Contents

Chapter 1

Introduction

Actually, fractional calculus is a part of real analysis that studies all the topics assuming arbitrary real powers α of the differential operator; so, in the present situation researchers are showing more interest to work in the field of fractional calculus as a generalization of the ordinary calculus; which is one of our main targets in this thesis, besides its various applications in physics, bio engineering, see [11],and [20] and recently a climate change model has been studied et al.[7].

For many years, many definitions of fractional derivative have been introduced by various researchers, The most known are the Riemann Liouville definition and the Caputo definition, see [18], and [19], for some applications refer to [27], [25], and [28]. The most well-known ones are:

1. Riemann-Liouville definition. For $\alpha \in [n-1, n)$, the α-derivative of f is

$$D^{\alpha}_{x_0}(f)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{x_0}^{x} \frac{f(t)}{(x-t)^{\alpha-n+1}} dt.$$

2. Caputo definition. For $\alpha \in [n-1, n)$, the α-derivative of f is

$$D^{\alpha}_{x_0}(f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_{x_0}^{x} \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt.$$

However, the following are some of the setbacks of one definition or the other:

1. The Riemann-Liouville derivative does not satisfy $D^{\alpha}_{a}(1) = 0 \left(D^{\alpha}_{a}(1) = 0\right.$ for the Caputo derivative), if α is not a natural number,

2. The two fractional derivatives do not satisfy the known formula of the derivative of the product of two functions: $D^{\alpha}_{a}(fg) = f\,(D^{\alpha}_{a} g) + g\,(D^{\alpha}_{a} f)$,

3. The two fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions: $D^{\alpha}_{a}\left(\frac{f}{g}\right) = \frac{g\left(D^{\alpha}_{a} f\right) - f\left(D^{\alpha}_{a} g\right)}{g^2}$,

4. The two fractional derivatives do not satisfy the chain rule:
   $D_a^\alpha(f \circ g) = f^{(\alpha)}(g(x))g^{(\alpha)}(x)$,

5. The two fractional derivatives do not satisfy: $D^\alpha D^\beta(f) = D^{\alpha+\beta}(f)$, in general.

Khalil et al. [15] has introduced a new derivative called the conformable fractional derivative of f of order $\alpha \in (0,1]$ and for all $x > 0$ is defined by

$$T_\alpha f(x) = \lim_{h \to 0} \frac{f\left(x + hx^{1-\alpha}\right) - f(x)}{h},$$

which is a natural extension and almost satisfies all the classical properties of the usual first derivative.
Abdeljawad has a long history and an important contribution in this field, for instance, et al. [1], [2], and [30], and other articles, they have developed and implemented the fundamentals of the analytic theory of the conformable fractional calculus and studied different topics and applications in the general fractional calculus. Recently, they have combined fuzzy calculus, and conformable calculus to introduce the fuzzy conformable calculus et al. [30].

Dixit and Ujlayan in [9] and [8] have introduced a U-D fractional derivative as a convex combination of the function and its first derivative, where $(D^\alpha f)(x) = (1-\alpha)f(x)+\alpha f'(x)$ for $\alpha \in (0,1]$, they have studied the main results of this operator.

Kajouni et al. [14] introduced a new fractional derivative based on an exponential function defined as follows $(D^\alpha f)(x) = e^{(\alpha-1)x} f'(x)$.

Alhribat and Abu Hasheesh in [4], have used the limit approach definition that is associated with the hyperbolic function to define a ne fractional de,rivative as follows $(D^\alpha f)(x) = \cosh((1-\alpha)x)f'(x)$.; so, one can work on hyperbolic functions which is a rich area of identities and nice properties that facilitates the computations, especially in solving fractional differential equations.

Many types of fractional differential equations are studied and solved with respect to the conformable fractional differential operator, and other fractional operators, for instance, refer to [16], [17], and[3].

This thesis is organized as follows: In Chapter 2, we study the conformable fractional l derivative that is undefined by Khalil et.al [15] which is more natural and effective than previous definitions nd can be generalized to include any α,we study the main proprieties and theorems related to the definition.

In chapter 3, many types of fractional versions of differential equations according to the conformable fractional operator are discussed and solved.
In particular, α- order conformable linear, Bernoulli, exact, and homogeneous fractional differential equations of order α are studied. Moreover, the fractional differential equations of order 2α are generalized and investigated, for instance, fractional generalization of Undetermined Coefficients, and Variation of Parameters methods are presented.
Also, a linear system of fractional conformable differential equations is investigated and solved.

Finally, in Chapter 4, other recent types of fractional derivatives such as UD-fractional derivative, Exponential fractional derivative, and Hyperbolic fractional derivative are introduced as a limit approach derivative, and they are associated with the classical first derivative with different base(coefficient) functions.
On one side their Calculus and their main properties are studied and some comparisons with the conformable fractional derivatives and other fractional derivatives are made. On the other side, many applications to fractional differential equations are solved based on these fractional differential operators.

Chapter 2

Conformable fractional derivative

In this chapter, we study the conformable fractional derivative introduced by Khalil .et.al [15] which is more natural and effective than previous definitions and can be generalized to include any α, we study the main proprieties and theorems related to the definition.

2.1 Properties and main theorems

In this section, we discuss, improve and complement some recent results of the conformable derivative introduced and established by Khalil et.al. [15]. Among other things we show that each function f defined on $(a, b), a > 0$ has a conformable derivative if and only if it has a classical first derivative. [26]

Definition 2.1. *Given a function* $f : [0, \infty) \to \mathbb{R}$*. Then the conformable fractional derivative of* f *of order* α*,* $0 < \alpha \leq 1$*is defined by*

$$(T_\alpha f)(x) = \lim_{\varepsilon \to 0} \frac{f\left(x + \varepsilon x^{1-\alpha}\right) - f(x)}{\varepsilon}$$

for all $x > 0$*. If* f *is* α*-differentiable in some* $(0, a), a > 0$*, and* $\lim_{x \to 0^+} (T_\alpha f)(x)$ *exist, then*

$$(T_\alpha f)(0) = \lim_{x \to 0^+} (T_\alpha f)(x)$$

We sometimes, write $f^\alpha(t)$ *for* $(T_\alpha f)(t)$*, to denote the conformable fractional derivative of order* α*. In addition, if the conformable fractional derivative of* f *of order* α *exists, then we say* f *is an* α*- differentiable function.*

As a consequence of the above definition, the following useful theorem is obtained.

Proof. Since f is $\alpha-$ differentiable function at $x = x_0$, we know that

$$(T_\alpha f)(x_0) = \lim_{\varepsilon \to 0} \frac{f\left(x_0 + \varepsilon x_0^{1-\alpha}\right) - f(x_0)}{\varepsilon}$$

exists. If we next assume $x \neq x_0$ we can write the following

$$f(x_0 + \varepsilon x_0^{1-\alpha})) - f(x_0) = \frac{f\left(x_0 + \varepsilon x_0^{1-\alpha}\right) - f(x_0)}{\varepsilon}\varepsilon$$

as $\varepsilon \to 0$, then

$$\lim_{\varepsilon \to 0} f\left(x_0 + \varepsilon x_0^{1-\alpha}\right) - f(x_0) = f'(x_0).0$$

let $h = x_0^{1-\alpha}$, then

$$\lim_{h \to 0} f(x_0 + h) = f(x_0)$$

Hence, f is continuous at x_0 □

The following theorem is an important result to prove the next consequences.

Theorem 2.1. *(The Result Theorem of Conformable Fractional Derivative) If $b > a > 0$ and $f : [a, b] \to \mathbb{R}$ is differentiable function then f is α-differentiable function at $x > a$, then*

$$f^{(\alpha)}(x) = x^{1-\alpha} \frac{d}{dx} f(x)$$

Proof. take $h = \epsilon x^{1-\alpha}$ then,

$$\begin{aligned} T_\alpha(f)(x) &= \lim_{\varepsilon \to 0} \frac{f\left(x + \varepsilon x^{1-\alpha}\right) - f(x)}{\varepsilon} \\ &= x^{1-\alpha} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= x^{1-\alpha} \frac{df(x)}{dx} \end{aligned}$$

since by assumption, f is differentiable at $x > 0$. This completes the proof of the theorem. □

It can easily be shown that $T_\alpha$satisfies all proprieties in the following theorem.

Proposition 2.1. *Properties of Conformable Fractional Derivative.*

Let $\alpha \in (0, 1]$ and f, g be $\alpha-$ differentiable at a point $x > 0$. Then

(i) *$T_\alpha(c) = 0.c \in \mathbb{R}$ for all constant functions $f(x) = c$.*

(ii) $T_\alpha(x^p) = px^{p-\alpha}$, *for all* $p \in \mathbb{R}$.

(iii) $T_\alpha(af+bg)(x) = aT_\alpha(f)(x) + bT_\alpha(g)(x)$, *for all* $a, b \in \mathbb{R}$.

(iv) $T_\alpha(fg)(x) = g(x)T_\alpha(f)(x) + f(x)T_\alpha(g)(x)$.

(v) $T_\alpha\left(\frac{f}{g}\right)(x) = \frac{g(x)T_\alpha(f)(x) - f(x)T_\alpha(g)(x)}{[g(x)]^2}$.

(vi) $T_\alpha(f \circ g)(x) = f'(g(x))T_\alpha(g)(x)$ *(Chain Rule).*

Proof. Using Theorem 2.2, all properties will be proven consecutively. Now, for fixed $\alpha \in (0,1]$, it is easily seen that

$$x^{1-\alpha}\frac{d}{dx}(c) = x^{1-\alpha} \cdot 0 = 0$$

This proves property (i).
Secondly, for the property (ii),

$$T_\alpha(x^p) = x^{1-\alpha}\frac{d}{dx}(x^p) = x^{p-\alpha}$$

this propriety coincided happen with the same traditional definition of Riemann Louisville and of Caputo on polynomials.
Using similar arguments applied to property (iii)

$$\begin{aligned}
\mathrm{T}_\alpha(af+bg)(x) &= x^{1-\alpha}(af+bg)'(x)\\
&= x^{1-\alpha}(af+bg)'(x)\\
&= x^{1-\alpha}\left(af'(x) + bg'(x)\right)\\
&= x^{1-\alpha}af'(x) + x^{1-\alpha}bg'(x)\\
&= aT_\alpha(f)(x) + bT_\alpha(g)(x)
\end{aligned}$$

Hence, the definition satisfies linearity property. Then property (iv) is proven by

$$\begin{aligned}
T_\alpha(fg)(x) &= x^{1-\alpha}(fg)'(x)\\
&= x^{1-\alpha}\left(f'g + fg'\right)(x)\\
&= x^{1-\alpha}\left(f'g\right)(x) + x^{1-\alpha}\left(fg'\right)(x)\\
&= x^{1-\alpha}f'(x)g(x) + x^{1-\alpha}f(x)g'(x)\\
&= \left(T_\alpha f(x)\right)g(x) + \left(T_\alpha g(x)\right)f(x)
\end{aligned}$$

Then, for (v)

$$T_\alpha\left(\frac{f}{g}\right)(x) = x^{1-\alpha}\left(\frac{f}{g}\right)'(x)$$

$$\begin{aligned}
T_\alpha\left(\frac{f}{g}\right)(x) &= x^{1-\alpha}\frac{\big(g(x)f'(x) - f(x)g'(x)\big)}{[g(x)]^2} \\
&= \frac{\big(g(x)f'(x)x^{1-\alpha} - f(x)g'(x)x^{1-\alpha}\big)}{[g(x)]^2} \\
&= \frac{g(x)T_\alpha(f)(x) - f(x)T_\alpha(g)(x)}{[g(x)]^2}
\end{aligned}$$

Finally, property (vi) will be proven using Theorem 2.2 as the following

$$\begin{aligned}
T_\alpha(f \circ g)(x) &= t^{1-\alpha}(f \circ g)'(x) \\
&= x^{1-\alpha}f'\big(g(x)\big)g'(x) \\
&= f'(g(x))t^{1-\alpha}g'(x) \\
&= f'(g(x))\big(T_\alpha(g(x))
\end{aligned}$$

This completed the proof of the theorem. □

Corollary 2.1. *Corollary of Quotient Property*
Let $\alpha \in (0, 1]$ and f be $\alpha-$ differentiable at a point $x > 0$. Then

$$T_\alpha\left(\frac{1}{f(x)}\right) = -\frac{f^\alpha(x)}{[f(x)]^2}$$

Proof. Using Theorem 2.2 and the property (v) of proposition 2.1, then

$$\begin{aligned}
T_\alpha\left(\frac{1}{f(x)}\right) &= x^{1-\alpha}\left(\frac{1}{f(x)}\right)' \\
&= x^{1-\alpha}\frac{\big(f(x)(0) - f'(x)\big)}{[f(x)]^2} \\
&= -\frac{f'(x)x^{1-\alpha}}{[f(x)]^2} \\
&= -\frac{f^\alpha(x)}{[f(x)]^2}.
\end{aligned}$$

□

Corollary 2.2. *Let $\alpha \in (0, 1]$ and f be $\alpha-$ differentiable at a point $x > 0$. Then*

$$T_\alpha(f(x))^2 = 2\big(f(x)f^\alpha(x)\big)$$

Proof. Using theorem 2.2, we get

$$\begin{aligned} T_\alpha(f(x))^2 &= x^{1-\alpha}\left(f(x)^2\right)' \\ &= x^{1-\alpha}(2f(x)f'(x)) \\ &= 2\left(f(x)f^\alpha(x)\right) \end{aligned}$$

□

Further, many functions behave as in the usual derivative. Here are some formulas

Example 2.1. *Conformable fractional derivative of certain functions:*

(i) $T_\alpha\left(\frac{1}{\alpha}x^\alpha\right) = 1$

(ii) $T_\alpha\left(\sin\frac{1}{\alpha}x^\alpha\right) = \cos\frac{1}{\alpha}x^\alpha$.

(iii) $T_\alpha\left(\cos\frac{1}{\alpha}x^\alpha\right) = -\sin\frac{1}{\alpha}x^\alpha$.

(iv) $T_\alpha\left(e^{\frac{1}{\alpha}x^\alpha}\right) = e^{\frac{1}{\alpha}x^\alpha}$.

We should notice that a function could be α- differentiable function for some α at a point but not differentiable.

Example 2.2. *let $f(x) = 2\sqrt{x}$, then $T_{1/2}(f)(0) = \lim_{x\to 0^+} T_{1/2}(f)(x) = 1$ where $T_{1/2}(f)(0) = 1$ for $x > 0$. But $T_1(f)(0)$ does not exist. This is not the case for the known classical fractional derivatives.*

Ramark 2.1. *The index law, that is, $T_\alpha T_\beta(f) = T_{\alpha+\beta}(f)$ for any α, β does not hold in general.*

Indeed, if $f(x) = x^2, \alpha = \frac{1}{2}, \beta = \frac{1}{3}$ then $T_{\frac{1}{2}}T_{\frac{1}{3}}(f) = \frac{10}{3}x^{\frac{7}{6}}$, while $T_{\frac{1}{2}+\frac{1}{3}}(f) = T_{\frac{5}{6}}(f) = 2x^{\frac{7}{6}}$. Hence, $T_{\frac{1}{2}}T_{\frac{1}{3}}(f) \neq T_{\frac{1}{2}+\frac{1}{3}}(f)$.

Next, we consider the possibility of $\alpha \in (n, n+1]$, for some $n \in \mathbb{N}$. We have the following definition.

Definition 2.2. *Let $\alpha \in (n, n+1]$, and f be an n-differentiable at x, where $x > 0$. Then the conformable derivative of f of order α is defined as*

$$T_\alpha(f)(x) = \lim_{\varepsilon\to 0} \frac{f^{(\lceil\alpha\rceil-1)}\left(x + \varepsilon x^{(\lceil\alpha\rceil-\alpha)}\right) - f^{(\lceil\alpha\rceil-1)}(x)}{\varepsilon},$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

Ramark 2.2. *As a direct consequence of the above definition one can show that*

$$T_\alpha(f)(x) = x^{(\lceil\alpha\rceil-\alpha)}f^{\lceil\alpha\rceil}(x)$$

where $\alpha \in (n, n+1]$, and f is $(n+1)$-differentiable at $x > 0$.

Ramark 2.3. *For $\alpha = 1$, that is, $\alpha = n$ we obtain that the conformable derivative* (FD) *coincides with the classical first derivative $\frac{df(x)}{dx}$, that is, with $\frac{d^n f(x)}{dx^n}$. However, the zero order derivative $T_0(f)(x) = f^{(0)}(x)$ of a function $f(x)$ does not return the function $f(x)$, because $T_0(f)(x) = x^{1-0}\frac{df}{dx}(x) = x\frac{df}{dx}(x) \neq f(x)$ in general case.where $\alpha \in (n, n+1]$ and f is $(n+1)$ differentiable at $x > 0$.*

The previous definitions of fractional derivative Riemann-Liouville and Caputo do not enable us to study the analysis of α-differentiable functions. However, In definition2.1 makes it possible to prove basic analysis theorems such as Rolle's theorem and the mean value theorem[5].

Theorem 2.2. *(Rolle's Theorem for Conformable Fractional Differentiable Functions) Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a function satisfying the following*

(i) f continuous on $[a, b]$,

(ii) f is $\alpha-$ differentiable for some $\alpha \in (0, 1)$,

(iii) $f(a) = f(b)$,

Then, there exists $c \in (a, b)$ such that $f^{(\alpha)}(c) = 0$.

Proof. Since f is continuous on $[a, b]$ and $f(a) = f(b)$, there exists $c \in (a, b)$ at which the function has a local extreme.With no loss of generality, assume c is a point of local minimum. So

$$T_\alpha(f(c)) = \lim_{\epsilon \to 0^+} \frac{f\left(c + (\epsilon c^{1-\alpha}\right) - f(c)}{\epsilon}$$

$=$

$$T_\alpha(f(c)) = \lim_{\epsilon \to 0^-} \frac{f\left(c + (\epsilon c^{1-\alpha}\right) - f(c)}{\epsilon}$$

But, the two limits have opposite signs, so $T_\alpha(f(c)) = 0$. □

Theorem 2.3. *(Mean Value Theorem for Conformable Fractional differentiable functions).*

Let $a > 0$ and $f : [a, b] \longrightarrow \mathbb{R}$, be a given function that satisfies

(i) f is continuous on $[a, b]$,

(ii) f is α-differentiable for some $\alpha \in (0, 1)$.

Then, there exists $c \in (a, b)$ such that

$$f^{(\alpha)}(c) = \frac{f(b) - f(a)}{(1/\alpha)b^{(\alpha)} - (1/\alpha)a^\alpha}.$$

Proof. The equation of the secant through $((a, (f(a))$and $(b, (f(b))$is

$$y - f(a) = \frac{f(b) - f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}\left(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha\right),$$

which we can write as

$$y = \frac{f(b) - f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}\left(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha\right) + f(a).$$

Define $g(x)$ as

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{\frac{1}{\alpha}b^\alpha - \frac{1}{\alpha}a^\alpha}\left(\frac{1}{\alpha}x^\alpha - \frac{1}{\alpha}a^\alpha\right).$$

then, $g(a) = g(b) = 0$, g is continuous on $[a.b]$ and differentiable on (a, b), so by Roll's theorem, there exist c in (a, b) such that $g^\alpha(c) = 0$, but

$$g^{(\alpha)}(x) = f^{(\alpha)}(x) - \frac{f(b) - f(a)}{(1/\alpha)b^\alpha - (1/\alpha)a^\alpha}$$

hence,

$$g^{(\alpha)}(c) = f^{(\alpha)}(c) - \frac{f(b) - f(a)}{(1/\alpha)b^\alpha - (1/\alpha)a^\alpha} = 0.$$

Then, the function g satisfies the conditions of the fractional Rolle's theorem. Hence there exists $c \in (a, b)$, such that $g^{(\alpha)}(c) = 0$.Using the fact $T_\alpha\left(\frac{1}{\alpha}t^\alpha\right) = 1$,
we obtain,

$$f^{(\alpha)}(c) = \frac{f(b) - f(a)}{(1/\alpha)b^{(\alpha)} - (1/\alpha)a^\alpha}$$

□

Theorem 2.4. *(Cauchy Theorem for Conformable Differentiable Functions).*
Let $a > 0$ and $f, g : [a, b] \to \mathbb{R}$ be given functions that satisfy

(i) f, g are continuous on $[a, b]$,

(ii) f, g are α-differentiable for some $\alpha \in (0, 1)$ and $g^{(\alpha)}(t) \neq 0$ for all $t \in (a, b)$,
Then, there exists $c \in (a, b)$, such that

$$\frac{f^{(\alpha)}(c)}{g^{(\alpha)}(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof. Consider the function

$$F(t) = f(t) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(t) - g(a))$$

It is clear that the function $F(t)$ satisfies the conditions of Rolle's theorem for Conformable Differentiable functions equivalent to classical differentiable functions on

(a,b). Hence, according to Theorem 2.3, there exists $c \in (a,b)$, such that $F'(c) = 0$, that is,

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Since

$$\frac{f'(c)}{g'(c)} = \frac{c^{\alpha-1} f^{(\alpha)}(c)}{c^{\alpha-1} g^{(\alpha)}(c)} = \frac{f^{(\alpha)}(c)}{g^{(\alpha)}(c)}$$

Hence the result is obtained. □

Theorem 2.5. *(Darboux's Theorem for Conformable Differentiable Functions) Let $b > a > 0$ and $f : (0, +\infty) \to \mathbb{R}$ be a given function that satisfies*

(i) f is α-differentiable for some $\alpha \in (0, 1)$.

(ii) $f^{(\alpha)}(a) \cdot f^{(\alpha)}(b) < 0$.

Then, there exists $c \in (a,b)$, such that $f^{(\alpha)}(c) = 0$.

Proof. According to Theorem, the function f is differentiable on $(0, +\infty)$. Further, from as well as by we have that

$$a^{1-\alpha} b^{1-\alpha} f'(a) f'(b) < 0$$

that is,

$$f'(a) f'(b) < 0$$

Now, using classical Darboux's theorem for differentiable functions,we obtain that there exists $c \in (a,b)$ such that $f'(c) = 0$, that is, $f^{(\alpha)}(c) = 0$.
Hence the proof is complete. □

Finally, we have the following results:

Theorem 2.6. *Let $f : [a,b] \to \mathbb{R}$ be α-differentiable for some $\alpha \in (0,1)$.*

(i) If $f^{(\alpha)}$ is bounded on $[a,b]$ where $a > 0$. Then f is uniformly continuous on $[a,b]$, and hence f is bounded.

(ii) If $f^{(\alpha)}$ is bounded on $[a,b]$ and continuous at a. Then f is uniformly continuous on $[a,b]$, and hence f is bounded.

It is well-known that if $f'(x)$ is bounded on $I = [a,b]$, then f is uniformly continuous on I. However, the converse need not be true. To see this, consider $f(x) = 2\sqrt{x}$ on $I = [0,1]$. Then f is uniformly continuous on [0, 1] but $f'(x)$ is not bounded there. However, boundedness of $f^{(\alpha)}(x)$ for $0 < \alpha < 1$ and the continuity of f on I (continuity of f at 0 in the subspace topology is equivalent to right continuity of f at 0), which implies, by the above theorem2.7, the uniform continuity of f on I

2.2 Conformable Fractional Integral

In the case of the integration, the most important class of functions where the integral is defined is the space of continuous functions. The conformable fractional integral is discussed as follows, see Abdeljawad [1].

Definition 2.3. *Let $f : [a, \infty) \to \mathbb{R}$ be continuous function. Let $a \geq 0$, and $\alpha \in (0,1)$. Then the conformable fractional integral $I_\alpha^a(f)(x)$ exists*

$$I_\alpha^a(f)(x) = I_1^a\left((x-a)^{\alpha-1} f(x)\right) = \int_a^x \frac{f(t)}{(t)^{1-\alpha}} dt$$

If the Riemann improper integral exists.
For the sake of simplicity, let's consider $a = 0$, hence we work on

$$I_\alpha^0(f)(x) = I^\alpha(f)(x)$$

Example 2.3. *Find the following integral $\boldsymbol{I}_{\frac{1}{2}}^0(\cos 2\sqrt{x})$*
Solution:

$$I_{\frac{1}{2}}^0(\cos 2\sqrt{x}) = \int_0^x \frac{\cos 2\sqrt{t}}{t^{1-\frac{1}{2}}} dt$$

Let u $= 2\sqrt{x}$,*then $du = \frac{1}{\sqrt{x}}dx$, so the integral becomes*

$$\int_0^t \cos u du = \sin(2\sqrt{x})$$

The following theorem explains that $\alpha-$ fractional derivative and $\alpha-$ fractional integral are inverse of each other.

Theorem 2.7. *Inverse Property*
Let $a \geq 0$ and $\alpha \in (0,1)$. Also, let f be a continuous function such that $I^\alpha(f(x))$ exists. For all $x > a$, then

$$T_\alpha\left[I^\alpha f(x)\right] = f(x).$$

Proof. Since f is continuous, then $I^\alpha(f)$ is certainly differentiable. Using theoremx, then

$$\begin{aligned} T_\alpha\left[I^\alpha(f(x))\right] &= x^{1-\alpha}\frac{d}{dx}\left[I^\alpha(f(x))\right] \\ &= x^{1-\alpha}\frac{d}{dx}\int_a^x \frac{f(t)}{t^{1-\alpha}}dt \\ &= x^{1-\alpha}\frac{f(x)}{x^{1-\alpha}} \\ &= f(x) \end{aligned}$$

□

In the right case we can similarly prove:
The following results are the conformable fractional versions of the mean value theoerem for integrals

Theorem 2.8. *(Mean Value Theorem for Fractional Integral)*
If $f : [a, b] \to \mathbb{R}$ is a continuous function on $[a, b]$. Then, there exists c in $[a, b]$ such that,

$$\int_a^b \frac{f(x)}{x^{1-\alpha}} dx = f(c) \left(\frac{1}{\alpha} b^\alpha - \frac{1}{\alpha} a^\alpha \right)$$

Proof. Since $f(t)$ is continuous on $[a, b]$, then $I^\alpha f(t) = \int_a^x \frac{f(x)}{x^{1-\alpha}} dx$ is continuous on $[a, b], \alpha-$ differentiable on (a, b) and $T_\alpha \left[I^\alpha f(t) \right] = f(t)$. So by the mean value theorem for conformable derivatives Theorem 2.4, there is a number c such that $a < c < b$ and

$$I^\alpha f(b) - I^\alpha f(a) = T_\alpha \left[I^\alpha f(x) \right] \left(\frac{1}{\alpha} b^\alpha - \frac{1}{\alpha} a^\alpha \right).$$

However, it is known that

$$T_\alpha \left[I^\alpha f(c) \right] = f(c),$$

$$I^\alpha f(b) = \int_a^b \frac{f(x)}{x^{1-\alpha}} dx = \int_a^b \frac{f(t)}{t^{1-\alpha}} dt,$$

and

$$I^\alpha f(a) = \int_a^a \frac{f(t)}{t^{1-\alpha}} dt = 0.$$

Thus

$$\int_a^b \frac{f(t)}{t^{1-\alpha}} dt = f(c) \left(\frac{1}{\alpha} b^\alpha - \frac{1}{\alpha} a^\alpha \right)$$

□

Theorem 2.9. *(Second Mean Value Theorem for Fractional Integral)*
Let f and g be functioned satisfying the following conditions: continuous and integrable, bounded on $[a, b]$.Let $m = \inf\{f(x) : x \in [a, b]\}$ and $M = \sup\{f(x) : x \in [a, b]$. Then, there exists a number $c \in (a, b)$ such that

$$\int_a^b \frac{f(x)g(x)}{x^{1-\alpha}} dx \leq c \int_a^b \frac{g(x)}{x^{1-\alpha}} dx$$

Proof. If $m = \inf f, M = \sup f$ and $g(x) \geq 0$ in $[a, b]$, then

$$mg(x) < f(x)g(x) < Mg(x) \tag{2.1}$$

Divide(2.1) by $x^{1-\alpha}$ and then integrate with respect to x over (a, b), resulting

$$m \int_a^b \frac{g(x)}{t^{1-\alpha}} dx \leq \int_a^b \frac{f(x)g(x)}{x^{1-\alpha}} dx \leq M \int_a^b \frac{g(x)}{x^{1-\alpha}} dx.$$

Then there exists a number c in $[m, M]$ such that

$$\int_a^b \frac{f(x)g(t)}{t^{1-\alpha}}dt \leq c\int_a^b \frac{g(t)}{x^{1-\alpha}}dx.$$

□

2.3 Partial Conformable Fractional derivative

In this section, the conformable partial fractional derivative of a real-valued function with several variables is defined. As an application, the α - exact and homogeneous conformable fractional differential equations, see [23] and [24].

Definition 2.4. *Let f be a real valued function with n variables and $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ be a point whose $i - th$ component is positive. Then the limit*

$$\lim_{\varepsilon \to 0} \frac{f\left(a_1, \ldots, a_i + \varepsilon a_i^{1-\alpha}, \ldots, a_n\right) - f\left(a_1, \ldots, a_n\right)}{\varepsilon}$$

If exists, is denoted $\frac{\partial^\alpha}{\partial t_i^\alpha} f(a)$, and called the $i - th$ conformable partial derivative of f of order $\alpha \in (0, 1]$ at a.

Definition 2.5. *If a real valued function f with n variables has all conformable partial derivatives of order $\alpha \in (0, 1]$ at $\boldsymbol{a} = (a_1, \ldots, a_n) \in R^n$, each $a_i > 0$, then the conformable gradient of f of the order α at $\boldsymbol{a}$ is*

$$\nabla^\alpha f(\boldsymbol{a}) = \left(\frac{\partial^\alpha}{\partial x_1^\alpha} f(\boldsymbol{a}), \ldots, \frac{\partial^\alpha}{\partial x_n^\alpha} f(\boldsymbol{a})\right)$$

Example 2.4. *Consider*

$$f(x, y) = 2x^{2\alpha} - 4y^{2+\alpha}$$

find $\nabla^\alpha f(1, 2)$
solution

$$\nabla^\alpha f(1, 2) = (4\alpha, -32 - 16\alpha)$$

Definition 2.6. *Let $\alpha \in (0, 1]$ and f be a real valued function with n variables defined on an open set $D \subset R^n$, such that, for all $(x_1, \ldots, x_n) \in D$, each $x_i > 0$, a function f is said to be in $C_\alpha(D, R)$ if all its conformable fractional partial derivatives of order α exists and are continuous on D.*

The following Theorem for conformable partial derivatives fractional orders presented as follows:

Theorem 2.10. *Let α, β be positive constants such that $0 < \alpha, \beta < 1$. Assume That $f(x, y)$ is function for which $\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{\partial^\beta f(x,y)}{\partial y^\beta}\right)$ and $\frac{\partial^\beta}{\partial y^\beta}\left(\frac{\partial^\alpha f(x,y)}{\partial x^\alpha}\right)$ exists and are continuous over a domain $X \subset R^n$ such that for all $(x, y) \in X, \quad x, y > 0$, then*

$$\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{\partial^\beta f(x, y)}{\partial y^\beta}\right) = \frac{\partial^\beta}{\partial y^\beta}\left(\frac{\partial^\alpha f(x, y)}{\partial x^\alpha}\right)$$

For all $(x, y) \in X$

Example 2.5. *Consider*

$$f(x, y) = 2x^{2\beta} + 4xy^{\beta+\alpha},$$

then

$$\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{\partial^\beta(2x^{2\beta} - 4y^{\beta+\alpha})}{\partial y^\beta}\right) = 4(\beta + \alpha)y^\alpha x^{1-\alpha}$$

and

$$\frac{\partial^\beta}{\partial y^\beta}\left(\frac{\partial^\alpha(2x^{2\beta} - 4y^{2+\alpha})}{\partial x^\alpha}\right) = 4(\beta + \alpha)y^\alpha x^{1-\alpha}$$

The α - exact conformable fractional differential equation is introduced as follows:

Definition 2.7. *Let $0 < \alpha \leq 1$. A first-order differential equation of the form $M(x, y)dx + N(x, y)dy = 0$ is called α - exact if there exists a function $\Phi(x, y)$ such that*

$$\frac{\partial^\alpha \Phi}{\partial y^\alpha} = M \text{ and } \frac{\partial^\alpha \Phi}{\partial x^\alpha} = N$$

Consequently,

$$d^\alpha \Phi(x, y) = M(x, y)dx + N(x, y)dy = 0$$

from the properties of the conformable fractional derivative, we get Φ as a constant function.

Example 2.6. *Consider*

$$2x^\alpha y^\alpha dx + \left(x^{2\alpha} - 4y^{2+\alpha}\right) dy = 0$$

for some $\alpha \in (0, 1]$.
Solution: Here

$$\frac{\partial^\alpha \left(2x^\alpha y^\alpha\right)}{\partial y^\alpha} = 2\alpha x^\alpha \text{ and } \frac{\partial^\alpha \left(x^{2\alpha} - 4y^{2+\alpha}\right)}{\partial x^\alpha} = 2\alpha x^\alpha$$

Thus the equation is conformable $\alpha-$ exact.

Chapter 3

Conformable Fractional Differential Equations

In this chapter, we will study and solve many types of fractional differential equations according to conformable fractional differential operators[12], [13], [24], [6], [10].

3.1 Conformable $\alpha-$ order fractional differential equations

In this section, we will study different methods for solving various types of fractional differential equations of the first-order linear and nonlinear fractional equations[12], [6].

3.1.1 Conformable linear, Bernoulli, and Ricatti fractional differential equations

Definition 3.1. *The first order linear conformable fractional differential equation is defined as*

$$T_\alpha(y) + h(x)y = k(x) \tag{3.1}$$

Where $0 < \alpha < 1, T_\alpha(y)$ describe the conformable derivative of y and $h(x), k(x)$ are α-differentiable Functions.

If $k(x) = 0$, then

$$T_\alpha(y) + h(x)y = 0 \tag{3.2}$$

is called homogeneous.

Ramark 3.1. *When $\alpha = 1$ we get better the classical differential equations of first order expressed as $y' + h(x)(y) = k(x)$*

To solve (3.1), it can be transformed to

$$\begin{aligned} x^{1-\alpha}y' + h(x)y = k(x) \\ y' + \frac{h(x)}{x^{1-\alpha}}y = \frac{k(x)}{x^{1-\alpha}} \end{aligned} \tag{3.3}$$

Now equation (3.3) is an ordinary first-order differential equation that has a general solution

$$y = e^{-\int \frac{h(x)}{x^{1-\alpha}}dx}\left[\int \frac{k(x)}{x^{1-\alpha}} e^{\int \frac{h(x)}{x^{1-\alpha}}dx} dx + c\right]$$

which can be written as:

$$y = e^{-I_\alpha h(x)}\left[I_\alpha\left(k(x)e^{I_\alpha h(x)}\right) + c\right] \tag{3.4}$$

On the other side, one can obtain the solution in equation (3.4) as a sum of the homogeneous and particular solution as follows:

Ramark 3.2. *The general solution for the differential equation (3.1) is given by:*

$$y(x) = y_h(x) + y_p(x)$$

Theorem 3.1. *The homogeneous solution of the conformable differential equation (3.2) is indicated by*

$$y_h(x) = ce^{-I_\alpha^0 h(x)}$$

Where y_h *is homogeneous solution and* h *is any continuous function in the domain of* $I_\alpha^{\mathbf{0}}$.

Proof. we have just verified that

$$y(x) = ce^{-I_\alpha^0 h(x)}$$

We get by substituting into equation (3.3)

$$\begin{aligned} T_\alpha(y) + h(x)y &= cx^{1-\alpha}\frac{d}{dx}\left[e^{-I_\alpha^0 h(x)}\right] + ch(x)e^{-I_\alpha^0 h(x)} \\ &= -cx^{1-\alpha}\frac{d}{dx}\left[I_\alpha^0 h(x)\right]e^{-I\alpha_\alpha^0 h(x)} + ch(x)e^{-I_\alpha^0 h(x)} \\ &= -cx^{1-\alpha}\frac{h(x)}{x^{1-\alpha}}e^{-I_\alpha^0 h(x)} + ch(x)e^{-I_\alpha^0 h(x)} \\ &= 0 \end{aligned}$$

□

Theorem 3.2. *The particular solution of the conformable differential equation (3.1) is given by*

$$y_p(x) = \lambda(x)e^{-I_\alpha^0 h(x)}$$

Where $h(x)$ is any continuous function in the field of I_α^0 and the function $\lambda : R \to R$ is obtained through the following condition,

$$\lambda(x) = I_\alpha^0 \left(k(x)e^{I_\alpha^0 k(x)}\right)$$

Proof. Substituting

$$y_p(x) = \lambda(x)e^{-I_\alpha^0 h(x)}$$

in equation (3.1), We obtain

$$T_\alpha(y) + h(x)y = T_\alpha\left(I_\alpha^0\left(k(x)e^{I_\alpha^0 h(x)}\right)e^{-I_\alpha^0 h(x)}\right) + h(x)\left(I_\alpha^0\left(k(x)e^{I_\alpha^0 h(x)}\right)e^{-I_\alpha^0 h(x)}\right.$$

$$= k(x)e^0 - I_\alpha^0\left(k(x)e^{I_\alpha^0 h(x)}\right)h(x)e^{-I_\alpha^0 h(x)} + h(x)\left(I_\alpha^0\left(k(x)e^{I_\alpha^0 h(x)}\right)e^{-I_\alpha^0 h(x)}\right.$$

$$= k(x).$$

□

Example 3.1. *Find a particular and the homogeneous solution to the following fractional differential equation*

$$T_{\frac{1}{2}}(y) - y = 5e^{2\sqrt{x}}$$

Solution : The homogeneous solution is

$$y_h = ce^{-I_1^0(-1)} = ce^{2\sqrt{x}}$$

and a particular solution is

$$y_p = \lambda(x)e^{-I_1^0(-1)} = 10\sqrt{t}e^{2\sqrt{x}}$$

Example 3.2. *Solve the first-order linear fractional differential equation*

$$T_{\frac{1}{2}}(y) + y = x^2 + 2x^{\frac{3}{2}}$$

Solution: we have $h(x) = 1$ and $k(x) = x^2 + 2x^{\frac{3}{2}}$ then by substituting in the general solution (3.4),we get

$$y = e^{-I_\alpha h(x)}\left[I_\alpha k(x)e^{-I_\alpha h(x)} + c\right]$$

$$y = e^{-2x^{\frac{1}{2}}}\left[x^2 e^{2x^{\frac{1}{2}}} + c\right]$$

$$y = x^2 + ce^{-2x^{\frac{1}{2}}}$$

Definition 3.2. *The Bernoulli fractional differential equations can be expressed in the following way.*

$$T_\alpha(y) + h(x)y = k(x)(y)^n$$

where $h(x)$ and $k(x)$ are α-differentiable function, and y is a function that to be determined.
This equation can be transformed as

$$x^{1-\alpha}(y') + h(x)y = k(x)(y)^n,$$

which implies that

$$y' + \frac{h(x)}{x^{1-\alpha}}y = \frac{k(x)}{x^{1-\alpha}}(y)^n \tag{3.5}$$

It is clear that it will be linear for $n = 0$ or $n = 1$. By changing the dependent variable, it can be reduced to a linear ordinary equation for every other value of n.

Let $z = y^{(1-n)}$, then $z' = (1-n)y^{-n}y'$,
hence equation (3.11) is reduced to linear equation

$$z' + (1-n)\frac{h(x)}{x^{1-\alpha}}z = (1-n)\frac{k(x)}{x^{1-\alpha}}$$

According to the results the general solution is as follows

$$y = \left(e^{-I_\alpha((1-n)h(x))}\left[I_\alpha\left((1-n)k(x)e^{I_\alpha((1-n)h(x))}\right) + c\right]\right)^{\frac{1}{1-n}}$$

Example 3.3. *Solve the first order linear conformable fractional differential equation*

$$T_{\frac{1}{2}}(y) + \sqrt{x}y = \left(xe^{-2x}\right)(y)^{-1}$$

Solution:
We have Bernoulli fractional differential equations, where $h(x) = \sqrt{x}$ and $k(x) = xe^{-2x}$.So the general solution is

$$y = \left(e^{-I_\alpha((1-n)h(x))}\left[I_\alpha\left((1-n)k(x)e^{I_\alpha((1-n)h(x))}\right) + c\right]\right)^{\frac{1}{1-n}}$$

$$y = \left(e^{-2x}\left(\frac{4}{3}x^{\frac{3}{2}} + c\right)\right)^{\frac{1}{2}}$$

Definition 3.3. *The Riccati fractional differential equations are a natural extension of a first order fractional differential equation.*

$$T_\alpha(y) = h(x) + k(x)y + u(x)(y)^2,$$

where $h(x), k(x)$ and $u(x)$ are α-differentiable functions, and y is a function that isn't recognized.

If a specific solution y_1 is known, then there's the general solution, which comes in the form of $y = y_1 + z$ where z is an all-encompassing solution to the following Bernoulli Fractional differential equation.

$$T_\alpha(z) + \big(-k(x) - 2u(x)y_1\big)\, z = u(x)(z)^2$$

Example 3.4. *Solve the Riccati fractional differential equations*

$$T_{\frac{1}{2}}(y) = -x\sqrt{x} + \frac{1}{2\sqrt{x}}y + \sqrt{x}(y)^2, y_1 = \sqrt{x}$$

Solution: We first solve the Bernoulli Fractional differential equation.

$$T_\alpha(z) + \left(-\frac{1}{2\sqrt{x}} - 2\sqrt{x}\sqrt{x}\right) z = \sqrt{x}(z)^2$$

The general solution is

$$z = \left(e^{-I_\alpha\left((-1)\left(-\frac{1}{2\sqrt{x}}-2x\right)\right)}\left[I_{\frac{1}{2}}\left((-1)\sqrt{x}e^{-I_\alpha\left((-1)\left(-\frac{1}{2\sqrt{x}}-2x\right)\right)}\right) + c\right]\right)^{-1}.$$

So

$$z = \frac{2\sqrt{x}e^{\frac{4}{3}x^{\frac{1}{2}}}}{c - e^{\frac{4}{3}x^{\frac{3}{2}}}}.$$

Therefore

$$y = \frac{c - e^{\frac{4}{3}x^{\frac{3}{2}}}}{2\sqrt{x}e^{\frac{4}{3}x^{\frac{1}{2}}}}.$$

3.1.2 Method of undetermined coefficients

In this section, a fractional version of the method of undetermined coefficients is presented under some specific conditions. Let us start with the following definition.

Definition 3.4. *Let $n \in \mathbb{N}$, the set of natural numbers, and $\alpha \in (0, 1)$. We call α a factor of n if there exists $k \in \mathbb{N}$ such that $k\alpha = n$.*

Example 3.5. *$\frac{1}{2}$ is a factor of 2 with $k = 4$, and $\frac{1}{3}$ is a factor of 1 with $k = 3$. But $\frac{3}{5}$ is not a factor of 1 .*

Definition 3.5. *A fractional polynomial of degree n and factor α is a function of the form*

$$P(x) = a_n x^n + a_{n-1} x^{n-\alpha} + \ldots. + a_{n-k-1} x^{\alpha} + a_{n-k},$$

where $a_j \in \mathbb{R}$, the set of real numbers.
We write $P(x)$ is an $(n, \alpha)-$ fractional polynomial. If $a_n = 0$, we take n to be the smallest n for which α is a factor.

Example 3.6. *$P(x) = x + x^{\frac{1}{2}} - 4$ is a $\left(1, \frac{1}{2}\right) -$ fractional polynomial, and $Q(X) = 2x^{\frac{3}{2}} + 5x - x^{\frac{1}{2}} + 7$ is a $\left(2, \frac{1}{2}\right) -$ fractional polynomial.*
In addition, $H(x) = x + x^{\frac{2}{3}} - 1$ is a $\left(1, \frac{1}{3}\right) -$ fractional polynomial. Here the coefficient of $x^{\frac{1}{3}}$ is 0 .

Definition 3.6. *Let $J(\alpha)$ be the set of all (n, α)-polynomials for all $n \in \mathbb{N}$ and fixed α;*

clearly, $J(\alpha)$ is a subspace of the space of all continuous functions on $[0, \infty)$.

Let $G(\alpha)$ be the space of all functions of the form $c_1 \sin\left(\frac{x^{\alpha}}{\alpha}\right) + c_2 \cos\left(\frac{t^{\alpha}}{\alpha}\right)$, where $c_1, c_2 \in \mathbb{R}$.

$E(\alpha)$ be the space of all functions of the form $ce^{\frac{x^{\alpha}}{\alpha}}$, where $c \in \mathbb{R}$;

and let $M(\alpha)$ be the space of all functions of the form $e^{\frac{x^{\alpha}}{\alpha}} \left(P_1(x) \sin\left(\frac{x^{\alpha}}{\alpha}\right) + P_2(x) \cos\left(\frac{x^{\alpha}}{\alpha}\right)\right)$, where $P_1, P_2 \in J(\alpha)$.

Definition 3.7. *A differential equation is called an (n, α) - fractional differential equation if it is of the form*

$$a_n y^{(n)} + a_{n-1} y^{(n-\alpha)} + \ldots + a_{n-k-1} y^{(\alpha)} + a_{n-k} y = f(x),$$

where α is a factor of the natural number n. It is called a fractional differential equation of order n and factor α. If $a_n = 0$, we take n to be the smallest n for which α is a factor.

In this section, we consider linear equations of the form

$$y^{(\alpha)} + ay = f(x) \tag{3.6}$$

where $f(x)$ is an element of one of the spaces $J(\alpha), G(\alpha), E(\alpha)$, and $M(\alpha)$.

Let us write y_h for the solution of the homogeneous equation $y^{(\alpha)} + ay = 0$, and y_p for any particular solution of $y^{(\alpha)} + ay = f(x)$. Then as in the case of ordinary differential equations, the general solution is $y_g = y_h + y_p$.

Ramark 3.3. *It is interesting to observe that the general solution of the homogeneous part of the equation (3.8) is $y_h = e^{r\frac{1}{\alpha}x^\alpha}$, where r is the solution of the auxiliary equation $r + a = 0$.*
Now, to find y_p using the method of undetermined coefficients, the function $f(x)$ must be in one of the spaces $J(\alpha), G(\alpha), E(\alpha)$, and $M(\alpha)$.

Consequently, if $f(x) = ce^{x^\alpha}$, then y_p must be be^{x^α}, where b is to be determined by substituting be^{x^α} in $y^{(\alpha)} + ay = ce^{x^\alpha}$.

If $f(x) = c_1 \sin x^\alpha + c_2 \cos x^\alpha$, then y_p must be $b_1 \sin x^\alpha + b_2 \cos x^\alpha$, where b_1 and b_2 are to be determined by substituting in the equation $y^{(\alpha)} + ay = c_1 \sin x^\alpha + c_2 \cos x^\alpha$.

Similarly if $f(x)$ is an (n, α)-fractional polynomial.

Example 3.7. *Let us consider the general solution of the following fractional differential equations:*

(a) $y^{\left(\frac{1}{2}\right)} + 2y = \sin\sqrt{x}$.
(b) $y^{\left(\frac{1}{3}\right)} - y = x$.
(c) $y^{\left(\frac{1}{2}\right)} - 3y = xe^{\sqrt{x}} \cos\sqrt{x}$.

Solution(1):The auxiliary equation of

$$y^{\left(\frac{1}{2}\right)} + 2y = 0$$

is $r + 2 = 0$, so $r = -2$.
Hence

$$y_h = be^{-2(2)\sqrt{x}} = be^{-4\sqrt{x}}.$$

and,

$$y_p = A\sin\sqrt{x} + B\cos\sqrt{x}$$

Noting there is no similarity between y_h and any of the terms of y_p.
Substituting y_p in the equation

$$y^{\left(\frac{1}{2}\right)} + 2y = \sin\sqrt{x},$$

we get. $A = \frac{8}{17}$ and $B = -\frac{2}{17}$
Hence

$$y = be^{-4\sqrt{t}} + \frac{8}{17}\sin\sqrt{x} - \frac{2}{17}\cos\sqrt{x}.$$

Solution(2): The auxiliary equation of

$$y^{\left(\frac{1}{3}\right)} - y = 0$$

is $r-1=0$, *so* $r=1$.
Hence

$$y_h = be^{3\sqrt[3]{x}}$$

Again,

$$y_p = ax + bx^{\frac{2}{3}} + cx^{\frac{1}{3}} + d$$

Since there is no similarity between any of the terms of y_p *and* y_h. *Substituting* y_p *in the equation* $y^{\left(\frac{1}{3}\right)} - y = x$, *we get*

$$a = -1, b = 1, c = \frac{2}{3}, d = \frac{2}{9}.$$

Solution(3): One can easily see that $y_h = be^{6\sqrt{x}}$. *As for* y_p, *the form is*

$$y_p = e^{\sqrt{x}}\left(c_1x + c_2\sqrt{x} + c_3\right)(A\sin\sqrt{x} + B\cos\sqrt{x}),$$

Noting that there is no similarity between y_h *and any of the terms of* y_p. *If there is a similarity between* y_h *and any of the terms of* y_p *we give this example:*

Example 3.8. *Solve*

$$y^{\left(\frac{1}{2}\right)} - y = 5e^{2\sqrt{x}}$$

Here $y_h = e^{2\sqrt{x}}$, *and the form of* y_p *is* $y_p = be^{2\sqrt{x}}$. *But if we substitute* y_p *in the equation, we will not be able to determine* b.
Hence we try

$$y_p = b\sqrt{x}e^{2\sqrt{x}}.$$

Substitute such y_p *in the equation to get*

$$b\sqrt{x}e^{2\sqrt{x}} + \frac{1}{2}be^{2\sqrt{x}} - b\sqrt{x}e^{2\sqrt{x}} = 5e^{2\sqrt{x}}$$

Hence $b = 10$. *So* $y_g = ce^{2\sqrt{x}} + 10\sqrt{x}e^{2\sqrt{x}}$.

Thus in case of similarity between any of the terms of y_p *and* y_h *we multiply the assumed form of* y_p *by* x^α, *whenever the equation is* $y^{(\alpha)} + by = f(x)$.

3.1.3 Conformable Fractional (reduced to) Exact Equations

As a continuation of (Section 2.3), we solve some fractional differential equations reduced to a fractional alpha-exact,also the alpha- homogeneous fractional differential equations is presented and solved, see [24], and[23].

We start with the following result which is a generalization of the classical one.

Theorem 3.3. *Let* $0 < \alpha \leq 1$. *Let* M, N *be real-valued functions with two variables defined on a set* D *and the class* C_α *on* D. *Then* $M(x,y)dx + N(x,y)dy = 0$ *is* $\alpha-$ *exact if and only if*

$$\frac{\partial^\alpha N}{\partial x^\alpha} = \frac{\partial^\alpha M}{\partial y^\alpha}, \forall (x,y) \in D$$

As a consequence of the above theorem $\alpha-$ exact fractional differential equations can be solved in the same as exact ordinary differential equations.
Next, we discuss the solution of non-$\alpha-$ exact fractional differential equations it could be reduced to $\alpha-$ exact.

Definition 3.8. *Let $0 < \alpha \leq 1$. Let M, N, μ be real-valued functions with two variables defined on a set D and class C_α on D, with $x, y > 0, are for all (x, y) \in D$. The function $\mu(x, y)$ is an integrating factor to the fractional differential equation $M(x, y)dx + N(x, y)dy = 0$, if the fractional differential equation*

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

is α-exact.

Ramark 3.4. *To find an integrating factor $\mu(x, y)$, apply the $\alpha-$ exactness condition on the equation*

$$\frac{\partial^\alpha(\mu N)}{\partial x^\alpha} = \frac{\partial^\alpha(\mu M)}{\partial y^\alpha}$$

That is,

$$\mu\left(\frac{\partial^\alpha M}{\partial y^\alpha} - \frac{\partial^\alpha N}{\partial x^\alpha}\right) = N(x, y)\frac{\partial^\alpha \mu}{\partial x^\alpha} - M(x, y)\frac{\partial^\alpha \mu}{\partial y^\alpha} \tag{3.7}$$

This is a fractional partial differential equation for the unknown function $\mu(x, y)$, which is more difficult to solve than the original fractional ordinary differential equation. However, for some special cases can be solve for an integrating factor.

1) μ is a function of x or y

If μ is a function of x only, that is, $\mu = \mu(x)$, then

$$\frac{\partial^\alpha \mu}{\partial x^\alpha} = T_\alpha \mu, \frac{\partial^\alpha \mu}{\partial y^\alpha} = 0$$

And equation (3.7) becomes

$$NT_\alpha \mu = \mu\left(\frac{\partial^\alpha M}{\partial y^\alpha} - \frac{\partial^\alpha N}{\partial x^\alpha}\right)$$

This implies that

$$\frac{1}{\mu}T_\alpha \mu = \frac{1}{N}\left(\frac{\partial^\alpha M}{\partial y^\alpha} - \frac{\partial^\alpha N}{\partial x^\alpha}\right)$$

Since $\mu(x)$ is a function of x only, the left-hand side is a function of x only. Hence, if an integrating factor of the form $\mu(x)$ is to exit, the right-hand side must be is a function of x only. Then the integrating factor is,

$$\mu(x) = \exp\left[\int \frac{1}{N}\left(\frac{\partial^\alpha M}{\partial y^\alpha} - \frac{\partial^\alpha N}{\partial x^\alpha}\right)x^{\alpha-1}dx\right] \tag{3.8}$$

Interchanging M and N, and x and y in equation (3.8), one obtains an integrating factor for another special case

$$\mu(y) = \exp[\int \underbrace{\frac{1}{M}\left(\frac{\partial^\alpha N}{\partial x^\alpha} - \frac{\partial^\alpha M}{\partial y^\alpha}\right)}_{\text{function of } y \text{ only}} y^{\alpha-1} dy]$$

Example 3.9. *Find an integrating factor of the following equation*

$$2\sqrt{xy}dx + \left(2x - 4\sqrt{y}\right) dy = 0$$

Solution. Here

$$\frac{\partial^{1/2}\left(2\sqrt{xy}\right)}{\partial y^{/2}} = \sqrt{x} \text{ and } \frac{\partial^{1/2}\left(2x - 4\sqrt{y}\right)}{\partial x^{1/2}} = 2\sqrt{x}$$

Thus the equation is not conformable $\alpha-$ exact. So,

$$\frac{\partial^{1/2}\left(2x - 4\sqrt{y}\right)}{\partial x^{1/2}} - \frac{\partial^{1/2}\left(2\sqrt{xy}\right)}{\partial y^{1/2}} = \sqrt{x}$$

Thus

$$\frac{1}{\mu}T_\alpha \mu = \frac{\sqrt{x}}{2\sqrt{xy}} = \frac{1}{2\sqrt{y}}$$

So now it is matter routine to solve the equation noticing that

$$\mu(y) = \exp\left[\int \frac{1}{2\sqrt{y}} y^{-1/2} dy\right] = \sqrt{y}$$

2. μ is a function of x and y
If μ is a function of z only, where $z = z(x, y)$, then

$$\frac{\partial^\alpha \mu}{\partial x^\alpha} = \left(T_\alpha \mu\right)(z) \cdot z^{\alpha-1} \cdot \frac{\partial^\alpha z}{\partial x^\alpha}, \frac{\partial^\alpha \mu}{\partial y^\alpha} = \left(T_\alpha \mu\right)(z) \cdot z^{\alpha-1} \cdot \frac{\partial^\alpha z}{\partial y^\alpha}$$

And equation (3.4) becomes

$$\left(T_\alpha \mu\right)(z) \cdot z^{\alpha-1} \cdot \left(N \cdot \frac{\partial^\alpha Z}{\partial x^\alpha} - M \cdot \frac{\partial^\alpha Z}{\partial y^\alpha}\right) = \mu\left(\frac{\partial^t heleft - handM}{\partial y^\alpha} - \frac{\partial^\alpha N}{\partial x^\alpha}\right)$$

This implies the right-hand

$$\frac{1}{\mu(z)}\left(T_\alpha \mu\right)(z) = \frac{\frac{\partial^\alpha M}{\partial y^\alpha} - \frac{\partial^\alpha N}{\partial x^\alpha}}{z^{\alpha-1} \cdot \left(N \cdot \frac{\partial^\alpha z}{\partial x^\alpha} - M \cdot \frac{\partial^\alpha z}{\partial y^\alpha}\right)}$$

Since $\mu(z)$ is a function of z only, left hand side is a function of z only. Hence, if an integrating factor of the form $\mu(z)$ is to exit, right hand side must be is a function of z only. Then the integrating factor is,

$$\mu(z)=\exp\left[\int\frac{\frac{\partial^{\alpha}M}{\partial y^{\alpha}}-\frac{\partial^{\alpha}N}{\partial x^{\alpha}}}{z^{\alpha-1}\cdot\left(N\cdot\frac{\partial^{\alpha}z}{\partial x^{\alpha}}-M\cdot\frac{\partial^{\alpha}}{\partial y^{\alpha}}\right)}\cdot z^{\alpha-1}\cdot dz\right] \tag{3.9}$$

Example 3.10. *Find an integrating factor of the form $\mu(z)$, where $z=xy$, of the following equation*

$$\left(4y^{1+\alpha}-6xy^{\alpha}\right)dx+\left(6x^{\alpha}y-4x^{1+\alpha}\right)dy=0$$

for Some $\alpha\in(0,1]$.

Solution. Here

$$\frac{\partial^{\alpha}\left(4y^{\alpha+1}-6xy^{\alpha}\right)}{\partial y^{\alpha}}=4(1+\alpha)y-6\alpha x$$

And

$$\frac{\partial^{\alpha}\left(6x^{\alpha}y-4x^{\alpha+1}\right)}{\partial x^{\alpha}}=6\alpha y-4(1+\alpha)x$$

Thus, the equation is not α - exact.

Now, computing the conformable fractional partial derivative of function $\mu(z)$ with respect to x and y, then

$$\frac{\partial^{\alpha}\mu}{\partial x^{\alpha}}=\left(T_{\alpha}\mu\right)(z)\cdot z^{\alpha-1}\cdot\frac{\partial^{\alpha}z}{\partial x^{\alpha}}=y^{\alpha}\cdot\left(T_{\alpha}\mu\right)(z)$$

And

$$\frac{\partial^{\alpha}\mu}{\partial y^{\alpha}}=\left(T_{\alpha}\mu\right)(z)\cdot z^{\alpha-1}\cdot\frac{\partial^{\alpha}z}{\partial y^{\alpha}}=x^{\alpha}\cdot\left(T_{\alpha}\mu\right)(z)$$

Substituting these derivatives in equation (3.9) so that

$$\frac{1}{\mu(z)}\left(T_{\alpha}\mu\right)(z)=\frac{(2-\alpha)}{z^{\alpha}}$$

Finally, applying the fractional integral with respect to z on both sides of above equation, an integrating factor is obtained

$$\mu(x,y)=x^{2-\alpha}y^{2-\alpha}$$

Example 3.11. *Find an integrating factor of the form $\mu(z)$, where $z=x+y$, of the following equation*

$$\left(3x^{\frac{1}{2}}-y^{\frac{1}{2}}\right)dx+\left(3y^{\frac{1}{2}}+x^{\frac{1}{2}}\right)dy=0$$

Solution: Here

$$\frac{\partial^{\frac{1}{2}}\left(3x^{\frac{1}{2}}-y^{\frac{1}{2}}\right)}{\partial y^{\frac{1}{2}}}=-\frac{1}{2}\ \textit{and}\ \frac{\partial^{\frac{1}{2}}\left(3y^{\frac{1}{2}}+x^{\frac{1}{2}}\right)}{\partial x^{\frac{1}{2}}}=\frac{1}{2}$$

Now, computing the conformable fractional partial derivative of function $\mu(z)$ with respect to x and y, then

$$\frac{\partial^{\frac{1}{2}}\mu}{\partial x^{\frac{1}{2}}} = \left(T_{\frac{1}{2}}\mu\right)(z)\cdot z^{-\frac{1}{2}}\cdot\frac{\partial^{\frac{1}{2}}Z}{\partial x^{\frac{1}{2}}} = x^{\frac{1}{2}}\cdot z^{-\frac{1}{2}}\cdot\left(T_{\frac{1}{2}}\mu\right)(z)$$

And

$$\frac{\partial^{\frac{1}{2}}\mu}{\partial y^{\frac{1}{2}}} = \left(T_{\frac{1}{2}}\mu\right)(z)\cdot z^{-\frac{1}{2}}\cdot\frac{\partial^{\frac{1}{2}}Z}{\partial y^{\frac{1}{2}}} = y^{\frac{1}{2}}\cdot z^{-\frac{1}{2}}\cdot\left(T_{\frac{1}{2}}\mu\right)(z)$$

so that

$$\frac{1}{\mu(z)}\left(T_{\frac{1}{2}}\mu\right)(z) = -\frac{1}{\sqrt{z}}$$

Finally, applying the fractional integral with respect to z on both sides of above equation, an integrating factor is obtained

$$\mu(x,y) = \frac{1}{x+y}$$

3.1.4 Homogeneous fractional differential equation

In Section2.3, the homogeneous conformable fractional functions is studied, here homogeneous fractional conformable differential equations is introduced and solved by reduce it to α-exact.

Theorem 3.4. *Let $\alpha \in (0,1]$ and M, N are real valued functions with two variables defined on an open set D for witch $(tx,ty)\in D$ whenever $t>0$ and $(x,y)\in D$, with $x,y>0$, that satisfies*

1. *M, N are homogeneous function of degree r*
2. *$M, N \in C_\alpha(D,R)$ Then an integrating factor of the homogeneous differential equation $M(x,y)dx+ N(x,y)dy = 0$, is given by*

$$\mu(x,y) = \frac{1}{x^\alpha\cdot M(x,y) + y^\alpha\cdot N(x,y)}$$

providing

$$x^\alpha\ M(x,y) + y^\alpha\cdot N(x,y) \neq 0, \forall (x,y)\in D$$

Proof. In fact, computing the conformable fractional partial derivatives of functions μM and μN with respect to x and y, respectively, then

$$\frac{\partial^\alpha}{\partial y^\alpha}\left(\frac{M(x,y)}{x^\alpha\cdot M(x,y)+y^\alpha\cdot N(x,y)}\right)$$
$$= \frac{y^\alpha\cdot N(x,y)\cdot\frac{\partial^\alpha M(x,y)}{\partial y^\alpha} - \alpha M(x,y)N(x,y) - y^\alpha\cdot M(x,y)\cdot\frac{\partial^\alpha N(x,y)}{\partial y^\alpha}}{(x^\alpha\cdot M(x,y)+y^\alpha\cdot N(x,y))^2}.$$

And

$$\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{N(x,y)}{x^\alpha \cdot M(x,y) + y^\alpha \cdot N(x,y)}\right)$$
$$= \frac{x^\alpha \cdot M(x,y) \cdot \frac{\partial^\alpha N(x,y)}{\partial x^\alpha} - \alpha M(x,y)N(x,y) - x^\alpha \cdot N(x,y) \cdot \frac{\partial^\alpha M(x,y)}{\partial x^\alpha}}{(x^\alpha \cdot M(x,y) + y^\alpha \cdot N(x,y))^2}.$$

Finally, using the Conformable Euler's Theorem on homogeneous functions, the result is followed. □

Example 3.12. *Consider*

$$\left(x^{2+\alpha} + yx^{1+\alpha}\right) dx + \left(2x^{2+\alpha} + 3y^{2+\alpha}\right) dy = 0$$

For some $\alpha \in (0, 1]$.

Since $M(x,y) = x^{2+\alpha} + yx^{1+\alpha}$ *and* $N(x,y) = 2x^{2+\alpha} + 3y^{2+\alpha}$ *are homogeneous functions of degree* $2+\alpha$ *and class* C_α *on open set* D*, with* $x, y > 0$*, then above equation is a homogeneous differential equation and*

$$\mu(x,y) = \frac{1}{x^\alpha\left(x^{2+\alpha} + yx^{1+\alpha}\right) + y^\alpha\left(2x^{2+\alpha} + 3y^{2+\alpha}\right)} = \frac{1}{x^{2+2\alpha} + yx^{1+2\alpha} + 2x^{2+\alpha}y^\alpha + 3y^{2+2\alpha}}$$

is an integrating factor of it.

3.2 General solution of second order fractional differential equations

In this section, a general method for solving second order fractional differential equations has been presented based on conformable fractional derivative. This method realizes on determining a general solution of homogeneous and a particular solution of second order linear fractional differential equations.[13]
First, the main theory of this fractional equations is presented.

Definition 3.9. *Consider the general second order linear fractional differential equation based on conformable fractional derivative as follows*

$$T_\alpha T_\alpha(y) + P(x)T_\alpha(y) + Q(x)y = R(x) \tag{3.10}$$

Where $P(x), Q(x)$*, and* $R(x)$ *are* α*-differentiable functions and* y *is an unknown function.*
If $R(x)$ *is identically zero, then equation (3.10) reduces to the homogeneous fractional equation.*

$$T_\alpha T_\alpha(y) + P(x)T_\alpha(y) + Q(x)y = 0 \tag{3.11}$$

Theorem 3.5. *If $y_h(x)$ is the general solution of fractional (3.11) and $y_p(x)$ is any particular solution of fractional (3.10), then $y_h\big(x) + y_p(x)$ is a general solution of fractional (3.10).*

Proof. Suppose that $y(x)$ is a solution of (3.10), since $y_p(x)$ is any particular solution of fractional (3.10), then we want to show that $y(x) - y_p(x)$ is a solution of (3.11):

$$\begin{aligned} T_\alpha T_\alpha(y-y_p) + P(x)T_\alpha(y-y_p) + Q(x)(y-y_p) &= T_\alpha T_\alpha(y) + P(x)T_\alpha(y) - T_\alpha T_\alpha(y_p) + P(x)T_\alpha(y_p) \\ +Q(x)y_p) &= R(x) - R(x) \\ &= 0 \end{aligned}.$$

Since y_h is a general solution to (3.11), it results that

$$y(x) - y_p(x) = y_h(x)$$

or

$$y(x) = y_h(x) + y_p(x)$$

□

Theorem 3.6. *If $y_1(x)$ and $y_2(x)$ are any two solutions of fractional (3.11), then $C_1y_1(x) + C_2y_2(x)$ is also a solution for any constants C_1 and C_2.*

Proof. The statement follows immediately from the fact that

$$\begin{aligned} &T_\alpha T_\alpha\big(C_1y_1(x) + C_2y_2(x)\big) + P(x)T_\alpha\big(C_1y_1(x) + C_2y_2(x)\big) + Q(x)\big(C_1y_1(x) + C_2y_2(x)\big) \\ &= C_1\Big(\ T_\alpha T_\alpha\big(y_1(x)\big) + P(x)T_\alpha\big(y_1(x)\big) + Q(x)y_1(x)\Big) + \\ &C_2(T_\alpha T_\alpha\Big(y_2(x) + P(x)T_\alpha\big(y_2(x)\big) + Q(x)y_2(x)\Big) \\ &= C_1 \cdot 0 + C_2 \cdot 0 = 0 \end{aligned}$$

Since by assumption, $y_1(x)$ and $y_2(x)$ are solutions of equation (3.11) .

Definition 3.10. *The fractional Wronskian of two functions $f(x)$ and $g(x)$ is defined by*

$$\begin{aligned} W_\alpha(f(x), g(x)) &= \begin{vmatrix} f(x) & g(x) \\ T_\alpha(f(x)) & T_\alpha(g(x)) \end{vmatrix} \\ &= f(x) \cdot T_\alpha(g(x)) - g(x).T_\alpha(f(x)). \end{aligned}$$

Theorem 3.7. *If $y_1(x)$ and $y_2(x)$ are any two solutions of fractional (3.11) on an interval $[a, b]$, then their fractional Wronskian $W = W_\alpha(y_1(x), y_2(x))$ is either identically zero or never zero on $[a, b]$.*

Proof. We begin by observing that

$$T_\alpha(W) = y_1\ T_\alpha T_\alpha(y_2) - y_2\ T_\alpha T_\alpha(y_2)$$

Next, since $y_1(x)$ and $y_2(x)$ are both solutions of fractional equation, we have

$$T_\alpha T_\alpha(y_1(x)) + P(x)T_\alpha(y_1(x)) + Q(x)y_1(x) = 0$$

$$T_\alpha T_\alpha(y_2(x)) + P(x)T_\alpha(y_2(x)) + Q(x)y_2(x) = 0.$$

First equation multiplying by y_2 subtract to the second equation by y_1 result in

$$(y_1\ T_\alpha T_\alpha(y_2) - y_2\ T_\alpha T_\alpha(y_2)) + P(x)(y_1\ T_\alpha(y_2) - y_2\ T_\alpha(y_1)) = 0$$

or

$$T_\alpha(W) + P(x)W = 0.$$

The general solution of this first order fractional differential equation based on conformable fractional derivative is

$$W = W_\alpha\left(x_0\right) e^{-\mathrm{I}_\alpha(P(x))},$$

and since the exponential factor is never zero, the proof is completed □

Theorem 3.8. *If $y_1(x)$ and $y_2(x)$ are any two solutions of fractional equation(3.10) on an interval $[a, b]$, then they are linearly dependent on this interval if and only if their fractional Wronskian $W = W_\alpha(y_1(x), y_2(x))$, is identically zero.*

Proof. Let $y_1(x)$ and $y_2(x)$ are linearly dependent, and we show

$$W_\alpha(y_1(x), y_2(x)) = 0.$$

First, assume without loss of generality, that $y_2 = Cy_1$, for some constant, so

$$T_\alpha(y_2) = C\ T_\alpha(y_1)$$

By elimination C, from this equation, we obtain

$$y_1\ T_\alpha(y_2) - y_2\ T_\alpha(y_1) = 0,$$

which proves this half of the theorem.

We now assume that the $W_\alpha\left(y_1(x), y_2(x)\right) = 0$, and prove linearly dependent. If $y_1(x)$ is identically zero on $[a, b]$, then the functions are linearly dependent.

We may therefore assume that $y_1(x)$, does not vanish identically on $[a, b]$, from which it follows by continuity that $y_1(x)$ does not vanish at all on some sub-interval $[c, d]$ of $[a, b]$.

Since the Wronskian is identically zero on $[a, b]$, we can divide it by y_1^2 to get

$$\frac{y_1\ T_\alpha(y_2) - y_2\ T_\alpha(y_1)}{y_1^2} = 0$$

on $[c, d]$. which written as

$$T_\alpha(\frac{y_2}{y_1}) = 0,$$

and by conformable fractional integrating we obtain

$$\frac{y_2}{y_1} = C,$$

or

$$y_2 = Cy_1,$$

for some constant C, and all x, in $[c, d]$.
Finally, since y_2, and Cy_1, have equal value in $[c, d]$, they have equal conformable fractional derivative, so

$$y_2 = Cy_1,$$

For all x, in $[a, b]$, which concludes the argument. □

Theorem 3.9. *Let $y_1(x)$ and $y_2(x)$ be linearly dependent of the homogeneous fractional equation (3.11) on the interval $[a, b]$. Then $C_1y_1(x) + C_2y_2(x)$ is the general solution of the fractional equation on this interval.*

Proof. Let $y(x)$, be any solution of (3.11(on $[a, b]$. We must show that constant C_1, C_2, can be found so that

$$y(x) = C_1y_1(x) + C_2y_2(x),$$

for all x in $[a, b]$. Since $C_1y_1(x) + C_2y_2(x)$, and $y(x)$ are both solutions of (3.11) on $[a, b]$, it suffices to show that for some point x_0, in $[a, b]$, we can find C_1, C_2 so that

$$C_1y_1(x_0) + C_2y_2(x_0) = y(x_0)$$

and

$$C_1\ T_\alpha(y_1(x_0)) + C_2\ T_\alpha(y_2(x_0)) = T_\alpha(y(x_0))$$

For this system to be solvable for C_1, C_2, it suffices that the following determinant be non-zero.

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ T_\alpha(y_1(x_0)) & T_\alpha(y_2(x_0)) \end{vmatrix}$$

$$= y_1(x_0) \cdot T_\alpha(y_2(x_0)) - y_2(x_0) \cdot T_\alpha(y_1(x_0))$$

This leads us to investigate the function Wronskian of that $W_\alpha(y_1(x_0), y_2(x_0))$, have a value different from zero.

Next, we study some special cases. □

3.2.1 The general solution of the homogeneous fractional equation when one solution is known

Assume that $y_1(x)$, is a known nonzero solution of Equation (3.11), and let $y_2(x) = v(x)y_1(x)$, is a solution of the equation where $v(x)$ an unknown function.
So

$$T_\alpha\big(y_2(x)\big) = v(x)\cdot T_\alpha\big(y_1(x)\big) + y_1(x)\cdot T_\alpha(v(x))$$

$$T_\alpha T_\alpha\big(u_2(x)\big) = v(x)\cdot T_\alpha T_\alpha\big(u_1(x)\big) + 2\ T_\alpha\big(y_1(x)\big)\cdot T_\alpha(v(x)) + y_1(x)\cdot T_\alpha T_\alpha\big(y_1(x)\big)$$

By substituting the above results into Equation 3.11, we get

$$v(x)\Big(T_\alpha T_\alpha\big(y_1(x)\big) + P(x)T_\alpha\big(y_1(x)\big) + Q(x)y_1(x)\Big) + y_1(x)T_\alpha T_\alpha(v(x)) + T_\alpha(v(x))\big(P(x)y_1(x) + 2\ T_\alpha\big(y_1(x)\big) = 0.$$

Since $y_1(x)$ is a solution of Eq.(3.11) , It reduces

$$y_1(x)T_\alpha T_\alpha(v(x)) + T_\alpha(v(x))\Big(P(x)y_1(x) + 2\ T_\alpha\big(y_1(x)\big) = 0$$

or

$$\frac{T_\alpha T_\alpha(v(x))}{T_\alpha(v(x))} = -2\frac{T_\alpha\big(y_1(x)\big)}{y_1(x)} - P(x)$$

A fractional integration now gives $\ln\big(T_\alpha(v(x))\big) = -2\ln\big(y_1(x)\big) - I_\alpha(P(x))$

$$T_\alpha(v(x)) = \frac{1}{y_1^2(x)}e^{-I_\alpha(P(x))}$$

and

$$v(x) = I_\alpha\left(\frac{1}{y_1^2(x)}e^{-I_\alpha(P(x))}\right).$$

Consequently, the general solution of homogeneous fractional equation (3.11) is as follow,

$$y_h(x) = C_1 y_1(x) + C_2\left(I_\alpha\left(\frac{1}{y_1^2(x)}e^{-I_\alpha(P(x))}\right) y_1(x)\right.$$

Example 3.13. *If $y_1(x) = 3\sqrt[3]{x}$ is a solution of the following homogeneous equation*

$$9\sqrt[3]{x^2}\ T_{\frac{2}{3}}\ T_{\frac{2}{3}}(y(x)) - 6\sqrt[3]{x}\ T_{\frac{2}{3}}(y(x)) + 2y(x) = 0$$

Then, we have

$$v(x) = 3\sqrt[3]{x}$$

Therefore

$$y_2(x) = 9\sqrt[3]{x^2}$$

So the general solution is as follows,

$$y_h(x) = C_1\sqrt[3]{x} + C_2\sqrt[3]{x^2}$$

Example 3.14. *We know that* $u_1(x) = x$ *is a solution of*

$$2xT_{\frac{1}{2}}\ T_{\frac{1}{2}}(u(x)) + \sqrt{x}\ T_{\frac{1}{2}}(u(x)) - 2u(x) = 0.$$

solution:

According to the previous approach $v(x)$ *and second solution* $u_2(x)$ *are obtained as follows,*

$$v(x) = -\frac{x^{-2}}{2}, u_2(x) = -\frac{1}{2x}.$$

Therefore a general solution of above equation has the following form,

$$u_h(x) = C_1x + C_2x^{-1}.$$

3.2.2 The homogeneous fractional equations with constant coefficients

We are now in a position to give a complete discussion of the homogeneous equation of Eq.(3.11) for the special case in which p *and* q *are constants.*

$$T_\alpha T_\alpha(y(x)) + p\ T_\alpha(y(x)) + qy(x) = 0 \tag{3.12}$$

Our starting point is the fact that the exponential function $e^{m(\frac{1}{\alpha}x^\alpha)}$, *has the property that its conformable fractional derivative are all constant multiples of the function itself. It leads us to consider*

$$y(x) = e^{m\left(\frac{1}{\alpha}x^\alpha\right)} \tag{3.13}$$

as a possible solution for Equation(3.12), we have

$$T_\alpha(y(x)) = me^{m(\frac{1}{\alpha}x^\alpha)} \tag{3.14}$$

and

$$T_\alpha T_\alpha(y(x)) = m^2e^{m(\frac{1}{\alpha}x^\alpha)} \tag{3.15}$$

Substituting Equations (3.13), (3.14),and (3.15) into (3.12) yields

$$(m^2 + pm + q)e^{m(\frac{1}{\alpha}x^\alpha)} = 0 \tag{3.16}$$

and since $e^{m(\frac{1}{\alpha}x^\alpha)}$ *is never zero, (3.16) holds if and only if* m *satisfies the following auxiliary equation,*

$$m^2 + pm + q = 0. \tag{3.17}$$

It is clear that the roots m_1 and m_2 of Equation (3.17) are distinct real numbers if and only if $p^2 - 4q > 0$. In this case, we get the two solutions

$$y_1(x) = e^{m_1(\frac{1}{\alpha}x^\alpha)}$$

and

$$y_2(x) = e^{m_2(\frac{1}{\alpha}x^\alpha)}$$

Since the ratio

$$\frac{e^{m_1(\frac{1}{\alpha}x^\alpha)}}{e^{m_2(\frac{1}{\alpha}x^\alpha)}} = e^{(m_1-m_2)(\frac{1}{\alpha}x^\alpha)}$$

is not constant, these solutions are linearly independent and

$$y_h(x) = C_1 e^{m_1(\frac{1}{\alpha}x^\alpha)} + C_2 e^{m_2(\frac{1}{\alpha}x^\alpha)}$$

is the general solution of Equation(3.12).

If $m_1 = m_2$, then we obtain only one solution $y_1(x) = e^{m_1\left(\frac{1}{\alpha}x^\alpha\right)}$. Therefore, we can easily find a second linearly independent solution by the previous method as the following form

$$y_2(x) = \left(\frac{1}{\alpha}x^\alpha\right) e^{m_1\left(\frac{1}{\alpha}x^\alpha\right)}$$

and the general solution of Equation (3.12) is

$$y_h(x) = \left(C_1 + C_2\left(\frac{1}{\alpha}x^\alpha\right)\right) e^{m_1\left(\frac{1}{\alpha}x^\alpha\right)}$$

If the roots m_1 and m_2 are distinct complex numbers, then they can be written in the form $a \pm Ib$ and our two real solutions of Equation (3.12) are as follows

$$y_1(x) = e^{a(\frac{1}{\alpha}x^\alpha)}(\cos b(\frac{1}{\alpha}x^\alpha))$$

$$y_2(x) = e^{a(\frac{1}{\alpha}x^\alpha)}(\sin b(\frac{1}{\alpha}x^\alpha)).$$

So the solution of Equation (3.12) will be obtained as the following

$$y_h(x) = e^{a\left(\frac{1}{\alpha}x^\alpha\right)}\left(C_1 \cos b\left(\frac{1}{\alpha}x^\alpha\right) + C_2 \sin b\left(\frac{1}{\alpha}x^\alpha\right)\right)$$

Example 3.15. *Consider the following homogeneous fractional differential equation*

$$T_\alpha T_\alpha(y(x)) - 3\ T_\alpha(y(x)) + 2y(x) = 0.$$

The auxiliary equation is as follows

$$m^2 - 3m + 2 = 0$$

with the roots $m_1 = 1$, and $m_2 = 2$.
The general solution of the above equation is as follows

$$y_h(x) = C_1 e^{\left(\frac{1}{\alpha}x^\alpha\right)} + C_2 e^{2\left(\frac{1}{\alpha}x^\alpha\right)}.$$

Example 3.16. *The general solution of homogeneous equation of*

$$T_\alpha T_\alpha(u(x)) + 4\ T_\alpha(y(x)) + 4y(x) = 0$$

will be obtained as follows

$$y_h(x) = \left(C_1 + C_2\left(\frac{1}{\alpha}x^\alpha\right)\right)e^{-2\left(\frac{1}{\alpha}x^\alpha\right)}$$

Example 3.17. *Let's consider the following homogeneous fractional differential equation*

$$T_\alpha T_\alpha(y(x)) - 2\ T_\alpha(y(x)) + 3y(x) = 0.$$

Using the above result, we get

$$y_h(x) = e^{\left(\frac{1}{\alpha}x^\alpha\right)}\left(C_1 \cos\sqrt{2}\left(\frac{1}{\alpha}x^\alpha\right) + C_2 \sin\sqrt{2}\left(\frac{1}{\alpha}x^\alpha\right)\right).$$

.

3.2.3 Euler's conformable fractional equation

A conformable version of Cauchy-Euler fractional differential equation is studied.

Definition 3.11. *The homogeneous fractional differential equation,*

$$\left(\frac{1}{\alpha}x^\alpha\right)^2\ T_\alpha T_\alpha(y(x)) + p\left(\frac{1}{\alpha}x^\alpha)T_\alpha(y(x)\right) + qy(x) = 0, \quad x > 0(3.18)$$

where p, q are constant, is called Euler's fractional l equation.

By using the change independent variable.

$$z = ln(\frac{1}{\alpha}x^\alpha)$$

we have

$$T_\alpha(y(z)) = (\frac{1}{\alpha})x^\alpha)^{-1}\frac{dy}{dz} \tag{3.19}$$

$$T_\alpha T_\alpha(y(z)) = -(\frac{1}{\alpha}x^\alpha)^{-2}\frac{dy}{dz} + (\frac{1}{\alpha}x^\alpha)^{-2}\frac{d^2y}{dz^2} \tag{3.20}$$

Substituting Equations (3.19) and (3.20) into (3.18), we get

$$\frac{d^2y}{dz^2} + (p-1)\frac{dy}{dz} + qy = 0 \tag{3.21}$$

That equation (3.21) is an ordinary differential equation with constant coefficient, and according to this approach the auxiliary equation has the following form

$$m^2 + (p-1)m + q = 0 \tag{3.22}$$

Suppose m_1 *and* m_2 *are roots of Equations(3.22). If they are distinct real numbers, then the following solution of (3.18) can be obtained as,*

$$y_h(x) = C_1(\frac{1}{\alpha}x^\alpha)^{m_1} + C_2(\frac{1}{\alpha}x^\alpha)^{m_2}.$$

If

$$m_1 = m_2,$$

we derive the general solution as:

$$y_h(x) = (C_1 + C_2\ln(\frac{1}{\alpha}x^\alpha))(\frac{1}{\alpha}x^\alpha)^{m_1}.$$

And if m_1 *and* m_2 *are distinct complex numbers then the general solution of Equation (3.18) will be derived as follows:*

$$y_h(x) = (\frac{1}{\alpha}x^\alpha)^a[C_1\cos b\ln(\frac{1}{\alpha}x^\alpha) + C_2\sin b\ln(\frac{1}{\alpha}x^\alpha)]$$

Example 3.18. *Let's consider the following homogeneous fractional differential equation*

$$(\frac{1}{\alpha}x^\alpha)^2\ T_\alpha T_\alpha(y(x)) - 2(\frac{1}{\alpha}x^\alpha)T_\alpha(y(x)) + 2y(x) = 0$$

The auxiliary equation is as follows

$$m^2 - 3m + 2 = 0$$

with the roots $m_1 = 1$*, and* $m_2 = 2$.

So the general solution is

$$y_h(x) = C_1\left(\frac{1}{\alpha}x^\alpha\right) + C_2\left(\frac{1}{\alpha}x^\alpha\right)^2.$$

Example 3.19. *consider following homogeneous equation*

$$(\frac{1}{\alpha}x^\alpha)^2\ T_\alpha T_\alpha(y(x)) - 3(\frac{1}{\alpha}x^\alpha)T_\alpha(y(x)) + 4y(x) = 0.$$

The root of auxiliary equation are

$$m_1 = m_2 = 2.$$

Thus the general solution is as follows:

$$y_h(x) = (C_1 + C_2 \ln(\frac{1}{\alpha}x^\alpha))(\frac{1}{\alpha}x^\alpha)^2$$

Example 3.20. *Let's consider Euler's fractional equation as follows*

$$(\frac{1}{\alpha}x^\alpha)^2\ T_\alpha T_\alpha(y(x)) + 3(\frac{1}{\alpha}x^\alpha)T_\alpha(y(x)) + 2y(x) = 0$$

The root of auxiliary equation are

$$m_1 = -1 + i, m_2 = -1 - i.$$

Consequently, we have,

$$y_h(x) = (\frac{1}{\alpha}x^\alpha)^{-1}[C_1 \cos(\ln(\frac{1}{\alpha}x^\alpha)) + C_2 \sin(\ln(\frac{1}{\alpha}x^\alpha))]$$

3.3 Determining a particular solution of nonhomogeneous fractional equation

In this section, we have introduced variation of parameters and undetermined coefficients methods for determining a particular solution of nonhomogeneous fractional equations.

3.3.1 Variation of parameters or Lagrange approach

Assume that $u_1(x), u_2(x)$ are two linearly independent solution homogeneous fractional differential equation of the second order fractional differential equation (3.11), we suppose that the particular solution $u_p(x)$ of equation (3.10), is

$$u_p(x) = \vartheta_1(x)u_1(x) + \vartheta_2(x)u_2(x), \tag{3.23}$$

where $\vartheta_1(x), \vartheta_2(x)$ are two unknown functions. By computing the conformable fractional derivative of (3.23), we derive

$$T_\alpha(u_p(x)) = (u_1\ T_\alpha(\vartheta_1(x)) + u_2\ T_\alpha(\vartheta_2(x))) + (\vartheta_1\ T_\alpha(u_1(x)) + \vartheta_2\ T_\alpha(u_2(x)).$$

Now,suppose that

$$u_1\ T_\alpha(\vartheta_1(x) + u_2\ T_\alpha(\vartheta_2(x) = 0. \tag{3.24}$$

So

$$T_\alpha(u_p(x)) = \vartheta_1\ T_\alpha(u_1(x)) + \vartheta_2\ T_\alpha(u_2(x)). \tag{3.25}$$

Therefore,

$$T_\alpha T_\alpha(u_p(x)) = \vartheta_1\ T_\alpha T_\alpha(u_1(x) + T_\alpha(\vartheta_1(x))T_\alpha\left(u_1(x)\right) + T_\alpha(\vartheta_2(x)\right)T_\alpha\left(u_2(x)\right) + \vartheta_2\ T_\alpha T_\alpha(u_2(x)). \tag{3.26}$$

By substituting (3.23), (3.25) and (3.26) into equation (3.10), and some manipulation, we get

$$\begin{gathered}\vartheta_1\Big(\ T_\alpha T_\alpha\left(u_1(x)\right) + P(x)T_\alpha\left(u_1(x)\right) + Q(x)u_1(x)\Big)\\ +\vartheta_2\Big(\ T_\alpha T_\alpha\left(u_2(x)\right) + P(x)T_\alpha\left(u_2(x)\right) + Q(x)u_2(x)\Big)\\ +T_\alpha\left(\vartheta_1(x)\right)T_\alpha\left(u_1(x)\right) + T_\alpha\left(\vartheta_2(x)\right)T_\alpha\left(u_2(x)\right) = R(x)\end{gathered}$$

Since $u_1(x)$, and $u_2(x)$ are solutions of 3.11, the two expressions in parentheses are equal to zero, and so the above equation reduces to

$$T_\alpha\left(\vartheta_1(x)\right)T_\alpha\left(u_1(x)\right) + T_\alpha\left(\vartheta_2(x)\right)T_\alpha\left(u_2(x)\right) = R(x). \tag{3.27}$$

Considering equation (3.27) and equation (3.24) together, we obtain the following results

$$\vartheta_1(x) = I_\alpha\left(\frac{-u_2(x)R(x)}{W_\alpha\left(u_1(x), u_2(x)\right)}\right),$$

$$\vartheta_2(x) = I_\alpha\left(\frac{u_1(x)R(x)}{W_\alpha\left(u_1(x), u_2(x)\right)}\right)$$

so

$$u_p(x) = u_1(x) \cdot I_\alpha\left(\frac{-u_2(x)R(x)}{W_\alpha\left(u_1(x), u_2(x)\right)}\right) + u_2(x) \cdot I_\alpha\left(\frac{u_1(x)R(x)}{W_\alpha\left(u_1(x), u_2(x)\right)}\right).$$

Example 3.21. *Let's consider the following fractional equation*

$$9\sqrt[3]{x^2}T_{\frac{2}{3}}T_{\frac{2}{3}}(u(x)) - 6\sqrt[3]{x}T_{\frac{2}{3}}(u(x)) + 2u(x) = 9x^2\sqrt[3]{x^2}.$$

By using the previous, the homogeneous solutions are as follows,

$$u_1(x) = 3\sqrt[3]{x}, u_2(x) = 9\sqrt[3]{x^2}$$

, and a particular solution is

$$u_p(x) = -\frac{39}{56}x^3.$$

So the general solution

$$u(x) = C_1\sqrt[3]{x} + C_2\sqrt[3]{x^2} - \frac{39}{56}x^3.$$

Example 3.22. *Find the general solution of*

$$2x\ T_{\frac{1}{2}}\ T_{\frac{1}{2}}(u(x)) + \sqrt{x}\ T_{\frac{1}{2}}(u(x)) - 2u(x) = 4x^3$$

The homogeneous solutions are $u_1(x) = x$*, and* $u_2(x) = -\frac{1}{2x}$
, and a particular solution is

$$u_p(x) = \frac{1}{4}x^3$$

So the general solution of an equation is

$$u(x) = C_1x + C_2x^{-1} + 0.25x^3.$$

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3.3.2 The method of undetermined coefficients

In this section, we use the Undetermined coefficients procedure to find $u_p(x)$ for a special case of the non-homogeneous equation of the homogeneous equation (3.12). In particular, we consider

$$T_\alpha T_\alpha(u(x)) + p\ T_\alpha(u(x)) + qu(x) = R(x)$$

, where p, q are constant and

$$R(x) = \left(a_0 + a_1\left(\tfrac{1}{\alpha}x^\alpha\right) + a_2\left(\tfrac{1}{\alpha}x^\alpha\right)^2 + \cdots + a_n\left(\tfrac{1}{\alpha}x^\alpha\right)^n\right) e^{\beta\left(\frac{1}{\alpha}x^\alpha\right)} \sin\gamma\left(\tfrac{1}{\alpha}x^\alpha\right) \text{ or}$$

$$R(x) = \left(a_0 + a_1\left(\frac{1}{\alpha}x^\alpha\right) + a_2\left(\frac{1}{\alpha}x^\alpha\right)^2 + \cdots + a_n\left(\frac{1}{\alpha}x^\alpha\right)^n\right) e^{\beta\left(\frac{1}{\alpha}x^\alpha\right)} \cos\gamma\left(\frac{1}{\alpha}x^\alpha\right).$$

We choose a particular solution in the following form

$$u_p(x) = \Bigg[\left(A_0 + A_1\left(\tfrac{1}{\alpha}x^\alpha\right) + A_2\left(\tfrac{1}{\alpha}x^\alpha\right)^2 + \cdots + A_n\left(\tfrac{1}{\alpha}x^\alpha\right)^n\right) e^{\beta\left(\frac{1}{\alpha}x^\alpha\right)} \sin\gamma\left(\tfrac{1}{\alpha}x^\alpha\right) +$$
$$\left(B_0 + B_1\left(\tfrac{1}{\alpha}x^\alpha\right) + B_2\left(\tfrac{1}{\alpha}x^\alpha\right)^2 + \cdots + B_n\left(\tfrac{1}{\alpha}x^\alpha\right)^n\right) e^{\beta\left(\frac{1}{\alpha}x^\alpha\right)} \sin\gamma\left(\tfrac{1}{\alpha}x^\alpha\right)\Bigg]\left(\tfrac{1}{\alpha}x^\alpha\right)^m$$

that $A_0, A_1, \ldots, A_n, B_0, B_1, \ldots, B_n$, unknown coefficients and m is the lowest non-negative integer number, that removes homogeneous solutions, in choosing $u_p(x)$.

Example 3.23. *Consider the following nonhomogeneous fractional equation*

$$T_{\frac{2}{3}}\ T_{\frac{2}{3}}(u(x)) - 2\ T_{\frac{2}{3}}(u(x)) = 18\sqrt[3]{x^2} - 10.$$

Solution: The homogeneous solutions are

$$u_1(x) = 1, u_2(x) = e^{\frac{4\sqrt[3]{x^2}}{3}}$$

, And a particular solution is

$$u_p(x) = 3\sqrt[3]{x^2} - \frac{27}{4}\left(\sqrt[3]{x^2}\right)^2$$

So the general solution is

$$u(x) = C_1 + C_2 e^{\frac{4\sqrt[3]{x^2}}{3}} + 3\sqrt[3]{x^2} - \frac{27}{4}\left(\sqrt[3]{x^2}\right)^2$$

Example 3.24. *Find the general solution of*

$$T_{\frac{1}{2}}\ T_{\frac{1}{2}}(u(x)) - 2\ T_{\frac{1}{2}}(u(x)) + u(x) = 2\sqrt{x}e^{2\sqrt{x}}$$

Solution:

$$u(x) = C_1 e^{2\sqrt{x}} + C_2\sqrt{x}e^{2\sqrt{x}} + \frac{4}{3}x\sqrt{x}e^{2\sqrt{x}}$$

Example 3.25. *Consider non-homogeneous fractional equation as follows*

$$T_{\frac{1}{2}}\ T_{\frac{1}{2}}(u(x)) + 4u(x) = 4\cos 4\sqrt{x} + 32x - 8\sqrt{x}.$$

This equation has a general solution such as,

$$u(x) = C_1\cos 4\sqrt{x} + C_2\sin 4\sqrt{x} + 2\sqrt{x}\sin 4\sqrt{x} + 8x - 2\sqrt{x} - 2.$$

3.4 Systems of Linear Fractional Differential Equations

In this section we discuss systems of conformable linear differential equations with constant coefficients. We give full solution for homogeneous and non-homogeneous systems[10].

3.4.1 The Solution of the homogeneous system

Consider the square systems of order 2. Let y_1, y_2 be any two functions of the variable x such that

$$y_1^{(\alpha)} = a_{11}y_1 + a_{12}y_2 + f_1$$
$$y_2^{(\alpha)} = a_{21}y_1 + a_{22}y_2 + f_2$$

with $\alpha \in (0, 1]$. This system can be written in the form

$$Y^{(\alpha)} = AY + F$$

where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, *and* $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

Theorem 3.10. *Let A has two real distinct real eigenvalues λ_1 and λ_2 say, with corresponding eigenvectors E_1 and E_2 respectively. Then the system*

$$Y^{(\alpha)} = AY. \tag{3.28}$$

has two independent solutions

$$Y_1 = E_1 e^{\frac{\lambda_1 x^\alpha}{\alpha}}$$

and

$$Y_2 = E_2 e^{\frac{\lambda_2 x^\alpha}{\alpha}}$$

and the general solution of the system (3.28) is

$$Y_h = c_1 Y_1 + c_2 Y_2,$$

where Y_h is the solution of the homogeneous system (3.28).

Proof.

$$Y_1^{(\alpha)} = \frac{d^\alpha}{dx^\alpha} E_1 e^{\frac{\lambda_1 x^\alpha}{\alpha}} = E_1 \frac{d^\alpha}{dx^\alpha} e^{\frac{\lambda_1 x^\alpha}{\alpha}} = E_1 \lambda_1 e^{\frac{\lambda_1 x^\alpha}{\alpha}} = A E_1 e^{\frac{\lambda_1 x^\alpha}{\alpha}} = AY_1$$

Similarly for Y_2.
Since system (3.28) is linear and homogeneous then the result follows.

□

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Example 3.26. *Consider the system*

$$Y^{(\alpha)} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} Y$$

Solution:

The eigenvalues of the matrix are $\lambda_1 = 2$, *and* $\lambda_2 = -2$. *The corresponding eigenvectors are* $E_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ *and* $E_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$*.Hence*

$$Y_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2\frac{x^\alpha}{\alpha}}$$

and

$$Y_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2\frac{x^\alpha}{\alpha}}.$$

So, the general solution is

$$Y_h = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2\frac{x^\alpha}{\alpha}} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2\frac{x^\alpha}{\alpha}}.$$

Theorem 3.11. *Let* A *has a complex eigenvalue* $\lambda = c + id$ *with corresponding eigenvectors* E*,and* $\bar{\lambda} = c - id$ *is an eigenvalue with eigenvector* $\bar{E}$. *Then the system*

$$Y^{(\alpha)} = AY$$

has two independent solutions:

$$Y_1 = e^{\frac{cx^\alpha \alpha}{\alpha}} \left(G \cos d\frac{x^\alpha}{\alpha} + H \sin d\frac{x^\alpha}{\alpha} \right)$$

and

$$Y_2 = e^{\frac{cx^\alpha}{\alpha}} \left(H \cos d\frac{x^\alpha}{\alpha} - G \sin d\frac{x^\alpha}{\alpha} \right)$$

where

$$G = \frac{E + \bar{E}}{2}, \quad H = i\frac{E - \bar{E}}{2}$$

Further

$$Y_h = c_1 Y_1 + c_2 Y_2.$$

Proof. Follows directly, by substituting Y_1, and Y_2 in equation (3.28) as in Theorem (3.10), and the facts:

$$T_\alpha \left(\sin \frac{1}{\alpha} x^\alpha \right) = \cos \left(\frac{1}{\alpha} x^\alpha \right)$$

and

$$T_\alpha \left(\cos \frac{1}{\alpha} x^\alpha \right) = -\sin \left(\frac{1}{\alpha} x^\alpha \right)$$

Example 3.27. *Consider the system* $Y^{(\alpha)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Y$.

The eigenvalues are i *and* $-i$. *The eigenvectors are* $E = \begin{pmatrix} i \\ 1 \end{pmatrix}$, *and* $\bar{E} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$. *So* $G = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, *and* $H = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$.*So*

$$Y_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos \frac{x^\alpha}{\alpha} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin \frac{x^\alpha}{\alpha} = \begin{pmatrix} -\sin \frac{x^\alpha}{\alpha} \\ \cos \frac{x^\alpha}{\alpha} \end{pmatrix}$$

and

$$Y_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos \frac{x^\alpha}{\alpha} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin \frac{x^\alpha}{\alpha} = \begin{pmatrix} -\cos \frac{x^\alpha}{\alpha} \\ -\sin \frac{x^\alpha}{\alpha} \end{pmatrix}$$

Theorem 3.12. *Let* A *has a repeated real eigenvalue* λ *say, with an eigenvector* E. *Then the system* $Y^{(\alpha)} = AY$ *has two linearly independent solutions*

$$Y_1 = E e^{\frac{\lambda x^\alpha}{\alpha}}$$

and

$$Y_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \frac{x^\alpha}{\alpha} e^{\frac{\lambda x^\alpha}{\alpha}} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\frac{\lambda x^\alpha}{\alpha}},$$

where $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ *and* $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ *are to be determined through substituting* Y_2 *in the system (3.28). Further* $Y_h = c_1 Y_1 + c_2 Y_2$.

Proof. The proof follows the same lines as in the classical case. □

Example 3.28. *Consider the system*

$$Y^{(\alpha)} = \begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix} Y$$

The eigenvalues of A are 1 and 1 . An eigenvector for A corresponding to the eigenvalue 1 is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. *Hence*

$$Y_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{\frac{x^\alpha}{\alpha}}.$$

For Y_2, we put

$$Y_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \frac{x^\alpha}{\alpha} e^{\frac{x^\alpha}{\alpha}} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\frac{x^\alpha}{\alpha}}$$

Using the properties of the conformable fractional derivative when substituting in the system we get
$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$, *and* $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. *Hence*

$$Y_2 = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \frac{x^\alpha}{\alpha} e^{\frac{x^\alpha}{\alpha}} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{\frac{x^\alpha}{\alpha}}$$

3.4.2 The particular Solution of the non-homogeneous system

Consider the system $Y^{(\alpha)} = AY + F$. we have found the solution of the homogeneous system $Y^{(\alpha)} = AY$. In this case we try to find a particular solution of the non-homogeneous system $Y^{(\alpha)} = AY + F$, say Y_p. Then the general solution will be $Y_g = Y_h + Y_p$

Let Y_1 and Y_2 be the two independent solutions of the system $Y^{(\alpha)} = AY$, where A is a 2 by 2 matrix. We Form the matrix

$$\phi(t) = \begin{pmatrix} Y_1 & Y_2 \end{pmatrix}$$

, whose columns are Y_1 and Y_2. Since Y_1 and Y_2 are independent, then $\phi(t)$ is invertible. Let us denote the conformable integral of any function g by $I_{(\alpha)}(g)$, where

$$I_{(\alpha)}(g) = \int_a^x \frac{f(x)}{x^{1-\alpha}} dx.$$

Now, we introduce the following theorem.

Theorem 3.13. *A particular solution of the system $Y^{(\alpha)} = AY + F$ is $Y_p = \phi(x) I_{(\alpha)} \left(\phi^{-1}(x) F(x)\right)$*

Proof. The proof is just the method of variation of parameters. Now,

$$Y_h = c_1 Y_1 + c_2 Y_2 = \phi(t) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

So we replace the parameters c_1 and c_2 by functions say u_1 and u_2. Following the classical procedure to get

$$Y_p = \phi(x) I_{(\alpha)} \left(\phi^{-1}(x) F(x)\right.$$

Now let us check that such Y_p satisfies $Y_p^{(\alpha)} = AY_p + F$:

Since the conformable derivative satisfies the multiplication rule for differentiation, we get:

$$\begin{aligned}\left(\phi(x) I_{(\alpha)} \left(\phi^{-1}(x) F(x)\right)^{(\alpha)}\right. &= \phi^{(\alpha)}(x) I_{(\alpha)} \left(\phi^{-1}(x) F(x) + \phi(x)\phi^{-1}(x) F(x)\right. \\ &= \phi^{(\alpha)}(x) I_{(\alpha)} \left(\phi^{-1}(x) F(x) + F(x)\right.\end{aligned} \tag{3.29}$$

But since both Y_1 and Y_2 satisfy the equation $Y_i^{(\alpha)} = AY_i$, then it follows that $\phi^{(\alpha)}(x) = A\phi(x)$. Consequently,

$$\begin{aligned}\phi^{(\alpha)}(x) I_{(\alpha)} \left(\phi^{-1}(x) F(x)\right. &= A\phi(x) I_{(\alpha)} \left(\phi^{-1}(x) F(x)\right. \\ &= AY_p\end{aligned}$$

Thus we get from equation (3.29) that $Y_p^{(\alpha)} = AY_p + F$. □

Example 3.29. *Consider the system*

$$Y^{(\alpha)} = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} Y - \begin{pmatrix} 15 \\ 4 \end{pmatrix} x e^{-2\frac{x^\alpha}{\alpha}}$$

with $Y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The eigenvalues of the matrix are -2 *and* 5, *and the corresponding homogeneous solution is*

$$Y_h = c_1 \begin{pmatrix} e^{-2\frac{x^\alpha}{\alpha}} \\ -3e^{-2\frac{x^\alpha}{\alpha}} \end{pmatrix} + c_2 \begin{pmatrix} 2e^{5\frac{x^\alpha}{\alpha}} \\ -3e^{5\frac{x^\alpha}{\alpha}} \end{pmatrix}$$

Now, the fundamental matrix $\phi(t) = \begin{pmatrix} e^{-2\frac{x^\alpha}{\alpha}} & 2e^{5\frac{x^\alpha}{\alpha}} \\ -3e^{-2\frac{t^\alpha}{\alpha}} & -3e^{5\frac{x^\alpha}{\alpha}} \end{pmatrix}$. *The inverse of* $\phi(x)$ *is*

$$\phi^{-1}(x) = \frac{1}{7} e^{-3\frac{x^\alpha}{\alpha}} \begin{pmatrix} e^{5\frac{x^\alpha}{\alpha}} & -2e^{5\frac{x^\alpha}{\alpha}} \\ 3e^{-2\frac{x^\alpha}{\alpha}} & e^{-2\frac{x^\alpha}{\alpha}} \end{pmatrix}$$

Now,

$$Y_p = \phi(x) \int_0^x \frac{\left(\phi^{-1}(s) F(s)\right)}{s^{1-\alpha}} ds$$

where

$$F(x) = -\begin{pmatrix} 15 \\ 4 \end{pmatrix} xe^{-2\frac{x^\alpha}{\alpha}}$$

One can see that

$$\phi^{-1}(s)F(s) = \begin{pmatrix} -s \\ -7se^{-7\frac{s^\alpha}{\alpha}} \end{pmatrix}$$

Hence

$$\frac{\left(\phi^{-1}(s)F(s)\right)}{s^{1-\alpha}} = \begin{pmatrix} -s^\alpha \\ -7s^\alpha e^{-7\frac{s^\alpha}{\alpha}} \end{pmatrix}$$

Consequently,

$$Y_p = \phi(x)\int_0^x \frac{\left(\phi^{-1}(s)F(s)\right)}{s^{1-\alpha}}ds = \phi(x)\int_0^x \begin{pmatrix} -s^\alpha \\ -7s^\alpha e^{-7\frac{s^\alpha}{\alpha}} \end{pmatrix} ds$$

Chapter 4

Other fractional derivatives with applications

In this chapter, we study three types of fractional derivatives that are also defined by using the limit approach, but they have different s, structures, namely the UD, Exponential, and Hyperbolic fractional derivatives will be considered, as their main properties and their basic calculus results, in addition, fractional differential equations concerning these fractional operators are studied and solved.

4.1 The UD fractional derivative

Ajay Dixit, Amit Ujlayan in [9] and [8] have introduced a U-D fractional derivative as a convex combination of the function and its first derivative, where $(D^\alpha f)(x) = (1-\alpha)f(x)+\alpha f'(x)$ *for* $\alpha \in (0,1]$*, they have studied the main results of this operator.*

4.1.1 Definition and main properties of the UD derivative

Definition 4.1. *For a given function* $f:[0,\infty)\to\mathbb{R}$ *and* $\alpha\in[0,1]$*, the UD derivative of order* α *is defined as*

$$D^\alpha f(x) = \lim_{\epsilon\to 0}\frac{e^{\epsilon(1-\alpha)}f\left(xe^{\frac{\epsilon\alpha}{x}}\right)-f(x)}{\epsilon}$$

If this limit exists, then $D^\alpha f(x)$ *is called the UD derivative of* f *for* $\alpha\in[0,1]$*, with the understanding that* $D^\alpha f(x) = \frac{d^\alpha f(x)}{dx^\alpha}$*. Also, if* f *is UD differentiable in the interval* $(0,x)$ *for* $x>0$ *and* $\alpha\in[0,1]$ *such that* $\lim_{x\to 0^+} f^\alpha(x)$ *exist then,*

$$f^\alpha(0) = \lim_{x\to 0^+} f^\alpha(x)$$

Now, we study the main results and properties of the UD-derivative

Theorem 4.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a differentiable function and $\alpha \in [0,1]$. Then, f is UD differentiable and*

$$D^{\alpha} f = (1-\alpha) f + \alpha D f \tag{4.1}$$

Proof. We have

$$\begin{aligned}
D^{\alpha} f(x) &= \lim_{\varepsilon \to 0} \frac{e^{\varepsilon(1-\alpha)} f\left(x e^{\frac{\varepsilon\alpha}{x}}\right) - f(x)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\left\{1+\varepsilon(1-\alpha)+o\left(\varepsilon^2\right)\right\}\left[f\left\{x+\varepsilon\alpha+o\left(\varepsilon^2\right)\right\}\right] - f(x)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\{1+\varepsilon(1-\alpha)\}\left[f(x)+f'(x)\{\varepsilon\alpha\}\right] - f(x)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{f(x)+\varepsilon(1-\alpha)f(x)+\varepsilon\alpha f'(x) - f(x)}{\varepsilon} \\
&= (1-\alpha) f(x) + \alpha f'(x),
\end{aligned}$$

where $\alpha \in [0,1]$. □

Proposition 4.1. *Let f and g be two differentiable functions in $[0,\infty)$ and $0 \leq \alpha, \gamma \leq 1$, then the following properties hold:*

1. $D^{\alpha}(af+bg) = aD^{\alpha}f + bD^{\alpha}g$ *(linearity)*
2. $D^{\alpha}(fg) = (D^{\alpha}f)\, g + \alpha(Dg)f$ *(product rule)*
3. $D^{\alpha}\left(\frac{g}{h}\right) = \left(\frac{(D^{\alpha}g)h - \alpha(Dh)g}{h^2}\right)$ *(quotient rule)*
4. $D^{\alpha}(g.h) \neq (D^{\alpha}g)\,.h + (D^{\alpha}h)\, g$ *(violation of Leibnitz's rule)*
5. $D^{\alpha}(D^{\gamma})\, g = D^{\gamma}(D^{\alpha})\, g$ *(commutativity)*

Ramark 4.1. *The UD derivative of order $\alpha, \alpha \in [0,1]$, as given in Definition 2.1. violets the Leibnitz's rule for fractional derivatives,*

$$D^{\alpha}(fg) \neq gD^{\alpha}f + fD^{\alpha}g$$

It also violets the law of indices,

$$D^{\alpha}(D^{\gamma})\, f \neq D^{\alpha+\gamma} f$$

.

Example 4.1. *The UD derivatives of order $\alpha, \alpha \in [0,1]$, of some elementary real-valued differentiable functions in $[0,\infty)$, can be given as following:*

(i) $D^{\alpha}\left((bx)^m\right) = (1-\alpha)(bx)^m + bm\alpha(bx)^{m-1}$

(ii) $D^{\alpha}\left(e^{bx}\right) = ((1-\alpha) + b\alpha)e^{bx}$

(iii) $D^{\alpha}(\sin(bx)) = (1-\alpha)\sin(bx) + b\alpha\cos(bx)$

(iv) $D^{\alpha}(\cos(bx)) = (1-\alpha)\cos(bx) - b\alpha\sin(bx)$

(v) $D^{\alpha}(\log(bx)) = b\alpha(bx)^{-1} + (1-\alpha)\log(bx)$

(vi) $D^{\alpha}(c) = (1-\alpha)c$, *where c is constant.*

Ramark 4.2. *The equation (4.1) asserts that the UD derivative of order $\alpha, \alpha \in [0,1]$, of a differentiable function $f : [0,\infty) \to \mathbb{R}$, is a convex combination of the function and the first derivative itself.*
Also,

$$D^{\alpha}f(x) = f(x)$$

for $\alpha = 0$ and

$$D^{\alpha}f(x) = f'(x),$$

for $\alpha = 1$, i.e., the UD derivative posses conformable property of conformable fractional derivatives.

Definition 4.2. *Let $f : [0,\infty) \to \mathbb{R}$ is a n times differentiable function. Then, the UD derivative of of order $\alpha, \alpha \in (n, n+1]$, is defined as*

$$D^{\alpha}f(x) = \lim_{\varepsilon \mapsto 0} \frac{e^{\varepsilon(1-\alpha)} f^{\lceil\alpha\rceil-1}\left(xe^{\frac{\varepsilon\alpha}{x}}\right) - f^{\lceil\alpha\rceil-1}(x)}{\varepsilon}$$

where $\lceil\alpha\rceil$ represents the smallest integer greater than or equal to α.

4.1.2 The main results about The UD derivative

The definition of the UD-derivative enable us to study the basic analysis of this derivative.

Theorem 4.2. *Let the function f is not unbounded in $[0,\infty)$. If f is UD differentiable for some $\alpha \in [0,1]$ at $x = a$, then f continuous at $x = a$.*

Proof. We show that $\lim_{\epsilon\mapsto 0} f(x+\epsilon\alpha) = f(x)$.

$$\begin{aligned}
\lim_{\epsilon\mapsto 0} f(x+\epsilon\alpha) - f(x) &= \lim_{\epsilon\mapsto 0}\left(\frac{(1+\epsilon(1-\alpha))f(x+\epsilon\alpha) - \epsilon(1-\alpha)f(x+\epsilon\alpha) - f(x)}{\epsilon}\right)\epsilon \\
&= \lim_{\epsilon\mapsto 0}\left(\frac{(1+\epsilon(1-\alpha))f(x+\epsilon\alpha) - f(x)}{\epsilon}\right)\epsilon - \lim_{\epsilon\mapsto 0}\epsilon(1-\alpha)f(x+\epsilon\alpha) \\
&= \lim_{\epsilon\mapsto 0}(D^\alpha f)\,\epsilon - \lim_{\epsilon\mapsto 0}\epsilon(1-\alpha)f(x+\epsilon\alpha) \\
&= 0(\text{ as } f \text{ is not unbounded for all } 0 \le x < \infty)
\end{aligned}$$

□

Theorem 4.3. *(Rolle's theorem for the UD derivative).*

Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ is a given function such that

1. *f is continuous in $[a,b]$,*
2. *f is differentiable in (a,b),*
3. *$f(a) = f(b)$.*

Then, there exists a point $c \in (a,b)$, such that

$$D^\alpha f(c) = (1-\alpha)f(c),$$

where $\alpha \in [0,1]$. □

Proof. As we know that $D^\alpha f(c) = (1-\alpha)f(c) + \alpha f'(c)$ and from classical Rolle's theorem of $f, f'(c) = 0$ implies $D^\alpha f(c) = (1-\alpha)f(c)$. □

Theorem 4.4. *(Mean Value theorem for the UD derivative).*

Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ is a given function such that

1. *f is continuous in $[a,b]$,*
2. *f is differentiable in (a,b). Then, there exists a point $c \in (a,b)$, such that*

$$D^\alpha f(c) = (1-\alpha)f(c) + \alpha\frac{f(b)-f(a)}{b-a}$$

where $\alpha \in [0,1]$.

Proof. The Result follows from the classical mean value theorem of f. □

Theorem 4.5. *Let f, g are two functions such that*

1. f, g are continuous in $[a, b]$,

2. f, g are UD differentiable in]a,b[for $0 \leq \alpha \leq 1$,

3. $D^\alpha f(x) = D^\alpha g(x)$ for all x in (a, b) and $0 < \alpha \leq 1$. Then,

$$(f - g)(x) = \eta e^{\frac{(\alpha-1)x}{\alpha}},$$

where η is a constant.

Proof. Let $y(x) = f(x) - g(x)$ for all $x \in (a, b)$, then

$$\begin{aligned} &D^\alpha (f - g)(x) = 0 \\ &D^\alpha y = 0 \\ &y(x) + \alpha D y(x) = 0 \\ \Rightarrow &y(x) = \eta e^{\frac{(\alpha-1)x}{\alpha}} \end{aligned}$$

□

Corollary 4.1. *If $D^\alpha f(x) = 0$, then f is not constant, in fact $f(x) = \eta e^{\frac{(\alpha-1)}{\alpha}x}$.*

Lemma 4.1. *(Sturm separation theorem)*

If $u(x)$ and $v(x)$ are two non-trivial continuous linearly independent solutions to a homogeneous second-order linear differential equation with x_0 and x_1 being successive roots of $u(x)$, then $v(x)$ has exactly one root in the open interval (x_0, x_1)

The following theorem is the UD-fractional version of the Sturm separation theorem.

Theorem 4.6. *Let y_1, y_2 be two linearly independent solutions of*

$$D^\alpha D^\alpha y(x) + p(x) D^\alpha y(x) + q(x) y(x) = 0$$

where $p(x), q(x)$ are continuous functions on (a, b) and $\alpha \in (0, 1]$. Then y_1 has a zero between any two consecutive zeroes of y_2. that is, zeros of y_1, y_2 occur alternately.

Proof.

$$y''(x) + P(x) y'(x) + Q(x) y(x) = 0,$$

where

$$P(x) = \frac{1}{\alpha}(2(1 - \alpha) + p(x)), Q(x) = \frac{1}{\alpha^2}\left((1 - \alpha)^2 + (1 - \alpha) p(x) + q(x)\right)$$

and the result follows from (Sturm separation theorem of ordinary differential equations). □

4.1.3 Geometrical interpretation of the UD derivative

The geometrical interpretation of the UD derivative of order $\alpha \in [0,1]$ of some real-valued differentiable functions can be visualized as follows [9]:

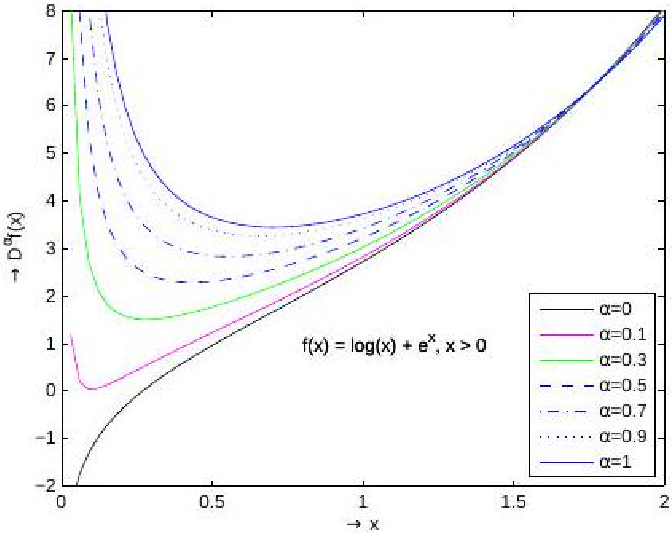


Figure 4.1: D^{α} operating on $f(x) = log(x) + e^{x}, x > 0$

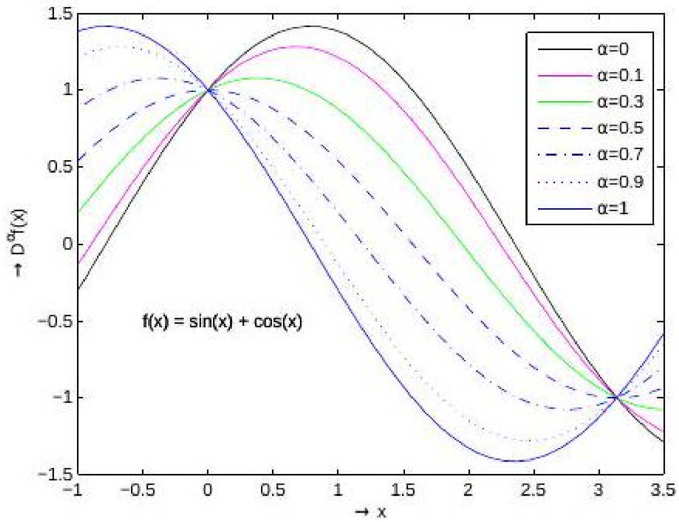


Figure 4.2: D^{α} operating on $f(x) = sin(x) + cos(x)$

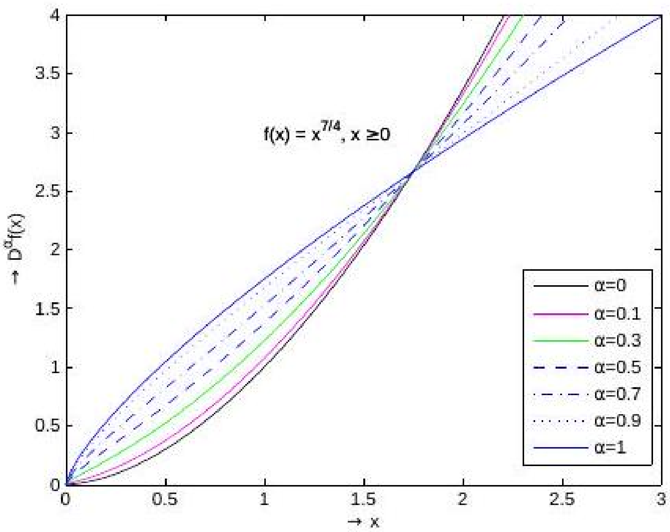


Figure 4.3: D^α operating on $f(x) = x^{7/2}, x > 0$

For $\alpha \in (0, 1]$, *the above graphs show that* $D^\alpha f(x) = f(x)$ *for* $\alpha = 0$ *and* $D^\alpha f(x) = f'(x)$ *for* $\alpha = 1$. *Also,* $D^\alpha f(x)$ *tends to* $f'(x)$ *uniformly as* α *tends from* 0 *to* 1.

4.1.4 The UD Integral Or Anti-UD Derivative Of Order α

In this section, we define the UD integral of order $\alpha, \alpha \in (0, 1]$, *which is an inverse operator of the proposed UD derivative.*

Let g *be a UD differentiable function as needed and* $0 < \alpha \leq 1$. *Then ,*

$$\begin{aligned}
&D^\alpha g(x) = f(x), \ \textit{where } D^\alpha \equiv \frac{d^\alpha}{dx^\alpha} \\
\Rightarrow &(1-\alpha)g(x) + \alpha D(g(x)) = f(x); \ \textit{where } D \equiv \frac{d}{dx} \\
\Rightarrow &\frac{dg(x)}{dx} + \frac{(1-\alpha)}{\alpha} g(x) = \frac{1}{\alpha} f(x) \\
\Rightarrow &g(x) = \frac{1}{\alpha} e^{\frac{(\alpha-1)}{\alpha}x} \int f(x) \cdot e^{\frac{(1-\alpha)}{\alpha}x} dx + Ce^{\frac{(\alpha-1)}{\alpha}x}
\end{aligned}$$

where C *is constant.*

Also $g(x) = I^\alpha(f(x))$, *is called anti-UD derivative of* $f(x)$ *for* $\alpha \in (0, 1]$. *One may verify that, at* $\alpha = 1$, *this integral coincides with the classical integral.*

Definition 4.3. *Let* f *be a continuous function in* $[a, b]$. *The UD integral,* $I_a^\alpha f$, *is defined as follows:*

$$I_a^\alpha f(x) = \frac{1}{\alpha} \int_a^x e^{\frac{(1-\alpha)}{\alpha}(t-x)} f(t) dt, \qquad \textit{where } \alpha \in (0, 1]$$

Theorem 4.7. *If g be a continuous function in (a,b) then for $x \geq a$*

1. $D^{\alpha}\left(I_a^{\alpha} g(x)\right) = g(x)$
2. $I_a^{\alpha}\left(D^{\alpha} g\right) = g(x) - e^{\frac{1-\alpha}{\alpha}(a-x)} g(a)$

Proof. (1)

$$\begin{aligned} D^{\alpha}\left(I_a^{\alpha} g(x)\right) &= (1-\alpha) I_a^{\alpha} g(x) + D\left(\int_a^x e^{\frac{(1-\alpha)(t-x)}{\alpha}} g(t)dt\right) \\ &= (1-\alpha) I_a^{\alpha} g(x) + \left[\frac{(1-\alpha)}{\alpha}\int_a^x e^{\frac{(1-\alpha)(t-x)}{\alpha}} g(t)dt + e^{\frac{(\alpha-1)x}{\alpha}} e^{\frac{(1-\alpha)x}{\alpha}} g(x)\right] \\ &= (1-\alpha) I_a^{\alpha} g(x) + (\alpha-1) I_a^{\alpha} g(x) + g(x) = g(x) \end{aligned}$$

(2)

$$\begin{aligned} I_a^{\alpha}\left(D^{\alpha} g(x)\right) &= (1-\alpha) I_a^{\alpha} g(x) + \int_a^x e^{\frac{(1-\alpha)(t-x)}{\alpha}} g'(t)dt \\ &= (1-\alpha) I_a^{\alpha} g(x) + e^{\frac{(\alpha-1)}{\alpha}x}\left(e^{\frac{(1-\alpha)t}{\alpha}}\right)_a^x - (1-\alpha) I_a^{\alpha} g(x) \\ &= g(x) - e^{\frac{1-\alpha}{\alpha}(a-x)} g(a) \end{aligned}$$

□

Corollary 4.2. *If $g(a) = 0$ then $D^{\alpha}\left(I_a^{\alpha} g\right) = I_a^{\alpha}\left(D^{\alpha} g\right) = g$.*

Some properties of the UD-integral are presented in the following theorem.

Theorem 4.8. *If g and h are continuous functions then the operator I_a^{α} possesses the following properties:*

1. $I_a^{\alpha}(\lambda g + \mu h) = \lambda I_a^{\alpha} g + \mu I_a^{\alpha} h$
2. $I_a^{\alpha}\left(I_a^{\gamma} g\right) = I_a^{\gamma}\left(I_a^{\alpha} g\right), \alpha, \gamma \in (0,1]$
3. $I_a^{\alpha}\left(I_a^{\alpha} g\right) \neq I_a^{2\alpha}(g)$
4. *If* $g \leqslant h$ *then* $I_a^{\alpha}(g) \leqslant I_a^{\alpha}(h)$
5. $|I_a^{\alpha} g| \leq I_a^{\alpha}|g|$
6. *If* $g \geq 0$ *then* $I_a^{\alpha}(g) \geq 0$
7. *Neither* $D^{\alpha}(g) = D\left(I_a^{(1-\alpha)}(g)\right)$ *nor* $I\left(D^{\alpha}(g)\right) = I_a^{(1-\alpha)}(g)$

where $0 < \alpha, \gamma \leq 1, x > 0, D$ and I be the usual differentiation and integration respectively.

Proof. (1), (2), (5) and (6) are obvious.

(2)

$$\begin{aligned} I_a^{\alpha} I_a^{\gamma} g(x) &= \frac{1}{\alpha}\int_a^s e^{\frac{(1-\alpha)}{\alpha}(\tau-x)}\left(\frac{1}{\gamma}\int_a^{\tau} e^{\left(\frac{1-\gamma}{\gamma}\right)(\xi-\tau)} g(\xi)\right)d\tau \\ &= \frac{1}{\alpha\gamma} e^{\frac{(\alpha-1)}{\alpha}x}\int_a^x\int_a^{\tau} e^{\left(\frac{1-\alpha}{\alpha}\right)(\tau-x)} e^{\left(\frac{1-\gamma}{\gamma}\right)(\xi-\tau)} g(\xi)d\xi d\tau \end{aligned}$$

$$= \frac{1}{\alpha\gamma} e^{\frac{(\alpha-1)}{\alpha}x} \int_a^x \int_\xi^x e^{\left(\frac{1-\alpha}{\alpha}\right)(\tau-x)} e^{\left(\frac{1-\gamma}{\gamma}\right)(\xi-\tau)} g(\xi) d\tau d\xi$$

(changing order of integration)

$$= \frac{1}{(1-\alpha)\gamma - \alpha(1-\gamma)} \left(\int_a^x e^{\left(\frac{1-\gamma}{\gamma}\right)(\xi-x)} g(\xi) d\xi - \int_a^x e^{\left(\frac{1-\alpha}{\alpha}\right)(\xi-x)} g(\xi) \right).$$

$$= \frac{1}{\alpha-\gamma} \left(\alpha I_a^\alpha - \gamma I_a^\gamma\right) g(x) = I_a^\gamma I_a^\alpha g(x)$$

(3) For $\alpha = \frac{1}{2}, a = 0$ and $g(x) = x$

We have $I_0^{1/2}\left(I_0^{1/2} g(x)\right) = 4e^{-x}\left[(x-2) + (x+2)e^{-x}\right]$ and $I_0 g(x) = \frac{s^2}{2}$.

(4) For $\alpha = \frac{1}{2}, a = 0$ and $g(x) = x$

We have $I_0^{1/2} g(x) = 2\left[(x-1) + e^{-x}\right], D^{1/2} g(x) = \frac{1}{2}(x+1)\ D\left(I_0^{1/2} g(x)\right) = 2 - e^{-x}$ and $I_0\left(D^{1/2} g(x)\right) = \frac{1}{2}\left(\frac{x^2}{2} + x\right)$

We end up the section with a list of UD Integrals of some elementary functions in the following proposition. □

Proposition 4.2. *Anti UD derivative of some elementary functions:*

1. $I^\alpha(\lambda) = \frac{\lambda}{(1-\alpha)} + Ce^{\frac{(\alpha-1)}{\alpha}x}$
2. $I^\alpha\left((bx+c)^m\right) = \frac{1}{(1-\alpha)} \sum_{r=0}^m (-1)^m \frac{m!}{(m-r)!} (bx+c)^{m-r} \left(\frac{\alpha b}{(1-\alpha)}\right)^r + Ce^{\frac{(\alpha-1)}{\alpha}x}$
3. $I^\alpha\left(e^{bx+c}\right) = \frac{e^{bx+c}}{\alpha b + (1-\alpha)} + Ce^{\frac{(\alpha-1)}{\alpha}x}$
4. $I^\alpha(\sin(bx+c)) = \frac{(1-\alpha)\sin(bx+c) - \alpha b \cos(bx+c)}{\alpha^2 b^2 + (1-\alpha)^2} + Ce^{\frac{(\alpha-1)}{\alpha}x}$
5. $I^\alpha(\cos(bx+c)) = \frac{(1-\alpha)\cos(bx+c) + \alpha b \sin(bx+c)}{\alpha^2 b^2 + (1-\alpha)^2} + Ce^{\frac{(\alpha-1)}{\alpha}x}$
   where λ and C are constants

4.1.5 Some UD-fractional differential equations

In this section, it is assumed that $D^\alpha \equiv \frac{d^\alpha}{dx^\alpha}$, the functions involved in the considered fractional differential equations are all differentiable in $[0, \infty)$, range of the fractional order lies in $(0, 1]$, and the UD derivative is used as given by the equation 4.1. First, we introduce and solve the linear, Bernulli, and Riccati UD-fractional deferential equations.

Definition 4.4. *The general form of the first order linear UD-fractional differential equation is defined as:*

$$D^\alpha(y) + h(x)y = k(x) \tag{4.2}$$

When $\alpha \in (0,1]$*,If* $k(x) = 0$ *then*

$$D^{\alpha}(y) + h(x)y = 0 \tag{4.3}$$

Is called a homogeneous UD-fractional equation.
Equation (4.2) can be rewritten as

$$D^{\alpha}(y) + h(x)y = k(x)$$

$$(1-\alpha)y + \alpha y' + h(x)y = k(x)$$

$$y' + \frac{[1-\alpha+h(x)]}{\alpha}y = \frac{k(x)}{\alpha}. \tag{4.4}$$

So, the general solution of the first-order linear ordinary differential equations (4.4) is given as follows:

$$y = e^{-\int \frac{(1-\alpha)+h(x)}{\alpha}dx}\left[\int \frac{k(x)}{\alpha} e^{\int \frac{(1-\alpha)+h(x)}{\alpha}dx} dx + c\right]$$

Example 4.2. *Consider the fractional differential equation,*

$$D^{1/3}y(x) + 2y(x) = e^{-3x}.$$

This equation can be written as:

$$\frac{dy(x)}{dx} + 10y(x) = 3e^{-3x}$$

which has the general solution:

$$y(x) = e^{-8x}\left(\int 3xe^{-3x}e^{8x}dx + c\right),$$

Hence

$$y(x) = 3ce^{-8x} + 3e^{-3x}$$

where c is an arbitrary constant.

Definition 4.5. *The generalized UD Bernoulli fractional differential equations can be expressed as*

$$D^{\alpha}y + p(x)D^{\beta}y = f(x)y^{n\alpha}, \quad \alpha, \beta \in (0,1], n \in \mathbb{N}, \tag{4.5}$$

where $p(x), f(x)$ *are* α*-differentiable functions and* $y(x)$ *is an unknown function to be found.*

For $n = 0$ *or* $n = 1$*, the equation becomes linear, otherwise, it is nonlinear.*
Note that:

(i) when $\alpha \neq \beta$ in equation (4.5), we have

$$[\alpha + \beta p(x)]y' + [(1-\alpha) + +p(x)[(1-\beta)]y] = f(x)y^{n\alpha}$$

Multiply both sides by $\left(y^{-n\alpha}\right)$, to get

$$[\alpha + \beta p(x)]y^{-n\alpha}y' + [(1-\alpha) + (1-\beta)]p(x)y^{1-n\alpha} = f(x) \tag{4.6}$$

Now, let $u = y^{1-n\alpha}$,then, $u' = (1-n\alpha)y^{-n\alpha}y'$.
The equation (4.6) becomes

$$u' + \frac{(1-n\alpha)}{\alpha+\beta p(x)}[(1-\alpha) + (1-\beta)p(x)]u = \frac{(1-n\alpha)}{\alpha+\beta p(x)}f(x)$$

Where $\alpha + \beta p(x) \neq 0$.
which is linear first ODE that has the general solution

$$y = \left(e^{-\int \frac{(1-n\alpha)}{\alpha+\beta p(x)}[(1-\alpha)+(1-\beta)p(x)]dx}\left[\int e^{\int \frac{(1-n\alpha)}{\alpha+\beta p(x)}[(1-\alpha)+(1-\beta)p(x)]dx}\frac{(1-n\alpha)}{\alpha+\beta p(x)}f(x)dx + c\right]\right)^{\frac{1}{1-n\alpha}},$$

where $n\alpha \neq 1$.

Example 4.3. *Consider the generalized Bernoulli UD-fractional differential equation:*

$$D^{\frac{1}{2}}y + D^{\frac{1}{4}}y = y^2$$
$$y(0) = 0$$

Solution: According to the above formula, the general solution of the equation, considering $y(0) = 0$ to get $c = -\frac{4}{5}$ is given by

$$y(x) = \left(-\frac{4}{5}e^{\frac{5}{3}x} + \frac{4}{5}\right)^{-1}$$

(ii) As a special case, when $n\alpha \neq 1, \beta = 0$ in equation (4.5), we have

The UD fractional Bernoulli differential equation of order α can be represented by

$$D^{\alpha}y + p(x)y = f(x)y^{n\alpha} \quad , \alpha \in [0,1]$$

This equation can be transformed by UD property to a fractional differential equation as

$$(1-\alpha)y + \alpha y' + p(x)y = f(x)y^{n\alpha}$$

Simplify this equation, we get

$$\alpha y' + [(1-\alpha) + p(x)]y = f(x)y^{n\alpha}$$

Multiplying both sides of the equation by $y^{-n\alpha}$, to get the bernoulli equation

$$\alpha y' y^{-n\alpha} + [(1-\alpha) + p(x)]y^{1-n\alpha} = f(x)$$

Now, let $u = y^{1-n\alpha}$,then, $u' = (1-n\alpha)y^{-n\alpha}y'$, where $[\alpha + \beta p(x) \neq 0]$.
Thus,it is transformed to linear equation

$$u' + \frac{(1-n\alpha)}{\alpha}[(1-\alpha) + p(x)]u = \frac{(1-n\alpha)}{\alpha}f(x),$$

that has the following solution:

$$y = \left(e^{-\int \frac{(1-n\alpha)}{\alpha}[(1-\alpha)+p(x)]dx}\left[\int e^{\int \frac{(1-n\alpha)}{\alpha}[(1-\alpha)+p(x)]dx}\frac{(1-n\alpha)}{\alpha}f(x)dx + c\right]\right)^{\frac{1}{1-n\alpha}},$$

where $n\alpha \neq 1, n \neq 0$.

Example 4.4. *Consider the Bernoulli fractional differential equation*

$$y^{\left(\frac{1}{2}\right)} + \frac{y}{2} = -\frac{2}{5}\sin x y^{\frac{7}{2}}$$

y(0)=0

Solution:

Using the above formula with $y(0) = 0$ to get $c = -\frac{9}{13}$. Hence the general solution of the equation is given by:

$$y(x) = \left(-\frac{13}{9}e^{-\frac{13}{9}x} - \frac{9}{13}\right)^{-\frac{4}{3}}$$

.

Moreover, we provide various types of UD-fractional differential equations.

Example 4.5. *Consider the fractional differential equation,*

$$D^{\gamma}\left(D^{\alpha}\right)y(x) = 0. \tag{4.7}$$

Using the UD derivative property, the equation (4.7) can be written as,

$$\alpha\gamma y''(x) + (\alpha + \gamma - 2\alpha\gamma)y'(x) + (1-\alpha)(1-\gamma)y(x) = 0$$

This yields the solution,

$$y(x) = Ae^{\left(\frac{\alpha-1}{\alpha}\right)x} + Be^{\left(\frac{\gamma-1}{\gamma}\right)x}$$

where A, B are arbitrary constants.

It should be noted that the solution depends on parameter α only (when $x \neq 0$) if one uses a Conformable derivative.

Example 4.6. *The relaxed equation in fractional space is described by the equation*

$$\frac{d^\alpha y(x)}{dx^\alpha} + c^\alpha y(x) = 0, \tag{4.8}$$

where $c > 0, x > 0$ and $0 < \alpha < 1$ has the solution,

$$y(x) = E_\alpha \left(-c^\alpha x^\alpha\right)$$

in terms of Mittag-Leffler function.

And, using Conformable derivative, the solution of the fractional differential equation (4.8) is given as

$$y(t) = A \exp\left(-\frac{(ct)^\alpha}{\alpha}\right)$$

where A is an arbitrary constant.

Again, using the proposed UD- derivative, the solution of (4.8) is given as

$$y(x) = \mathrm{B} \exp\left(\frac{(\alpha - 1) - c^\alpha}{\alpha}\right) x$$

where B is an arbitrary constant.

Above all the obtained solutions coincide at $\alpha = 1$.

Example 4.7. *Using the UD derivative, the fractional initial value problem for the damped simple harmonic oscillator*

$$\frac{d^2 y(x)}{dx^2} + b\frac{d^\alpha y(x)}{dx^\alpha} + \omega_0^2 y(x) = f(x); 0 < \alpha < 1, x > 0, \text{ with}$$

$$y(0) = c_0, \frac{d^\alpha y(0)}{dx^\alpha} = c_1$$

may be written as

$$\frac{d^2 y(x)}{dx^2} + \lambda \frac{dy(x)}{dx} + \mu y(x) = f(x)$$

Which complementary function of the solution will be

$$e^{-\lambda x/2}\left(A \sin\sqrt{\mu - \frac{\lambda^2}{4}}x + B\cos\sqrt{\mu - \frac{\lambda^2}{4}}x\right)$$

where $\lambda = \alpha b$ and $\mu = b(1 - \alpha) + \omega_0^2$.Ans well, the particular integral and the values of arbitrary constants A, B can be found using initial conditions when $f(x)$ is known.

Example 4.8. *Consider*

$$2xD^{1/2}y + y = 6\sin x; D^{1/2} \equiv \frac{d^{1/2}}{dx^{1/2}}, s > 0$$

we may write

$$\frac{1}{2}xy + \frac{1}{2}\left(xy' + y\right) = 3\sin x$$

which is

$$D^{1/2}(xy) = 3\sin x$$

so that

$$xy = I^{1/2}(3\sin x) + c = 6e^{-x}\int e^{x}\sin x dx + Ce^{-x}$$

hence

$$xy = 3(\sin x - \cos x) + Ce^{-x}$$

Example 4.9. *Consider another problem*

$$3D^{1/3}y + y = 3xe^{x}; D^{1/3} \equiv \frac{d^{1/3}}{dx^{1/3}}, x > 0$$

multiplying by $\exp(x)$ *and using UD differential operator, we get*

$$\frac{2}{3}e^{x}y + \frac{1}{3}\left(y + y'\right)e^{x} = xe^{2x}$$

which can be written as,

$$D^{1/3}\left(e^{x}y\right) = xe^{2x}$$

operating $I^{1/3}$ *we get*

$$y = \frac{3}{4}e^{x}\left(x - \frac{1}{4}\right) + Ce^{-3x}$$

Ramark 4.3. *In the above two problems, if one uses one of the existing analytic definitions of fractional derivative, for example, the following Conformable derivative*

$$D^{\beta}f = s^{1-\beta}\frac{df}{ds}; \beta \in (0,1]$$

we obtain the following answers

$$ze^{-s^{-1/2}} = 3\int e^{-s^{-1/2}}s^{-3/2}\sin s + C$$

and

$$ze^{s^{1/3}} = \int s^{\frac{1}{3}}e^{s+s^{1/3}} + C$$

respectively. This shows the proposed derivative yields a better result.

4.2 Exponential Fractional Derivative

Kajouni et al.[14] have introduced a new fractional derivative based on exponential function, the main properties and results, and some applications to fractional differential equations are studied in this section.

4.3 Definition and basic properties

Definition 4.6. *Given a function $f : [0, \infty) \longrightarrow \mathbb{R}$, and then the exponential fractional derivative of f order α is defined by*

$$(D^{\alpha} f)(x) = \lim_{h \longrightarrow 0} \frac{f\left(x + he^{(1-\alpha)x}\right) - f(x)}{h}$$

for all $x > 0$, and $\alpha \in (0, 1)$. If f is α differentiable in some $(0, a), a > 0$, and $\lim_{x \longrightarrow 0^{+}} (D^{\alpha} f)(x)$ exists, then define

$$(D^{\alpha} f)(0) = \lim_{x \longrightarrow 0^{+}} (D^{\alpha} f)(x).$$

Now, we study the basic analytic results and main properties of this derivative.

Theorem 4.9. *If a function $f : [0, +\infty) \longrightarrow \mathbb{R}$ and α differentiable at $x_0 > 0$, then f is continuous at x_0.*

Proof. Since

$$f\left(x_0 + he^{(1-\alpha)x_0}\right) - f(x_0) = (f(x_0 + he^{(1-\alpha)x_0} - f(x_0))/h \times h$$

then

$$\lim_{h \longrightarrow 0} \left[f\left(x_0 + he^{(1-\alpha)x_0}\right) - f(x_0) \right] = \lim_{h \longrightarrow 0} \left(\left(f\left(t_0 + he^{(1-\alpha)x_0}\right) - f(x_0) \right) / h \right) \times \lim_{h \longrightarrow 0} h.$$

Then,

$$\lim_{\varepsilon \longrightarrow 0} \left[f(x_0 + \varepsilon) - f(x_0) \right] = f^{(\alpha)}(x_0) \times 0$$

which implies that

$$\lim_{\varepsilon \longrightarrow 0} f(x_0 + \varepsilon) = f(x_0).$$

Hence, f is continuous at x_0. □

Theorem 4.10. *Let $0 < \alpha \leq 1$ and f, g be α differentiable at a point $x > 0$. Then,*

1. *$D^{\alpha}(af + bg) = a(D^{\alpha} f) + b(D^{\alpha} g)$, for all $a, b \in\in \mathbb{R}$.*
2. *$D^{\alpha}(x^{p}) = pe^{(1-\alpha)x} x^{p-1}$ for all $p \in \mathbb{R}$.*
3. *$D^{\alpha}(\lambda) = 0$, for all constant functions $f(x) = \lambda$.*

4. $(D^{\alpha} fg) = f(D^{\alpha} g) + g(D^{\alpha} f)$.

5. $\left(D^{\alpha}(f/g)\right) = \left(f(D^{\alpha} g) + g(D^{\alpha} f)\right)/g^2$.

6. If in addition, f is differentiable, then $(D^{\alpha} f)(x) = e^{(1-\alpha)x} f'(x)$.

Proof. We prove only (6), and the others can be shown as consequences.

$$\begin{aligned}
(D^{\alpha} f)(x) &= \lim_{h \longrightarrow 0} \frac{f\left(x + he^{(1-\alpha)x}\right) - f(x)}{h} \\
&= \lim_{\varepsilon \longrightarrow 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon e^{(\alpha-1)} x} \\
&= e^{(1-\alpha)x} \lim_{\varepsilon \longrightarrow 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \\
&= e^{(1-\alpha)x} f'(x)
\end{aligned}$$

□

The following are the Exponential fractional derivative of some basic functions

Example 4.10. *Let $b, c, p \in \mathbb{R}$ and $0 < \alpha \leq 1$. Then :*

1. $D^{\alpha}(x^p) = \left(pe^{(1-\alpha)x}\right) x^{p-1}$ *for all* $p \in \mathbb{R}$.

2. $D^{\alpha}(1) = 0$.

3. $D^{\alpha}(e^{cx}) = ce^{(1-\alpha)x} e^{cx}$, *for all* $c \in \mathbb{R}$.

4. $D^{\alpha}(\sin(bx)) = be^{(1-\alpha)x} \cos(bx), b \in \mathbb{R}$.

5. $D^{\alpha}(\cos(bx)) = -be^{(1-\alpha)x} \sin(bx), b \in \mathbb{R}$.

Theorem 4.11. *the following exponential fractional derivatives of certain functions:*

1. $D^{\alpha}\left(\sin\left(1/1 - \alpha e^{(1-\alpha)x}\right)\right) = \cos\left(1/1 - \alpha e^{(1-\alpha)x}\right)$.

2. $D^{\alpha}\left(\cos\left(1/1 - \alpha e^{(1-\alpha)x}\right)\right) = -\sin\left(1/1 - \alpha e^{(1-\alpha)x}\right)\Bigg)$.

3. $D^{\alpha}\left(e^{1/1-\alpha e^{(1-\alpha)x}}\right) = e^{1/1-\alpha e^{(1-\alpha)x}}$.

4. $D^{\alpha}\left(1/1 - \alpha e^{(1-\alpha)x}\right) = 1$.

Definition 4.7. *Let $\alpha \in (n, n+1]$, for some $n \in \mathbb{N}$, and f function be an n differentiable at $x > 0$. Then the α- fractional derivative of f is defined by*

$$(D^{\alpha} f)(x) = \lim_{h \longrightarrow 0} \frac{f^{(n)}\left(x + he^{(n+1-\alpha)x}\right) - f^{(n)}(x)}{h}$$

if the limit exists.

Ramark 4.4. *As a direct consequence of Definition 4.7, we can show that*

$$(D^{\alpha} f)(x) = e^{n+1-\alpha} f^{(n+1)}(x),$$

where $\alpha \in (n, n+1]$ and f is $(n+1)$ differentiable at $x > 0$.

Our definition makes it possible to prove basic analysis theorems such as Rolle's theorem and the mean value theorem.

Theorem 4.12. *Rolle's theorem for exponential fractional differentiable functions*
Let $a > 0$ and $f : [a, b] \longrightarrow \mathbb{R}$ be a given function that satisfies

1. *f is continuous on $[a, b]$,*

2. *f is α-differentiable for some $\alpha \in (0, 1)$,*

3. *$f(a) = f(b)$.*
   Then, there exists $c \in]a, b[$, such that $f^{(\alpha)}(c) = 0$.

Proof. Since f is continuous on $[a, b]$, and $f(a) = f(b)$, there is $c \in]a, b[$ which is a point of local extrema. With no loss of generality, assume c is a point of local minimum. So

$$\begin{aligned} D^{\alpha} f(c) &= \lim_{\varepsilon \longrightarrow 0^+} \frac{f\left(c + \varepsilon c^{1-\alpha}\right) - f(c)}{\varepsilon} \\ &= \lim_{\varepsilon \to 0^-} \frac{f\left(c + \varepsilon c^{1-\alpha}\right) - f(c)}{\varepsilon}. \end{aligned}$$

But,the first limit is non-negative ,and the second limit is non positive.

Hence $f^{(\alpha)}(c) = 0$. □

Theorem 4.13. *Mean value theorem for exponential fractional differentiable functions*
Let $a > 0$ and $f : [a, b] \longrightarrow \mathbb{R}$ be a given function that satisfies

1. *f is continuous on $[a, b]$,*

2. *f is α-differentiable for some $\alpha \in]0, 1[$,*

Then, there exists $c \in]a, b[$, such that

$$f^{(\alpha)}(c) = \frac{f(b) - f(a)}{(1/1-\alpha)e^{(1-\alpha)b} - (1/1-\alpha)e^{(1-\alpha)a}}.$$

Proof. Consider the function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{(1/1-\alpha)e^{(1-\alpha)b} - (1/1-\alpha)e^{(1-\alpha)a}} \left(\frac{1}{1-\alpha} e^{(1-\alpha)x} - \frac{1}{1-\alpha} e^{(1-\alpha)a} \right)$$

$$g(a) = f(a) - f(a) - \frac{f(b) - f(a)}{(1/1-\alpha)e^{(1-\alpha)b} - (1/1-\alpha)e^{(1-\alpha)a}} \left(\frac{1}{1-\alpha} e^{(1-\alpha)a} - \frac{1}{1-\alpha} e^{(1-\alpha)a} \right) = 0$$

and

$$g(b) = f(b) - f(a) - \frac{f(b) - f(a)}{(1/1-\alpha)e^{(1-\alpha)b} - (1/1-\alpha)e^{(1-\alpha)a}} \left(\frac{1}{1-\alpha} e^{(1-\alpha)b} - \frac{1}{1-\alpha} e^{(1-\alpha)a} \right) = 0.$$

Thus $g(a) = g(b)$. By Rolle's theorem, there exists $c \in (a, b)$, such that $g^{(\alpha)}(c) = 0$. Using the fact that $D^{\alpha}\left(1/1 - \alpha e^{(1-\alpha)x}\right) = 1$, the result follows.

□

Along the same lines in basic analysis, one can use the present mean value theorem to prove the following proposition.

Proposition 4.3. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be α differentiable for some $\alpha \in (0, 1)$.*

1. *If $f^{(\alpha)}$ is bounded on $[a, b]$, where $a > 0$, then f is uniformly continuous on $[a, b]$, and hence, f is bounded.*

2. *If $f^{(\alpha)}$ is bounded on $[a, b]$ and continuous at a, then f is uniformly continuous on $[a, b]$, and hence, f is bounded.*

4.3.1 Exponential Fractional Integral

Definition 4.8. *Let $\alpha \in (0, 1)$ and $a \geq 0$, let f be a function defined on $(a, x]$, Then, the α fractional integral of f is defined by*

$$I_{\alpha}^{a}(f)(x) = \int_{a}^{x} e^{(1-\alpha)s} f(s) ds$$

Next, we can present the Inverse result theorem of this operator.

Theorem 4.14. *If $f : [a, \infty) \longrightarrow \mathbb{R}$ is any continuous function in the domain of I_{α} and $0 < \alpha \leq 1$. Then, for $x > a$, we have*

$$D_{a}^{\alpha} I_{\alpha}^{a} f(x) = f(x)$$

Proof. Since f is continuous, then $I_\alpha^a f(t)$ is clearly differentiable.
Hence,

$$\begin{aligned} D_a^\alpha I_\alpha^a f(x) &= e^{(1-\alpha)x}\frac{d}{\mathrm{d}x} I_\alpha^a(f)(x) \\ &= e^{(1-\alpha)x}\frac{d}{\mathrm{d}x}\int_a^x e^{(\alpha-1)s} f(s)\mathrm{d}s \\ &= e^{(1-\alpha)x}e^{(\alpha-1)x} f(x) \\ &= f(x) \end{aligned}$$

□

Also, we can show the other inverse result as follows:

Ramark 4.5. *Let $f : (a,b) \longrightarrow \mathbb{R}$ be the function differentiable and $0 < \alpha < 1$. Then, for $x > a$, we have*

$$I_\alpha^a D_a^\alpha f(x) = f(x) - f(a).$$

4.3.2 Exponential fractional differential equations

In this section, we solve some Exponential fractional equations by reducing it to well-known ordinary differential equations .

Example 4.11. *Solve the equation*

$$y^{(\alpha)} + y = 0$$

Solution:

the auxiliary equation for

$$y^{(\alpha)} + y = 0$$

is

$$(\alpha - 1)r + 1 = 0$$

so that the solution is given by $y(x) = e^{(1/1-\alpha)e^{(\alpha-1)x}}$.

Example 4.12. *Consider the following equation:*

$$y^{(1/2)} + e^{-x/2}y = xe^{-x}$$

By multiplying it by e^x, we obtain $e^x y^{(1/2)} + e^{x/2}y = x$ and take advantage of the product rule for this fractional derivative

$$(e^x y)^{(1/2)} = x.$$

By using the fractional integral, we have

$$e^x y = \int_0^x te^{t/2}\ \mathrm{d}t + C.$$

Therefore,

$$y(x) = (2x - 4)e^{-x/2} + ce^{-x}.$$

Example 4.13. *Find the solution y which verifies this equation.*

$$y^{(1/2)} = \frac{-y^2 - x}{ye^{x/2}}$$

Thus, the fractional differential becomes

$$e^{-x/2}y'(x) = \frac{-y^2 - x}{y}e^{-x/2},$$

which brings back to

$$y' + y = -\frac{x}{y}.$$

This is a differential equation of Bernoulli and can be solved to

$$y = (-x^2e^{-x/2} + ce^{-x/2})^{1/2}$$

4.4 Hyperbolic Fractional Derivative

Recently a new fractional derivative based on hyperbolic functions has been introduced by Iyad Alhribat, and Amer Abu hasheesh in [4], its main properties and the basic calculus of this fractional derivative with applications to certain fractional differential equations have been studied.

4.4.1 Definition and main properties

Definition 4.9. *Given a function* $f : [0, \infty) \to \mathbb{R}$*, and* $\alpha \in (0, 1]$*, the hyperbolic fractional derivative of order* α *is defined by*

$$(D^{\alpha} f)(x) = \lim_{h \to 0} \frac{f(x + h\cosh(1-\alpha)x) - f(x)}{h}, \text{ for all } x > 0, \alpha \in (0, 1].$$

We will, sometimes, write $f^{(\alpha)}$ *for* $(D^{\alpha} f)(x)$*, to denote the hyperbolic fractional derivatives of* f *of order* α*. In addition, if the Hyperbolic fractional derivative of* f *of order* α *exists, then we simply say* f *is* α*-differentiable.*

If f *is hyperbolic* α*-differentiable in tus he interval* $(0, a)$ *for* $a > 0$ *and* $\alpha \in (0, 1]$ *such that* $\lim_{x \to 0^+} (D^{\alpha} f)(x)$ *exists, then* $(D^{\alpha} f)(0) = \lim_{x \to 0^+} (D^{\alpha} f)(x)$*.*

In the case of the conformable fractional derivative proposed in [15],we have two important remarks that are considered as the main motivation for our definition:

Ramark 4.6. *A function could be* α *- differentiable at a point but not differentiable, for example, take* $f(x) = 2\sqrt{x}, T_{\frac{1}{2}}(f)(0) = \lim_{x \to 0^+} T_{\frac{1}{2}}(f)(x) = 1$ *where* $T_{\frac{1}{2}}(f)(x) = 1$ *for* $x > 0$*. But* $T_1(f)(0)$ *does not exist. While for our definition, hyperbolic* α*-differentiable implies differentiable which is an advantage of our derivative.*

Ramark 4.7. *If* f *and* $T_{\alpha} f(x)$ *are differentiable, we have*

$$\frac{d}{\mathrm{d}x} T_{\alpha} f(x) = (1-\alpha) x^{-\alpha} \frac{d}{\mathrm{d}x} f(x) + x^{1-\alpha} \frac{d^2}{\mathrm{d}x^2} f(x).$$

Therefore,this expression tends to infinity when x *is very small, but this brings regularities in several mathematical problems especially when seeks to bounded* $T_{\alpha} f(x)$*.*

Now, we present the main properties and results concern to our new fractional derivative.

Theorem 4.15. *If a function* $f : [0, \infty) \to \mathbb{R}$ *is hyperbolic* α*-differentiable at* $x_0 > 0$*, then* f *is continuous at* x_0*.*

Proof. Since $f\left(x_0 + h\cosh(1-\alpha)x_0\right) - f(x_0) = \frac{f\left(x_0 + h\cosh(1-\alpha)x_0\right) - f(x_0)}{h} \cdot h$, then

$$\lim_{h \to 0} f\left(x_0 + h\cosh(1-\alpha)x_0\right) - f(x_0) = \lim_{h \to 0} \frac{f\left(x_0 + h\cosh(1-\alpha)x_0\right) - f(x_0)}{h} \cdot \lim_{h \to 0} h.$$

Let $\varepsilon = h\cosh\big((1-\alpha)x_0\big)$, then

$$\lim_{h\to 0} f\left(x_0 + h\cosh(bounda)x_0\right) - f\left(x_0\right),$$

$$= \cosh\big((1-\alpha)x_0\big)\lim_{\varepsilon\to 0}\frac{f\left(x_0+\varepsilon\right)-f\left(x_0\right)}{\varepsilon}\cdot\lim_{h\to 0} h,$$

$$= \cosh\big((1-\alpha)x_0\big)\, f'\left(x_0\right)\cdot 0 = 0.$$

Which implies $\lim_{\varepsilon\to 0} f\left(x_0+\varepsilon\right) = f\left(x_0\right)$,hence f is continuous at x_0. □

Theorem 4.16. *Let f, g be hyperbolic α-differentiable at a point $x > 0$, then for $0 < \alpha \le 1$*

1. $D^\alpha(af+bg) = a\left(D^\alpha f\right) + b\left(D^\alpha g\right)$ *for all* $a, b \in \mathbb{R}$,
2. $D^\alpha\left(x^p\right) = p\cosh((1-\alpha)x)x^{p-1}$ *for all* $p \in \mathbb{R}$,
3. $D^\alpha(\lambda) = 0$ *for all constant function* $f(x) = \lambda$,
4. $D^\alpha(fg) = f\left(D^\alpha g\right) + g\left(D^\alpha f\right)$,
5. $D^\alpha\left(\frac{f}{g}\right) = \frac{g(D^\alpha f) - f(D^\alpha g)}{g^2}$,
6. $D^\alpha(f\circ g)(x) = f'(g(x))D^\alpha(g)(x)$,
7. *In addition, if f is differentiable, then* $\left(D^\alpha f\right)(x) = \cosh((1-\alpha)x)f'(x)$.

Proof. We need only to prove (7) and (4), since the other rules are direct consequences.

(7):

$$\left(D^\alpha f\right)(x) = \lim_{h\to 0}\frac{f(x+h\cosh(1-\alpha)x) - f(x)}{h}.$$

Let $\varepsilon = h\cosh((1-\alpha)x)$
Therefore

$$\begin{aligned}\left(D^\alpha f\right)(x) &= \cosh((1-\alpha)x)\lim_{\varepsilon\to 0}\frac{f(x+\varepsilon)-f(x)}{\varepsilon},\\ &= \cosh((1-\alpha)x)f'(x).\end{aligned}$$

(4):

$$\left(D^\alpha fg\right)(x) = \lim_{h\to 0}\frac{f(x+h\cosh(1-\alpha)x)g(x+h\cosh(1-\alpha)x) - f(x)g(x)}{h},$$

$$= \lim_{h\to 0}\frac{f(x+h\cosh(1-\alpha)x)g(x+h\cosh(1-\alpha)x) - f(x)g(x+h\cosh(1-\alpha)x) + f(x)g(x+h\cosh(1-\alpha)x) - f(x)g(x)}{h},$$

$$= \lim_{h\to 0}\frac{f(x+h\cosh(1-\alpha)x) - f(x)}{h}g(x+h\cosh(1-\alpha)x) + f(x)\lim_{h\to 0}\frac{g(x+h\cosh(1-\alpha)x) - g(x)}{h},$$

$$= (D^\alpha f)(x) \lim_{h\to 0} g(x + h\cosh(1-\alpha)x) + f(x)(D^\alpha g)(x).$$

Since g is continuous at x, then $\lim_{h\to 0} g(x + h\cosh(1-\alpha)x) = g(x)$,hence

$$D^\alpha(fg) = f(D^\alpha g) + g(D^\alpha f).$$

□

Example 4.14. *It is worth noting the following hyperbolic fractional derivatives of certain functions for $\alpha \in (0, 1]$.*

1. $D^\alpha(sinh(1-\alpha)x) = (1-\alpha)\cosh^2((1-\alpha)x)$,
2. $D^\alpha(tanh(1-\alpha)x) = (1-\alpha)\,\text{sech}((1-\alpha)x)$,
3. $D^\alpha(2\cosh(1-\alpha)x) = (1-\alpha)\sinh(2(1-\alpha)x)$,
4. $D^\alpha(2\tan^{-1}\left(e^{(1-\alpha)x}\right)) = 1-\alpha$.

We generalize the definition of hyperbolic fractional derivative for $\alpha \in (n, n+1], n \in \mathbb{N}$.

Let $\alpha \in (n, n+1]$, for some $n \in \mathbb{N}$, and f is n-differentiable at $x > 0$, then the hyperbolic $\alpha-$ fractional derivative is defined by:

$$(D^\alpha f)(x) = \lim_{h\to 0} \frac{f^{(n)}(x + h\cosh(1-\alpha)x) - f^{(n)}(x)}{h}$$

if the limit exists.

Ramark 4.8. *As a direct consequence of this definition, we can show that*

$$(D^\alpha f)(x) = \cosh(((n+1)-\alpha)x)f^{(n+1)},$$

where $\alpha \in (n, n+1]$ and f is $(n+1)$- differentiable at $x > 0$.

The previous definitions of fractional derivative Riemann–Liouville and Caputo do not enable us to study the analysis of $\alpha-$ differentiable functions. However, our definition makes it possible to prove basic analysis theorems such as Rolle's theorem and the mean value theorem.

Theorem 4.17. *Rolle's theorem for hyperbolic fractional differentiable functions.*

Let $a > 0$ and $f : [0, \infty) \to \mathbb{R}$ be a given function that satisfies

1. *f is continuous on $[a, b]$,*
2. *f is α-differentiable for some $\alpha \in (0, 1]$,*
3. *$f(a) = f(b)$.*

Then there exists $c \in (a, b)$, such that $f^{(\alpha)}(c) = 0$.

Proof. Suppose f is continuous on $[a, b]$, and $f(a) = f(b)$, then there is a local extreme point $c \in (a, b)$. Without loss of generality, assume c is a point of local minimum. So,

$$\begin{aligned}(D^\alpha f)(c) &= \lim_{h \to 0^+} \frac{f(c + h\cosh(1-\alpha)c) - f(c)}{h} \\ &= \lim_{h \to 0^-} \frac{f(c + h\cosh(1-\alpha)c) - f(c)}{h}.\end{aligned}$$

But, the first limit is nonnegative, and the second limit is nonpositive. Hence $(D^\alpha f)(c) = 0$. □

Theorem 4.18. *Mean value theorem for hyperbolic fractional differentiable functions.*

Let $a > 0$ and $f : [0, \infty) \to \mathbb{R}$, be a given function that satisfies

1. *f is continuous on $[a, b]$,*
2. *f is α-differentiable for some $\alpha \in (0, 1]$.*

Then there exists $c \in (a, b)$, such that

$$(D^\alpha f)(c) = \frac{f(b) - f(a)}{\frac{2}{1-\alpha}\left[\tan^{-1}\left(e^{(1-\alpha)b}\right) - \tan^{-1}\left(e^{(1-\alpha)a}\right)\right]}.$$

Proof. Consider the function: $g(x) = f(x) - f(a) -$

$$\frac{f(b) - f(a)}{\frac{2}{1-\alpha}\left[\tan^{-1}\left(e^{(1-\alpha)b}\right) - \tan^{-1}\left(e^{(1-\alpha)a}\right)\right]}\left[\frac{2}{1-\alpha}\tan^{-1}\left(e^{(1-\alpha)x}\right) - \frac{2}{1-\alpha}\tan^{-1}\left(e^{(1-\alpha)a}\right)\right].$$

Then,

$$g(a) = g(b) = 0,$$

hence by Rolle's theorem, there exists $c \in (a, b)$, such that

$$g^{(\alpha)}(c) = 0.$$

Using the

$$D^\alpha\left(\frac{2\tan^{-1}\left(e^{(1-\alpha)x}\right)}{1-\alpha}\right) = 1,$$

we get

$$(D^\alpha f)(c) = \frac{f(b) - f(a)}{\frac{2}{1-\alpha}\left[\tan^{-1}\left(e^{(1-\alpha)b}\right) - \tan^{-1}\left(e^{(1-\alpha)a}\right)\right]}.$$

□

To work on the calculus of the hyperbolic fractional derivative, we need to define a corresponding anti-derivative.

4.4.2 Hyperbolic Fractional Integral and applications

We introduce the hyperbolic α-fractional integral as follows:

Definition 4.10. *Let $\alpha \in (0,1]$ and $a \geq 0$, let f be a function defined on $(a,x]$, then the hyperbolic $\alpha-$ fractional integral of f is defined by:*

$$I^a_\alpha(f)(x) = \int_a^x \operatorname{sech}((1-\alpha)s)f(s)ds$$

Example 4.15. *Evaluate the following hyperbolic fractional integrals.*

1. $I^1_{\frac{1}{2}}\left(\cosh\left(\frac{1}{2}x\right)\right) = \int_1^x 1ds = x - 1.$
2. $I_{concerning}(1) = \int_0^x \operatorname{sech}\left(\frac{1}{2}s\right)ds = 4\tan^{-1}\left(e^{\frac{1}{2}x}\right) - \pi.$

Ramark 4.9. *Since $\operatorname{sech}((1-\alpha)s)$ is continuous and bounded, then if $f(s)$ is continuous and bounded on $(a,x]$, then $I^a_\alpha(f)(x) = \int_a^x \operatorname{sech}((1-\alpha)s)f(s)ds$ is convergent; which is an extra advantage to the hyperbolic fractional integral.*

The following result shows the inverse property of the hyperbolic fractional operator.

Theorem 4.19. *If $f : [0,\infty) \to \mathbb{R}$ is any continuous function in the domain of I_α and $0 < \alpha \leq 1$. then, for $x > a$, we have $D^\alpha_a(I^a_\alpha f(x)) = f(x)$.*

Proof. Since f is continuous on, then $I^a_\alpha f(x)$ is clearly differentiable. Hence,

$$\begin{aligned} D^\alpha_a(I^a_\alpha f(x)) &= \cosh((1-\alpha)x)\frac{d}{dx}\left(I^a_\alpha f(x)\right) \\ &= \cosh((1-\alpha)x)\frac{d}{dx}\int_a^x \operatorname{sech}((1-\alpha)s)f(s)ds, \\ &= \cosh((1-\alpha)x)\operatorname{sech}((1-\alpha)x)f(x), \\ &= f(x). \end{aligned}$$

□

As an application to solve certain well-known fractional differential equations with respect to our hyperbolic differential operator with $\alpha \in (0,1]$.

Definition 4.11. *The general form of the linear hyperbolic fractional differential equation of order α is given by:*

$$y^{(\alpha)} + p(x)y = f(x), \tag{4.9}$$

where $p(x)$ and $f(x)$ are α-differentiable functions.
Clearly equation ((4.9))is equivalent to

$$\cosh((1-\alpha)x)y' + p(x)y = f(x). \tag{4.10}$$

If we Divide equation ((4.10)) by $\cosh((1-\alpha)x)$*, we get*

$$y' + p(x)\operatorname{sech}((1-\alpha)x)y = f(x)\operatorname{sech}((1-\alpha)x). \tag{4.11}$$

Now, equation ((4.11)) is a first order linear ordinary differential equation that has the general solution

$$y = \frac{1}{\mu(x)}\left(I^{\alpha}(f(x)\mu(x)\right),$$

where $\mu(x)$ *is the integrating factor given by:*

$$\mu(x) = e^{I^{\alpha}(p(x))}.$$

Example 4.16. *Solve the hyperbolic fractional differential equation* $y^{(\alpha)} + (1-\alpha)\sinh((1-\alpha)x)y = 1.$

This equation is transformed to the linear equation

$$y' + (1-\alpha)\tanh((1-\alpha)x)y = \operatorname{sech}((1-\alpha)x).$$

We compute the integrating factor

$$\mu(x) = e^{\int (1-\alpha)\tanh((1-\alpha)x)dx} = e^{\ln(\cosh((1-\alpha)x)} = \cosh((1-\alpha)x).$$

Hence, the general solution is given by

$$\begin{aligned} y &= \operatorname{sech}((1-\alpha)x)\int 1dx = \operatorname{sech}((1-\alpha)x)(x+c), \\ &= x\operatorname{sech}((1-\alpha)x) + c\operatorname{sech}((1-\alpha)x). \end{aligned}$$

Definition 4.12. *The general form of the Bernoulli hyperbolic fractional differential equation of order* α *is given by:*

$$y^{(\alpha)} + p(x)y = f(x)y^n, n \neq 0, 1. \tag{4.12}$$

where $p(x)$ *and* $f(x)$ *are* α*-differentiable functions.*

To solve equation ((4.12)), we use the substitution $z = y^{1-n}$ *that reduce it to linear*

hyperbolic fractional differential equation

$$z^{(\alpha)} + (1-n)p(x)z = (1-n)f(x), \tag{4.13}$$

that has a general solution

$$y = [\frac{1}{\mu(x)}(I^{\alpha}((1-n)f(x)\mu(x))]^{\frac{1}{1-n}},$$

where the integrating factor

$$\mu(x) = e^{I^{\alpha}((1-n)p(x))}.$$

Example 4.17. *Solve the hyperbolic fractional differential equation*

$$y^{(\alpha)} + (1-\alpha)\operatorname{sech}((1-\alpha)x)y = (1-\alpha)\frac{\operatorname{sech}((1-\alpha)x)}{y}.$$

The substitution $z = y^2$ reduces the equation into linear hyperbolic fractional differential equation

$$z^{(\alpha)} + 2(1-\alpha)\operatorname{sech}((1-\alpha)x)z = 2(1-\alpha)\operatorname{sech}((1-\alpha)x).$$

We compute the integrating factor

$$\mu(x) = e^{I^{\alpha}(2(1-\alpha)\operatorname{sech}((1-\alpha)x)} = e^{\int 2(1-\alpha)\operatorname{sech}^2((1-\alpha)x)dx} = e^{2\tanh((1-\alpha)x)},$$

hence, the general solution is given by

$$y = \left(e^{-2\tanh((1-\alpha)x)}\left(I^{\alpha}\left(2(1-\alpha)e^{2\tanh((1-\alpha)x)}\operatorname{sech}((1-\alpha)x)\right)\right)^{\frac{1}{2}},$$

$$= \left(e^{-2\tanh((1-\alpha)x)}\left(\int\left(2(1-\alpha)e^{2\tanh((1-\alpha)x)}\operatorname{sech}^2((1-\alpha)x)\right)\right)^{\frac{1}{2}},$$

$$= \left(e^{-2\tanh((1-\alpha)x)}\left(e^{2\tanh((1-\alpha)x)} + c\right)\right)^{\frac{1}{2}} = \sqrt{1 + ce^{-2\tanh((1-\alpha)x)}}.$$

Definition 4.13. *The general form of the Riccati hyperbolic fractional differential equation of order α is given by:*

$$y^{(\alpha)} = h(x) + k(x)y + u(x)y^2, \tag{4.14}$$

where $h(x), k(x)$, and $u(x)$ are α-differentiable functions.

To solve equation ((4.14)).If a specific solution y_1 is known, then the general solution, which comes in the form of $y = y_1 + z$, where z is the general solution to the following Bernoulli hyperbolic fractional differential equation

$$z^{(\alpha)} + \left(-k(x) - 2u(x)y_1\right)z = u(x)z^2. \tag{4.15}$$

Example 4.18. *Find the general solution of the hyperbolic fractional differential equation*

$$y^{(\alpha)} = \frac{\left(-2x^4 + x^2y + y^2\right)\left(e^{(1-\alpha)x} + e^{(\alpha-1)x}\right)}{2x^3},$$

given that $y_1 = -x^2$ is a solution.

We can simplify the equation to get

$$y^{(\alpha)} = \frac{\left(-2x^4 + x^2y + y^2\right)}{x^3}\cosh((1-\alpha)x) = \cosh((1-\alpha)x)\left(-2x + x^{-1}y + x^{-3}y^2\right),$$

which is Riccati equation. To solve it, we solve first the corresponding Bernoulli equation ((4.15)). After doing all the simplifications, we get

$$z^{(\alpha)} + \left(x^{-1}\cosh((1-\alpha)x)\right) z = x^{-3}\cosh((1-\alpha)x)z^2,$$

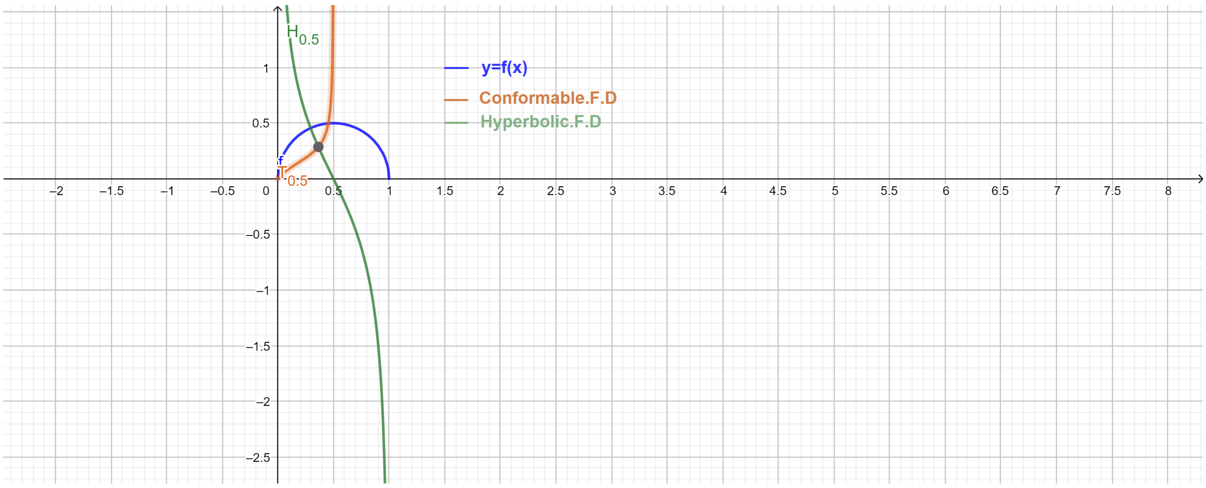
that has a general solution

$$z = \left(x(I^{\alpha}(-x^{-4}\cosh((1-\alpha)x)\right)^{-1} = \left(x(\frac{x^{-3}}{3} + c\right)^{-1} = \frac{3x^2}{1+3cx^3},$$

so the general solution for Riccati equation is

$$y = y_1 + z = -x^2 + \frac{3x^2}{1+3cx^3} = \frac{2x^2 - 3cx^5}{1+3cx^3}.$$

Figure(1)



Figure(2)

Finally,the above figures represent some graphical comparison between the conformable fractional derivative vs. the hyperbolic fractional derivative for two functions with different values of α.

Figure(1): shows the graphs of $f(x) = x^{\frac{3}{2}}$, *and its conformable fractional derivative vs. its hyperbolic fractional derivative with* $\alpha = 0.1$.

Figure(2): shows the graphs of $f(x) = \sqrt{x - x^2}$, *and its conformable fractional derivative vs. its hyperbolic fractional derivative with* $\alpha = 0.5$.

Bibliography

[1] Abdeljawad, Thabet. "On conformable fractional calculus." Journal of computational and Applied Mathematics 279 (2015): 57-66.

[2] Abdeljawad, Thabet, Qasem M. Al-Mdallal, and Fahd Jarad. "Fractional logistic models in the frame of fractional operators generated by conformable derivatives." Chaos, Solitons Fractals 119 (2019): 94-101.

[3] Al Horani, Mohammed, M. Abu Hammad, and Roshdi Khalil. "Variation of parameters for local fractional nonhomogenous linear differential equations." J. Math. Comput. Sci 16.2 (2016): 147-153.

[4] Alhribat, Iyad and Abu Hasheesh, Amer. "Hyperbolic fractional differential operator." Accepted in Palestine Journal of Mathematics(2022)

[5] Birgani, O. Taghipour, et al. "A note on some recent results of the conformable fractional derivative." Advances in the Theory of Nonlinear Analysis and its Application 3.1 (2018): 11-17.

[6] Dababneh, Amer, et al. "Conformable Fractional Bernoulli Differential Equation with Applications." 2019 IEEE Jordan International Joint Conference on Electrical Engineering and Information Technology (JEEIT). IEEE, 2019.

[7] Din, Anwarud, et al. "The mathematical study of climate change model under nonlocal fractional derivative." Partial Differential Equations in Applied Mathematics 5 (2022): 100204.

[8] Dixit, Ajay, Amit Ujlayan, and Priyanka Ahuja. "On the properties of The UD differential and integral operator." Journal—MESA-Nonlinear Studies 11 (2020): 291-300.

[9] Dixit, Ajay, and Amit Ujlayan. "The theory of UD derivative and its applications." (2021).

[10] Hammad, M. Abu, and R. Khalil. "Systems of linear fractional differential equations." Asian Journal of Mathematics and Computer Research 12.2 (2016): 120-126.

[11] Hilfer, Rudolf, ed. Applications of fractional calculus in physics. World Scientific, 2000.

[12] Ilie, M., J. Biazar, and Z. Ayati. "General solution of Bernoulli and Riccati fractional differential equations based on conformable fractional derivative." Int. J. Appl. Math. Res 6.2 (2017): 49-51.

[13] Ilie, Mousa, Jafar Biazar, and Zainab Ayati. "General solution of second order fractional differential equations." International Journal of Applied Mathematical Research 7.2 (2018): 56-61.

[14] Kajouni, Ahmed, et al. "A new conformable fractional derivative and applications." International Journal of Differential Equations 2021 (2021): 1-5.

[15] Khalil, Roshdi, et al. "A new definition of fractional derivative." Journal of Computational and applied mathematics 264 (2014): 65-70.

[16] Khalil, Roshdi, Mohammed Al Horani, and Douglas Anderson. "Undetermined coefficients for local fractional differential equations." J. Math. Comput. Sci 16 (2016): 140-146.

[17] Khalil, Roshdi, Mohammed Al Horani, and Douglas Anderson. "Undetermined coefficients for local fractional differential equations." J. Math. Comput. Sci 16 (2016): 140-146.

[18] Kilbas, Anatoliĭ Aleksandrovich, Hari M. Srivastava, and Juan J. Trujillo. Theory and applications of fractional differential equations. Vol. 204. elsevier, 2006.

[19] Miller, Kenneth S., and Bertram Ross. "An introduction to the fractional calculus and fractional differential equations." (1993).

[20] Magin, Richard. "Fractional calculus in bioengineering, part3." Critical Reviews™ in Biomedical Engineering 32.34 (2004).

[21] Mahmood, Salim S., et al. "Bernoulli and Riccati fractional differential equations can be solved analytically by using conformable derivatives." Turkish Journal of Computer and Mathematics Education (TURCOMAT) 12.13 (2021): 7158-7165.

[22] Martínez, F., et al. "Solving systems of conformable linear differential equations via the conformable exponential matrix." Ain Shams Engineering Journal 12.4 (2021): 4075-4080.

[23] Martínez, Francisco, Inmaculada Martínez-Vidal, and Silvestre Paredes. "Conformable Euler's theorem on homogeneous functions." Computational and Mathematical Methods 1.5 (2019): e1048.

[24] Martínez, Francisco, Inmaculada Martínez, and Silvestre Paredes. "A note on integrating factors of a conformable fractional differential equation." International Journal of Applied Physics and Mathematics 10.4 (2020): 143-152.

[25] Miller, Kenneth S., and Bertram Ross. "An introduction to the fractional calculus and fractional differential equations." (1993).

[26] Musraini, M., et al. "Classical properties on conformable fractional calculus." Pure and Applied Mathematics Journal 8.5 (2019): 83.

[27] Oldham, Keith, and Jerome Spanier. The fractional calculus theory and applications of differentiation and integration to arbitrary order. Elsevier, 1974.

[28] Podlubnv, I. "Fractional differential equations academic press." San Diego, Boston 6 (1999).

[29] Younas, Awais, et al. "Linear conformable differential system and its controllability." Advances in Difference Equations 2020 (2020): 1-26.

[30] Younus, Awais, et al. "Some fundamental results on fuzzy conformable differential calculus." Journal of Fractional Calculus and Nonlinear Systems 2.2 (2021): 31-61.