $\alpha = w(K_{11})$, $\beta = w(K_{22})$, and $\gamma = \|K_{12}\|$ we get the result.

□

In the coming theorem, we use the same method in proof of Theorem 3.24 to provide new bounds for eigenvalues of $P(z)$.

Theorem 5.4. *If λ is any eigenvalue of $P(z)$, then*

$$\begin{aligned}
|\lambda| &\leq 1 + \frac{1}{2}(1+\sqrt{m-1})\\
&+\frac{1}{2}\left(\rho(A_m) + 1 + \sqrt{\sum_{i=2}^{m-1}\left(\rho(A_i) - \rho(A_{i+1})\right)^2 + \left(\rho(A_m)+1\right)^2 + \left(\rho(A_1) - \rho(A_2) - 2\right)^2}\right).
\end{aligned}$$

Proof. At first, we write V as $V = C_1 + C_2 + U$, where

$$C_1 = \begin{bmatrix} \rho(A_m)+1 & \rho(A_{m-1})-\rho(A_m) & \cdots & \rho(A_2)-\rho(A_3) & \rho(A_1)-\rho(A_2)-2 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix}, \text{ and } U = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} |\lambda| &\leq w(V) \\ &= w(C_1 + U + C_2) \\ &\leq w(C_1) + w(U) + w(C_2). \end{aligned}$$

Since U is unitary, then $w(U) = 1$. To estimate $w(C_1)$ we use Theorem 2.17 and we get

$$W(C_1) \leq \frac{1}{2}\left(\rho(A_m) + 1 + \sqrt{\sum_{i=2}^{m-1}\left(\rho(A_i) - \rho(A_{i+1})\right)^2 + \left(\rho(A_m)+1\right)^2 + \left(\rho(A_1) - \rho(A_2) - 2\right)^2}\right).$$

Now, we want to estimate $w(C_2)$,

$$\begin{aligned} w(C_2) &= w\left(\begin{bmatrix} 0 & u \\ 0 & -1 \end{bmatrix}\right) \quad (\text{where } u = [0 \ -1 \ -1 \ \cdots \ -1]^T), \\ &\leq w\left(\begin{bmatrix} 0 & \|u\| \\ 0 & 1 \end{bmatrix}\right) \quad (\text{by Theorem 2.9 (1)}), \\ &= w\left(\begin{bmatrix} 0 & \sqrt{m-2} \\ 0 & 1 \end{bmatrix}\right) \end{aligned}$$

$$= \frac{1}{2}\rho\left(\begin{bmatrix} 0 & \sqrt{m-2} \\ \sqrt{m-2} & 2 \end{bmatrix}\right) \quad \text{(by Theorem 2.8)},$$
$$= \frac{1}{2}\left(1+\sqrt{m-1}\right) \quad \text{(by Remark 2.1)}.$$

Thus,

$$|\lambda| \leq 1+\frac{1}{2}(1+\sqrt{m-1}) + \frac{1}{2}\left(\rho(A_m)+1+\sqrt{\sum_{i=2}^{m-1}\left(\rho(A_i)-\rho(A_{i+1})\right)^2+\left(\rho(A_m)+1\right)^2+\left(\rho(A_1)-\rho(A_2)-2\right)^2}\right).$$

□

By using the same steps in the proof of Theorem 3.28, we establish other new bound the eigenvalues of $P(z)$ as it is documented the following theorem.

Theorem 5.5. *If λ is any eigenvalue of $P(z)$, then*

$$\begin{aligned} |\lambda| \leq\ & \frac{1}{2}\left(\rho(A_m)+1+\cos\frac{\pi}{m-1}+1\right) \\ +\ & \frac{1}{4}\left(\sqrt{4(\rho(A_m)+1)^2+(1+|\rho(A_{m-1})-\rho(A_m)|)^2+\sum_{i=2}^{m-2}(\rho(A_i)-\rho(A_{i+1}))^2+\alpha^2}\right) \\ +\ & \frac{1}{4}\left(\sqrt{\alpha^2+m+5}\right) \\ +\ & \frac{1}{4}\left(\sqrt{4\cos^2\frac{\pi}{m-1}+(1+|\rho(A_{m-1})-\rho(A_m)|)^2+\sum_{i=2}^{m-2}(\rho(A_i)-\rho(A_{i+1}))^2+m+1}\right), \end{aligned}$$

where $\alpha = \rho(A_1)-\rho(A_2)-1$.

Here, we take the same matrix polynomial that mentioned in Example 3.6 to apply our new bounds for the eigenvalues of matrix polynomials to it and compare them with some other result.

Example 5.1. *Consider the monic matrix polynomial $P(z) = Iz^3 + A_3z^2 + A_2z + A_1$, where*

$$A_1 = \begin{bmatrix} 6 & 2 & 2 \\ 2 & 6 & 2 \\ 2 & 2 & 6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we have the following table.

Table 5.1: Comparison of several upper bounds

Theorem 5.1	5.408668899902970
Theorem 5.2	5.804441399866167
Theorem 5.3	5.282546547149893
Theorem 5.4	5.579388104455560
Theorem 5.5	5.619549170505516

Through this example we not that Theorem 5.3 gives better upper bound for the eigenvalues of this matrix polynomial than other our results. But it did not give a better estimation for the eigenvalues than Theorem 3.23.

Now, we compare our new results with some of results mentioned previously by taking other numerical example of matrix polynomial of degree 3.

Example 5.2. *Consider the monic matrix polynomial* $P(z) = Iz^3 + A_3 z^2 + A_2 z + A_1$, *where*

$$A_1 = \begin{bmatrix} 5 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 5 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} \frac{1}{2} & 1 & 1 \\ 1 & \frac{1}{2} & 1 \\ 1 & 1 & \frac{1}{2} \end{bmatrix}.$$

Then the Table 5.2 provides upper bounds for eigenvalues of $P(z)$.

We note from Table 5.2 that our third bound (Theorem 5.3) gives better estimation for the eigenvalues of this matrix polynomial than other bounds and we have that for any $\lambda \in \sigma(P)$

$$|\lambda| \leq 4.518915173875558.$$

Table 5.2: Comparison of several upper bounds

Theorem	Upper bound
Theorem 3.8	6.057411647900729
Theorem 3.9	5.999999999999999
Theorem 3.10	7.545084971874736
Theorem 3.11	6.109531234081206
Theorem 3.15	6.336020108197149
Theorem 3.16	6.796728149279611
Theorem 3.17	8.719522630495298
Theorem 3.18	6.573907535246749
Theorem 3.19	6.578815026312832
Theorem 3.20	6.403943276465976
Theorem 3.21	7.523388082826173
Theorem 3.22	6.403943276465976
Theorem 3.23	4.871861997787114
Theorem 3.24	9.599771908979756
Theorem 3.25	11.249999999999996
Theorem 3.26	9.764903887933098
Theorem 3.27	11.249999999999996
Theorem 3.28	9.986423007418404
Theorem 5.1	5.957106781186547
Theorem 5.2	5.592257748812904
Theorem 5.3	4.518915173875558
Theorem 5.4	6.165046997768509
Theorem 5.5	5.921448480036181

Conclusion

The primary goal of this thesis is to explore and derive bounds for the eigenvalues of matrix polynomials. We started with reviewing fundamental concepts in matrix theory, including matrix norms, numerical radius, and spectral radius. These concepts provided the basic tool for establishing bounds of eigenvalues of matrix polynomials. Many known bounds for the eigenvalues were discussed in this thesis as well. Through the fact that similar matrices have the same spectral radius and using detailed analysis of Frobenius companion matrices we established our new bounds for the eigenvalues of matrix polynomials, particularly for matrix polynomials with commuting coefficients. Further related bounds can be obtained if various similarity matrices are used. It is worth mentioning that our results can be used in many applications in science and engineering.

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