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The Geometry of Planar Polygons in the Euclidean Space

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Dedication

To my family, my country, and Gaza country for their inspiration , love, support, and encouragement.

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Abstract

This thesis delves closely into the interesting field of polygons in $3\mathbb{R}^3$-dimensional space. We set out on an expedition to determine the basic components of three-dimensional geometry, such as lines, planes, and their distances. Equipped with these tools, we examine the subtleties of polygons, delving into their particular types, computing their areas by the application of the Shoelace Formula, and examining the generation issue of orthonormal bases for planes.
A key component of this research is the mapping of planar polygons from $\mathbb{R}^3$ to $\mathbb{R}^2$ and vice versa, which acts as a bridge between the 2D and 3D realms. Next, we explore the interesting idea of polygon-polygon overlapping, providing a foundation for classifying various scenarios of overlap.
The thesis tackles point inclusion methods in closed planar polygons, extending beyond simple visualization. We carefully assess three different approaches: directed ray, global, and ray tracing, which provide powerful tools for locating a point in a polygon.
Finally, we round up our investigation with the intriguing topic of 2D and 3D planar polygon smoothing. We present area-conserving smoothing approaches utilizing edge and single node relaxation techniques to achieve results that are both mathematically sound and visually stunning.
This thesis provides a comprehensive investigation of polygons in $\mathbb{R}^3$, providing experts and individuals to obtain an improved understanding of their characteristics, relationships, and interaction in three dimensions.

Contents

Chapter 1

Introduction

A building's sharp angles and a leaf's smooth curves combine to create a symphony of shapes in our surrounding environment. Ever think to consider the formation of these apparently complex structures?
Polygons are the fundamental building blocks of many shapes we see every day, and this thesis explores their fascinating realm in 3D.
Imagine a universe made up of complex 3D structures rather than just basic squares and circles. This thesis unveils the behaviors' hidden mysteries relative to these polygons. The foundation will be laid by studying lines, planes, and their interactions as well as the language of 3D geometry. Computational geometry plays a vast role in industry as it has many applications in our lives. It is an important branch of mathematics that relates mathematics with other fields like computer science that are shown in some applications like geographic information systems (deal with searching for a geometric location of an object like roads, country boundaries, and any spatial data), computer-aided design (the use of computers to create, modify, analyze, and optimize any design such that the designer has enhanced productivity, quality of the design, ...etc).
All of the above applications are needed in our daily life as it more reliable to robots and robotics science which need the simulation and visualization aspects that laid their foundations on meshing and grids and polygons but this process has roughness resulting from jaggedness and number of points needed (and if the point in, on, or out the poly-

gons as used in GIS) which can be solved by point inclusion problem algorithms and area conservative smoothing as any simulated object should preserve its physical characteristics.(Wikipedia)
We summarize the chapters one by one in the rest of the introduction.
In **Chapter two**, the reader is introduced to some geometric and topological concepts necessary for computational geometry. We touches on vectors, polygon geometry as well as its attributes such as distance, intersection, and area calculations. The chapter also has an examination of the idea of a basis and an orthonormal basis and how any plane could generate one. To finish it off, the center of mass (centroid problem) both in 2D and 3D cases is also explored in this chapter. All in all, this helps to establish a strong framework through which one can comprehend computations of geometric nature and mapping between 2D and 3D planes.
We investigate in **Chapter three** the topic of polygon-polygon overlap. we analyze the cases of planar polygons overlapping including overlapping of line-circle, circle-circle, line-triangle, triangle-triangle, line-polygon, and polygon-polygon. The chapter includes the results of this classification with every type has been thoroughly examined. In particular, attention is paid to polygon-polygon overlap results, where the chapter is provided by detailed insights into the interactions between polygons in both 2D and 3D spaces. Also, it includes an algorithm used for detecting the resulting shape of overlapping.
In Zi-qiang Li, Yan He (2012) , the authors proposed an algorithm to find the area of the resulting shape of overlapping irregular polygons that follows the **decomposition step**, which is the first step includes decomposing each irregular polygon into the minimum number of convex polygons. Then, **pairwise clipping step** that is conducted by the process of identification of every pair of overlapping convex polygons and make them into two sets. Finally, we use the **overlap area calculation step** that includes that for each identified overlapping convex polygon pair, the area of their overlap region using the formulas of area of polygon in $\mathbb{R}^2$ is calculated.
In **Chapter four** we addresses the point in polygon problem.This problem was first investigated in early computational geometry research.

One notable study by Sutherland et al. (1974) provides a comprehensive characterization of the ray casting algorithm while Galetzka and Glauner (2017) presented a simple and correct even-odd (ray casting) algorithm for complex polygons. In addition, Khamayseh and Kuprat (2008) proposed deterministic point inclusion methods like global point inclusion method algorithm which is based on find the most visible normal vector to the shot ray using either synthetic normal visible normal methods and determining the closest boundary point to the given query point.

In this chapter, some algorithms is presented that are used for solving this problem efficiently and robustly like the ray tracing point inclusion method, global point inclusion method, and the new one expressed by direct ray point inclusion method. In the end, the importance of developing computational methods for geometric queries is emphasized comprehensively.

The studies was continued to have the most robust and efficient algorithms as Schwinger et al. (2023) used vector geometric methods for efficient point-in-polygon calculations applicable to geospatial data. But we need algorithms for point in complex polygon problem with the needed characteristics and in 3D case and overlapped shapes.

Chapter five is built by taking up the topic of smooth closed curved planes in 2D and 3D space, and unveiling the idea of area-preserving smoothing, a technique adopted in computer graphics that smooths linear curves and surfaces piecewise yet maintains their full area intact. The rest of the chapter is based on pointing out the need to smooth surface meshes as a means to counteract unpleasant jagged or noisy artifacts which can affect physics-based simulations, leading to misinterpretation. In the conclusion, we present a 3D description of the region-preserving smoothing with illustrations on how curve smoothing happens in different planes. The paper encompass a comprehensive analyzing for how to smooth curves and surfaces in 3D without destroying the total area or any other quality of their shape.

Chapter 2

The Mathematical Properties of Polygons in $\mathbb{R}^3$

This chapter primarily paves the way for the following chapters. In the first section, we introduce the needed geometric and topological background for the next chapters. (Anton, Bivens, & Davis, (2012)), (Corral & Petrunin (2010)), (Faux & Pratt, (1985)), and (O'Rourke, (1994)).
The second section introducing the fundamental geometric objects including the vector and parametric equations of a line, and a plane and the ways that compute the unit normal vector for a plane in the three common cases. (Rudin, W.(1976)), (Corral & Petrunin (2010)), and (Weir, M. D., Hass, J., Thomas, G. B. (2016)).
In the third one, we discuss the concept of the distance between geometric objects, including the distance between a point and line, a point and a plane, and two lines. The fourth section delve into the intersection of the geometric objects encompassing the intersection between two lines, a line and a ray, and a line and a plane.(Abu-Munshar, (2013)).
We basically produce in the fifth section the background we seek to know of the geometry structure of the polygon defining its special types and the convexity property of the disk and triangle in $\mathbb{R}^3$. Then, we dig deeper to establish the foundations for finding the formula of the polygon's area in R^3 and $\mathbb{R}^2$, which is the shoelace formula, that including Green's and Stocks theorems to use them. (O'Rourke,(1994)),

(Weir, M. D., Hass, J., Thomas, G. B. (2016)), (K. Hormann and N. Sukumar (2015)), and (Strang, G. (2010), (S. L. Loney, 1900)).
The sixth section includes the algebra part of this thesis stated by presenting the suitable enough background of the orthonormal basis and expressing any vector using it. Also, we use this paved way to solve the issue of generating the orthonormal basis of any plane in all of it three cases of defining equation.
This laid the foundation to generate the map ψ that maps any polygon from $\mathbb{R}^3$ and $\mathbb{R}^2$ and vice versa. (S. Axler, (2015)), and (Strang, G. (2010)). The final section presenting the centroid (center of mass) of polygon concept and deriving its formula in $\mathbb{R}^2$ that concluded after producing the physical and mathematical background of the center of mass concept and how can we compute the centroid coordinates using the law of decomposition. In the conclusion of this section, we present the use of the mapping to find the centroid of any polygon in $\mathbb{R}^3$. (Marghitu, D. B., Dupac, M. (2012)).

2.1 Basic Topological and Geometrical Definitions

As we need a groundwork for the next chapters, so in this section we present some geometrical and topological definitions including the most main points which are the dot and cross product joined with their properties. Also we express the triple cross product property and the Lagrange's identity. The following definitions are grouped into two group , the first one is the topological definitions and the second one is the geometrical definitions.

- Topological Definitions

  Definition 2.1.1. We define the *open ball* of radius r with center $\mathbf{a}$ as $B_r(\mathbf{a}) = \{\mathbf{x} \in X \mid d(\mathbf{x}, \mathbf{a}) < r\}$, where (X, d) a metric space with the Euclidean metric $d(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in X$, where $X \subseteq \mathbb{R}^3$. And $\mathbf{a} = (a_1, a_2, a_3) \in X$ and $0 \leq r \in \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers
  If $r = \varepsilon > 0$ then the open ball $B_r(\mathbf{a})$ is called the $\epsilon-$neighborhood

of the point $\mathbf{a}$ and denoted by $N_\epsilon(\mathbf{a})$ that is expressed by $N_\epsilon(\mathbf{a}) = \{\mathbf{x} \in X, d(\mathbf{x}, \mathbf{a}) < \epsilon\}$.

Based on the open ball, we can define the terms open set and closed set.

Definition 2.1.2. We define a subset U of X as an *open set* if and only if for every point $\mathbf{u}$ in a set U, and for some $\varepsilon > 0$, there exists an ε-neighborhood $N_\varepsilon(\mathbf{u})$ of $\mathbf{u}$ such that $N_\varepsilon(\mathbf{u})$ is a subset of U. In other words, every point in U has a small enough ball around it that is still entirely contained in U. And its complement is called a *closed set*.

As we know, for any given set U, it has its interior points set, exterior points set, and boundary points set that we define in the next definitions.

Definition 2.1.3. The set of all points having an $\epsilon-$neighborhood contained in U for some $\epsilon > 0$ is called the *interior points set* of a set U. And denoted by U°. And if the $\epsilon-$neighbourhood is contained in the complement of U then the set is called the *exterior points set*, denoted by $\text{Ext}(U)$.

Definition 2.1.4. The *boundary* of a subset U of a metric space X, denoted by ∂U, is the set of all points $\mathbf{x} \in X$ such that every neighborhood of $\mathbf{x}$ contains both points in U and points in $X \setminus U$.

For any two sets A and B, then these sets either separated or connected which are defined in the following two definitions.

Definition 2.1.5. Two subsets A and B of a metric space X are said to be *separated* if both $A \cap B$ and $A \setminus B$ are empty.

Definition 2.1.6. A set $E \subseteq X$ is said to be *connected* if E is not a union of two nonempty separated sets.

In the rest of the topological definitions, we define the flowing terms:

1. Region

   Definition 2.1.7. A *region* R in a topological space X is a nonempty connected open set.

2. Connected Region

   Definition 2.1.8. We call a region R a *connected region* if and only if any two points in R can be joined by a finite number of line segments that lie entirely in R.

3. Surface

   Definition 2.1.9. A *surface* is the boundary of a three-dimensional figure.

4. Plane

   Definition 2.1.10. A flat surface that extends in all directions without bounds (infinitely) in two dimensions is called a *plane*. In higher dimensions, it is called a hyperplane.

5. Point

   Definition 2.1.11. A *point* in $\mathbb{R}^3$ is defined as the precise location in space.
   Any point $\mathbf{x} = (x_1, x_2, x_3)$ is represented by a position vector $\mathbf{x}$ where the origin point is its initial point.

6. Line

   Definition 2.1.12. A *line* is a one-dimensional figure that is infinitely long and without thickness. A line is called a line segment if and only if it has two distinct endpoints $\mathbf{a}$ and $\mathbf{b}$.

7. Collinear Points

   Definition 2.1.13. Three points are said to be *collinear* if they are contained in the same straight line.

8. Curve

   Definition 2.1.14. A *curve* is an object similar to a line but not required to be straight.

9. Simple Curve

   Definition 2.1.15. A *simple curve* is a curve that does not self-intersect.

10. Simple Closed Curve

    Definition 2.1.16. A *simple closed curve* is a simple curve where the initial point and the terminal point of the curve coincide.

- Geometrical Definitions

  Definition 2.1.17. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$. The *dot product* of $\mathbf{u}$ and $\mathbf{v}$ denoted by $\mathbf{u} \cdot \mathbf{v}$ is defined by the formula $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{3} u_i v_i$.

  Remark. 1. Any two vectors, $\mathbf{v}$ and $\mathbf{u}$, are said to be *perpendicular* if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

  2. We say that $\mathbf{v}$ and $\mathbf{u}$ are *parallel* if there exists a number c such that $\mathbf{v} = c\mathbf{u}$.
  3. The geometric definition's formula of the dot product of the vectors $\mathbf{u}$ and $\mathbf{v}$ is defined by the formula
     $\mathbf{u} \cdot \mathbf{v} = \| u \| \| v \| \cos\theta, \quad 0^\circ \leq \theta \leq 180^\circ$.
  4. The squared *length* of the vector $\mathbf{u}$ using the dot product is defined by the formula $\mathbf{u}.\mathbf{u} = \| \mathbf{u} \|^2$.
  5. If θ is the angle between two nonzero vectors $\mathbf{v}$ and $\mathbf{u}$, then

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} > 0 & \text{for } 0^\circ \leq \theta < 90^\circ \\ 0 & \text{for } \theta = 90^\circ \\ < 0 & \text{for } 90^\circ < \theta \leq 180^\circ \end{cases}$$

Definition 2.1.18. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$. The *cross product* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \times \mathbf{v}$, is a vector defined by the formula

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\hat{\imath} - (u_1v_3 - u_3v_1)\hat{\jmath} + (u_1v_2 - u_2v_1)\hat{k}$$

The magnitude of the cross-product can be computed as follows:

$$\| \mathbf{u} \times \mathbf{v} \| = \| \mathbf{u} \| \| \mathbf{v} \| \sin\theta, \quad 0 \leq \theta \leq 180$$

Remarks:

1. $\| \mathbf{u} \times \mathbf{v} \| =$ the area of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$.
2. The direction of $\mathbf{u} \times \mathbf{v}$ is the normal vector to both vectors, which is determined by the right-hand rule.
3. The cross product is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.
4. $\sin\theta$ can be interpreted as the following formula:

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{(\mathbf{u}.\mathbf{v})^2}{\| \mathbf{u} \|^2 \| \mathbf{v} \|^2}}$$

5. The Lagrange's identity.

$$\| \mathbf{u} \times \mathbf{v} \|^2 = (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v})^2 \tag{2.1}$$

6. The cross product between three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is called **Triple cross product property**, which is expressed as the following formulas:

(a)

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{w} \cdot \mathbf{u})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} = \mathbf{v}(\mathbf{w} \cdot \mathbf{u}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v}) \tag{2.2}$$

(b)

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{v}(\mathbf{w} \cdot \mathbf{u}) - \mathbf{u}(\mathbf{v} \cdot \mathbf{w}) \tag{2.3}$$

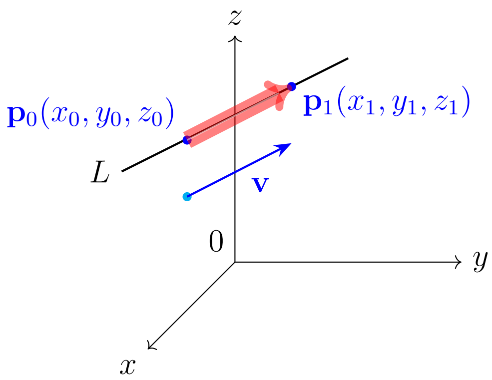


Figure 2.1: **Clarification of vector equation for a line.**

Definition 2.1.19. The vector projection of $\mathbf{u}$ onto $\mathbf{v}$ is defined by

$$\mathbf{proj_v u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\parallel \mathbf{v} \parallel^2}\right)\mathbf{v} \tag{2.4}$$

which is a vector in the direction of $\mathbf{v}$ and length of $\mathbf{u}$.

2.2 Fundamental Geometric Objects in Space

In this section, we started our journey to demonstrate the 3D realm by exploring point, line with its parametric and vector equations, and plane joined by its vector, parametric, and Cartesian equations.

2.2.1 The Straight Line Equation in $\mathbb{R}^3$

Line Vector Equation

In the Euclidean space $\mathbb{R}^3$, there are two formulas for the vector equation of the line $\mathbf{L}$ depending on what is given as follows:

1. Point and direction formula (if a point and a direction vector are given) as shown in figure 2.1.
   The vector equation of a line $\mathbf{L}$ passes through point $\mathbf{p_0} = (x_0, y_0, z_0)$and parallel to vector $\mathbf{v}$ is

$$\mathbf{L}(t) = \mathbf{p}_0 + t\mathbf{v}$$

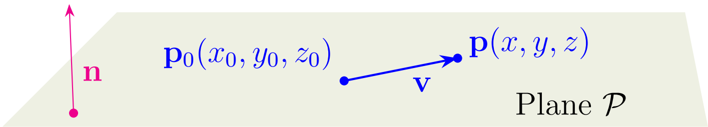


Figure 2.2: **Clarification of vector equation of a plane in the Euclidean space.**

The parameter t takes different values for each point: $-\infty < t < \infty$. And we call the vector $\mathbf{v}$ as a direction vector of $\mathbf{L}$.

2. Two points formula (If two points are given) The vector equation of a line that passes through two points $\mathbf{p}_0 = (x_0, y_0, z_0)$ and $\mathbf{p}_1 = (x_1, y_1, z_1)$ is defined by

$$\mathbf{L}(t) = \mathbf{p_0} + t(\mathbf{p_1} - \mathbf{p_0})$$

where $-\infty < t < \infty$.

Remarks:

1. A line segment is a line where $0 \leq t \leq 1$ and has two endpoints.
2. A line is called a **ray** if $0 \leq t < \infty$, i.e., it has one endpoint and extends infinitely from the other direction.

2.2.2 Plane Equation in $\mathbb{R}^3$

In the Euclidean space $\mathbb{R}^3$, the equation of a plane $\mathcal{P}$ can be defined by one of the following:

1. **Plane vector equation**
   A point $\mathbf{p}_0(x_0, y_0, z_0)$ in the plane $\mathcal{P}$ and a non-zero normal vector $\mathbf{n}$ to the plane, and this is called the normal form of its vector equation, which is expressed by:

$$(\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{n} = \mathbf{0}, \tag{2.5}$$

   where $\mathbf{p} = (x, y, z)$ is any point in $\mathcal{P}$ as shown in figure2.2. In addition, the normal vector can be calculated in different ways depending on what is given, as follows:

(a) $\mathbf{n} = (\mathbf{q} - \mathbf{p}) \times (\mathbf{r} - \mathbf{q})$, where $\mathbf{q}, \mathbf{p}, \mathbf{r}$ are any three points in the plane. Also, we can use any one of these points in the equation (2.5).

(b) $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_0$, where $\mathbf{v}_1, \mathbf{v}_0$ are any two intersected vectors in the plane. And the point can be found by the vectors' intersection point.

(c) $\mathbf{n} = \mathbf{L} \times \mathbf{v}$, where $\mathbf{L}$ is a line in the plane with endpoints $\mathbf{p}$ contained in the line and $\mathbf{q}$ in plane but not on line, and $\mathbf{v}$ is any vector in the plane.

2. **Plane parametric equation**
Another way to present a plane $\mathcal{P}$ using any three points in the plane $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ is by its parametric equation:

$$\mathbf{X}(s,t) = \mathbf{p}_0 + s\mathbf{u} + t\mathbf{v} = \mathbf{p}_0 + s\left(\mathbf{p}_1 - \mathbf{p}_0\right) + t\left(\mathbf{p}_2 - \mathbf{p}_0\right) \tag{2.6}$$

where $s, t \in \mathbb{R}, -\infty < s, t < \infty$

3. **The Cartesian equation of plane**
This equation is derived from the equation (2.5). From case 1, we have the point $\mathbf{p}_0 = (x_0, y_0, z_0)$, $\mathbf{p} = (x, y, z)$ and the unit normal $\hat{\mathbf{n}} = (a, b, c)$ of the plane $\mathcal{P}$, thus we have

$$\begin{aligned} (\mathbf{p} - \mathbf{p}_0) \cdot \hat{\mathbf{n}} &= 0 \\ (x - x_0, y - y_0, z - z_0) \cdot (a, b, c) &= 0 \\ ax - ax_0 + by - by_0 + cz - cz_0 &= 0 \\ ax + by + cz = ax_0 + by_0 + cz_0 &= d \end{aligned} \tag{2.7}$$

Hence, the Cartesian equation of the plane $\mathcal{P}$ is:

$$ax + by + cz = d \tag{2.8}$$

where $d = ax_0 + by_0 + cz_0$ is a constant.

2.3 Distance Between Geometric Objects

We investigate the issue of finding the minimum distance between a point and a line, two lines, and a point and plane using its parametric and vector equations. In addition, we depend on the following theorem to achieve our goal for finding the minimum distance.

Theorem 2.3.1. If a function $f(x) > 0$, then the critical points of $f(x)$ are the same as the critical points of $f^2(x)$.

Proof. Let $f(x)$ be a positive function. The critical points of $f^2(x)$ are where $(f^2(x))' = 0$ or $(f^2(x))'$ does not exist.
By using the chain rule to differentiate $f^2(x)$ with respect to x, thus $(f^2(x))' = 2f(x)f'(x)$.
As $(f^2(x))' = 0$, then either $(f(x))' = 0$, or $f(x) = 0$. But $f(x)$ is positive, thus $(f(x))' = 0$. Also, $(f^2(x))'$ does not exist if and only if $(f(x))'$ does not exist. Hence, the critical points of $f(x)$ are the same as those of $f^2(x)$. ■

2.3.1 Point-Line Distance

Let $\mathbf{L}$ be a line segment defined by its vector equation $\mathbf{L}(t) = \mathbf{u} + t\mathbf{v}$, and $\mathbf{q}$ be the projection of $\mathbf{p}$ on the line $\mathbf{L}$. As $\mathbf{q}$ is on $\mathbf{L}$, then $\mathbf{q} = \mathbf{u}+t\mathbf{v}$ for some t.

Theorem 2.3.2. The minimum distance between $\mathbf{p}$ and $\mathbf{L}$ is $\|\mathbf{p} - \mathbf{q}\|$.

Proof. By using Theorem 2.3.1, we can minimize $\| (\mathbf{p} - \mathbf{q}) \|$ by minimizing
$\| (\mathbf{p} - \mathbf{q}) \|^2$.
Let $g(t) =\| (\mathbf{p} - \mathbf{q}) \|^2$.
Thus

$$\begin{aligned} g(t) &=\| (\mathbf{p} - \mathbf{q}) \|^2 \\ &=\| \mathbf{p} - (\mathbf{u} + t\mathbf{v}) \|^2 \\ &= \Big(\mathbf{p} - (\mathbf{u} + t\mathbf{v})\Big) \cdot \Big(\mathbf{p} - (\mathbf{u} + t\mathbf{v})\Big) \end{aligned}$$

.

thus, differentiate $g(t)$ with respect to t and equate the derivative by zero to find the critical point. Thus, we obtain the following:

$$\begin{aligned}\frac{dg}{dt} &= 2\Bigg(\Big(\mathbf{p} - (\mathbf{u} + t\mathbf{v})\Big) \cdot \Big(\mathbf{p} - (\mathbf{u} + t\mathbf{v})\Big)\Bigg) \\ &= 2\Bigg(\Big(\mathbf{p} - (\mathbf{u} + t\mathbf{v})\Big) \cdot (\text{-}\mathbf{v})\Bigg) \\ &= 0\end{aligned}$$

.

By solving the equation, we obtain $t = \frac{(\mathbf{p}-\mathbf{u})\cdot\mathbf{v}}{\|\mathbf{v}\|^2}$ which is the critical point of $g(t)$. It follows that, $\frac{d^2g}{dt^2} = (\text{-}\mathbf{v}) \cdot (\text{-}\mathbf{v}) = \| \mathbf{v} \|^2 > 0$. Thus, by the second derivative test, $g(t)$ has a minimum value at $t = \frac{(\mathbf{p}-\mathbf{u})\cdot\mathbf{v}}{\|\mathbf{v}\|^2}$. Therefore, recognizing that the perpendicular distance is the shortest distance, which is only formed between the point and its projection on the line, if $(\mathbf{p} - \mathbf{q}) \cdot \mathbf{v} = 0$ with this value of t, then $\|\mathbf{p} - \mathbf{q}\|$ is the shortest distance.
Substitute t in $(\mathbf{p} - \mathbf{q}) \cdot \mathbf{v}$ as follows:

$$(\mathbf{p} - \mathbf{q}) \cdot \mathbf{v} = \Bigg(\mathbf{p} - \mathbf{u} - \Big(\frac{((\mathbf{p} - \mathbf{u}) \cdot \mathbf{v})}{\|\mathbf{v}\|^2}\Big)\mathbf{v}\Bigg) \cdot \mathbf{v} \tag{2.9}$$

$$= \Bigg((\mathbf{p} - \mathbf{u}) \cdot \mathbf{v} - \Big(\frac{((\mathbf{p} - \mathbf{u}) \cdot \mathbf{v})}{\|\mathbf{v}\|^2}\Big)\|\mathbf{v}\|^2\Bigg) \tag{2.10}$$

$$= (\mathbf{p} - \mathbf{u}) \cdot \mathbf{v} - (\mathbf{p} - \mathbf{u}) \cdot \mathbf{v} \tag{2.11}$$

$$= 0 \tag{2.12}$$

Hence, with this value of t we guarantee that $\|\mathbf{p} - \mathbf{q}\|$ is the shortest distance. ■

Remark. There is another form of the shortest distance, as follows:

$$d = \frac{|(\mathbf{p} - \mathbf{u}) \cdot \mathbf{v}|}{\|\mathbf{v}\|} \tag{2.13}$$

2.3.2 Line-Line Distance

Let $\mathbf{L}_1$, $\mathbf{L}_2$ are a line segments defined by their vector equations $\mathbf{L}_1(t) = \mathbf{u}_1 + t\mathbf{v}_1$, $\mathbf{L}_2(s) = \mathbf{u}_2 + s\mathbf{v}_2$ respectively. And $\mathbf{q}$ be the projection of $\mathbf{p} \in \mathbf{L}_1$ on the line $\mathbf{L}_2$. As $\mathbf{q}$ is on $\mathbf{L}_2$, then $\mathbf{q} = \mathbf{u}_2 + s\mathbf{v}_2$ for some t.

Theorem 2.3.3. The minimum distance between $\mathbf{p}$ and $\mathbf{L}$ is $\|\mathbf{p} - \mathbf{q}\|$.

Proof. By using Theorem 2.3.1, we can minimize $\parallel (\mathbf{p} - \mathbf{q}) \parallel$ by minimizing
$\parallel (\mathbf{p} - \mathbf{q}) \parallel^2$.
Let $g(t, s) =\parallel (\mathbf{p} - \mathbf{q}) \parallel^2$. Thus,

$$
\begin{aligned}
g(t, s) &=\parallel (\mathbf{p} - \mathbf{q}) \parallel^2 \\
&=\parallel (\mathbf{u}_1 + t\mathbf{v}_1) - (\mathbf{u}_2 + s\mathbf{v}_2) \parallel^2 \\
&= \Big((\mathbf{u}_1 + t\mathbf{v}_1) - (\mathbf{u}_2 + s\mathbf{v}_2)\Big) \cdot \Big((\mathbf{u}_1 + t\mathbf{v}_1) - (\mathbf{u}_2 + s\mathbf{v}_2)\Big)
\end{aligned}
$$

. thus, by differentiating partially $g(s, t)$ with respect to s and t and equalling both $\frac{\partial g}{\partial t}$ and $\frac{\partial g}{\partial s}$ by zero to find the critical points, we conclude:

$$\frac{\partial g}{\partial t} = 2\Bigg(\Big(\mathbf{u}_1 + t\mathbf{v}_1) - (\mathbf{u}_2 + s\mathbf{v}_2)\Big) \cdot (\mathbf{v}_1)\Bigg) = 0$$

.

$$\frac{\partial g}{\partial s} = 2\Bigg(\Big(\mathbf{u}_1 + t\mathbf{v}_1) - (\mathbf{u}_2 + s\mathbf{v}_2)\Big) \cdot (\text{-}\mathbf{v}_2)\Bigg) = 0$$

.

Therefore, we deduce a system of two equations with two unknowns, s and t. By arranging this system in matrices, we obtain:

$$\begin{bmatrix} (\mathbf{v}_1 \cdot \mathbf{v}_1) & -(\mathbf{v}_2 \cdot \mathbf{v}_1) \\ -(\mathbf{v}_1 \cdot \mathbf{v}_2) & (\mathbf{v}_2 \cdot \mathbf{v}_2) \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} (\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{v}_1 \\ (\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{v}_2 \end{bmatrix} \tag{2.14}$$

By solving this system, we obtain:

$$t = \frac{(\mathbf{v}_2 \cdot \mathbf{v}_2)\Big((\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{v}_1\Big) - (\mathbf{v}_2 \cdot \mathbf{v}_1)\Big((\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{v}_2\Big)}{(\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_2) - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2}$$

$$s = \frac{(\mathbf{v}_1 \cdot \mathbf{v}_1)\Big((\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{v}_2\Big) - (\mathbf{v}_1 \cdot \mathbf{v}_2)\Big((\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{v}_1\Big)}{(\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_2) - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2}$$

which are the critical points of $g(s,t)$.
Thus,since $\frac{\partial^2 g}{\partial t^2} = (\mathbf{v}_1) \cdot (\mathbf{v}_1) = \| \mathbf{v}_1 \|^2 > 0$, and $\frac{\partial^2 g}{\partial s^2} = (\mathbf{v}_2) \cdot (\mathbf{v}_2) = \| \mathbf{v}_2 \|^2 > 0$. Thus by second derivative test $g(s,t)$ has a minimum value at

$$t = \frac{(\mathbf{v}_2 \cdot \mathbf{v}_2)\Big((\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{v}_1\Big) - (\mathbf{v}_2 \cdot \mathbf{v}_1)\Big((\mathbf{u}_2 - \mathbf{u}_1) \cdot \mathbf{v}_2\Big)}{(\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_2) - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2}$$

$$s = \frac{(\mathbf{v}_1 \cdot \mathbf{v}_1)\Big((\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{v}_2\Big) - (\mathbf{v}_1 \cdot \mathbf{v}_2)\Big((\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{v}_1\Big)}{(\mathbf{v}_1 \cdot \mathbf{v}_1)(\mathbf{v}_2 \cdot \mathbf{v}_2) - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2}$$

It is clear that this is guaranteed with the values of the parameters; hence, $\|\mathbf{p} - \mathbf{q}\|$ is the minimum distance. ■

2.3.3 Point-Plane Distance

Let $\mathcal{P}$ be a plane that contains a point $\mathbf{p}_0$ and is parallel to two independent vectors $\mathbf{u}$ and $\mathbf{v}$. Also, it is defined by its vector equation $\mathbf{X}(t,s) = \mathbf{p}_0 + t\mathbf{u} + s\mathbf{v}$. Then, the minimal distance between any point $\mathbf{p}$ and $\mathcal{P}$ is the length of the line segment $\mathbf{L}$ that joins $\mathbf{p}$ and its projection $\mathbf{q}$ on $\mathcal{P}$.

Theorem 2.3.4. The minimum distance between a point $\mathbf{p}$ and a plane $\mathcal{P}$ is $\|\mathbf{p}-\mathbf{q}\|$.

Proof. By using Theorem 2.3.1, we can minimize $\| \mathbf{p}-\mathbf{q} \|$ by minimizing $\| \mathbf{p}-\mathbf{q} \|^2$.

$$\begin{aligned} \textit{Let } \; g(s,t) &= \| \mathbf{p}-\mathbf{q} \|^2 = \| \mathbf{p}-(\mathbf{p}_0+t\mathbf{u}+s\mathbf{v}) \|^2 \\ &= \Big(\mathbf{p}-(\mathbf{p}_0+t\mathbf{u}+s\mathbf{v}\Big)\cdot\Big(\mathbf{p}-(\mathbf{p}_0+t\mathbf{u}+s\mathbf{v})\Big) \end{aligned}$$

Differentiating partially $g(s,t)$ with respect to s and t and equalling both $\frac{\partial g}{\partial t}$ and $\frac{\partial g}{\partial s}$ by zero to find the critical points, implies that

$$\frac{\partial g}{\partial t} = 2\Bigg(\Big(\mathbf{p}-(\mathbf{p}_0+t\mathbf{u}+s\mathbf{v})\Big)\cdot(\text{-}\mathbf{u})\Bigg) = 0$$

.

$$\frac{\partial g}{\partial s} = 2\Bigg(\Big(\mathbf{p}-(\mathbf{p}_0+t\mathbf{u}+s\mathbf{v})\Big)\cdot(\text{-}\mathbf{v})\Bigg) = 0$$

.

Therefore , we have a system of two equations with two unknowns, s and t. By representing this system in matrices, we infer the following:

$$\begin{bmatrix} \mathbf{u}\cdot\mathbf{u} & \mathbf{v}\cdot\mathbf{u} \\ \mathbf{u}\cdot\mathbf{v} & \mathbf{v}\cdot\mathbf{v} \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} (\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{u} \\ (\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{v} \end{bmatrix} \tag{2.15}$$

By solving this system, we obtain:

$$t = \frac{(\mathbf{v}\cdot\mathbf{v})\Big((\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{u}\Big) - (\mathbf{u}\cdot\mathbf{v})\Big((\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{v}\Big)}{(\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{v})^{\mathbf{2}}} \tag{2.16}$$

$$s = \frac{-(\mathbf{v}\cdot\mathbf{u})\Big((\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{u}\Big) + (\mathbf{u}\cdot\mathbf{u})\Big((\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{v}\Big)}{(\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{v})^{\mathbf{2}}} \tag{2.17}$$

which are the critical points of $g(s,t)$.
Thus, since $\frac{\partial^2 g}{\partial t^2} = (\textbf{-u})\cdot(\textbf{-u}) = \parallel \mathbf{u} \parallel^2 > 0$,
and $\frac{\partial^2 g}{\partial s^2} = (\textbf{-v})\cdot(\textbf{-v}) = \parallel \mathbf{v} \parallel^2 > 0$. Thus by second derivative test $g(s,t)$ has a minimum value at

$$t = \frac{(\mathbf{v}\cdot\mathbf{v})\Big((\mathbf{p}-\mathbf{p_0})\cdot\mathbf{u}\Big) - (\mathbf{u}\cdot\mathbf{v})\Big((\mathbf{p}-\mathbf{p_0})\cdot\mathbf{v}\Big)}{(\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{v})^2} \tag{2.18}$$

$$s = \frac{-(\mathbf{v}\cdot\mathbf{u})\Big((\mathbf{p}-\mathbf{p_0})\cdot\mathbf{u}\Big) + (\mathbf{u}\cdot\mathbf{u})\Big((\mathbf{p}-\mathbf{p_0})\cdot\mathbf{v}\Big)}{(\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{v})^2} \tag{2.19}$$

Therefore, recognizing that the perpendicular distance is the minimum distance, which is only formed between the point and its projection on the plane.
If $(\mathbf{p}-\mathbf{q})\cdot\mathbf{v} = 0$ and $(\mathbf{p}-\mathbf{q})\cdot\mathbf{u} = 0$ with these values of t and s, then $\|\mathbf{p}-\mathbf{q}\|$ is the shortest distance. It is clear that this is guaranteed with the values of the parameters; hence, $\|\mathbf{p}-\mathbf{q}\|$ is the shortest distance. ■

The following theorem provides the minimum distance between a point and a plane without finding the values of parameters t and s.

Theorem 2.3.5. Let $\mathcal{P}$ be a plane with a point $\mathbf{p}_0$ contained in it, and a normal vector $\mathbf{n}$ of $\mathcal{P}$. Let $\mathbf{p}$ be any point in space, then the distance d between $\mathbf{p}$ and the plane is the minimum distance, and it is expressed as follows:

$$d = \frac{|(\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{n}|}{\|\mathbf{n}\|} \tag{2.20}$$

Proof. As shown in figure2.3

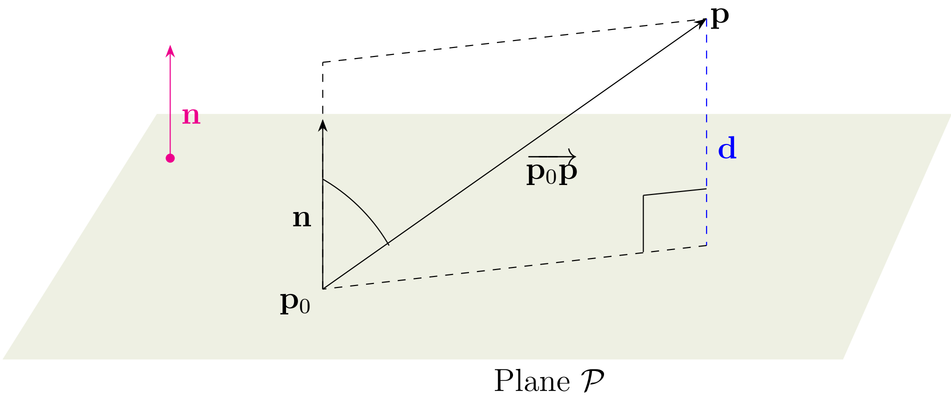


Figure 2.3: Distance between a plane and a point in space.

Let θ be the angle between $\mathbf{pp}_0$ and $\mathbf{n}$, then

$$\cos\theta = \frac{(\mathbf{pp}_0)\cdot\mathbf{n}}{\|\mathbf{pp}_0\|\ \|\mathbf{n}\|}$$
$$\frac{d}{\|\mathbf{pp}_0\|} = \frac{(\mathbf{pp}_0)\cdot\mathbf{n}}{\|\mathbf{pp}_0\|\ \|\mathbf{n}\|}$$
$$d = \frac{|(\mathbf{pp}_0)\cdot\mathbf{n}|}{\|\mathbf{n}\|} \quad \textbf{(taking the absolute value because of the distance)}$$

Since d is the magnitude of the projection of $\mathbf{pp}_0$ on $\mathbf{n}$, which expressed by the equation

$$d = \cos\theta\|\mathbf{pp}_0\| \tag{2.21}$$

Hence, the shortest distance is

$$d = \frac{|(\mathbf{pp}_0)\cdot\mathbf{n}|}{\|\mathbf{n}\|} \tag{2.22}$$

■

which is the magnitude of the perpendicular distance, and we can be sure that this is the minimum distance by Theorem 2.3.4, i.e., Theorem2.3.4 proves that the perpendicular distance is the minimum one.

2.4 Fundamental Geometric Objects Intersections

After finding the minimum distance in the previous section, in this one, we find the intersection between two lines, a line and a ray, and a line and a plane using its vector, parametric, and Cartesian equations, and find the intersection in every case of equations.

2.4.1 Line-Line Intersection

Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be two lines (line segments) in the 3-dimensional Euclidean space $\mathbb{R}^3$ defined by their vector equations $\mathbf{L}_1(t) = \mathbf{p} + t\mathbf{u}$, and $\mathbf{L}_2(s) = \mathbf{q} + s\mathbf{v}$ respectively.
To find the intersection, we need to find the values of the parameters t and s such that guarantee the intersection of the lines. We minimize the distance $\|\mathbf{L}_1 - \mathbf{L}_2\|$ by minimizing $\|\mathbf{L}_1 - \mathbf{L}_2\|^2$ based on using Theorem2.3.1 by partial derivative.

$$\begin{aligned} h(s,t) &= \Big\|\mathbf{L}_1 - \mathbf{L}_2\Big\|^2 \\ &= \Big(\mathbf{L}_1 - \mathbf{L}_2\Big)\cdot\Big(\mathbf{L}_1 - \mathbf{L}_2\Big) \end{aligned}$$

, then by using the chain rule, and as the intersection happens when $\Big\|\mathbf{L}_1 - \mathbf{L}_2\Big\|^2 = 0$, thus

$$\begin{aligned} 0 = \frac{\partial h}{\partial t} = 2\Big(\mathbf{p} + t\mathbf{u} - (\mathbf{q} + s\mathbf{v})\Big)\cdot\mathbf{u} \\ 0 = \frac{\partial h}{\partial s} = 2\Big(\mathbf{p} + t\mathbf{u} - (\mathbf{q} + s\mathbf{v})\Big)\cdot\mathbf{v} \end{aligned} \tag{2.23}$$

We got a system of two linear equations, as follows:

$$\begin{aligned} (\mathbf{q} - \mathbf{p})\cdot\mathbf{u} = t(\mathbf{u}\cdot\mathbf{u}) - s(\mathbf{v}\cdot\mathbf{u}) \\ (\mathbf{q} - \mathbf{p})\cdot\mathbf{v} = t(\mathbf{u}\cdot\mathbf{v}) - s(\mathbf{v}\cdot\mathbf{v}) \end{aligned} \tag{2.24}$$

Solving the system by matrices

$$\begin{bmatrix} \mathbf{u}\cdot\mathbf{u} & -\mathbf{v}\cdot\mathbf{u} \\ \mathbf{u}\cdot\mathbf{v} & -\mathbf{v}\cdot\mathbf{v} \end{bmatrix} \begin{bmatrix} t \\ s \end{bmatrix} = \begin{bmatrix} (\mathbf{q}-\mathbf{p})\cdot\mathbf{u} \\ (\mathbf{q}-\mathbf{p})\cdot\mathbf{v} \end{bmatrix}$$

Hence,

$$t = \frac{(\mathbf{u} \cdot \mathbf{v}) \cdot ((\mathbf{q} - \mathbf{p}) \cdot \mathbf{v}) - (\mathbf{v} \cdot \mathbf{v}) \cdot ((\mathbf{q} - \mathbf{p}) \cdot \mathbf{u})}{\|(\mathbf{u} \cdot \mathbf{v})\|^2 - \|\mathbf{u}\|^2 \|\mathbf{v}\|^2} \tag{2.25}$$

$$s = \frac{(\mathbf{u} \cdot \mathbf{u}) \cdot ((\mathbf{q} - \mathbf{p}) \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v}) \cdot ((\mathbf{q} - \mathbf{p}) \cdot \mathbf{u})}{\|(\mathbf{u} \cdot \mathbf{v})\|^2 - \|\mathbf{u}\|^2 \|\mathbf{v}\|^2} \tag{2.26}$$

2.4.2 Line-Ray Intersection

Let a ray $\mathbf{r}$, and a line segment $\mathbf{e}$ be defined by their vector equations as follows:

$$\begin{aligned} \mathbf{r}(s) &= \mathbf{q} + s\mathbf{v}, 0 \le \mathbf{s} < \infty \\ \mathbf{e}(t) &= \mathbf{p}_0 + t\mathbf{u}, 0 \le \mathbf{s} \le 1 \end{aligned}$$

respectively, where $\mathbf{u} = \mathbf{p}_1 - \mathbf{p}_0$. To find the intersection, we minimize the distance $\|\mathbf{r}(s) - \mathbf{e}(t)\|$ by minimizing $\|\mathbf{r}(s) - \mathbf{e}(t)\|^2$ based on using Theorem2.3.1 by partial derivative.

$$\begin{aligned} f(s,t) &= \Big\|\mathbf{r}(s) - \mathbf{e}(t)\Big\|^2 \\ &= \Big(\mathbf{r}(s) - \mathbf{e}(t)\Big) \cdot \Big(\mathbf{r}(s) - \mathbf{e}(t)\Big) \end{aligned}$$

Then, using the chain rule, and as the intersection happens when $\Big\|\mathbf{r}(t) - \mathbf{L}(s)\Big\|^2 = 0$, this implies

$$\begin{aligned} 0 &= \frac{\partial f}{\partial t} = 2\Big(\mathbf{q} + s\mathbf{v} - (\mathbf{p}_0 + t\mathbf{u})\Big) \cdot \mathbf{v} \\ 0 &= \frac{\partial f}{\partial s} = 2\Big(\mathbf{q} + s\mathbf{v} - (\mathbf{p}_0 + t\mathbf{u})\Big) \cdot \mathbf{u} \end{aligned} \tag{2.27}$$

We got a system of two linear equations, as follows:

$$\begin{aligned} (\mathbf{p}_0 - \mathbf{q}) \cdot \mathbf{v} &= s(\mathbf{v} \cdot \mathbf{v}) - t(\mathbf{u} \cdot \mathbf{v}) \\ (\mathbf{p}_0 - \mathbf{q}) \cdot \mathbf{u} &= s(\mathbf{v} \cdot \mathbf{u}) - t(\mathbf{u} \cdot \mathbf{u}) \end{aligned} \tag{2.28}$$

Solving the system by matrices

$$\begin{bmatrix} \mathbf{v}\cdot\mathbf{v} & -\mathbf{u}\cdot\mathbf{v} \\ \mathbf{u}\cdot\mathbf{v} & -\mathbf{u}\cdot\mathbf{u} \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} (\mathbf{p}_0-\mathbf{q})\cdot\mathbf{v} \\ (\mathbf{p}_0-\mathbf{q})\cdot\mathbf{u} \end{bmatrix}$$

we conclude

$$s = \frac{(\mathbf{u}\cdot\mathbf{v})\cdot((\mathbf{p}_0-\mathbf{q})\cdot\mathbf{u}) - (\mathbf{u}\cdot\mathbf{u})\cdot((\mathbf{p}_0-\mathbf{q})\cdot\mathbf{v})}{\|(\mathbf{u}\cdot\mathbf{v})\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{2.29}$$

$$t = \frac{(\mathbf{v}\cdot\mathbf{v})\cdot((\mathbf{p}_0-\mathbf{q})\cdot\mathbf{u}) - (\mathbf{u}\cdot\mathbf{v})\cdot((\mathbf{p}_0-\mathbf{q})\cdot\mathbf{v})}{\|(\mathbf{u}\cdot\mathbf{v})\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{2.30}$$

2.4.3 Line-Plane Intersection

Let L and $\mathcal{P}$ be any line and plane in $\mathbb{R}^3$, respectively. The intersection between them will result in three possible outcomes, as follows:

1. The whole line is contained in the plane.
2. They intersect at a single point.
3. They have no intersection; they are parallel.

We can prove this in two ways.

- **Algebraic Form.**
  Let L and $\mathcal{P}$ be a line and a plane in $\mathbb{R}^3$ defined by their vector equations $\mathbf{v} = \mathbf{v}_0 + t\mathbf{u}$, $(\mathbf{p} - \mathbf{p}_0)\cdot\mathbf{n} = 0$ respectively, where $-\inf < t < \inf$ and $\mathbf{n}$ is the normal vector of the plane. Then, if they are intersected, at least they intersect at a single point, say it is $\mathbf{p}$. Therefore, $\mathbf{p} = \mathbf{v}_0 + t\mathbf{u}$.
  Substituting the equation of $\mathbf{p}$ in the plane vector equation, hence

$$(\mathbf{v}_0 + t\mathbf{u} - \mathbf{p}_0)\cdot\mathbf{n} = 0$$
$$\Longrightarrow t(\mathbf{u}\cdot\mathbf{n}) + (\mathbf{v}_0 - \mathbf{p}_0)\cdot\mathbf{n} = 0$$

Solving for t we obtain

$$t = \frac{(\mathbf{v}_0 - \mathbf{p}_0) \cdot \mathbf{n}}{\mathbf{u} \cdot \mathbf{n}} \tag{2.31}$$

Based on the different possible values of t, we have the following results:

1. If $\mathbf{u} \cdot \mathbf{n} = 0$, then they are parallel (no intersection.).
2. If $(\mathbf{v}_0 - \mathbf{p}_0) \cdot \mathbf{n} = 0$, then the whole line is contained in the plane.
3. The intersection will result in a single point if $(\mathbf{v}_0 - \mathbf{p}_0) \cdot \mathbf{n} \neq 0$.

- **Parametric Form**
  Let a line L pass through a points $\mathbf{q}_0 = (x_0, y_0, z_0), \mathbf{q}_1 = (x_1, y_1, z_1)$ and $\mathcal{P}$ is a plane that contains three points $\mathbf{p}_0 = (x'_0, y'_0, z'_0), \mathbf{p}_1 = (x'_1, y'_1, z'_1)$,
  $\mathbf{p}_2 = (x'_2, y'_2, z'_2)$, which are defined by their parametric equations as follows:

$$\begin{aligned} \mathbf{L}(t) &= \mathbf{q}_0 + k(\mathbf{q}_1 - \mathbf{q}_0) && \textbf{(The line)} \\ \mathbf{X}(s,t) &= \mathbf{p}_0 + s(\mathbf{p}_1 - \mathbf{p}_0) + t(\mathbf{p}_2 - \mathbf{p}_0) && \textbf{(The plane)} \end{aligned}$$

where $k, t, s \in \mathbb{R}$. Thus,

$$\begin{aligned} \mathbf{q}_0 + k(\mathbf{q}_1 - \mathbf{q}_0) &= \mathbf{p}_0 + s(\mathbf{p}_1 - \mathbf{p}_0) + t(\mathbf{p}_2 - \mathbf{p}_0) \\ \mathbf{q}_0 - \mathbf{p}_0 &= -k(\mathbf{q}_1 - \mathbf{q}_0) + s(\mathbf{p}_1 - \mathbf{p}_0) + t(\mathbf{p}_2 - \mathbf{p}_0) \\ \mathbf{q}_0 - \mathbf{p}_0 &= k(\mathbf{q}_0 - \mathbf{q}_1) + s(\mathbf{p}_1 - \mathbf{p}_0) + t(\mathbf{p}_2 - \mathbf{p}_0) \end{aligned}$$

By representing this system in matrix language, we deduce

$$\begin{bmatrix}\mathbf{q}_0-\mathbf{q}_1 & \mathbf{p}_1-\mathbf{p}_0 & \mathbf{p}_2-\mathbf{p}_0\end{bmatrix}\begin{bmatrix}k\\ s\\ t\end{bmatrix}=\begin{bmatrix}\mathbf{q}_0-\mathbf{p}_0\end{bmatrix}$$

$$\begin{bmatrix}k\\ s\\ t\end{bmatrix}=\begin{bmatrix}\mathbf{q}_0-\mathbf{q}_1 & \mathbf{p}_1-\mathbf{p}_0 & \mathbf{p}_2-\mathbf{p}_0\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{q}_0-\mathbf{p}_0\end{bmatrix}$$

$$=\frac{1}{(\mathbf{q}_0-\mathbf{q}_1)\cdot\big((\mathbf{p}_1-\mathbf{p}_0)\times(\mathbf{p}_2-\mathbf{p}_0)\big)}\begin{bmatrix}((\mathbf{p}_1-\mathbf{p}_0)\times(\mathbf{p}_2-\mathbf{p}_0))\cdot(\mathbf{q}_0-\mathbf{p}_0)\\ ((\mathbf{p}_2-\mathbf{p}_0)\times(\mathbf{q}_0-\mathbf{q}_1))\cdot(\mathbf{q}_0-\mathbf{p}_0)\\ ((\mathbf{q}_0-\mathbf{q}_1)\times(\mathbf{p}_1-\mathbf{p}_0))\cdot(\mathbf{q}_0-\mathbf{p}_0)\end{bmatrix}$$

Hence, the values of k, s, t are given as follows:

$$\begin{aligned}k&=\frac{((\mathbf{p}_1-\mathbf{p}_0)\times(\mathbf{p}_2-\mathbf{p}_0))\cdot(\mathbf{q}_0-\mathbf{p}_0)}{(\mathbf{q}_0-\mathbf{q}_1)\cdot\big((\mathbf{p}_1-\mathbf{p}_0)\times(\mathbf{p}_2-\mathbf{p}_0)\big)}\\ s&=\frac{((\mathbf{p}_2-\mathbf{p}_0)\times(\mathbf{q}_0-\mathbf{q}_1))\cdot(\mathbf{q}_0-\mathbf{p}_0)}{(\mathbf{q}_0-\mathbf{q}_1)\cdot\big((\mathbf{p}_1-\mathbf{p}_0)\times(\mathbf{p}_2-\mathbf{p}_0)\big)}\\ t&=\frac{((\mathbf{q}_0-\mathbf{q}_1)\times(\mathbf{p}_1-\mathbf{p}_0))\cdot(\mathbf{q}_0-\mathbf{p}_0)}{(\mathbf{q}_0-\mathbf{q}_1)\cdot\big((\mathbf{p}_1-\mathbf{p}_0)\times(\mathbf{p}_2-\mathbf{p}_0)\big)}\end{aligned}\tag{2.32}$$

Remarks:

1. If $(\mathbf{q}_0-\mathbf{q}_1)\cdot\big((\mathbf{p}_1-\mathbf{p}_0)\times(\mathbf{p}_2-\mathbf{p}_0)\big)=0$, then they are parallel (no intersection).
2. If $((\mathbf{p}_1-\mathbf{p}_0)\times(\mathbf{p}_2-\mathbf{p}_0))\cdot(\mathbf{q}_0-\mathbf{p}_0)=0$ then the whole line is contained in the plane.
3. The intersection will result in a single one point if $((\mathbf{p}_1-\mathbf{p}_0)\times(\mathbf{p}_2-\mathbf{p}_0))\cdot(\mathbf{q}_0-\mathbf{p}_0)\neq 0$.

- **Cartesian Form**

  Let a line **L** and a plane $\mathcal{P}$ in $\mathbb{R}^3$ are defend by the equations respectively as follows:

$$\textbf{The line:}\ \begin{aligned} x &= x_1 + t(x_2 - x_1) \\ y &= y_1 + t(y_2 - y_1) \\ z &= z_1 + t(z_2 - z_1) \end{aligned} \tag{2.33}$$

  Where $(x_1, y_1, z_1), (x_2, y_2, z_2)$ are two points on the line and t is a parameter.

$$\textbf{The plane:}\ ax + by + cz = d \tag{2.34}$$

  Where a, b, c are the coordinates of the of the unit normal vector of the plane, and d is constants. Substitute x, y, z in the plane's equation, it follows that

$$\begin{aligned} a(x_1 + t(x_2 - x_1)) + b(y_1 + t(y_2 - y_1)) + c(z_1 + t(z_2 - z_1)) &= d \\ ax_1 + by_1 + cz_1 + t(a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1)) &= d \end{aligned}$$

  Hence

$$t = \frac{d - (ax_1 + by_1 + cz_1)}{(a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1))} \tag{2.35}$$

  Remarks:

  1. If the denominator $(a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1))$ is zero, then the line is parallel to the plane.
  2. If the denominator is nonzero and t is also nonzero, then the line intersects the plane at a single point. This point can be found by substituting the value of t into the equation of the line.
  3. If the denominator is nonzero and t is zero, then the line is already lying on the plane. Otherwise, there is no intersection.

2.5 Geometry Structure of Polygons

We begin, in this section, to delve into the geometry structure of the polygons in space, clarifying the polygon's special types, convexity property especially for circle and triangle. Also, focusing on the head-point, which represented by having a formula of the polygons' area in 2D and 3D based on using Green's Theorem to get the Shoelace formula in 2D, and using Stoke's Theorem to calculate the formula in 3D.

Definition 2.5.1. A finite collection of connected line segments forming a simple closed curve that forms the boundary of a region in a plane $\mathcal{P}$ is called a **polygon**. Any polygon consists of two fundamental geometric blocks, which are the following:

1. The edges, which are the line segments, say are $\{e_0, e_1, e_2, \cdots, e_n\}$.
2. The vertices say are $\{v_0, v_1, v_2, \cdots, v_n\}$ which are the intersection points of the adjacent edges.

2.5.1 Special Types of Polygons

Definition 2.5.2. A **simple (non-self-intersecting)** polygon is a polygon in which none of its edges intersect except at their endpoints. Otherwise, we call it **complex polygon**.

Definition 2.5.3. A polygon $\mathbb{P}$ is said to be **regular** if all its edges have equal length; otherwise, it is **irregular**.

Definition 2.5.4. A polygon $\mathbb{P}$ is said to be **planar** if it lies entirely in a single plane. i.e., all of its vertices lie in the same plane. Otherwise, it is a **non-planar** polygon.

Definition 2.5.5. A **convex** polygon $\mathbb{P}$ is a polygon such that any line segment connecting any two interior points of $\mathbb{P}$ lies entirely inside $\mathbb{P}$; otherwise, it is called **concave(non-convex)** polygon.

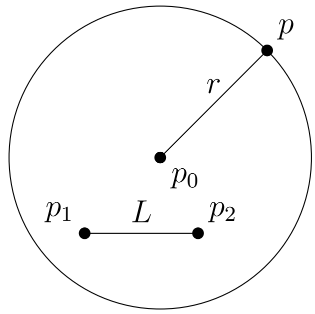


Figure 2.4: Clarifying the convexity property of the Disk in $\mathbb{R}^3$

Theorem 2.5.1. Every disk D in $\mathbb{R}^3$ with a radius r is a convex geometric object.

Proof. Let C be the circumference of a disk D radius r defined as $D = \{\mathbf{p} : |\mathbf{p} - \mathbf{p}_0| \leq r\}$ where $\mathbf{p} = (x, y, z)$ and $(\mathbf{p} - \mathbf{p}_0) \cdot \hat{\mathbf{n}} = 0$ where $\hat{\mathbf{n}}$ is the unit normal vector of the plane $\mathcal{P}$ that disk contained in.
Let $\mathbf{p}_1, \mathbf{p}_2, 0 \leq t \leq 1$ any two points belong to the disk connected by the line segment $\mathbf{L}$ defined as $\mathbf{L}(t) = (1-t)\mathbf{p}_1 + t\mathbf{p}_2$. Thus $|\mathbf{p}_1 - \mathbf{p}_0| \leq r$ and $|\mathbf{p}_2 - \mathbf{p}_0| \leq r$.
Hence,

$$\begin{aligned}
|\mathbf{L}(t) - \mathbf{p}_0| &= |(1-t)\mathbf{p}_1 + t\mathbf{p}_2 - \mathbf{p}_0| \\
&= |(1-t)\mathbf{p}_1 + t\mathbf{p}_2 - (1-t)\mathbf{p}_0 - t\mathbf{p}_0| \\
&\leq |(1-t)\mathbf{p}_1 - (1-t)\mathbf{p}_0| + |t\mathbf{p}_2 - t\mathbf{p}_0| \\
&\leq |(1-t)||\mathbf{p}_1 - \mathbf{p}_0| + |t||\mathbf{p}_2 - \mathbf{p}_0| \\
&\leq (1-t)r + tr \\
&= r
\end{aligned}$$

Therefore, every point of the line is an interior point of the disk (the line lies entirely in the disk). Whence, the disk is a convex object which is clarified in figure2.4. ∎

Theorem 2.5.2. Every triangle T in $\mathbb{R}^3$ is a convex geometric object.

Proof. Let T be a triangle defined by $T(s,t) = (\mathbf{p}_1 - \mathbf{p}_0)s + (\mathbf{p}_2 - \mathbf{p}_0)t + \mathbf{p}_0$, and $(\mathbf{x} - \mathbf{x}_0) \cdot \hat{\mathbf{n}} = 0$ where $0 \le s, t \le 1$ and $0 \le s + t \le 1$, $\mathbf{x}, \mathbf{x}_0$ are any two point of the triangle, and $\hat{\mathbf{n}}$ is the unit normal vector of the plane $\mathcal{P}$ the the triangle contained in.
Let the line $\mathbf{L} = (1-r)\mathbf{q}_0 + r\mathbf{q}_1$, where $0 \le r \le 1$, be a line that connects the points $\mathbf{q}_0$ and $\mathbf{q}_1$ that are interior points of the triangle T. We can show its convexity by showing that any point α contained in the line lies entirely in the polygon.
Since $\mathbf{q}_0$ and $\mathbf{q}_1$ are inside T, then $\mathbf{q}_0 = (\mathbf{p}_1 - \mathbf{p}_0)s_0 + (\mathbf{p}_2 - \mathbf{p}_0)t_0 + \mathbf{p}_0$, and
$\mathbf{q}_1 = (\mathbf{p}_1 - \mathbf{p}_0)s_1 + (\mathbf{p}_2 - \mathbf{p}_0)t_1 + \mathbf{p}_0$, where $0 \le s_0, t_0, s_1, t_1 \le 1$, $0 \le s_0 + t_0 \le 1$, and $0 \le s_1 + t_1 \le 1$.
As α lies in $\mathbf{L}$, then $\alpha = (1-r)\mathbf{q}_0 + r\mathbf{q}_1$, for some $0 \le r \le 1$. Therefore

$$\begin{aligned}
\alpha &= (1-r)\mathbf{q}_0 + r\mathbf{q}_1 \\
&= (1-r)\Big((\mathbf{p}_1 - \mathbf{p}_0)s_0 + (\mathbf{p}_2 - \mathbf{p}_0)t_0 + \mathbf{p}_0\Big) + r\Big((\mathbf{p}_1 - \mathbf{p}_0)s_1 + (\mathbf{p}_2 - \mathbf{p}_0)t_1 + \mathbf{p}_0\Big) \\
&= (\mathbf{p}_1 - \mathbf{p}_0)\Big((1-r)s_0 + rs_1\Big) + (\mathbf{p}_2 - \mathbf{p}_0)\Big((1-r)t_0 + rt_1\Big) + \mathbf{p}_0 \\
&= (\mathbf{p}_1 - \mathbf{p}_0)s^* + (\mathbf{p}_2 - \mathbf{p}_0)r^* + \mathbf{p}_0
\end{aligned} \tag{2.36}$$

where $s^* = (1-r)s_0 + rs_1$ **and** $t^* = (1-r)st_0 + rt_1$.
Thus, we need to show that $0 \le s^* \le 1$, $0 \le t^* \le 1$, and $0 \le s^* + t^* \le 1$ to show that α lies inside T.
Since $0 \le r \le 1$, then $0 \le 1 - r \le 1$, and since $0 \le s, t \le 1$, therefore $0 \le s^* \le 1$, and $0 \le t^* \le 1$.
To show that $0 \le s^* + t^* \le 1$.

$$\begin{aligned}
s^* + t^* &= (1-r)s_0 + rs_1 + (1-r)t_0 + rt_1 \\
&= (1-r)(s_0 + t_0) + r(s_1 + t_1) \\
&< 1 - r + r \\
&= 1
\end{aligned} \tag{2.37}$$

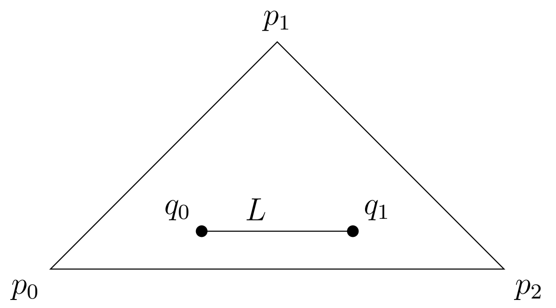


Figure 2.5: Triangle convexity property in $\mathbb{R}^3$

Since

$$0 \leq r \leq 1$$
$$-1 \leq r \leq 0$$
$$0 \leq 1 - r \leq 1$$

Hence $0 \leq s^* + t^* \leq 1$. Whence $T(s,t)$ is convex that is demonstrated in figure2.5. ■

2.5.2 Area of Polygons in $\mathbb{R}^3$

Definition 2.5.6 (Visibility)**.** Let **p**, and **q** be any two points on a polygon $\mathbb{P}$, then **q** is said to be visible to **p** (**p** can see **q**)if and only if the line segment $\overline{\mathbf{pq}}$ lies entirely in $\mathbb{P}$.

Definition 2.5.7. A diagonal of a polygon is a line segment connecting two nonadjacent vertices nowhere exterior to the polygon.

Definition 2.5.8. Three consecutive vertices **a,b,c** of a polygon form an ear of the polygon if **ac** is a diagonal.

Definition 2.5.9. The **curl** vector, denoted by ∇,is a vector defined by

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \tag{2.38}$$

The curl of any vector field $\mathbf{F} = M\hat{i} + N\hat{j} + P\hat{k}$ is defined by

$$curl\mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \nabla \times \mathbf{F} \tag{2.39}$$

Theorem 2.5.3 (**Green's Theorem**). Let **C** be a positively oriented piecewise smooth simple closed curve, and let R be the region boundary by **C**. If P and Q are functions of (x, y) having continuous first partial derivatives on D, where D is the domain, then,

$$\iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int\limits_C \mathbf{F}.dr \tag{2.40}$$

where $\mathbf{F}$=$P(x, y)\hat{i} + Q(x, y)\hat{j}$ and $dr = dx\hat{i} + dy\hat{j}$

Theorem 2.5.4 (**Stoke's Theorem**). Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth curve C. Let $\mathbf{F} = M\hat{i} + N\hat{j} + P\hat{k}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then the circulation of $\mathbf{F}$ around C in the direction counterclockwise with respect to the surface's unit normal vector $\hat{\mathbf{n}}$ equals the integral of $\nabla \times \mathbf{F}.\hat{\mathbf{n}}$ over S.

$$\iint\limits_S \nabla \times \mathbf{F}.\hat{\mathbf{n}} dA = \oint\limits_C \mathbf{F}.dr \tag{2.41}$$

Stoke's Theorem as shown in [1]figure2.6 is a higher-dimensional version of Green's Theorem. As Green's Theorem relates the integral over a region R to the line integral around the region's boundary curve, while Stoke's Theorem relates the integral over a surface S to the line integral around its boundary curve. And we can see the difference in

[1]This figure is taken from [8]

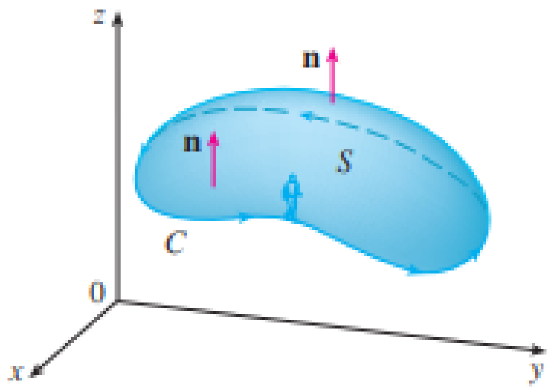


Figure 2.6: Clarification of Stoke's Theorem

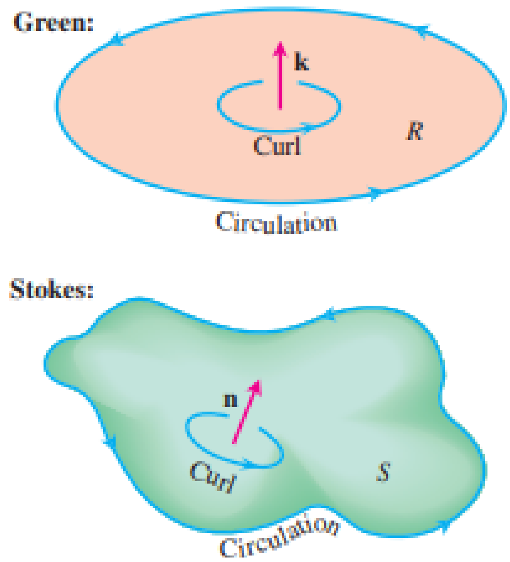


Figure 2.7: Green's comparing to Stoke's

[2]figure2.7. Under the condition that if the curve C in $xy-$plane. Then

$$\nabla \times \mathbf{F}.\hat{\mathbf{n}} = \nabla \times \mathbf{F}.\hat{\mathbf{k}} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \tag{2.42}$$

Thus, Stoke's Theorem will be as follows:

$$\oint_C \mathbf{F}.dr = \iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big) dxdy = \iint_R \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big) dA \tag{2.43}$$

which is the formula of Green's Theorem.
The area of polygon is computed by using the total sum of area of triangles that polygon partitioned into and this can be shown either by using cross product, or using Green's Theorem, and they will lead to same result.

[2]This figure is taken from [8]

Theorem 2.5.5. Let P be a polygon(concave or convex) with the vertices $\{\mathbf{p}_k\}_{k=1}^{n}$, $\mathbf{p}_{n+1} = \mathbf{p}_1$ in $\mathbb{R}^3$, and contained in a plane $\mathcal{P}$ with a unit normal vector $\hat{\mathbf{n}}$, then the area of P denoted by $A(P)$ is given by the absolute value of one half times the dot product of the normal vector $\hat{\mathbf{n}}$ and the sum over all edges of the polygon of the cross product between consecutive vertices:

$$A(P) = \frac{1}{2}|\hat{\mathbf{n}} \cdot (\sum_{k=1}^{n} \mathbf{p}_k \times \mathbf{p}_{k+1})| \tag{2.44}$$

.

Proof. Let P be a polygon with $n-$vertices, $\mathbf{p}_1 = (x_1, y_1, z_1)), \mathbf{p}_2 = (x_2, y_2, z_2), \mathbf{p}_3 = (x_3, y_3, z_3), \cdots, \mathbf{p}_n = (x_n, y_n, z_n)$, $\mathbf{p}_{n+1} = \mathbf{p}_0$ in $\mathbb{R}^3$, S the surface bounded the boundaries of P which is the curve C, edges of P, and $\mathbf{C}_k$ be the $\mathbf{k}^{th}$edge in P with endpoints $\mathbf{p}_k, \mathbf{p}_{k+1}$ that is defined as follows:

$$\begin{aligned} L(t) &= (1-t)\mathbf{p}_k + t\mathbf{p}_{k+1} \\ &= \big((1-t)x_k + tx_{x+1}, (1-t)y_k + ty_{x+1}, (1-t)z_k + tz_{x+1}\big), \quad 0 \leq t \leq 1 \end{aligned}$$

Let the vector field $\mathbf{F} = \frac{1}{2}\hat{\mathbf{n}} \times \mathbf{r}$, where $\hat{\mathbf{n}} = (n_1, n_2, n_3)$ is the unit normal to P, and

$$\begin{aligned} \mathbf{r} &= (x, y, z) \\ &= ((1-t)x_k + tx_{k+1}, (1-t)y_k + ty_{k+1}, (1-t)z_k + tz_{k+1}), 0 \leq t \leq 1 \end{aligned}$$

is any point on $\mathbf{C}_k$. Thus

$$\begin{aligned}
\hat{\mathbf{n}} \times \mathbf{r} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ n_1 & n_2 & n_3 \\ x & y & z \end{vmatrix} = (n_2 z - n_3 y)\,\hat{i} + (n_3 x - n_1 z)\,\hat{j} + (n_1 y - n_2 x)\,\hat{k} \\
\nabla \times \mathbf{F} &= \frac{1}{2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (n_2 z - n_3 y) & (n_3 x - n_1 z) & (n_1 y - n_2 x) \end{vmatrix} \\
&= n_1 \hat{i} + n_1 \hat{j} + n_1 \hat{k} \\
&= \hat{\mathbf{n}} \\
(\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} &= \hat{\mathbf{n}}.\hat{\mathbf{n}} \\
&= 1 \\
d\mathbf{r} &= \frac{dr}{dt} dt \\
&= \frac{d}{dt}(x, y, z) dt \\
&= (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) \\
&= \mathbf{p}_{k+1} - \mathbf{p}_k \\
&= \mathbf{C}_k
\end{aligned} \tag{2.45}$$

As $\iint_S 1 d\mathbf{A} = A$, thus by using Stoke's Theorem we possess the formula of the area by the line integral over the boundary curve.

Hence

$$\iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}} dA = \iint_S 1 dA = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \sum_{k=1}^{n} \int_{C_k} \frac{1}{2}(\hat{\mathbf{n}} \times \mathbf{r}) \cdot d\mathbf{r}$$

$$= \sum_{k=1}^{n} \int_{C_k} \frac{1}{2}(\hat{\mathbf{n}} \times \mathbf{r}) \cdot (dx, dy, dz)$$

$$= \frac{1}{2} \sum_{k=1}^{n} \Bigg[(x_{k+1} - x_k) n_2 \int_0^1 (1-t) z_k + t z_{k+1}) dt$$

$$- (x_{k+1} - x_k) n_3 \int_0^1 (1-t) y_k + t y_{k+1}$$

$$+ (y_{k+1} - y_k) n_3 \int_0^1 (1-t) x_k + t x_{k+1}) dt$$

$$- (y_{k+1} - y_k) n_1 \int_0^1 (1-t) z_k + t z_{k+1}$$

$$+ (z_{k+1} - z_k) n_1 \int_0^1 (1-t) y_k + t y_{k+1}) dt$$

$$- (z_{k+1} - z_k) n_2 \int_0^1 (1-t) x_k + t x_{k+1} \Bigg]$$

$$= \frac{1}{2} \Bigg[n_1 (y_k z_{k+1} - z_k y_{k+1})$$

$$+ n_2 (z_k x_{k+1} - x_k z_{k+1}) + n_3 (y_{k+1} x_k - y_k x_{k+1}) \Bigg]$$

$$= \frac{1}{2} \sum_{k=1}^{n} (n_1, n_2, n_3).(\mathbf{p}_k \times \mathbf{p}_{k+1})$$

$$= \frac{1}{2} \sum_{k=1}^{n} n \cdot \big(\mathbf{p}_k \times \mathbf{p}_{k+1}\big)$$

Whence, by applying the integration over every edge we obtain the following:

$$\begin{aligned} A(P) &= \frac{1}{2}\left|\sum_{k=1}^{n}\int_{C_k}\mathbf{F}.d\mathbf{r}\right| \\ &= \frac{1}{2}\left|\hat{\mathbf{n}}.\left(\sum_{k=1}^{n}\mathbf{p}_k\times\mathbf{p}_{k+1}\right)\right| \end{aligned} \tag{2.46}$$

■

2.5.3 The Shoelace Formula

The shoelace formula, also known as the Gauss area formula, was described by Albrecht Ludwig Friedrich Meister in 1976. It is a mathematical algorithm for computing a simple polygon's area in $xy-$plane in 2D($\mathbb{R}^2$) using cross-multiplying of the polygon's vertices coordinates.

It is called by this name since the coordinates constant cross-multiplying is like a threading shoelace. Also, this formula can be used for complex polygons (self-overlapping polygons).
Given a planar polygon with a sequence of oriented vertices $\mathbf{p}_k = (x_k, y_k),\ k = 1, 2, \cdots, n,\ \mathbf{p}_{n+1} = \mathbf{p}_1$ in the Cartesian plane, then the shoelace formula of the oriented (signed)area of the polygon denoted by $A_s(P)$ is computed as follows:

$$
\begin{aligned}
A_s(P) &= \frac{1}{2}\Big[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 + \cdots + x_{n-1}y_n - x_ny_{n-1} \\
&\quad + x_ny_1 - x_1y_n\Big] \\
&= \frac{1}{2}\left[\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & x_4 \\ y_3 & y_4 \end{vmatrix} + \cdots \begin{vmatrix} x_{n-1} & x_n \\ y_{n-1} & y_n \end{vmatrix} + \begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix}\right] \\
&= \frac{1}{2}\left[\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & \cdots x_{n-1} & x_n \\ y_1 & y_2 & y_3 & y_4 & \cdots y_{n-1} & y_n \end{vmatrix}\right] \quad \textbf{Horizontal Form} \\
&= \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ \vdots & \vdots \\ x_{n-1} & y_{n-1} \\ x_n & y_n \end{vmatrix} \quad \textbf{Vertical Form}
\end{aligned}
\tag{2.47}
$$

Hence, the area of the polygon is expressed as follows:

$$
A(P) = |A_s(P)| = \frac{1}{2}\left|\hat{\mathbf{n}}.\left(\sum_{k=1}^{n} \mathbf{p}_k \times \mathbf{p}_{k+1}\right)\right| \tag{2.48}
$$

where $\hat{\mathbf{n}} = \hat{\mathbf{k}}$.
Where in figure2.8, we clarify the mechanism of the shoelace formula.

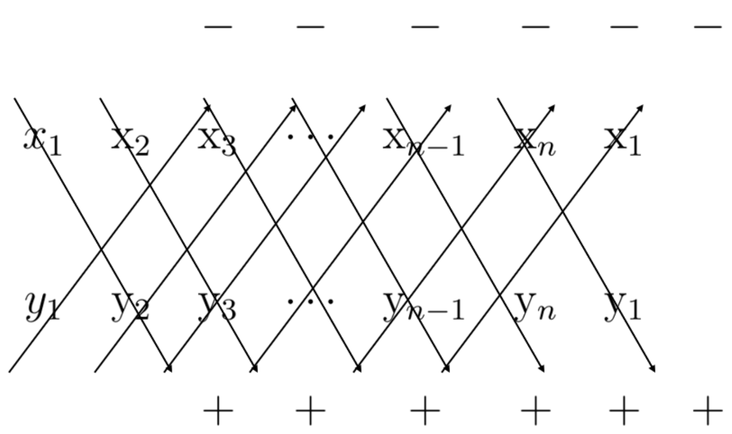


Figure 2.8: Shoelace formula mechanism.

2.6 Orthonormal Basis of Planes in $\mathbb{R}^3$

Every plane in any space need and orthonormal basis to define spanning set of this plane as we can find a formula to find any point on this plane. Thus, this section is provided by the exposing the concept of orthonormal basis, and then the process of generating an orthonormal basis in three cases as the following:

1. The plane is defined by a normal vector and a point.
2. The plane is defined by three points.
3. The plane is defined by its Cartesian equation.

Then, using these orthonormal basis, we define an isomorphic and isotropic map ψ that map a polygon from $\mathbb{R}^2$ and $\mathbb{R}^3$ and vice versa. We start with the definitions of the basis, orthonormal basis, and the standard basis the is the common usual basis in mathematics.

Definition 2.6.1. A basis B of a vector space V over a field F (such as the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$) is a subset B of V that satisfies the two following conditions:

1. **Linear independence**: For every finite subset $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ of B, if

$c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m = \mathbf{0}$ for some $c_1, \ldots, c_m \in F$,
then $c_1 = \cdots = c_m = 0$.

2. **Spanning property**: For every vector $\mathbf{v}$ in V, one can choose $a_1, \ldots, a_n$ in F and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in B such that $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$.

Definition 2.6.2. An orthonormal basis of a vector space V over a field F is a special type of basis that satisfies the following conditions:

1. **Orthogonality**: Every pair of distinct vectors $\mathbf{v}_i, \mathbf{v}_j$ in B are orthogonal, meaning that their inner product is zero: $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ for $i \neq j$.

2. **Normalization**: Every vector $\mathbf{v}_i$ in B has unit length, meaning that its inner product with itself is one: $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1$ for all i.

Definition 2.6.3. A standard basis is a special type of orthonormal vector basis where each basis vector has only one non-zero entry with a value of 1.
In an n-dimensional Euclidean space $\mathbb{R}^n$, the standard basis vectors are denoted as $\mathbf{e}_i$(or $\mathbf{e}^{(i)}$) for $i = 1, \ldots, n$, where n is the dimension of the vector space spanned by this basis.
Any vector $\mathbf{x}$ in this space can be expressed as a linear combination of the standard basis vectors as:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n.$$

For instance, in the Euclidean plane $\mathbb{R}^2$, the standard basis vectors are:

$$\hat{\mathbf{e}_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \hat{i}, \hat{\mathbf{e}_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \hat{j}$$

Likewise, in the Euclidean 3-space $\mathbb{R}^3$, the standard basis vectors are:

$$\hat{\mathbf{e}_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hat{i}, \hat{\mathbf{e}_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hat{j}, \hat{\mathbf{e}_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \hat{k}$$

The following theorem includes the process of using the orthonormal basis to write any vector $\mathbf{v}$ in $\mathbb{R}^n$.

Theorem 2.6.1. Let the set $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n\}$ be an orthonormal basis of a vector space $\mathbb{R}^n$ over a field R, then any vector $\mathbf{v} \in V$ can be written as

$$\mathbf{v} = \sum_{k=1}^{n} (\mathbf{v} \cdot \mathbf{u}_i)\mathbf{u}_i \tag{2.49}$$

Proof. Suppose $B = \mathbf{u}_1, \ldots, \mathbf{u}_n$ is an orthonormal basis for $\mathbb{R}^n$. Since B is a basis, then any vector $\mathbf{v}$ can be written uniquely as a linear combination of the vectors in B as

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + \cdots + c_n\mathbf{u}_n \tag{2.50}$$

Thus

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{u}_1 &= c_1\mathbf{u}_1 \cdot \mathbf{u}_1 + c_2\mathbf{u}_2 \cdot \mathbf{u}_1 + c_3\mathbf{u}_3 \cdot \mathbf{u}_1 + \cdots + c_n\mathbf{u}_n \cdot \mathbf{u}_1 \\
&= c_1 \times 1 + c_2 \times 0 + c_3 \times 0 + \cdots + c_n \times 0 \\
&= c_1 \\
\mathbf{v} \cdot \mathbf{u}_2 &= c_1\mathbf{u}_2 \cdot \mathbf{u}_2 + c_2\mathbf{u}_2 \cdot \mathbf{u}_2 + c_3\mathbf{u}_3 \cdot \mathbf{u}_2 + \cdots + c_n\mathbf{u}_n \cdot \mathbf{u}_2 \\
&= c_1 \times 0 + c_2 \times 1 + c_3 \times 0 + \cdots + c_n \times 0 \\
&= c_2 \\
\mathbf{v} \cdot \mathbf{u}_3 &= c_1\mathbf{u}_1 \cdot \mathbf{u}_3 + c_2\mathbf{u}_2 \cdot \mathbf{u}_3 + c_3\mathbf{u}_3 \cdot \mathbf{u}_3 + \cdots + c_n\mathbf{u}_n \cdot \mathbf{u}_3 \\
&= c_1 \times 0 + c_2 \times 0 + c_3 \times 1 + \cdots + c_n \times 0 \\
&= c_3 \\
&\vdots \\
\mathbf{v} \cdot \mathbf{u}_n &= c_1\mathbf{u}_1 \cdot \mathbf{u}_n + c_2\mathbf{u}_2 \cdot \mathbf{u}_n + c_3\mathbf{u}_3 \cdot \mathbf{u}_n + \cdots + c_n\mathbf{u}_n \cdot \mathbf{u}_n \\
&= c_1 \times 0 + c_2 \times 0 + c_3 \times 0 + \cdots + c_n \times 1 \\
&= c_n
\end{aligned} \tag{2.51}$$

Since the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \cdots, \mathbf{u}_n\}$ is an orthonormal basis thus

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Hence, substituting the values of $c_i, \ i = 1, 2, \cdots, n$ in equation (57) implies that

$$\begin{aligned} \mathbf{v} &= (\mathbf{v}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{v}\cdot\mathbf{u}_2)\mathbf{u}_2 + (\mathbf{v}\cdot\mathbf{u}_3)\mathbf{u}_3 + \cdots + (\mathbf{v}\cdot\mathbf{u}_n)\mathbf{u}_n \\ &= \sum_{k=1}^{n}(\mathbf{v}\cdot\mathbf{u}_i)\mathbf{u}_i \end{aligned} \tag{2.52}$$

■

Consider a plane $\mathcal{P}$ in 3-dimensional space defined either by $(\mathbf{p}_0, \mathbf{n})$ or by three points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$. Then, we can find the orthonormal basis $(\hat{\mathbf{n}}, \hat{\mathbf{e}_1}, \hat{\mathbf{e}_2})$ by the following steps according to the following cases:.

1. **Case (1): The plane $\mathcal{P}$ is defined by a point $\mathbf{p}_0$ and a normal vector $\mathbf{n}$**
   (a) Take any point in space say $\mathbf{p} = (x, y, z) = (a+h, b+h, c+h)$, $h \neq 0 \in \mathbb{R}$
   (b) Project $\mathbf{p}$ on the plane $\mathcal{P}$ by $\mathbf{p}^{proj} = \mathbf{p} + t\hat{\mathbf{n}}$, thus we formed a right-angled triangle.
   We want to find $t \neq 0 \in \mathbb{R}$. Using the dot product, it follows that

$$\begin{aligned} 0 &= (\mathbf{p}^{proj} - \mathbf{p}) \cdot (\mathbf{p}^{proj} - \mathbf{p}_0) \\ &= t\hat{\mathbf{n}} \cdot (\mathbf{p} - \mathbf{p}_0 - t\hat{\mathbf{n}}) && \textbf{(Divide both sides by t)} \\ &= \hat{\mathbf{n}} \cdot (-\mathbf{p} + \mathbf{p}_0 + t\hat{\mathbf{n}}) \\ &= \hat{\mathbf{n}} \cdot (\mathbf{p} - \mathbf{p}_0) + t && \textbf{(add the additive inverse } \hat{\mathbf{n}} \cdot (\mathbf{p} - \mathbf{p}_0)\textbf{)} \\ t &= \hat{\mathbf{n}} \cdot (\mathbf{p}_0 - \mathbf{p}) \\ &= -h(n_1 + n_2 + n_3) \end{aligned}$$

   Substitute t in $\mathbf{p}^{proj}$ to find it.
   In general, consider the point (x, y, z) and a plane $\mathcal{P}$ with normal vector $\mathbf{n} = (u, v, w)$, and a point (a, b, c) in it. We need to find t such that (a, b, c), $(x+tu, y+tv, z+tw)$, and (x, y, z) form a right triangle as shown in figure2.9.

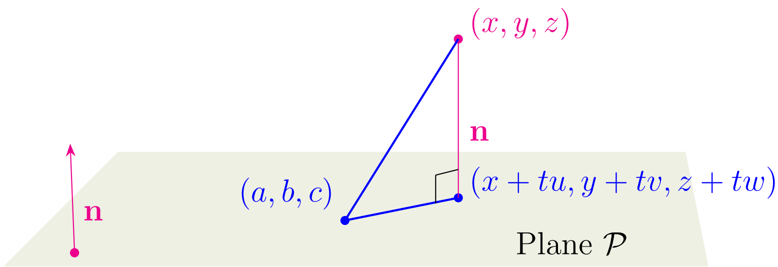


Figure 2.9: **Clarification of vector equation of a plane in the Euclidean space.**

Using the dot product implies the following:

$$
\begin{aligned}
0 &= (u, v, w).(a - (x + tu), b - (y + tv), c - (z + tw)) \\
&= au - xu - tu^2 + bv - yv - tv^2 + cw - zw - tw^2 \\
t(u^2 + v^2 + w^2) &= (au + bv + cw) - (xu + yv + zw) \\
t &= \frac{(au + bv + cw) - (xu + yv + zw)}{u^2 + v^2 + w^2}
\end{aligned}
$$

(c) The basis will be as:

$$
\begin{aligned}
\hat{\mathbf{n}} &= \frac{(\eta_1, \eta_2, \eta_3)}{\|(\eta_1, \eta_2, \eta_3)\|} \\
\hat{\mathbf{e}_1} &= \frac{\mathbf{p}^{proj} - \mathbf{p}_0}{\| \mathbf{p}^{proj} - \mathbf{p}_0 \|} \\
\hat{\mathbf{e}_2} &= \frac{\hat{\mathbf{n}} \times \hat{\mathbf{e}_1}}{\|\hat{\mathbf{n}} \times \hat{\mathbf{e}_1}\|}
\end{aligned}
$$

Then, the span of the plan

$$\mathbf{X}(s, t) = \mathbf{p}_0 + s\hat{\mathbf{e}_1} + t\hat{\mathbf{e}_2}, \quad s, t \in \mathbb{R} \tag{2.53}$$

2. **Case (2): The plane $\mathcal{P}$ is defined by a three points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$**

   (a) Find the unit normal vector.

$$\hat{\mathbf{n}} = \frac{(\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0)}{\| (\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0) \|} \tag{2.54}$$

(b) The orthonormal basis

$$\begin{aligned} \hat{\mathbf{n}} &= \frac{(\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0)}{\| (\mathbf{p}_1 - \mathbf{p}_0) \times (\mathbf{p}_2 - \mathbf{p}_0) \|} \\ \hat{\mathbf{e}_1} &= \frac{\mathbf{p}_1 - \mathbf{p}_0}{\| \mathbf{p}_1 - \mathbf{p}_0 \|} \\ \hat{\mathbf{e}_2} &= \frac{\hat{\mathbf{n}} \times \hat{\mathbf{e}_1}}{\|\hat{\mathbf{n}} \times \hat{\mathbf{e}_1}\|} \end{aligned} \tag{2.55}$$

Then, the span of the plane

$$\mathbf{X}(s,t) = \mathbf{p}_0 + s\hat{\mathbf{e}_1} + t\hat{\mathbf{e}_2}, \quad s,t \in \mathbb{R} \tag{2.56}$$

3. **Case (3): The plane $\mathcal{P}$ is defined by the Cartesian equation**

Consider a plane defined by $\mathbf{p} \cdot \mathbf{n} = d$.
First we need to find the reference point $\mathbf{p}_0$ by which here will be the closest point to the origin.

Lemma 2.6.2. The closest point $\mathbf{p}_0$ on a plane $\mathcal{P}$ to the origin is

$$\mathbf{p}_0 = \frac{d\mathbf{n}}{\|\mathbf{n}\|^2} \tag{2.57}$$

where $d = \mathbf{p}_0 \cdot \mathbf{n} = ax_0 + by_0 + cz_0$ is constant by the Cartesian equation of the plane , and $\mathbf{n} = (a,b,c)$ is the normal vector of the plane.

Proof. Suppose a given plane $\mathcal{P}$ defined by the Cartesian equation $d = \mathbf{p}_0 \cdot \mathbf{n}$, and a normal vector $\mathbf{n} = (a,b,c)$.
The closest point on a plane to the origin is the point where a line perpendicular to the plane, i.e., the line as a vector is parallel to the normal vector, and intersects the plane. As the minimum distance between a point and a plane is along a line perpendicular to the plane. Thus, this line L is defined by the equation

$$\mathbf{p} = t\mathbf{n}$$

Substituting the value of $\mathbf{p}$ in equation
$(\mathbf{p} - \mathbf{p}_0) \cdot \mathbf{n} = 0$,thus

$$\begin{aligned}(t\mathbf{n} - \mathbf{p}_0) \cdot \mathbf{n} &= 0 \\ t\|\mathbf{n}\|^2 &= \mathbf{p}_0 \cdot \mathbf{n} \\ t &= \frac{d}{\|\mathbf{n}\|^2}\end{aligned}$$

Notice that the point $\frac{d\mathbf{n}}{\|\mathbf{n}\|^2}$ is a point on the plane since

$$(\frac{d\mathbf{n}}{\|\mathbf{n}\|^2} - \mathbf{p}_0) \cdot \mathbf{n} = d - \mathbf{p}_0 \cdot \mathbf{n} = d - d = 0$$

,thus, the closest point $\mathbf{p}_0$ on a plane $\mathcal{P}$ to the origin is

$$\mathbf{p}_0 = t\mathbf{n} = \frac{d\mathbf{n}}{\|\mathbf{n}\|^2} = \frac{d\mathbf{n}}{a^2 + b^2 + c^2} \tag{2.58}$$

■

The orthonormal basis have two cases

1. **Case (a): if** $d \neq 0$.
   Use the same procedures in case that the plane $\mathcal{P}$ defined by a point $\mathbf{p}_0$ and a unit normal vector $\hat{\mathbf{n}} = (n_1, n_2, n_3)$, thus the orthonormal basis will be as

$$\begin{aligned}\hat{\mathbf{n}} &= (\eta_1, \eta_2, \eta_3) \\ \hat{\mathbf{e}_1} &= \frac{\mathbf{p}^{proj} - \mathbf{p}_0}{\| \mathbf{p}^{proj} - \mathbf{p}_0 \|} \\ \hat{\mathbf{e}_2} &= \frac{\hat{\mathbf{n}} \times \hat{\mathbf{e}_1}}{\|\hat{\mathbf{n}} \times \hat{\mathbf{e}_1}\|}\end{aligned} \tag{2.59}$$

   where $\mathbf{p}^{proj} = \mathbf{p} + t\mathbf{n}$ and

$$\mathbf{p}_0 = \frac{d\mathbf{n}}{\|\mathbf{n}\|^2} = \frac{d\mathbf{n}}{a^2 + b^2 + c^2} \tag{2.60}$$

2. **Case (b): if** $d = 0$ Then just take $\mathbf{p}_0 = (0, 0, 0)$ and do the same procedures in case (a).

Theorem 2.6.3. The set of vectors $\hat{\mathbf{n}}, \hat{\mathbf{e_1}}, \hat{\mathbf{e_2}}$ form an orthonormal basis of the plane $\mathcal{P}$ in $\mathbb{R}^3$.

Proof. Let $\mathbf{x}$ be any point in the plane $\mathcal{P}$ with a set of vectors $B = \hat{\mathbf{n}}, \hat{\mathbf{e_1}}, \hat{\mathbf{e_2}}$ in $\mathbb{R}^3$, then

1. The set B is a spanning set of the plane $\mathcal{P}$.
   Let $\mathbf{x}$ be any arbitrary point in $\mathcal{P}$ defined by the following equation:

$$(\mathbf{p} - \mathbf{p}_0) \cdot \hat{\mathbf{n}} = 0 \tag{2.61}$$

   Using the plane equation, and since $\mathbf{x} \in \mathcal{P}$, we obtain

$$\begin{aligned}
(\mathbf{x} - \mathbf{p}_0) \cdot \hat{\mathbf{n}} &= 0 \\
\mathbf{x} \cdot \hat{\mathbf{n}} &= \mathbf{p}_0 \cdot \hat{\mathbf{n}} \\
(\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} &= (\mathbf{p}_0 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} \\
(\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} &= (\mathbf{p}_0 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + c_2\hat{\mathbf{e}}_1 + c_3\hat{\mathbf{e}}_\mathbf{2} - (c_2\hat{\mathbf{e}}_\mathbf{1} + c_3\hat{\mathbf{e}}_\mathbf{2}) \\
(\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + c_2\hat{\mathbf{e}}_1 + c_3\hat{\mathbf{e}}_\mathbf{2} &= (\mathbf{p}_0 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + c_2\hat{\mathbf{e}}_1 + c_3\hat{\mathbf{e}}_\mathbf{2}
\end{aligned} \tag{2.62}$$

   Let

$$\begin{aligned}
\mathbf{x} &= (\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + c_2\hat{\mathbf{e}}_1 + c_3\hat{\mathbf{e}} && (2.63) \\
&= (\mathbf{p}_0 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + c_2\hat{\mathbf{e}}_1 + c_3\hat{\mathbf{e}} && (2.64) \\
&= c_1\hat{\mathbf{n}} + c_2\hat{\mathbf{e}}_\mathbf{1} + c_3\hat{\mathbf{e}}_\mathbf{2} && (2.65)
\end{aligned}$$

   To find the coefficients c_1, c_2, and c_3, we take the dot product of both sides of the equation with $\hat{\mathbf{n}}$, $\hat{\mathbf{e_1}}$, and $\hat{\mathbf{e_2}}$, respectively. This gives us a system of equations:

$$\begin{aligned}
\mathbf{x} \cdot \hat{\mathbf{n}} &= c_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}) + c_2(\hat{\mathbf{e_1}} \cdot \hat{\mathbf{n}}) + c_3(\hat{\mathbf{e_2}} \cdot \hat{\mathbf{n}}) \\
&= c_1 \qquad\qquad (\textbf{Also } \ c_1 = \mathbf{p}_0 \cdot \hat{\mathbf{n}}) \\
\mathbf{x} \cdot \hat{\mathbf{e_1}} &= c_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{e_1}}) + c_2(\hat{\mathbf{e_1}} \cdot \hat{\mathbf{e_1}}) + c_3(\hat{\mathbf{e_2}} \cdot \hat{\mathbf{e_1}}) \\
&= c_2 \\
\mathbf{x} \cdot \hat{\mathbf{e_2}} &= c_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{e_2}}) + c_2(\hat{\mathbf{e_1}} \cdot \hat{\mathbf{e_2}}) + c_3(\hat{\mathbf{e_2}} \cdot \hat{\mathbf{e_2}}) \\
&= c_3
\end{aligned} \tag{2.66}$$

Thus $-\infty < c_1, c_2, c_3 < \infty$.
Hence, we have expressed $\mathbf{x}$ as a linear combination of vectors in B, i.e., $\mathbf{x} = c_1\hat{\mathbf{n}} + c_2\hat{\mathbf{e}_1} + c_3\hat{\mathbf{e}_2}$. Since this holds for any point $\mathbf{x}$ in the plane $\mathcal{P}$, we have shown that the set B is a spanning set of the plane $\mathcal{P}$.

2. The set B is linearly independent.
To prove that the set $B = \hat{\mathbf{n}}, \hat{\mathbf{e}_1}, \hat{\mathbf{e}_2}$ is linearly independent, we need to show that the only solution to the equation $c_1\hat{\mathbf{n}}+c_2\hat{\mathbf{e}_1}+c_3\hat{\mathbf{e}_2} = \mathbf{0}$ is $c_1 = c_2 = c_3 = 0$.

Taking the dot product of both sides of the equation with $\hat{\mathbf{n}}$, $\hat{\mathbf{e}_1}$, and $\hat{\mathbf{e}_2}$, respectively, thus

$$\begin{aligned} 0 \cdot \hat{\mathbf{n}} &= (c_1\hat{\mathbf{n}} + c_2\hat{\mathbf{e}_1} + c_3\hat{\mathbf{e}_2}) \cdot \hat{\mathbf{n}} = c_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}) + c_2(\hat{\mathbf{e}_1} \cdot \hat{\mathbf{n}}) + c_3(\hat{\mathbf{e}_2} \cdot \hat{\mathbf{n}}) = c_1 \\ 0 \cdot \hat{\mathbf{e}_1} &= (c_1\hat{\mathbf{n}} + c_2\hat{\mathbf{e}_1} + c_3\hat{\mathbf{e}_2}) \cdot \hat{\mathbf{e}_1} = c_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{e}_1}) + c_2(\hat{\mathbf{e}_1} \cdot \hat{\mathbf{e}_1}) + c_3(\hat{\mathbf{e}_2} \cdot \hat{\mathbf{e}_1}) = c_2 \\ 0 \cdot \hat{\mathbf{e}_2} &= (c_1\hat{\mathbf{n}} + c_2\hat{\mathbf{e}_1} + c_3\hat{\mathbf{e}_2}) \cdot \hat{\mathbf{e}_2} = c_1(\hat{\mathbf{n}} \cdot \hat{\mathbf{e}_2}) + c_2(\hat{\mathbf{e}_1} \cdot \hat{\mathbf{e}_2}) + c_3(\hat{\mathbf{e}_2} \cdot \hat{\mathbf{e}_2}) = c_3 \end{aligned} \tag{2.67}$$

Since $\hat{\mathbf{n}}$, $\hat{\mathbf{e}_1}$, and $\hat{\mathbf{e}_2}$ are orthonormal, their dot products with themselves are 1, and their dot products with each other are 0. Therefore, we have:

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned}$$

Hence, the only solution to the equation $c_1\hat{\mathbf{n}} + c_2\hat{\mathbf{e}_1} + c_3\hat{\mathbf{e}_2} = \mathbf{0}$ is $c_1 = c_2 = c_3 = 0$, which implies that the set B is linearly independent.

Therefore, we conclude that the set B is a set of an orthonormal basis of the plane $\mathcal{P}$ in $\mathbb{R}^3$.

■

2.6.1 Mapping of Planar Polygons From $\mathbb{R}^3$ To $\mathbb{R}^2$

In 3D we have an extra degree of freedom, and by mapping to 2D we reduce the degrees of freedom by putting the third dimension equal to zero as a constraint.
Consider a polygon $P = \{\mathbf{x}_k\}$, $k = 1, 2, \cdots, n$ open or closed in plane $(\mathbf{p}_0, \mathbf{n})$ with orthonormal basis $(\hat{\mathbf{n}}, \hat{\mathbf{e}_1}, \hat{\mathbf{e}_2})$, then

$$\begin{aligned} \mathbf{x}_k &= (\mathbf{x}_k.\hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{x}_k.\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}_k.\hat{\mathbf{e}_2})\hat{\mathbf{e}_2} \\ &= (\mathbf{p}_0.\hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{x}_k.\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}_k.\hat{\mathbf{e}_2})\hat{\mathbf{e}_2} \end{aligned} \tag{2.68}$$

Since $(\mathbf{x}_k - \mathbf{p}_0).\hat{\mathbf{n}} = 0$. We can map any point from $\mathbb{R}^3$ to $\mathbb{R}^2$ by the map $\psi : \mathbb{R}^3 \to \mathbb{R}^2$,

$$\psi(\mathbf{x}_k) = (\mathbf{x}_k.\hat{\mathbf{e}_1}, \mathbf{x}_k.\hat{\mathbf{e}_2}) = (\xi_k, \eta_k) \tag{2.69}$$

Theorem 2.6.4. The map ψ is an isomorphism.

Proof. **First**, ψ is one to one map.
Let $\mathbf{x}_1$, $\mathbf{x}_2$ be any two points in a plane $\mathcal{P}$ in $\mathbb{R}^3$ with an orthonormal basis $(\hat{\mathbf{n}}, \hat{\mathbf{e}_1}, \hat{\mathbf{e}_2})$, thus the points can be expressed as

$$\begin{aligned} \mathbf{x}_1 &= (\mathbf{x}_1.\hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{x}_1.\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}_1.\hat{\mathbf{e}_2})\hat{\mathbf{e}_2} \\ \mathbf{x}_2 &= (\mathbf{x}_2.\hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{x}_2.\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}_2.\hat{\mathbf{e}_2})\hat{\mathbf{e}_2} \end{aligned}$$

, then

$$\begin{aligned} \psi(\mathbf{x}_1) &= (\mathbf{x}_1.\hat{\mathbf{e}_1}, \mathbf{x}_1.\hat{\mathbf{e}_2}) = (\mathbf{x}_1.\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}_1.\hat{\mathbf{e}_2})\hat{\mathbf{e}_2} \\ \psi(\mathbf{x}_2) &= (\mathbf{x}_2.\hat{\mathbf{e}_1}, \mathbf{x}_2.\hat{\mathbf{e}_2}) = (\mathbf{x}_2.\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}_2.\hat{\mathbf{e}_2})\hat{\mathbf{e}_2} \end{aligned}$$

Assume $\psi(\mathbf{x}_1) = \psi(\mathbf{x}_2)$, thus

$$\begin{aligned} (\mathbf{x}_1.\hat{\mathbf{e}_1}, \mathbf{x}_1.\hat{\mathbf{e}_2}) &= (\mathbf{x}_2.\hat{\mathbf{e}_1}, \mathbf{x}_2.\hat{\mathbf{e}_2}) \\ (\mathbf{x}_1.\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}_1.\hat{\mathbf{e}_2})\hat{\mathbf{e}_2} &= (\mathbf{x}_2.\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}_2.\hat{\mathbf{e}_2})\hat{\mathbf{e}_2} + 0 \\ (\mathbf{x}_1.\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}_1.\hat{\mathbf{e}_2})\hat{\mathbf{e}_2} &= (\mathbf{x}_2.\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}_2.\hat{\mathbf{e}_2})\hat{\mathbf{e}_2} + (\mathbf{p}_0.\hat{\mathbf{n}})\hat{\mathbf{n}} - (\mathbf{p}_0.\hat{\mathbf{n}})\hat{\mathbf{n}} \\ (\mathbf{x}_1.\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}_1.\hat{\mathbf{e}_2})\hat{\mathbf{e}_2} &= (\mathbf{x}_2.\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}_2.\hat{\mathbf{e}_2})\hat{\mathbf{e}_2} + (\mathbf{x}_2.\hat{\mathbf{n}})\hat{\mathbf{n}} - (\mathbf{x}_1.\hat{\mathbf{n}})\hat{\mathbf{n}} \\ (\mathbf{x}_1.\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}_1.\hat{\mathbf{e}_2})\hat{\mathbf{e}_2} + (\mathbf{x}_1.\hat{\mathbf{n}})\hat{\mathbf{n}} &= (\mathbf{x}_2.\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}_2.\hat{\mathbf{e}_2})\hat{\mathbf{e}_2} + (\mathbf{x}_2.\hat{\mathbf{n}})\hat{\mathbf{n}} \\ \mathbf{x}_1 &= \mathbf{x}_2 \end{aligned} \tag{2.70}$$

Since the equation of the plane $(\mathbf{p}-\mathbf{p}_0).\hat{\mathbf{n}} = 0$, and as the points $\mathbf{x}_1, \mathbf{x}_2$ are on the plane $\mathcal{P}$, thus $((\mathbf{x}_1 - \mathbf{p}_0).\hat{\mathbf{n}}) = 0$, and $((\mathbf{x}_2 - \mathbf{p}_0).\hat{\mathbf{n}}) = 0$. Therefore, the map ψ is one-to-one.
Second, ψ is onto map. Let $\mathbf{x}_1 = (\mathbf{x}_1 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{x}_1 \cdot \hat{\mathbf{e_1}})\hat{\mathbf{e_1}} + (\mathbf{x}_1 \cdot \hat{\mathbf{e_2}})\hat{\mathbf{e_2}}$ be any point in $\mathbb{R}^3$ and let $\mathbf{y} = (\xi, \eta) \in \mathbb{R}^2$ then

$$\psi(\mathbf{x}) = (\mathbf{x} \cdot \hat{\mathbf{e}}_1, \mathbf{x} \cdot \hat{\mathbf{e}}_2) = (\xi, \eta) = \mathbf{y} \tag{2.71}$$

Take $\xi = \mathbf{x}.\hat{\mathbf{e_1}}, \eta = \mathbf{x}.\hat{\mathbf{e_2}}$. Hence the map ψ is onto.
Finally, we need to prove that the map is linear. i.e.

$$\psi(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) = \alpha\psi(\mathbf{x}_1) + \beta\psi(\mathbf{x}_2) \tag{2.72}$$

for each $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{P}$.
As we mentioned before that $(\hat{\mathbf{n}}, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$ is the orthonormal basis of $\mathcal{P}$, thus the two points can be written as follows:

$$\begin{aligned}
\mathbf{x}_1 &= (\mathbf{x}_1.\hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{x}_1.\hat{\mathbf{e_1}})\hat{\mathbf{e_1}} + (\mathbf{x}_1.\hat{\mathbf{e_2}})\hat{\mathbf{e_2}} \\
\mathbf{x}_2 &= (\mathbf{x}_2.\hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{x}_2.\hat{\mathbf{e_1}})\hat{\mathbf{e_1}} + (\mathbf{x}_2.\hat{\mathbf{e_2}})\hat{\mathbf{e_2}}
\end{aligned}$$

Applying the map ψ on $\alpha\mathbf{x}_1 + \beta\mathbf{x}_2$ implies that

$$\begin{aligned}
\psi(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) &= ((\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) \cdot \hat{\mathbf{e_1}}, (\alpha\mathbf{x}_1 + \beta\mathbf{x}_2) \cdot \hat{\mathbf{e_2}}) \\
&= \alpha(\mathbf{x}_1 \cdot \hat{\mathbf{e_1}}, \mathbf{x}_1 \cdot \hat{\mathbf{e_2}}) + \beta(\mathbf{x}_2 \cdot \hat{\mathbf{e_1}}, \mathbf{x}_2 \cdot \hat{\mathbf{e_2}}) \\
&= \alpha\psi(\mathbf{x}_1) + \beta\psi(\mathbf{x}_2)
\end{aligned}$$

Hence, the map ψ is a linear map. Therefore, The map ψ is an isomorphism. ∎

The fact that this mapping is an isomorphism guarantees the existence of the inverse mapping $\psi^{-1} = \Phi : \mathbb{R}^2 \to \mathbb{R}^3$, and it is given by

$$\psi^{-1}(\xi_k, \eta_k) = (\mathbf{p}_0.\hat{\mathbf{n}})\hat{\mathbf{n}} + \xi_k\hat{\mathbf{e}}_1 + \eta_k\hat{\mathbf{e}}_2 \tag{2.73}$$

Theorem 2.6.5. The map ψ is isotropic.

Proof. Let $\mathbf{x} = \mathbf{v} - \mathbf{q}$ be any vector in a plane $\mathcal{P}$ in $\mathbb{R}^3$ with an orthonormal basis$(\hat{\mathbf{n}}, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$ and defined by the equation $(\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{n} = 0$, where the points $\mathbf{v}, \mathbf{q}$ are arbitrary points in $\mathcal{P}$, thus $(\mathbf{v}-\mathbf{p}_0)\cdot\hat{\mathbf{n}}=0, (\mathbf{q}-\mathbf{p}_0)\cdot\hat{\mathbf{n}}=0$. Therefore, $\mathbf{v},\mathbf{q}$ can be written as follows:

$$\begin{aligned}
\mathbf{v} &= (\mathbf{v}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{v}\cdot\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{v}\cdot\hat{\mathbf{e}_2})\hat{\mathbf{e}_2}\\
&= (\mathbf{p}_0\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{v}\cdot\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{v}\cdot\hat{\mathbf{e}_2})\hat{\mathbf{e}_2}\\
\mathbf{q} &= (\mathbf{q}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{q}\cdot\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{q}\cdot\hat{\mathbf{e}_2})\hat{\mathbf{e}_2}\\
&= (\mathbf{p}_0\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{q}\cdot\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{q}\cdot\hat{\mathbf{e}_2})\hat{\mathbf{e}_2}
\end{aligned}$$

and $\mathbf{x}$ can be written as follows:

$$\mathbf{x} = (\mathbf{x}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + (\mathbf{x}\cdot\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}\cdot\hat{\mathbf{e}_2})\hat{\mathbf{e}_2}$$

By applying the map on $\mathbf{x}$ we deduce

$$\psi(\mathbf{x}) = (\mathbf{x}\cdot\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}\cdot\hat{\mathbf{e}_2})\hat{\mathbf{e}_2}$$

Taking the squared norm of $\psi(\mathbf{x})$, it follows that

$$\begin{aligned}
\|\psi(\mathbf{x})\|^2 &= (\mathbf{x}\cdot\hat{\mathbf{e}_1})^2 + (\mathbf{x}\cdot\hat{\mathbf{e}_2})^2\\
&= \Big[(\mathbf{x}\cdot\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}\cdot\hat{\mathbf{e}_2})\hat{\mathbf{e}_2}\Big]\cdot\Big[(\mathbf{x}\cdot\hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{x}\cdot\hat{\mathbf{e}_2})\hat{\mathbf{e}_2}\Big]\\
&= \Big[(\mathbf{x}\cdot\hat{\mathbf{e}_1})\hat{\mathbf{e}_1}+(\mathbf{x}\cdot\hat{\mathbf{e}_2})\hat{\mathbf{e}_2}+\mathbf{p}_0\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}-\mathbf{p}_0\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}\Big]\cdot\Big[(\mathbf{x}\cdot\hat{\mathbf{e}_1})\hat{\mathbf{e}_1}+(\mathbf{x}\cdot\hat{\mathbf{e}_2})\hat{\mathbf{e}_2}+\mathbf{p}_0\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}-\mathbf{p}_0\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}\Big]\\
&= \Big[(\mathbf{x}\cdot\hat{\mathbf{e}_1})\hat{\mathbf{e}_1}+(\mathbf{x}\cdot\hat{\mathbf{e}_2})\hat{\mathbf{e}_2}+(\mathbf{v}-\mathbf{q})\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}\Big]\cdot\Big[(\mathbf{x}\cdot\hat{\mathbf{e}_1})\hat{\mathbf{e}_1}+(\mathbf{x}\cdot\hat{\mathbf{e}_2})\hat{\mathbf{e}_2}+(\mathbf{v}-\mathbf{q})\cdot\hat{\mathbf{n}}\hat{\mathbf{n}}\Big]\\
&= \Big[(\mathbf{x}\cdot\hat{\mathbf{e}_1})\hat{\mathbf{e}_1}+(\mathbf{x}\cdot\hat{\mathbf{e}_2})\hat{\mathbf{e}_2}+(\mathbf{x}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}\Big]\cdot\Big[(\mathbf{x}\cdot\hat{\mathbf{e}_1})\hat{\mathbf{e}_1}+(\mathbf{x}\cdot\hat{\mathbf{e}_2})\hat{\mathbf{e}_2}+(\mathbf{x}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}\Big] = \|\mathbf{x}\|^2
\end{aligned}$$

Hence, $\|\psi(\mathbf{x})\| = \|\mathbf{x}\|$. Whence the map ψ is an isotropic map. ■

Theorem 2.6.6. Let P be a planar polygon with $n-$vertices, $\mathbf{p}_1 = (x_1,y_1,z_1)),\mathbf{p}_2 = (x_2,y_2,z_2),\mathbf{p}_3 = (x_3,y_3,z_3),\cdots,\mathbf{p}_n = (x_n,y_n,z_n)$, $\mathbf{p}_{n+1}=\mathbf{p}_1$ in $\mathbb{R}^3$, then the mapping ψ is angle and area conservative.

Proof. We first find the orthonormal basis. Let

$$\begin{aligned}
\hat{\mathbf{n}} &= \hat{\mathbf{e}_1}\times\hat{\mathbf{e}_2}\\
\hat{\mathbf{e}_1} &= \hat{\mathbf{e}_2}\times\hat{\mathbf{n}}\\
\hat{\mathbf{e}_2} &= \hat{\mathbf{n}}\times\hat{\mathbf{e}_1}
\end{aligned} \tag{2.74}$$

Where $\hat{\mathbf{n}}$ is the unit normal of plane $\mathcal{P}$,with equation $(\mathbf{p}-\mathbf{p}_0)\cdot\mathbf{n} = 0$,of the polygon in 3D. For the area-conserving proof, we will start by computing the area of the polygon in 2D. Since the set $\hat{\mathbf{e_1}}, \hat{\mathbf{e_2}}, \hat{\mathbf{n}}$ is the orthonormal basis for the plane containing the polygon in 3D, then every point $\mathbf{p}_k$ in polygon can be written as

$$\begin{aligned} \mathbf{p}_k &= (\mathbf{p}_k \cdot \hat{\mathbf{e_1}})\hat{\mathbf{e_1}} + (\mathbf{p}_k \cdot \hat{\mathbf{e_2}})\hat{\mathbf{e_2}} + (\mathbf{p}_k \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} \\ &= (\mathbf{p}_k \cdot \hat{\mathbf{e_1}})\hat{\mathbf{e_1}} + (\mathbf{p}_k \cdot \hat{\mathbf{e_2}})\hat{\mathbf{e_2}} + (\mathbf{p}_0 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} \end{aligned} \tag{2.75}$$

Thus,

$$\psi(\mathbf{p}_k) = (\mathbf{p}_k \cdot \hat{\mathbf{e_1}}, \mathbf{p}_k \cdot \hat{\mathbf{e_2}}) = (\xi_k, \eta_k) \tag{2.76}$$

By Green's Theorem, we possess the area as the shoelace formula.

$$\begin{aligned} A(P_{2D}) &= \frac{1}{2}\left|\hat{\mathbf{k}} \cdot \sum_{k=1}^{n} (\xi_k, \eta_k) \times (\xi_{k+1}, \eta_{k+1})\right| \\ &= \frac{1}{2}\left|\sum_{k=1}^{n} (\xi_k, \eta_k) \times (\xi_{k+1}, \eta_{k+1})\right| \\ &= \frac{1}{2}\left|\sum_{k=1}^{n} (\mathbf{p}_k \times \mathbf{p}_{k+1}).\hat{\mathbf{n}}\right| \\ &= \frac{1}{2}\left|\hat{\mathbf{n}} \cdot \sum_{k=1}^{n} (\mathbf{p}_k \times \mathbf{p}_{k+1})\right| \\ &= A(P_{3D}) \end{aligned} \tag{2.77}$$

Note that

$$\begin{aligned} (\xi_k, \eta_k) \times (\xi_{k+1}, \eta_{k+1}) &= \xi_k \eta_{k+1} - \xi_{k+1} \eta_k \\ &= \mathbf{p}_k \cdot \hat{\mathbf{e_1}} \mathbf{p}_{k+1} \cdot \hat{\mathbf{e_2}} - \mathbf{p}_{k+1} \cdot \hat{\mathbf{e_1}} \mathbf{p}_k \cdot \hat{\mathbf{e_2}} \\ &= (\mathbf{p}_k \times \mathbf{p}_{k+1}) \cdot (\hat{\mathbf{e_1}} \times \hat{\mathbf{e_2}}) \\ &= (\mathbf{p}_k \times \mathbf{p}_{k+1}).\hat{\mathbf{n}} \end{aligned}$$

This result is computed using the vector identity

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \tag{2.78}$$

As known, the angle is constructed by the union of two lines, or types of lines. Since we are dealing with a polygon we have line segments

which are the edges of the polygon. For the second part of the proof (The proof that the map is angle conservative).
Let the angle θ is an arbitrary angle between an arbitrary two edges $\mathbf{e}_k = \mathbf{p}_k - \mathbf{p}_{k-1}$, $\mathbf{e}_{k+1} = \mathbf{p}_{k+1} - \mathbf{p}_k$ defined by their vector equations as follows:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{p}_{k+1} - \mathbf{p}_k \\
\mathbf{v}_2 &= \mathbf{p}_{k-1} - \mathbf{p}_k
\end{aligned}$$

Using the equation 2.68 to write the points $\mathbf{p}_k, \mathbf{p}_{k+1}$ as follows:

$$\begin{aligned}
\mathbf{p}_{k-1} &= (\mathbf{p}_{k-1} \cdot \hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{p}_{k-1} \cdot \hat{\mathbf{e}_2})\hat{\mathbf{e}_2} + (\mathbf{p}_0 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} \\
\mathbf{p}_k &= (\mathbf{p}_k \cdot \hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{p}_k \cdot \hat{\mathbf{e}_2})\hat{\mathbf{e}_2} + (\mathbf{p}_0 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} \\
\mathbf{p}_{k+1} &= (\mathbf{p}_{k+1} \cdot \hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{p}_{k+1} \cdot \hat{\mathbf{e}_2})\hat{\mathbf{e}_2} + (\mathbf{p}_0 \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}
\end{aligned}$$

Thus the equation of $\mathbf{v}_1, \mathbf{v}_2$ will be as follows:

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{p}_{k+1} - \mathbf{p}_k \\
&= (\mathbf{p}_{k+1} \cdot \hat{\mathbf{e}_1})\hat{\mathbf{e}_1} - (\mathbf{p}_k \cdot \hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{p}_{k+1} \cdot \hat{\mathbf{e}_2})\hat{\mathbf{e}_2} - (\mathbf{p}_k \cdot \hat{\mathbf{e}_2})\hat{\mathbf{e}_2} \\
&= (\mathbf{p}_{k+1} \cdot \hat{\mathbf{e}_1} - \mathbf{p}_k \cdot \hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{p}_{k+1} \cdot \hat{\mathbf{e}_2} - \mathbf{p}_k \cdot \hat{\mathbf{e}_2})\hat{\mathbf{e}_2} \\
&= (\mathbf{p}_{k+1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_1}\hat{\mathbf{e}_1} + (\mathbf{p}_{k+1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_2}\hat{\mathbf{e}_2} \\
\mathbf{v}_2 &= \mathbf{p}_{k-1} - \mathbf{p}_k \\
&= (\mathbf{p}_{k-1} \cdot \hat{\mathbf{e}_1})\hat{\mathbf{e}_1} - (\mathbf{p}_k \cdot \hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{p}_{k-1} \cdot \hat{\mathbf{e}_2})\hat{\mathbf{e}_2} - (\mathbf{p}_k \cdot \hat{\mathbf{e}_2})\hat{\mathbf{e}_2} \\
&= (\mathbf{p}_{k-1} \cdot \hat{\mathbf{e}_1} - \mathbf{p}_k \cdot \hat{\mathbf{e}_1})\hat{\mathbf{e}_1} + (\mathbf{p}_{k-1} \cdot \hat{\mathbf{e}_2} - \mathbf{p}_k \cdot \hat{\mathbf{e}_2})\hat{\mathbf{e}_2} \\
&= (\mathbf{p}_{k-1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_1}\hat{\mathbf{e}_1} + (\mathbf{p}_{k-1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_2}\hat{\mathbf{e}_2}
\end{aligned}$$

Thus

$$\begin{aligned}
(\mathbf{v}_1.\mathbf{v}_2) &= \Big((\mathbf{p}_{k+1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_1}, (\mathbf{p}_{k+1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_2}\Big) \cdot \Big((\mathbf{p}_{k-1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_1}, (\mathbf{p}_{k-1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_2}\Big) \\
&= (\mathbf{v}_1 \cdot \hat{\mathbf{e}}_1)(\mathbf{v}_2 \cdot \hat{\mathbf{e}}_1) + (\mathbf{v}_1 \cdot \hat{\mathbf{e}}_2)(\mathbf{v}_2 \cdot \hat{\mathbf{e}}_2)
\end{aligned} \tag{2.79}$$

Therefore, Let θ be the angle between the vectors $\mathbf{v}_1$, and $\mathbf{v}_2$ in 3D , and θ' be the angle between the vectors $(\mathbf{v}_1.\hat{\mathbf{e}_1}, \mathbf{v}_1.\hat{\mathbf{e}_2})$, and $(\mathbf{v}_2.\hat{\mathbf{e}_1}, \mathbf{v}_2.\hat{\mathbf{e}_2})$

in 2D using the mapping ψ. Then,

$$\begin{aligned}
\cos\theta' &= \frac{(\mathbf{v}_1.\hat{\mathbf{e}_1}, \mathbf{v}_1.\hat{\mathbf{e}_2}) \cdot (\mathbf{v}_2.\hat{\mathbf{e}_1}, \mathbf{v}_2.\hat{\mathbf{e}_2})}{\|(\mathbf{v}_1.\hat{\mathbf{e}_1}, \mathbf{v}_1.\hat{\mathbf{e}_2})\| \, \|(\mathbf{v}_2.\hat{\mathbf{e}_1}, \mathbf{v}_2.\hat{\mathbf{e}_2})\|} \\
&= \frac{(\mathbf{v}_1.\hat{\mathbf{e}}_1)(\mathbf{v}_2.\hat{\mathbf{e}}_1) + (\mathbf{v}_1.\hat{\mathbf{e}}_2)(\mathbf{v}_2.\hat{\mathbf{e}}_2) + 0}{\|\mathbf{v}_1\ \|\mathbf{v}_2\|} \\
&= \frac{(\mathbf{v}_1.\hat{\mathbf{e}}_1)(\mathbf{v}_2.\hat{\mathbf{e}}_1) + (\mathbf{v}_1.\hat{\mathbf{e}}_2)(\mathbf{v}_2.\hat{\mathbf{e}}_2) + (\mathbf{v}_1.\hat{\mathbf{n}})(\mathbf{v}_2.\hat{\mathbf{n}})}{\|\mathbf{v}_1\ \|\mathbf{v}_2\|} \\
&= \frac{(\mathbf{v}_1.\mathbf{v}_2).\hat{\mathbf{e}_1} + (\mathbf{v}_1.\mathbf{v}_2).\hat{\mathbf{e}_2} + (\mathbf{v}_1.\mathbf{v}_2).\hat{\mathbf{n}}}{\|\mathbf{v}_1\ \|\mathbf{v}_2\|} \\
&= \frac{(\mathbf{v}_1.\mathbf{v}_2)}{\|\mathbf{v}_1\| \, \|\mathbf{v}_2\|} \\
&= \cos\theta
\end{aligned} \tag{2.80}$$

Since

$$\begin{aligned}
\|(\mathbf{v}_1.\hat{\mathbf{e}_1}, \mathbf{v}_1.\hat{\mathbf{e}_2})\|^2 &= \Big((\mathbf{p}_{k+1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_1}\Big)^2 + \Big((\mathbf{p}_{k+1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_2}\Big)^2 \\
&= \Big((\mathbf{p}_{k+1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_1}\hat{\mathbf{e}_1} + (\mathbf{p}_{k+1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_2}\hat{\mathbf{e}_2} \\
&\quad + \mathbf{p}_0 \cdot \hat{\mathbf{n}}\hat{\mathbf{n}} - \mathbf{p}_0 \cdot \hat{\mathbf{n}}\hat{\mathbf{n}}\Big) \cdot \Big((\mathbf{p}_{k+1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_1}\hat{\mathbf{e}_1} + (\mathbf{p}_{k+1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_2}\hat{\mathbf{e}_2} \\
&\quad + \mathbf{p}_0 \cdot \hat{\mathbf{n}}\hat{\mathbf{n}} - \mathbf{p}_0 \cdot \hat{\mathbf{n}}\hat{\mathbf{n}}\Big) \\
&= \Big((\mathbf{p}_{k+1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_1}\hat{\mathbf{e}_1} + (\mathbf{p}_{k+1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_2}\hat{\mathbf{e}_2} \\
&\quad + \mathbf{p}_{k+1} \cdot \hat{\mathbf{n}}\hat{\mathbf{n}} - \mathbf{p}_k \cdot \hat{\mathbf{n}}\hat{\mathbf{n}}\Big) \cdot \Big((\mathbf{p}_{k+1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_1}\hat{\mathbf{e}_1} + (\mathbf{p}_{k+1} - \mathbf{p}_k) \cdot \hat{\mathbf{e}_2}\hat{\mathbf{e}_2} \\
&\quad + \mathbf{p}_{k+1} \cdot \hat{\mathbf{n}}\hat{\mathbf{n}} - \mathbf{p}_k \cdot \hat{\mathbf{n}}\hat{\mathbf{n}}\Big) \\
&= \|\mathbf{v}_1\|^2 \\
\|(\mathbf{v}_2.\hat{\mathbf{e}_1}, \mathbf{v}_2.\hat{\mathbf{e}_2})\|^2 &= \|\mathbf{v}_2\|^2 \quad \textbf{(Also, in the same way)}
\end{aligned}$$

Hence

$$\begin{aligned}
\|(\mathbf{v}_1.\hat{\mathbf{e}_1}, \mathbf{v}_1.\hat{\mathbf{e}_2})\| &= \|\mathbf{v}_1\| \\
\|(\mathbf{v}_2.\hat{\mathbf{e}_1}, \mathbf{v}_2.\hat{\mathbf{e}_2})\| &= \|\mathbf{v}_2\|
\end{aligned}$$

Therefore, the map ψ is angle conservative. ■

It is clarified in the figures2.10 and 2.11 that the mapping ψ maps a multiple polygons , convex and non convex, from 2D to 3D and vices versa as it isomorphic and isotropic map.

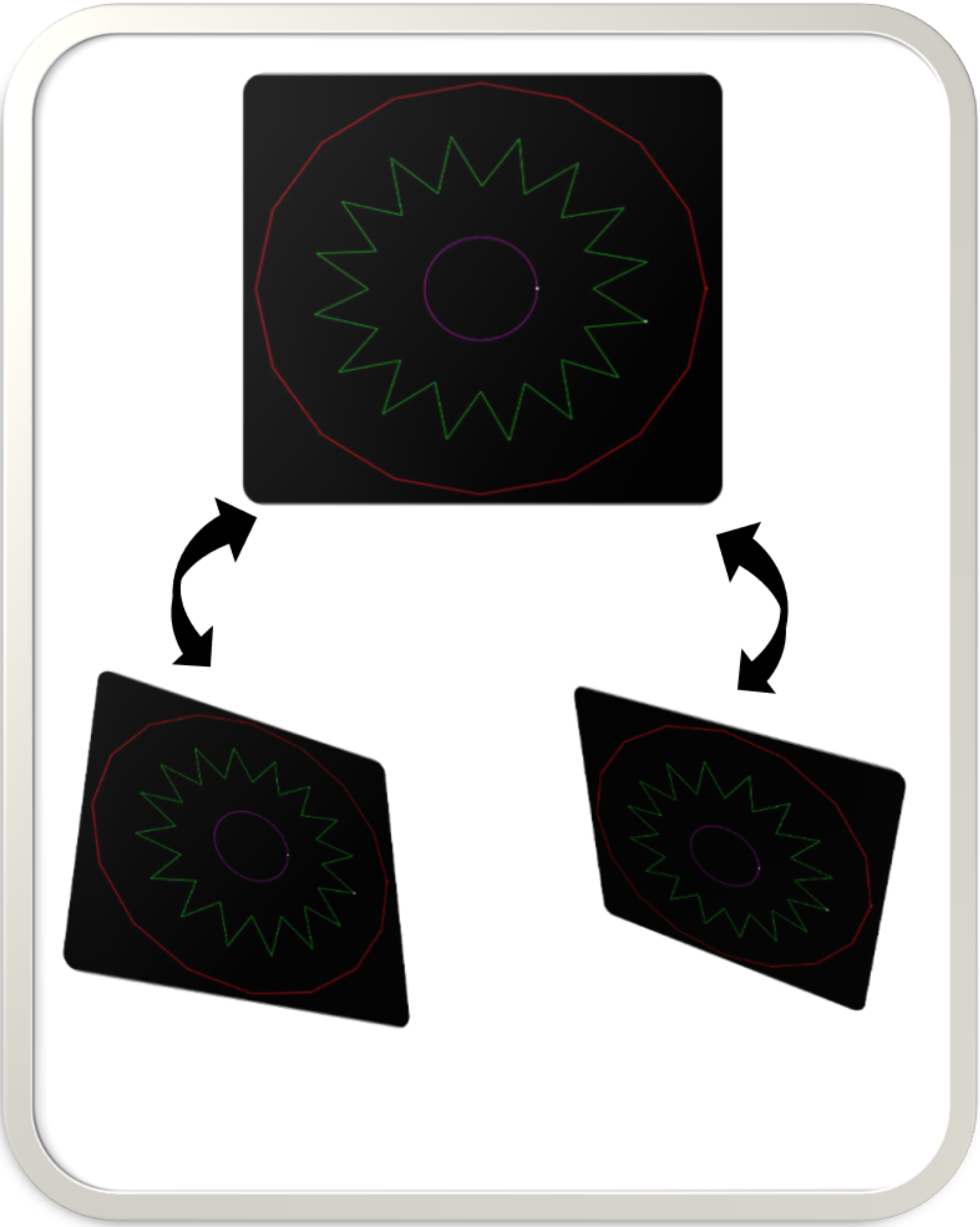


Figure 2.10: **Mapping polygons from 2D to 3D and vice versa.**

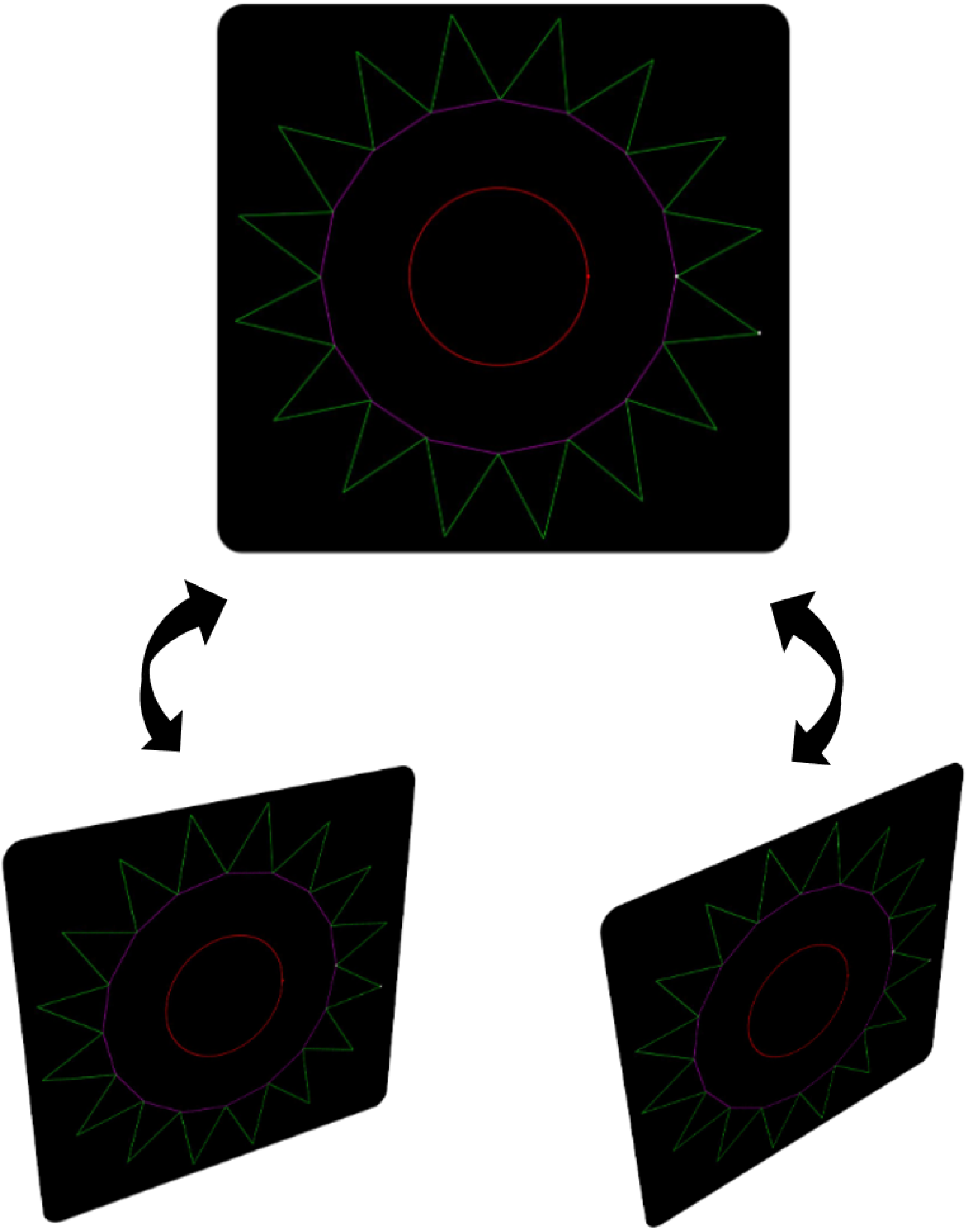


Figure 2.11: **Mapping polygons from 2D to 3D and vice versa using another planes.**

2.7 Centroid of Polygon

This section is built by exploring the centroid of the polygon including an introduction of it, its definition, and its formula in $\mathbb{R}^2$ and then using the map ψ to find the centroid in $\mathbb{R}^3$ based on the law of decomposition and the formula of the centroid of a triangle.

Definition 2.7.1. The **first moment** of a point $\mathbf{p}$ with respect to a point $\mathbf{O}$ is the vector $\mathbf{M} = s\mathbf{r}_p$, where $\mathbf{r}_p$ is the position vector of $\mathbf{p}$ relative to $\mathbf{O}$, and s is a scalar associated with $\mathbf{p}$ which called the **strength** of $\mathbf{p}$.

Definition 2.7.2. The **centroid** is defined as the point defining the geometric center of a system or an object. The centroid is unique.
Suppose we have a set of points $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \cdots, \mathbf{p}_n\}$, then the point $\mathbf{c}$ concerning which the sum of the first moments of the points of S is equal to zero is the centroid of S.

Let $\mathbf{O}$ be a reference point and $\mathbf{r}_c$ and $\mathbf{r}_i$ be the position vectors of the centroid $\mathbf{c}$ and the points $\mathbf{p}_i, i = 1, 2, 3 \cdots, n$ relative to $\mathbf{O}$, respectively. Then, the position vector of $\mathbf{p}_i, ; i = 1, 2, 3 \cdots, n$ relative to $\mathbf{c}$ is given by $\mathbf{r}_i - \mathbf{r}_c$, and the first moment of each point $\mathbf{p}_i$ with respect to $\mathbf{c}$ is $s_i(\mathbf{r}_i - \mathbf{r}_c)$.
If $\mathbf{c}$ is the centroid, then the sum of the first moments is equal to zero, thus

$$
\begin{aligned}
\sum_{i=1}^{n} s_i(\mathbf{r}_i - \mathbf{r}_c) = 0 \\
\sum_{i=1}^{n} s_i\mathbf{r}_i = \sum_{i=1}^{n} s_i\mathbf{r}_c \\
\mathbf{r}_c = \frac{\sum_{i=1}^{n} s_i\mathbf{r}_i}{\sum_{i=1}^{n} s_i}
\end{aligned}
\tag{2.81}
$$

Remarks:

1. If $\sum_{i=1}^{n} s_i = 0$, then the centroid is not defined.

2. The location of the centroid is independent of the choice of the reference point $\mathbf{O}$.

3. If $s_i = m_i, i = 1, 2, 3, \cdots, n$, where m_i is a mass then we call the centroid by the **mass center** of the body.

Using the standard basis of $\mathbb{R}^3$ and the Cartesian coordinates of the centroid $\mathbf{c} = (x_c, y_c, z_c)$, thus we can express $\mathbf{r}_c$ as follows:

$$\mathbf{r}_c = \mathbf{x}_c\hat{i} + \mathbf{y}_c\hat{j} + \mathbf{z}_c\hat{k} \tag{2.82}$$

, therefore, we can compute the x_c, y_c, and z_c coordinates of the centroid **C** using the following equations:

$$\begin{aligned} \mathbf{x}_c &= \frac{\sum_{i=1}^n s_i x_i}{\sum_{i=1}^n s_i} \\ \mathbf{y}_c &= \frac{\sum_{i=1}^n s_i y_i}{\sum_{i=1}^n s_i} \\ \mathbf{z}_c &= \frac{\sum_{i=1}^n s_i z_i}{\sum_{i=1}^n s_i} \end{aligned} \tag{2.83}$$

In terms of integral, then the position vector $\mathbf{r}_c$ is

$$\mathbf{r}_c = \frac{\int_\zeta \mathbf{r}\, d\zeta}{\int_\zeta d\zeta} = \frac{\int_\zeta \mathbf{r}\, d\zeta}{\zeta} \tag{2.84}$$

, and the Cartesian coordinates can be expressed as follows:

$$\begin{aligned} \mathbf{x}_c &= \frac{\int_\zeta x\, d\zeta}{\int_\zeta d\zeta} \\ \mathbf{y}_c &= \frac{\int_\zeta y\, d\zeta}{\int_\zeta d\zeta} \\ \mathbf{z}_c &= \frac{\int_\zeta z\, d\zeta}{\int_\zeta d\zeta} \end{aligned} \tag{2.85}$$

As a result, the position vector of the centroid of a curve, surface, or solid relative to the reference point **O** is

$$\mathbf{r}_c = \frac{\int_\zeta \mathbf{r}\, d\zeta}{\int_\zeta d\zeta} = \frac{\int_\zeta \mathbf{r}\, d\zeta}{\zeta} \tag{2.86}$$

where

1. ζ is a curve, surface, or solid.
2. $\mathbf{r}$ is the position vector of ζ.
3. dζ is the length of the curve, area of the surface, and volume of the solid. Thus,

$$d\zeta = \begin{cases} d\text{L} & \text{if } \zeta \text{ is curve} \\ d\text{A} & \text{if } \zeta \text{ is surface} \\ d\text{V} & \text{if } \zeta \text{ is solid} \end{cases}$$

4.

$$\int_{\zeta} d\zeta = \begin{cases} \text{L}= (\textit{the total length of the curve}) & \text{if } \zeta \text{ is curve} \\ \text{A}= (\textit{the total area of the surface}) & \text{if } \zeta \text{ is surface} \\ \text{V}= (\textit{the total volume of the solid}) & \text{if } \zeta \text{ is solid} \end{cases}$$

The process of finding the centroid of an object can be accomplished through a technique called "**The method of decomposition**". This method involves the following steps:

1. Divide the body into a number of simpler body shapes, which may be particles(points), curves, surfaces, or solids. The holes can be treated as pieces with a negative size, mass, or weight.
2. Locate the coordinates $\mathbf{x}_{c_i}, \mathbf{y}_{c_i}, \mathbf{z}_{c_i}, i = 1, 2, 3, \cdots, n$ of the centroid of each simpler shape.
3. Find the centroid $\mathbf{c}$ coordinates $\mathbf{x}_c, \mathbf{y}_c, \mathbf{z}_c$ of the whole object as follows:

$$\begin{aligned} \mathbf{x}_c &= \frac{\sum_{i=1}^{n} \int_{\zeta} x \, d\zeta}{\sum_{i=1}^{n} \int_{\zeta} d\zeta} \\ \mathbf{y}_c &= \frac{\sum_{i=1}^{n} \int_{\zeta} y \, d\zeta}{\sum_{i=1}^{n} \int_{\zeta} d\zeta} \\ \mathbf{z}_c &= \frac{\sum_{i=1}^{n} \int_{\zeta} z \, d\zeta}{\sum_{i=1}^{n} \int_{\zeta} d\zeta} \end{aligned} \tag{2.87}$$

And using the equation(85), thus equation(90) will be as follows:

$$\begin{aligned} \mathbf{x}_c &= \frac{\sum_{i=1}^{n} \mathbf{x}_{c_i}\zeta_i}{\sum_{i=1}^{n} \zeta_i} \\ \mathbf{y}_c &= \frac{\sum_{i=1}^{n} \mathbf{y}_{c_i}\zeta_i}{\sum_{i=1}^{n} \zeta_i} \\ \mathbf{z}_c &= \frac{\sum_{i=1}^{n} \mathbf{z}_{c_i}\zeta_i}{\sum_{i=1}^{n} \zeta_i} \end{aligned} \tag{2.88}$$

For centroid of surface with area A, then $\zeta = A$, $d\zeta = dA$, $\zeta_i = A_i$ and

$$\begin{aligned} \mathbf{x}_c &= \frac{\sum_{i=1}^{n} \mathbf{x}_{c_i}A_i}{\sum_{i=1}^{n} A_i} \\ \mathbf{y}_c &= \frac{\sum_{i=1}^{n} \mathbf{y}_{c_i}A_i}{\sum_{i=1}^{n} A_i} \\ \mathbf{z}_c &= \frac{\sum_{i=1}^{n} \mathbf{z}_{c_i}A_i}{\sum_{i=1}^{n} A_i} \end{aligned} \tag{2.89}$$

where $\sum_{i=1}^{n} = A$

Theorem 2.7.1. Let P be a planar polygon with $n-$vertices: $\mathbf{p}_1 = (x_1, y_1)), \mathbf{p}_2 = (x_2, y_2), \mathbf{p}_3 = (x_3, y_3), \cdots, \mathbf{p}_n = (x_n, y_n)$, $\mathbf{p}_{n+1} = \mathbf{p}_1$ in $\mathbb{R}^2$, then the centroid of P is given as follows:

$$\frac{1}{6A}\left(\sum_{i=1}^{n}(x_i + x_{i+1})(x_i y_{i+1} - x_{i+1}y_i), \sum_{i=1}^{n}(y_i + y_{i+1})(x_i + x_{i+1})(x_i y_{i+1} - x_{i+1}y_i)\right) \tag{2.90}$$

Proof. Using the method of decomposition, divide the polygon into $n-$triangles then using the fact that the centroid coordinates of the triangle are given by the following equations:

$$\begin{aligned} \mathbf{x}_c &= \frac{1}{3}\sum_{i=1}^{3}(x_i + x_{i+1}) \\ \mathbf{y}_c &= \frac{1}{3}\sum_{i=1}^{3}(y_i + y_{i+1}) \end{aligned} \tag{2.91}$$

and as the vector area of the triangle is given by

$$A(P) = \frac{1}{2}\left(\sum_{i=1}^{n}(x_i y_{i+1} - x_{i+1}y_i)\right) \tag{2.92}$$

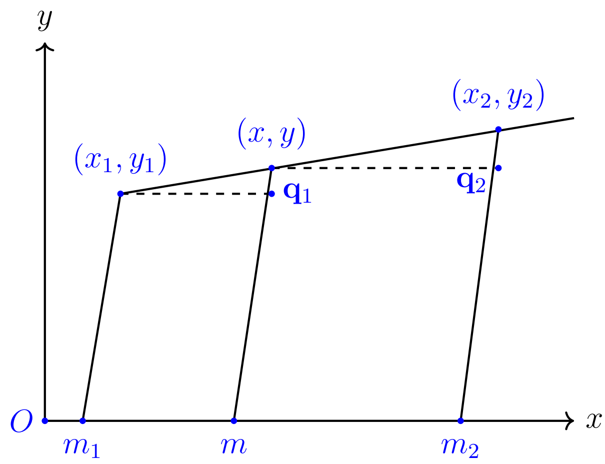


Figure 2.12: **Section formula clarification.**

thus, the centroid coordinates of the polygon P are given by

$$\mathbf{x}_c = \frac{\sum_{i=1}^{n} \mathbf{x}_{c_i} A_i}{\sum_{i=1}^{n} A_i} \qquad \mathbf{y}_c = \frac{\sum_{i=1}^{n} \mathbf{y}_{c_i} A_i}{\sum_{i=1}^{n} A_i} \tag{2.93}$$

it follows that,

$$\mathbf{x}_c = \frac{\sum_{i=1}^{n} \frac{1}{3}(x_i + x_{i+1})\frac{1}{2}\left(x_i y_{i+1} - x_{i+1} y_i\right)}{\sum_{i=1}^{n} A_i} = \frac{\sum_{i=1}^{n}(x_i + x_{i+1})\left(x_i y_{i+1} - x_{i+1} y_i\right)}{6A}$$
$$\mathbf{y}_c = \frac{\sum_{i=1}^{n} \frac{1}{3}(y_i + y_{i+1})\frac{1}{2}\left(x_i y_{i+1} - x_{i+1} y_i\right)}{\sum_{i=1}^{n} A_i} = \frac{\sum_{i=1}^{n}(y_i + y_{i+1})\left(x_i y_{i+1} - x_{i+1} y_i\right)}{6A} \tag{2.94}$$

The rest of the proof is just to prove that the centroid coordinates of a triangle T with vertices $\mathbf{p}_1 = (x_1, y_1), \mathbf{p}_2 = (x_2, y_2), \mathbf{p}_3 = (x_3, y_3)$, $\mathbf{p}_4 = \mathbf{p}_1$ in $\mathbb{R}^3$, are given by

$$\mathbf{x}_c = \frac{1}{3}\sum_{i=1}^{3}(x_i + x_{i+1}) \qquad \mathbf{y}_c = \frac{1}{3}\sum_{i=1}^{3}(y_i + y_{i+1}) \tag{2.95}$$

Lemma 2.7.2 (). If $\mathbf{L}$ is a line segment joining two point $\mathbf{p}_1 = (x_1, y_1)$ and $\mathbf{p}_2 = (x_2, y_2)$, and $\mathbf{q}$ is a point on $\mathbf{L}$ dividing it in a ratio $m : n$, then the coordinates of the point $\mathbf{q}$ are given by

$$\left(\frac{mx_1 + nx_2}{m+n}, \frac{my_1 + ny_2}{m+n}\right) \tag{2.96}$$

Proof. Let $\mathbf{L}$ be a line joining two points $\mathbf{p}_1$ and $\mathbf{p}_2$, and $\mathbf{q}$ be a point on $\mathbf{L}$ dividing it in a ration $m : n$. it is clear that $\mathbf{Om_1} = x_1$, $\mathbf{m_1p_1} = y_1$, $\mathbf{Om_2} = x_2$, $\mathbf{m_2p_2} = y_2$, $\mathbf{Om} = x$, and $\mathbf{mq} = y$.
In addition, we can see that the line segments $\overline{\mathbf{mp_1m_1}}$, $\overline{\mathbf{qm}}$ are parallel, and the same property for $\overline{\mathbf{qm}}$, and $\overline{\mathbf{p_2m_2}}$. Also, $\overline{\mathbf{m_1m}}$, $\overline{\mathbf{p_1q_1}}$ are parallel, and the same property for $\overline{\mathbf{mm_2}}$, and $\overline{\mathbf{qq_2}}$.
Thus,

$$\begin{aligned}
\mathbf{p_1q_1} &= \mathbf{m_1m} = \mathbf{Om} - \mathbf{Om_1} = x - x_1 \\
\mathbf{qq_2} &= \mathbf{mm_2} = \mathbf{Om_2} - \mathbf{Om} = x_2 - x \\
\mathbf{q_1q} &= \mathbf{qm} - \mathbf{p_1m_1} = y - y_1 \\
\mathbf{q_2p_2} &= \mathbf{q_2m_2} - \mathbf{qm} = y_2 - y
\end{aligned}$$

Since the two triangles $\mathbf{p_1q_1q}$, and $\mathbf{qq_2p_2}$ are similar, thus

$$\begin{aligned}
\frac{m}{n} &= \frac{\mathbf{p_1q}}{\mathbf{qp_2}} = \frac{\mathbf{p_1q_1}}{\mathbf{qq_2}} = \frac{x - x_1}{x_2 - x} \\
m(x_2 - x) &= n(x - x_1) \\
x &= \frac{mx_1 + nx_2}{m + n}
\end{aligned}$$

Also

$$\begin{aligned}
\frac{m}{n} &= \frac{\mathbf{p_1q}}{\mathbf{q_1p_2}} = \frac{\mathbf{q_1q}}{\mathbf{q_2p_2}} = \frac{y - y_1}{y_2 - y} \\
m(x_2 - x) &= n(x - x_1) \\
y &= \frac{my_1 + ny_2}{m + n}
\end{aligned}$$

Therefor the coordinates of the point $\mathbf{q}$ are give by

$$\left(\frac{mx_1 + nx_2}{m + n}, \frac{my_1 + ny_2}{m + n}\right) \tag{2.97}$$

□

Using equation2.87 we determine that the centroid c of the triangle is located at the point where the three medians of the triangle intersect. This point corresponds to $\mathbf{h}/3$ where h is the height of the triangle that is the median connecting the vertex (x_3, y_3) with the midpoint of the base $\overline{\mathbf{p_2p_1}}$ call it D. Thus, as c partitions the median into ration of 2:1, thus by using lemma2.7.2 we infer the following:

$$\begin{aligned}
c &= (x_c, y_c) && (2.98) \\
&= \left(\frac{2\frac{(x_1+x_2)}{2} + 1 \cdot x_3}{2 + 1}, \frac{2\frac{(y_1+y_2)}{2} + 1 \cdot y_3}{2 + 1}\right) && (2.99) \\
&= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right) && (2.100)
\end{aligned}$$

We prove that

$$\begin{aligned}\frac{1}{6A}\left(\sum_{i=1}^{n}(x_i+x_{i+1})(x_iy_{i+1}-x_{i+1}y_i)\right)&=\frac{1}{3}\sum_{i=1}^{3}(x_i+x_{i+1})\\ \frac{1}{6A}\left(\sum_{i=1}^{n}(y_i+y_{i+1})(x_iy_{i+1}-x_{i+1}y_i\right)&=\frac{1}{3}\sum_{i=1}^{3}(y_i+y_{i+1})\end{aligned}\tag{2.101}$$

We will prove the x-component of the centroid, and the other one can be proved using the same procedures.

$$\frac{1}{6A}\left(\sum_{i=1}^{n}(x_i+x_{i+1})(x_iy_{i+1}-x_{i+1}y_i)\right)$$

Add and subtract $x_1(x_2y_3-y_2x_3), x_2(x_3y_1-y_3x_1),$ and $x_3(x_1y_2-y_1x_2)$:

$$=\frac{1}{6A}\left(\sum_{i=1}^{n}(x_i+x_{i+1})(x_iy_{i+1}-x_{i+1}y_i)+\Big[x_1(x_2y_3-y_2x_3)+x_2(x_3y_1-y_3x_1)+x_3(x_1y_2-y_1x_2)\right.$$
$$\left.-\big(x_1(x_2y_3-y_2x_3)+x_2(x_3y_1-y_3x_1)+x_3(x_1y_2-y_1x_2)\big)\Big]\right.$$

Using the shoelace formula for the area A :

$$A=\frac{1}{2}\left|\sum_{i=1}^{n}(x_iy_{i+1}-x_{i+1}y_i)\right|$$

$$=\frac{1}{6A}\big((x_1+x_2)(x_1y_2-y_1x_2)+(x_2+x_3)(x_2y_3-y_2x_3)+(x_3+x_1)(x_3y_1-y_3x_1)\big)$$

Recognizing that each term $(x_iy_{i+1}-x_{i+1}y_i)$ contributes twice the area:

$$\begin{aligned}&=\frac{1}{6A}\big((x_1+x_2+x_3)(2A)\big)\\&=\frac{1}{6A}\cdot(x_1+x_2+x_3)\cdot 2A\\&=\frac{1}{3}(x_1+x_2+x_3)\end{aligned}$$

Thus

$$\begin{aligned} x_{c_i} &= \frac{1}{3}\sum_{i=1}^{n}(x_i + x_{i+1}) \\ y_{c_i} &= \frac{1}{3}\sum_{i=1}^{n}(y_i + y_{i+1}) \end{aligned} \tag{2.102}$$

Hence, the centroid of P is

$$\mathbf{c} = \frac{1}{6A}\left(\sum_{i=1}^{n}(x_i + x_{i+1})(x_i y_{i+1} - x_{i+1} y_i), \sum_{i=1}^{n}(y_i + y_{i+1})(x_i y_{i+1} - x_{i+1} y_i)\right) \tag{2.103}$$

□

The centroid vector equation of the polygon in $\mathbb{R}^2$ is given as follows:

$$\mathbf{c} = x_c + y_c = (\mathbf{c}\cdot\hat{\mathbf{e_1}})\hat{\mathbf{e_1}} + (\mathbf{c}\cdot\hat{\mathbf{e_2}})\hat{\mathbf{e_2}} \tag{2.104}$$

Using the inverse map ψ^{-1}, we possess the vector equation of the centroid of the polygon P in $\mathbb{R}^3$ as follows:

$$\begin{aligned} \mathbf{c} &= \psi^{-1}(x_c, y_c) \\ &= \mathbf{p}_0 \cdot \hat{\mathbf{n}} + x_c + y_c \end{aligned} \tag{2.105}$$

where the triple $(\hat{\mathbf{n}}, \hat{\mathbf{e_1}}, \hat{\mathbf{e_2}})$ is the orthonormal basis of the plane containing the polygon P, and $\mathbf{p}_0$ is a fixed point in the plane.

Chapter 3

Polygon-Polygon Overlapping

In this chapter, We investigate the possible results of the overlapping of line and a circle , two circles that we discuss in the first and second sections. In the third section, we express the triangle-triangle overlapping, including the possible outcomes and the cases of the overlapping. (Sabharwal, C. L., Leopold, J. L., McGeehan, D. (2013)).
The fourth section also has the same pattern of the previous sections in delving into the overlapping process but of two polygons, which is the general case of the previous ones. The final section presents the headline of this chapter. An algorithm is provided that includes a test of overlapping and a method to find the shape resulting from the overlapping process.

3.1 Line-Circle Overlapping

In this section, we demonstrate the cases of the overlapping process between a line and a circle and the possible overlapping results.
Let $\mathbf{L}_1$ be a line segment in Euclidean space defined by a vector equation $\boldsymbol{\ell}_1 = \mathbf{p}_0 + t\mathbf{u}, 0 \leq t \leq 1$ where $\mathbf{u}$ is vector parallel to $\mathbf{L}_1$. And let $\mathbf{A}_1$,$\mathbf{A}_2$ are circles defined by vector equations $(x-a)^2 + (y-b)^2 = s_1^2$, $(x-c)^2 + (y-d)^2 = s_2^2$ respectively.
The results of a line-circle intersection are as follows:

1. One point(the line is a tangent of the circle).
2. A segment of the line(Two point intersection).

Remark. The aforementioned results occur frequently in any intersection between a line and any closed geometric shape.

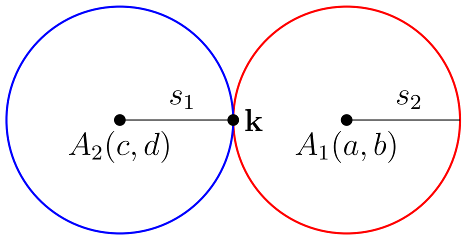


Figure 3.1: Clarification of one-point intersection.

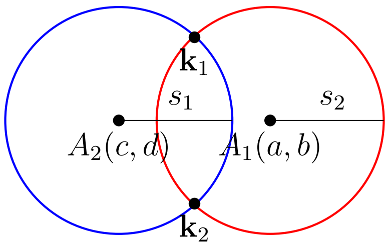


Figure 3.2: Clarification for two-point intersection

3.2 Circle-circle Overlapping

Extending to a more general case than the previous section, here, we introduce the circle-circle overlapping cases and possible results.
In general, the circle-circle intersection has two outcomes as follows:

- **One Point overlapping.** The two circles are tangential to each other and intersect in a point **k** as shown in Figure 3.1.
- **Two Point Overlapping.** This intersection results in a geometric shape, possibly like a lens or one of them inside the other which has a shorter radius as shown in Figure3.2.

3.3 Triangle-Triangle Overlapping

The overlapping of any two triangles has three categories as we investigate in this section that include the followings:

1. Single point overlapping.
2. Line overlapping.
3. Area overlapping.

In addition, we focus just on the area overlapping since it come up with shapes that can be used as polygons that have areas and other geometric properties.
Let **ABC**, and **PQR** are two co-planar(i.e, in the same plane)triangles defined by a vector equations $T_1 = p_1 + sM + tN$,$T_2 = p'_1 + sM' + tN'$ respectively, where $0 \leq s, t \leq 1,\ 0 \leq s + t \leq 1, M = p_2 - p_1, N = p_3 - p_2, M' = p'_2 - p'_1$
$N' = p'_3 - p'_2$ and p_1, p_2, p_3 are the vertices of T_1, and p'_1, p'_2, p'_3 are the vertices of T_2.
Three categories included all of the triangles' intersection possible results.

Category 1:Single point overlapping
This category includes the following intersections as shown in figure 3.3-3.4:

- Vertex-vertex intersection (figure3.3)
- Vertex-edge interior intersection (figure 3.4)

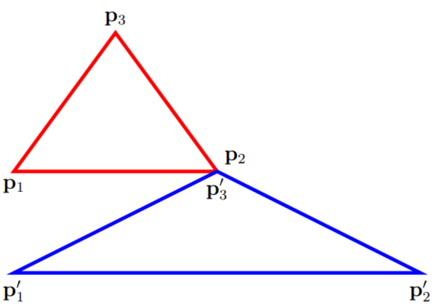


Figure 3.3: Vertex-vertex intersection

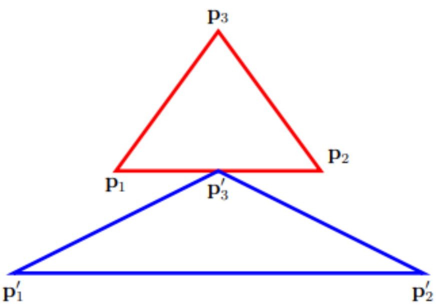


Figure 3.4: Vertex-edge interior intersection

Category 2: Line overlapping
It encompasses the next results, as clarified in figure 3.5-3.7:

- Edge-edge collinear intersection (figure 3.7).
- Edge-triangle interior intersection (figure 3.5).
- Triangle interior-triangle interior intersection (figure 3.6).

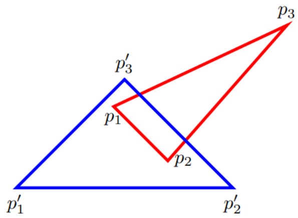


Figure 3.5: Edge-triangle interior intersection

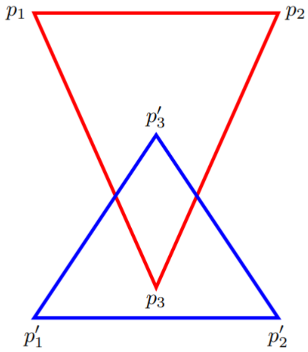


Figure 3.6: Triangle interior-triangle interior intersection

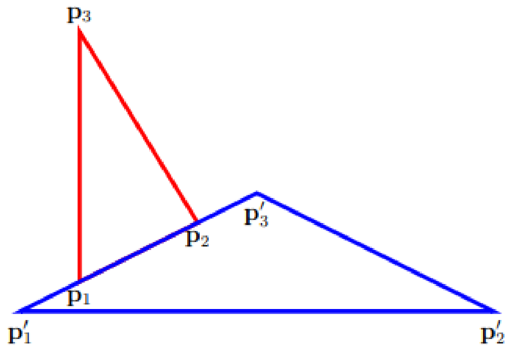


Figure 3.7: Edge interior-edge interior cross intersection

Category 3:Area Overlapping
This one includes the following outcomes, as explained in figures 3.8-3.10:

- Vertex-triangle interior intersection (figure 3.8).
- Edge interior-edge interior cross intersection (figure 3.9).
- Triangle interior-triangle interior intersection (figure 3.10).

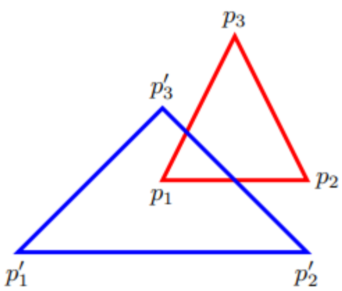


Figure 3.8: Vertex-triangle interior intersection

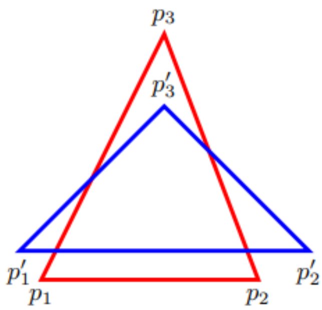


Figure 3.9: Edge interior-edge interior cross intersection

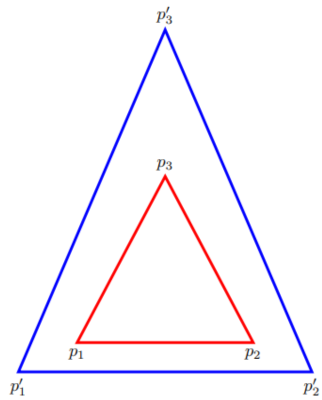


Figure 3.10: Triangle interior-triangle interior intersection

In general, in our study, we will take into account only the intersections that construct an area greater than zero.

3.4 Polygon-Polygon Overlapping

This section is provided by the general case of shapes that are overlapped which are the polygons and their overlapping represented by polygon-polygon overlapping with the possible overlapping cases and the possible outcomes of this process.
As before, in the case of an intersection between a line **L** and a polygon P, the result is either one point or a segment of the line.
The second case is the intersection between polygons, which resulted in three results as follows (shown in the figures 3.11-3.15:

- A point that may be a shared vertex or an interior point of an edge.
- A line segment that may be a shared edge.
- A new polygon may also contain one of the previous results or one of the polygons.

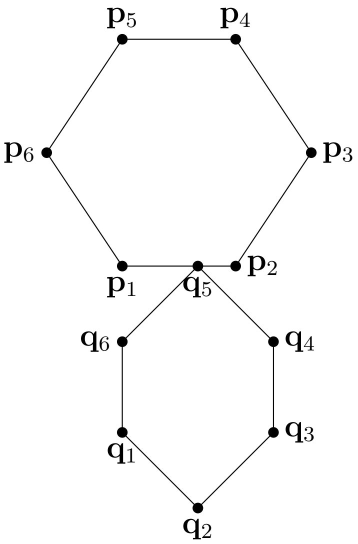


Figure 3.11: Point interior of an edge case.

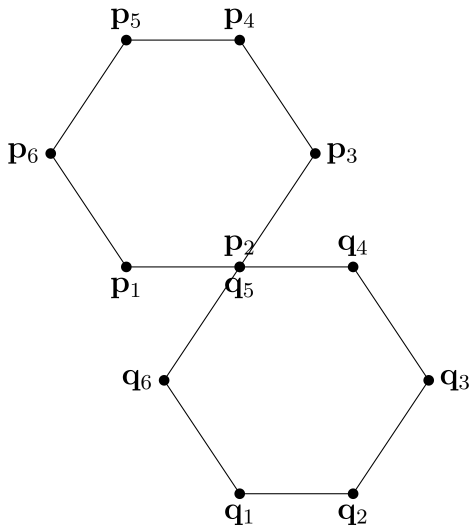


Figure 3.12: Shared vertex case.

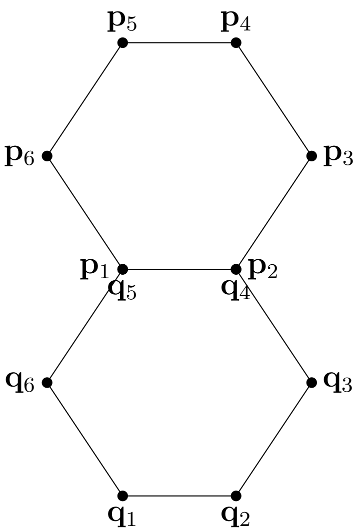


Figure 3.13: Shared edge case.

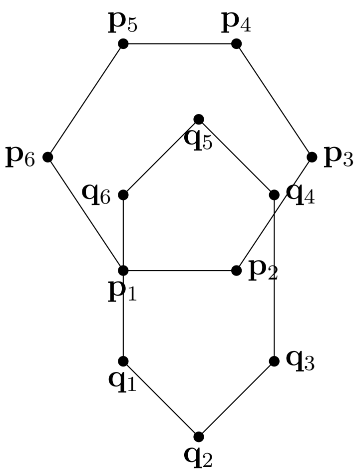


Figure 3.14: New formed polygon case.

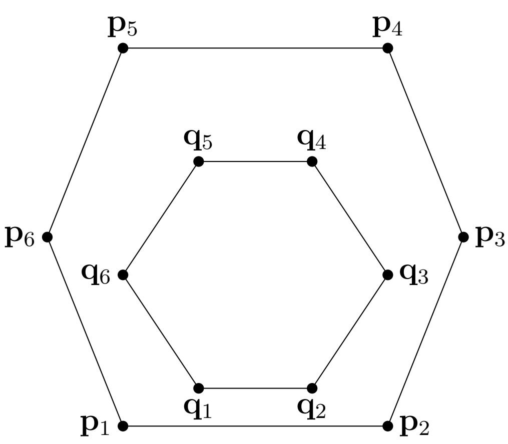


Figure 3.15: One polygon inside the other one case.

3.5 Polygon-Polygon Overlapping Shape

Every shape in 2D can be considered a polygon even a circle, thus in this section, we delve into two following topics:

1. The overlapping Test.
2. The overlapping Resulting shape.

which can be done in polygon-polygon overlapping resulting shape algorithm.

Algorithm 1 **Polygon-Polygon overlapping Resulting Shape Algorithm**

Require: Two polygon $\mathbf{P}_1, \mathbf{P}_2$, vertices $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots, \mathbf{p}_n\}, \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \ldots, \mathbf{q}_n\}$ of $\mathbf{P}_1, \mathbf{P}_2$ respectively.

1: Find intersection points using Line-Line intersection by testing the intersection of every edge of $\mathbf{P}_1$ with every edge of $\mathbf{P}_2$.
2: **return** New formed polygons $\mathbf{P}'_1$ and $\mathbf{P}'_2$.
3: Find the midpoint of every edge of the new polygons.
4: **if** the midpoint of an edge $\mathbf{e}'_i$ of $\mathbf{P}'_1$ is an interior point of $\mathbf{P}'_2$ **then** save the edge
5: **else** eliminate edge $\mathbf{e}'_i$
6: **end if**
7: **if** the midpoint of an edge $\mathbf{e}'_i$ of $\mathbf{P}'_2$ is an interior point of $\mathbf{P}'_1$ **then** save the edge
8: **else** eliminate edge $\mathbf{e}'_i$
9: **end if**
10: Combine the edges resulting from steps 4-9.
11: **return** the overlapping polygon.

Remarks:

- The overlapping test can be accomplished by steps 1-2 of the algorithm.
- The overlapping resulting shape can be conducted by steps 3-10 of the algorithm.
- The figure3.16 clarifies the shaded overlapping polygon in 2D and 3D.

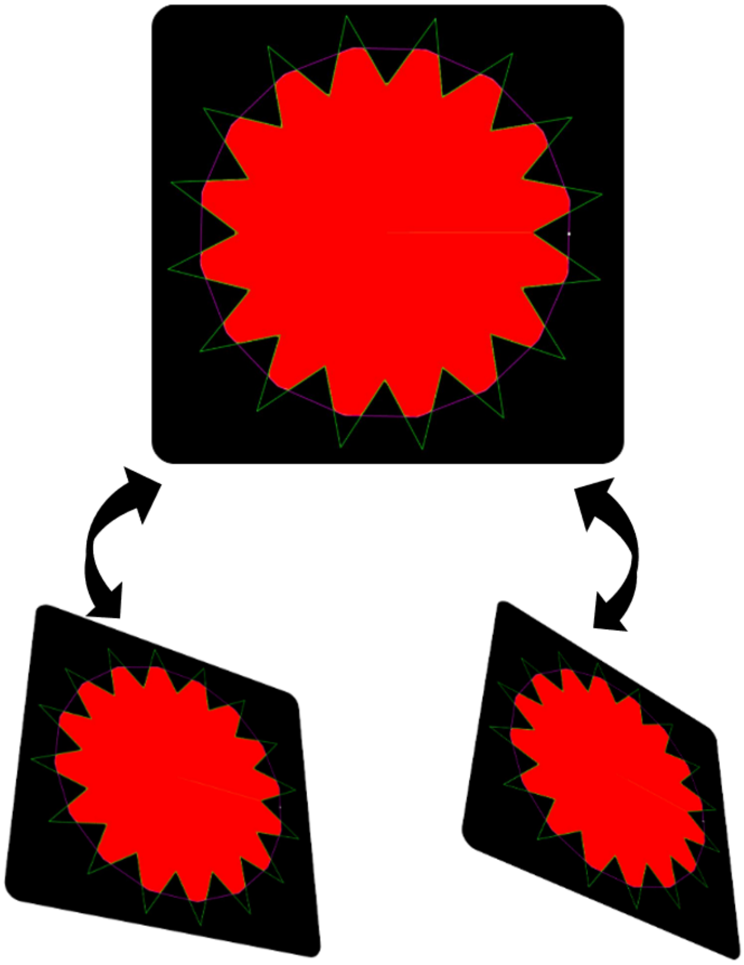


Figure 3.16: **Polygon-polygon overlapping (intersection) in 2D and 3D using the mapping ψ and ψ^{-1}.**

Chapter 4

Point Inclusion Methods for Closed Planar Polygons

The problem of determining if a query point **q** is contained in a given polygon is considered a vast issue in computational geometry applications, especially in computer-aided design (CAD), and geographic information systems(GIS). The basic form of this problem is to deal with the shape as divided into two regions and start to find in which region the given query point lies in.
In the first section we present the **ray tracing inclusion method** that used to check out if a query point lies in,on, or out of a closed planar polygon that started by shooting a ray from the query point, and count every intersection between an edge of the polygon and the shooted ray. If the accumulated count number is even, then the point lies out of the polygon, and if it is odd, then the query point lies inside the polygon. The second section discusses another method called by **global point inclusion method** which seeks to find the closest point of polygon to the query point and using the dot product between the ray and normal vector of the edge containing the closest one to determine the location of the query point.(Abu-Munshar, (2013)), (Kuprat, A., Khamayseh, A., George, D., Larkey, L. (1998))
We use the same reference for the third section that present a new algorithm for that vast issue titles by direct ray point inclusion method that gain the robustness (the algorithm produces a correct results) and efficiency (the algorithm is fast) properties which started by finding the closest point of the polygon to the ray by using line-ray intersection and then use the dot product to determine the closest normal vector to the ray and determine if the query point lies in, or out the polygon. The "Point in polygon problem" in 3D is a generalization of the "Point in polygon problem" in a 2D geometric polygon that involves determining whether a given query point **q** is inside or outside a polygon **P**. This problem has been widely studied. In this section, we will present some of the algorithms and the results of (Abu-Munshar2013) that are used in 2D. Then we will present a new algorithm that satisfies the robustness and efficiency properties in 3D.

4.1 Ray Tracing Point Inclusion Method

One of the first algorithms using for solve point n polygon problem is ray tracing method, which we investigate in this section with mechanism that it is based on, and its algorithm.
Given a closed simple polygon $\mathbf{P}$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{n-1}$, edges $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{n-1}$.
The ray tracing method algorithm is a useful technique but with limitations as it is based on shooting a ray in any direction from a query a point $\mathbf{q}$ and counts the intersection that occurred with the boundary of the polygon. If we have an odd number of intersections then the point in it, and if the number is even then it will be outside the polygon.
The mechanism of the Ray Tracing Point Inclusion test is that the shooted ray changes its position from inside to outside or vice versa continuously until finishing the intersections and being outside the polygon.
If the point is inside the shape then the ray starts from inside and after the first intersection, it will become outside the polygon as the role of the boundary is to separate the " in " and " out " parts of the polygon. Then, the next intersection returns it inside again, which will end up outside the polygon after some repeating intersections to get an odd number of intersections indicating that the point is inside the shape.
But, if the point is outside the polygon, the method differs in the cross results as the ray will be inside the polygon, next cross will return outside again. It will continuously do the crosses until it finishes also outside the shape to get an even number of intersections indicating that the point is outside the polygon.

Algorithm 2 Ray Tracing Point Inclusion Test Algorithm

```
input: query point q, vertices {v_0, v_1, v_2, ..., v_n} of P.
Let e_i be the edge joining the vertices v_i, v_{i+1}
Set count=0
Shoot a random ray r from q .
For i = 0 to n − 1
if e_i ∩ r ≠ ϕ &(e_i ∩ r ≠ v_i and e_i ∩ r ≠ v_{i+1})
Count++
If e_i ∩ r ≠ ϕ & e_i ∩ r = v_i
Then set r = q + t (p−q)/||p−q|| + (1/2) e_1 and repeat the previous steps
If e_i ∩ r ≠ ϕ & e_i ∩ r = v_{i+1}
Then set r = q + t (p−q)/||p−q|| + (1/2) e_1 and repeat the previous steps
If the count is odd, return inside
Else return outside.
```

The two other algorithms differ just in the search goal, as the global point inclusion method (clarified in the thesis) is based on finding the closest point $\mathbf{p}$ on the boundary of the polygon to the query point $\mathbf{q}$, while the direct point inclusion method (for more information see [28]) seeks to find the closest point $\mathbf{p}$ on the boundary to the given shot ray.

4.2 Global Point Inclusion Method

The global inclusion method is a method developed by two scientists (Khamayseh and Kuprat) in 2008, which is a robust method for giving correct results for the point in the polygon problem.
This method is based on the following three steps:

1. Finding the closest point $\mathbf{p}$ on the boundary of the polygon to the query point $\mathbf{q}$.
2. Finding the normal $\mathbf{n}_p$.
   If the point $\mathbf{p}$ is the interior point of the polygon, then $\mathbf{n}_p$ is the unit outward normal vector of the polygon. If the point is a boundary point (either it is a vertex shared by two edges or contained on an edge) then $\mathbf{n}_p$ can be computed by one of the following :
   - Synthetic normal method.
   - Visible normal method.
3. This step include finding a numerical value $d = (\mathbf{p} - \mathbf{q}).\mathbf{n}_p$. Then,
   - if $d > 0$ then the point is inside the polygon
   - if $d < 0$ then the point is outside the polygon
   - if $d = 0$ then $\mathbf{q} = \mathbf{p}$

Methods for finding the $\mathbf{n_p}$:

1. **Synthetic Normal Method:**
   This approach was initiated in 3D by Khamayseh and Kuprat (2008), as it calculates the weighted normal of normals in the solid ball (intersection of the neighborhood with the boundary of the object) of $\mathbf{p}$. In 2D, the same procedure will be conducted, but the neighborhood (the intersection of the neighborhood with the boundary of the object) is a circle, and we compute the average of the normals in the neighborhood of the point. As we assumed that it was a $\mathbf{C}^1$ continuous shape[1], the normal vector will be the limit of the normals as the

[1] A $\mathbf{C}^1$ continuous shape implies that it is continuously differentiable up to the first derivative.

normals change direction gradually.

According to those cases, the normal at $\mathbf{p}$ is defined as follows:

(a) If $\mathbf{p} \in \mathbf{e}_i$ then $\mathbf{n}_p = \mathbf{n}_i$

(b) If $\mathbf{p} \in \mathbf{e}_i \cap \mathbf{e}_j$ then

$$\mathbf{n}_p = \frac{\mathbf{n}_i + \mathbf{n}_j}{\| \mathbf{n}_i + \mathbf{n}_j \|} \tag{4.1}$$

where $\mathbf{n}_i = N \times \mathbf{e}_i$ is the normal at edge $\mathbf{e}_i$. Where N is the outward normal vector of the polygon.

Let $\mathbf{N}_\varepsilon(\mathbf{p}) = B_\varepsilon(\mathbf{p}) \bigcap \partial P$, then the synthetic normal is equal to the line integral:

$$\mathbf{n_p} = \lim_{\varepsilon \to 0} \frac{\int\limits_{\mathbf{N}_\varepsilon(\mathbf{p})} \mathbf{n} dl}{\left\| \int\limits_{\mathbf{N}_\varepsilon(\mathbf{p})} \mathbf{n} dl \right\|} \tag{4.2}$$

where $\mathbf{B}_\epsilon(\mathbf{p})$ is a closed ball with radius ϵ centered at $\mathbf{p}$ and $\mathbf{n}$ is the outward normal on the surface.
we can see that there is an equivalence between equations 124,125 as

$$\begin{aligned} \frac{\int\limits_{\mathbf{N}_\varepsilon(\mathbf{p})} \mathbf{n} dl}{\left\| \int\limits_{\mathbf{N}_\varepsilon(\mathbf{p})} \mathbf{n} dl \right\|} &= \frac{\int_0^\varepsilon \mathbf{n}_i d\tau + \int_0^\varepsilon \mathbf{n}_j d\tau}{\left\| \int_0^\varepsilon \mathbf{n}_i d\tau + \int_0^\varepsilon \mathbf{n}_j d\tau \right\|} \\ &= \frac{\left(\mathbf{n}_i + \mathbf{n}_j\right) \varepsilon}{\left\| \left(\mathbf{n}_i + \mathbf{n}_j\right) \varepsilon \right\|} = \frac{\left(\mathbf{n}_i + \mathbf{n}_j\right)}{\left\| \left(\mathbf{n}_i + \mathbf{n}_j\right) \right\|} \end{aligned} \tag{4.3}$$

2. **Visible Normal Method**
Khamayseh, Ortega, and Kuprat developed this method in 1995, and this method is based on the cases where the point $\mathbf{p}$ lies as explained by the following:

(a) if $\mathbf{p}$ is interior point of an edge $\mathbf{e}_i$, then the normal outward vector $\mathbf{n}_p = \mathbf{n}_i$, as defined in the synthetic normal method.

(b) if $\mathbf{p} \in \mathbf{e}_i \bigcap \mathbf{e}_j$ ($\mathbf{p}$ is a vertex) where $\mathbf{e}_i$ and $\mathbf{e}_j$ are two edges in the polygon, then $\mathbf{n}_p$ is chosen based on the following condition:

$$\begin{aligned} &\textbf{if} \;\; \left|(\mathbf{q} - \mathbf{p}) . \mathbf{n_i}\right| \geq \left|(\mathbf{q} - \mathbf{p}) . \mathbf{n_j}\right| \;\; \textbf{then} \;\; \mathbf{n_p} = \mathbf{n_i} \\ &\textbf{if} \;\; \left|(\mathbf{q} - \mathbf{p}) . \mathbf{n_j}\right| \geq \left|(\mathbf{q} - \mathbf{p}) . \mathbf{n_i}\right| \;\; \textbf{then} \;\; \mathbf{n_p} = \mathbf{n_j}, \end{aligned}$$

where $\mathbf{n}_i$, $\mathbf{n}_j$ are the outward normal vectors of the edges $\mathbf{e}_i$, $\mathbf{e}_j$ respectively, and $\mathbf{q}$ is a given query point.

Based on this condition, the resulted outward normal vector $\mathbf{n}_p$ is called the visible normal as it the normal vector of the visible edge

Algorithm 3 Global Point Inclusion Method Algorithm

Require: query point $\mathbf{q}$, vertices $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of P

```
Find a point p on P closest to q
r ← q − p
if ||r|| = 0 then
    return outside
end if
Calculate the normal vector n_p at p
if r · n_p > 0 then
    return outside
else
    return inside
end if
```

Finding The Closest Boundary Point

As aforementioned, the second main step of the global point inclusion method is that we seek to find the closest boundary point to the query point $\mathbf{q}$. The global point inclusion method checks the distance between the query point and every edge of the polygon to find the minimum distance that produces the closest boundary point which is an interior point to this edge.

4.3 Directed Ray Point Inclusion Method

As a result of the limitations of the aforementioned algorithms of solving the point in polygon problem, we express, in this section, a new algorithm entitled by direct ray point inclusion method that has the robustness and efficiency properties. Also, we provide algorithm of this method with all possible cases of the normal vector of closest boundary point of the shot ray that is initiated from a given query point.
This method is less expensive than the global point inclusion method as finding the closest boundary point here is faster as we seek to find the closest boundary point along a ray shooted from the query point **q**, while in the previous method, we need to find the closest one the query point. In addition, this way of finding the closest boundary point has two advantages as follows:

1. Trivial test rejection.
   If the shooted ray has no intersection with the polygon boundary then the point is obviously outside the polygon.

2. Closest point to the ray.
   Finding the closest boundary point along the shooted ray is less expensive than finding the closest one to the query point itself.

Algorithm 4 Directed Ray Point Inclusion Method Algorithm

Given a plane $\mathcal{P}$ defined by equation: $(\mathbf{x} - \mathbf{x}_0) \cdot \hat{\mathbf{n}} = 0$.
Given a closed simple polygon $\mathbf{P}$ contained in $\mathcal{P}$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{n-1}$, edges $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{n-1}$. Also, $\partial\mathbf{P}$ is the boundary of the polygon, and $\hat{\mathbf{n}}_i = \hat{\mathbf{n}} \times \mathbf{e}_i$ is the normal vector of the edge $\mathbf{e}_i$, where $\hat{\mathbf{n}} = \frac{(\mathbf{e}_i \times \mathbf{e}_{i+1})}{\|\mathbf{e}_i \times \mathbf{e}_{i+1}\|}$ for a fixed i.
Let $\mathbf{q}$ be the given query point and $\mathbf{p}$ be the closest boundary point along the shot ray $\mathbf{r}$ from $\mathbf{q}$.
if $(\mathbf{q} - \mathbf{x}_0) \cdot \hat{\mathbf{n}} \neq 0$ **then**
return $\mathbf{q}$ is outside.
else
Set $t_{\min} = \infty$, and the closest point $\mathbf{p}$ to the ray $\mathbf{r}$ by $\mathbf{q}$.
for $i = 0$ **to** $n - 1$ **do**
Compute $\mathbf{t}_i$ using Equation (1).
if $0 \leq \mathbf{t}_i$ **and** $t_i < t_{\min}$ **then**
Set $t_{\min} = t_i$ and $\mathbf{p} = \mathbf{q} + t_{\min}\mathbf{v}$.
else
$t_{\min} = \infty$.
end if
end for
Let the direction of the shot ray (the ray itself) be $\mathbf{r}^{dir} = -\mathbf{R} = \frac{(\mathbf{q}-\mathbf{p})}{\|\mathbf{q}-\mathbf{p}\|}$.
Let $\mathcal{A}$ be a set of the patches that $\mathbf{p}$ is contained in.
if $|\mathcal{A}| = 1$ **then**
if $\mathbf{r}^{dir} \cdot \hat{\mathbf{n}}_i > 0$ **then**
return $\mathbf{q}$ is outside.
else
return $\mathbf{q}$ is inside.
end if
else if $|\mathcal{A}| = 2$ **then**
if $\mathbf{r}^{dir} \cdot \hat{\mathbf{e}}_i > \mathbf{r}^{dir} \cdot \hat{\mathbf{e}}_{i+1}$ & $\mathbf{r}^{dir} \cdot \hat{\mathbf{e}}_i > \hat{e}_{\max}$ **then**
$\hat{e}_{\max} = \mathbf{r}^{dir} \cdot \hat{\mathbf{e}}_i$, $\hat{\mathbf{n}}_{\max} = \hat{\mathbf{n}}_i$.
else if $\mathbf{r}^{dir} \cdot \hat{\mathbf{e}}_{i+1} > \mathbf{r}^{dir} \cdot \hat{\mathbf{e}}_i$ & $\mathbf{r}^{dir} \cdot \hat{\mathbf{e}}_{i+1} > \hat{e}_{\max}$ **then**
$\hat{e}_{\max} = \mathbf{r}^{dir} \cdot \hat{\mathbf{e}}_{i+1}$, $\hat{\mathbf{n}}_{\max} = \hat{\mathbf{n}}_{i+1}$.
else
$t_{\max} = -\infty$.
end if
if $\mathbf{r}^{dir} \cdot \hat{\mathbf{n}}_i > \mathbf{r}^{dir} \cdot \hat{\mathbf{n}}_{i+1}$ & $\mathbf{r}^{dir} \cdot \hat{\mathbf{n}}_i > t_{\max}$ **then**
$t_{\max} = \mathbf{r}^{dir} \cdot \hat{\mathbf{n}}_i$, $\hat{\mathbf{n}}_{\max} = \hat{\mathbf{n}}_i$.
else if $\mathbf{r}^{dir} \cdot \hat{\mathbf{n}}_{i+1} > \mathbf{r}^{dir} \cdot \hat{\mathbf{n}}_i$ & $\mathbf{r}^{dir} \cdot \hat{\mathbf{n}}_{i+1} > t_{\max}$ **then**
$t_{\max} = \mathbf{r}^{dir} \cdot \hat{\mathbf{n}}_{i+1}$, $\hat{\mathbf{n}}_{\max} = \hat{\mathbf{n}}_{i+1}$.
else
$\hat{\mathbf{n}}_{\max} = \frac{\mathbf{n}_i + \mathbf{n}_{i+1}}{\|\mathbf{n}_i + \mathbf{n}_{i+1}\|}$.
end if
end if
if $\mathbf{r} \cdot \hat{\mathbf{n}}_{\max} \geq 0$ **then**
return $\mathbf{q}$ is outside.
end if
end if

Remarks:

- The figures4.1 and 4.2 demonstrate creating the structure of the geometry (data structure in programming domain) of the polygons before conducting the direct ray point inclusion algorithm in 2D and 3D.

Figure 4.1: Loops(polygons) geometric structure in 2D

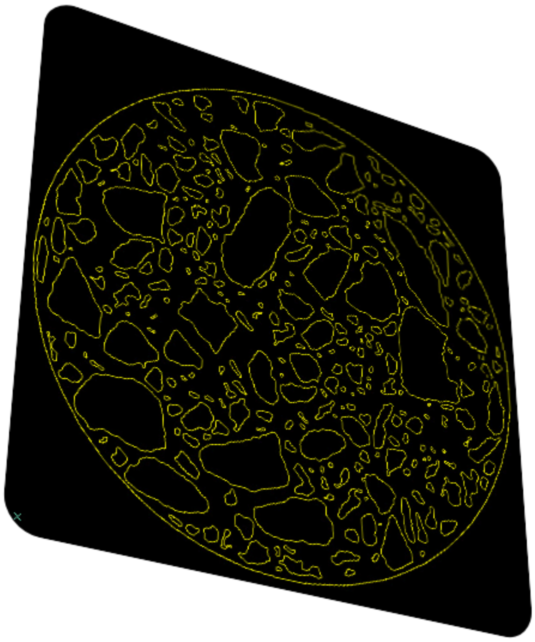


Figure 4.2: Loops(polygons) geometric structure in 3D

- The figures4.3 and 4.4 show the random dropping process of points in and out of polygons.

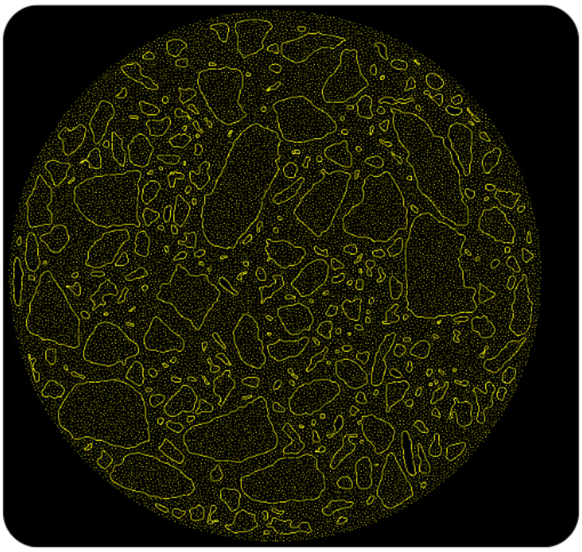


Figure 4.3: Points dropping phase in 2D

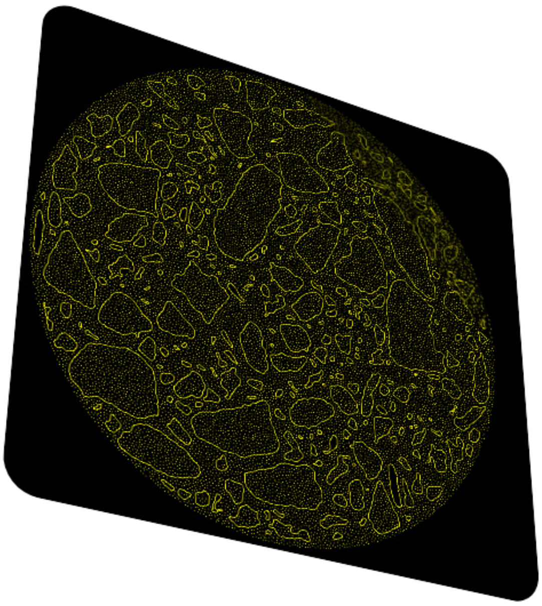


Figure 4.4: Points dropping phase in 3D

- The figures4.5 and 4.6 are clarifying the conduction of the direct ray point location method in 3D algorithm in 2D and 3D.

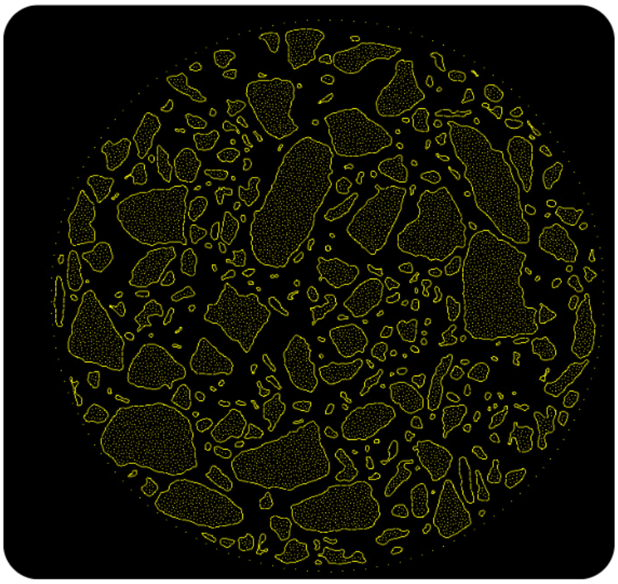


Figure 4.5: The points inside the loops in 2D

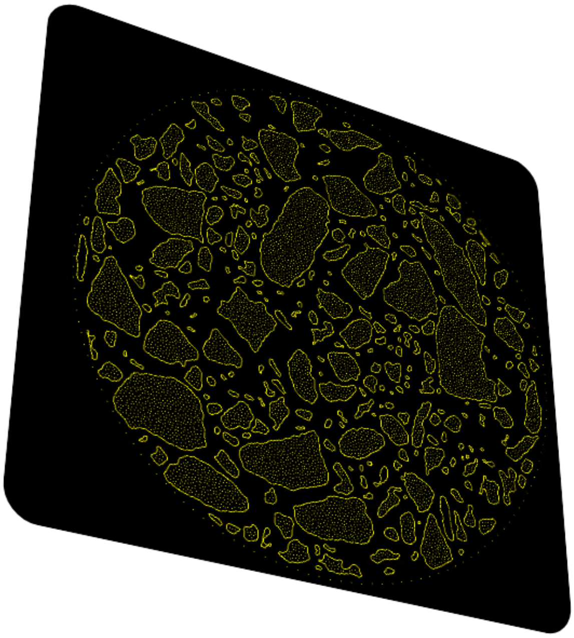


Figure 4.6: The points inside the loops in 3D

Chapter 5

Smoothing of Planar Polygons in 2D and 3D

The first section in this chapter delves into the process of smoothing planar polygons in $\mathbf{R}^2$ using single node relaxation and edge relaxation methods that include tangential and tangential and normal smoothing respectively to have smooth planar polygon without any jaggedness or any stair-step phenomena in simulation the any object on a grid. This process primarily seeks to an area-conserving smoothing process of the polygons as we just need to smooth the shape not to change an geometrical or physical property (in case in $\mathbb{R}^3$). The material of this section is taken from **(Noufal, B. (2019))**, **(J. Banks, J. S. Carson, and B. L. Nelson, (2010))**, **(Kuprat, A., Khamayseh, (1998))**.
The final section delves into area-conserving smoothing process in $\mathbb{R}^3$,after the first section paved the way for smoothing an object in $\mathbb{R}^3$, using the map ψ to downgrade the object from 3D to 2D and smooth it, then we use the inverse map ψ^{-1} to return it bake to its authentic habituate $\mathbb{R}^3$. This section's material derived from **(Noufal, B. (2019))**.

5.1 Area-Conserving Smoothing of Planar Polygons Plane in 2D

Area-conserving smoothing is a technique used in computer graphics to smooth piecewise linear curves and surfaces while preserving their total area. This method is useful in many applications, including computer-aided design, simulation, and visualization.

For our study, it is needed in simulation to smooth the surface grid to overcome the jaggedness phenomena problem. Also to avoid the noise in the grid since the noise will give us incorrect results for the physics quantities we simulate.

Definition 5.1.1. The process of building a model for an existing or proposed system to imitate it and its operation while making sure that we can test different scenarios or process changes is called by **the simulation process**.

Definition 5.1.2. The **polygonal mesh** is defined as the process of association of the parts of a polygon which include vertices(nodes, points), edges, and faces that are used to define the shape of an object.
The vertices characterized the geometry of the shape, and the faces always either are triangles(triangle mesh), quadrilaterals, or simple convex or non-convex polygons. The surface of the grid constructed by the mesh is called **surface grid**.

Definition 5.1.3. The process of removing the noise of the surface grid with minimal damage to the geometric object is called **smoothing** or **mesh smoothing**. Smoothing is carried out by moving the vertices of the geometric feature with the constraint of not changing the connectivity of the edges or losing or adding vertices to the polygon.

Physics-based simulations often result in curves or surfaces that are jagged or as we call non-smooth which may be unsuitable for the simulation that will be conducted after the initial simulation but in other conditions is called subsequent simulation. This jaggedness represented by the stair-step phenomena in simulation might generate incorrect results in the subsequent simulation unless the curve or the surface that is the interface of subsequent simulation is smoothed,
By smoothing the interface we smooth the surface grid, and to be this achieved, three conditions have to be satisfied:

1. **Adjacent facets of the surface grid have normals adjusted to vary more gradually:**
   When we smooth the surface grid, we ensure that the normals of the adjacent facets change their direction gradually, since this helps to create a more continuous and natural-locking surface.
2. **Nodes densities are equidistributed on the surface:**
   As we know node density refers to the number of nodes in a given area, thus, when the surface is smoothed, it is important to ensure that the node is uniform across the surface. i.e., the nodes are equidistributed on the surface, since the uneven distribution of nodes on a surface can result in irregularities, which may

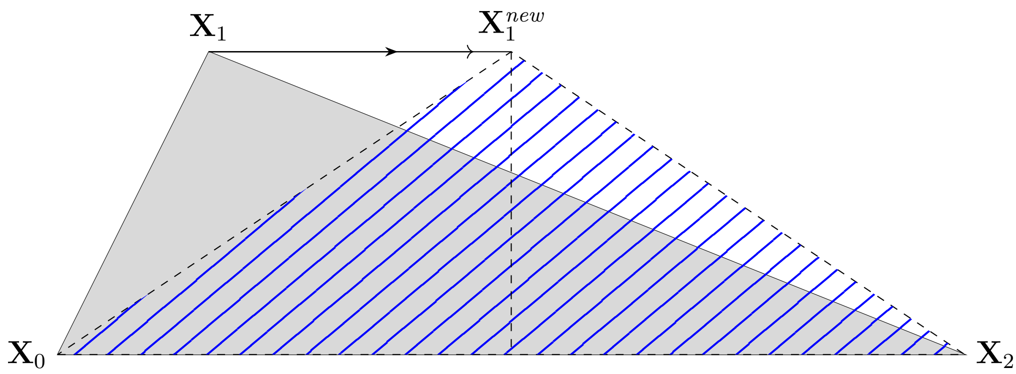


Figure 5.1: **Moving the node $\mathbf{x}_1$ in the tangential direction to the edge $\overrightarrow{\mathbf{x}_0\mathbf{x}_3}$**

cause the surface to stretch or compress in certain areas, ultimately leading to the appearance of noticeable defects or visual artifacts.

3. **The aspect ratios of facets are improved:**
   The aspect ratio of a facet refers to the ratio of its longest side to its shortest side. When smoothing a surface grid, it's important to ensure that the aspect ratios of the facets are improved. High aspect ratios can lead to stretched or distorted facets, which can create visible artifacts on the surface.

5.1.1 Area Conserving Smoothing Using Single Node Relaxation Method

Suppose $P = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \cdots, \mathbf{x}_n\}$ is a closed non-self intersecting curve(polygon) in $\mathbb{R}^2$ that encloses a region R with a signed area with counterclockwise orientation. We seek a conservative smoothing operation of this curve locally for each point $\mathbf{x}_i$ which means changing the position of the point's neighborhood set of points $\{\mathbf{x}_{i-s}, \mathbf{x}_{i-s+1}, \cdots, \mathbf{x}_{i+s}\}$, *s small.*
To smooth the curve P, we should do it in each local neighborhood point in the whole curve in some order, which is called **sweep**.
We aim to perform the smoothing operation without altering the area of the region R. Thus, the number of sweeps should be as small as possible.

As a surface grid consists of either triangles or polygons, we first will apply the smoothing operation on a triangle.
Suppose we have a triangle T with vertices $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ in $\mathbb{R}^2$ in, as shown in figure5.1, with a signed area A_1 based on a counterclockwise orientation, thus

$$\begin{aligned} A_{prev} &= \frac{1}{2}(\mathbf{x}_2 - \mathbf{x}_0)^{\perp} \cdot (\mathbf{x}_1 - \mathbf{x}_0) \\ &= \frac{1}{2}\|\mathbf{x}_2 - \mathbf{x}_1\|.h \end{aligned} \tag{5.1}$$

where $h = hieght\ of\ triangle$, and the vector $\mathbf{n}$ is the normal vector of the plane containing the triangle T, and the unit normal vector $\hat{n} = \frac{(x_2-x_0)^\perp}{\|(x_2-x_0)^\perp\|}$, where the perpendicular vector to any vector $\mathbf{m}$ is the vector $\mathbf{m}^\perp = (-\mathbf{m}_y, \mathbf{m}_x)$.
Thus

$$h = \frac{2\mathbf{A}_{prev}}{\|\mathbf{x}_2 - \mathbf{x}_0\|} \tag{5.2}$$

The determined new position of vertex x_1 can be calculated as $x_1^{\text{new}} = \frac{1}{2}(x_0 + x_2) + h\hat{\mathbf{n}}$. Then, by doing this process for all nodes sequentially ordered, we achieved the needed smoothing process. Therefore, we have an algorithm for smoothing the whole geometric object as follows:

Algorithm 5 The single node relaxation method for area-conserving smoothing of a closed plane curve.

for $i = 0$ to $n - 1$ **do**
  Perform the smoothing operation on the neighborhood $\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}$ (i.e. Relax the node $\mathbf{x}_{i+1}$)
  $A_{\text{prev}} \leftarrow \frac{1}{2}(\mathbf{x}_{i+2} - \mathbf{x}_i)^\perp \cdot (\mathbf{x}_{i+1} - \mathbf{x}_i)$
  $h \leftarrow \frac{2\mathbf{A}_{prev}}{\|\mathbf{x}_{i+2}-\mathbf{x}_i\|}$
  $\hat{n} \leftarrow \frac{(x_{i+2}-x_i)^\perp}{\|(x_{i+2}-x_i)^\perp\|}$
  $x_{i+1}^{\text{new}} \leftarrow \frac{1}{2}(x_i + x_{i+2}) + h\hat{\mathbf{n}}$
end for

Claim: Algorithm 4 is an area preserving algorithm.

Proof:

$$
\begin{aligned}
\mathbf{A}^{new} &= \frac{1}{2}(x_{i+2} - x_i)^{\perp} \cdot (x_{i+1}^{new} - x_i) \\
&= \frac{1}{2}(x_{i+2} - x_i)^{\perp}.(\frac{1}{2}(x_{i+2} + x_i) + h\hat{n} - x_i) \\
&= \frac{1}{2}(x_{i+2} - x_i)^{\perp}.(\frac{1}{2}(x_{i+2} - x_i) + h\hat{n}) \\
&= \frac{1}{2}(x_{i+2} - x_i)^{\perp}. \left(\frac{1}{2}(x_{i+2} + x_i) + h\frac{(x_{i+2} - x_i)^{\perp}}{\|(x_{i+2} - x_i)^{\perp}\|} \right) \\
&= \frac{1}{4}(x_{i+2} - x_i)^{\perp}.(x_{i+2} - x_i) + \frac{h}{2\|(x_{i+2} - x_i)^{\perp}\|}(x_{i+2} - x_i)^{\perp} \cdot (x_{i+2} - x_i)^{\perp} \\
&= \frac{h}{2\|(x_{i+2} - x_i)^{\perp}\|}\|(x_{i+2} - x_i)^{\perp}\|^2 \\
&= \frac{h}{2}\|(x_{i+2} - x_i)^{\perp}\| \\
&= \frac{h}{2}\|(x_{i+2} - x_i)\| \\
&= \frac{2\mathbf{A}^{prev}}{2\|(x_{i+2} - x_i)\|}\|(x_{i+2} - x_i)\| \\
&= \mathbf{A}^{prev}
\end{aligned}
$$

■

Remarks:

- If the curve is open, then we don't relax the endpoints, as the endpoints of a given curve are stable and fixed, so any perturbation of them will distort the curve.
- As the algorithm smooths in the tangential direction, the smoothing operation in the normal direction means changing the position of the node in the direction of the surface normal vector.
  Thus, if we take a star shape as in figure 5.2. Normal smoothing is forbidden because of the conservation area requirement. As a result, the single node relaxation method algorithm does not work for all shapes; thus, we need a new method for smoothing that guarantees normal smoothing, which is called the **edge relaxation method**.

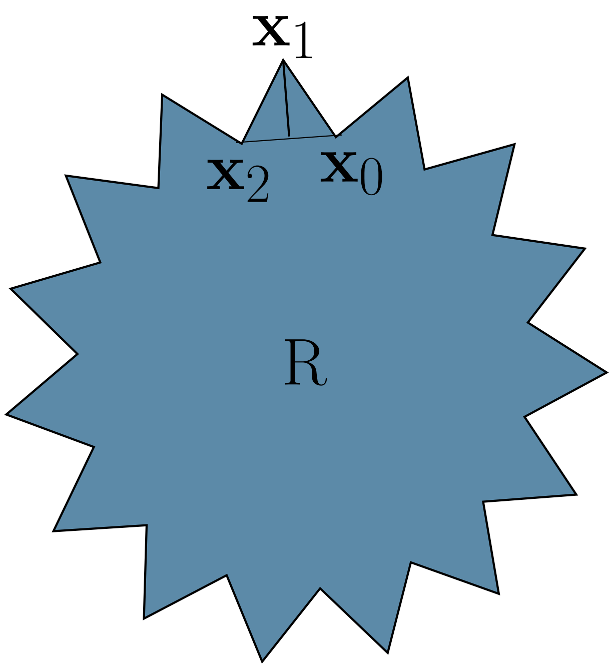


Figure 5.2: **Algorithm 1 does not change the Star-shape.**

5.1.2 Area Conserving Smoothing Using Edge Relaxation Method

The edge relaxation method smooths shapes in the tangential and normal directions while satisfying the condition of the conserving area.
Consider a closed curve ζ with four sequential points $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ as shown in the non-dashed shape in figure 5.3. Let the direction of the edge $\overrightarrow{\mathbf{x}_0\mathbf{x}_3}$ be the direction tangential to the curve, and the direction normal to ζ refers to the direction orthogonal to the edge. We have a single constraint for normal smoothing while there are two degrees of freedom (the normal components of $\mathbf{x}_1$ and $\mathbf{x}_2$). Therefore, to make normal smoothing allowed, we simultaneously change the positions of $\mathbf{x}_1$ and $\mathbf{x}_2$ while ensuring conserving area conditions.
The smoothing operation here is achieved by moving $\mathbf{x}_1$ and $\mathbf{x}_2$ such that their projection onto the edge $\overrightarrow{\mathbf{x}_0\mathbf{x}_3}$ are one third and two third of the length of $\overrightarrow{\mathbf{x}_0\mathbf{x}_3}$ respectively so that the projections are spaced equally and the distance between $\mathbf{x}_1$ and $\overrightarrow{\mathbf{x}_0\mathbf{x}_3}$ which is represented by h is the same between $\mathbf{x}_2$ and $\overrightarrow{\mathbf{x}_0\mathbf{x}_3}$.
As the quadrilateral $(\mathbf{x}_0, \mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_0)$ will become a trapezium by smoothing with two bases $\overline{\mathbf{x}_0\mathbf{x}_3}$ and $\overline{\mathbf{x}_1^{new}\mathbf{x}_2^{new}}$ with lengths l and $\frac{1}{3}l$ respectively. Thus, the area of the quadrilateral A_{quad}^{prev} is :

$$\begin{aligned} A_{quad}^{prev} &= \frac{1}{2}h(l + \frac{1}{3}l) \\ &= \frac{2}{3}hl \end{aligned} \tag{5.3}$$

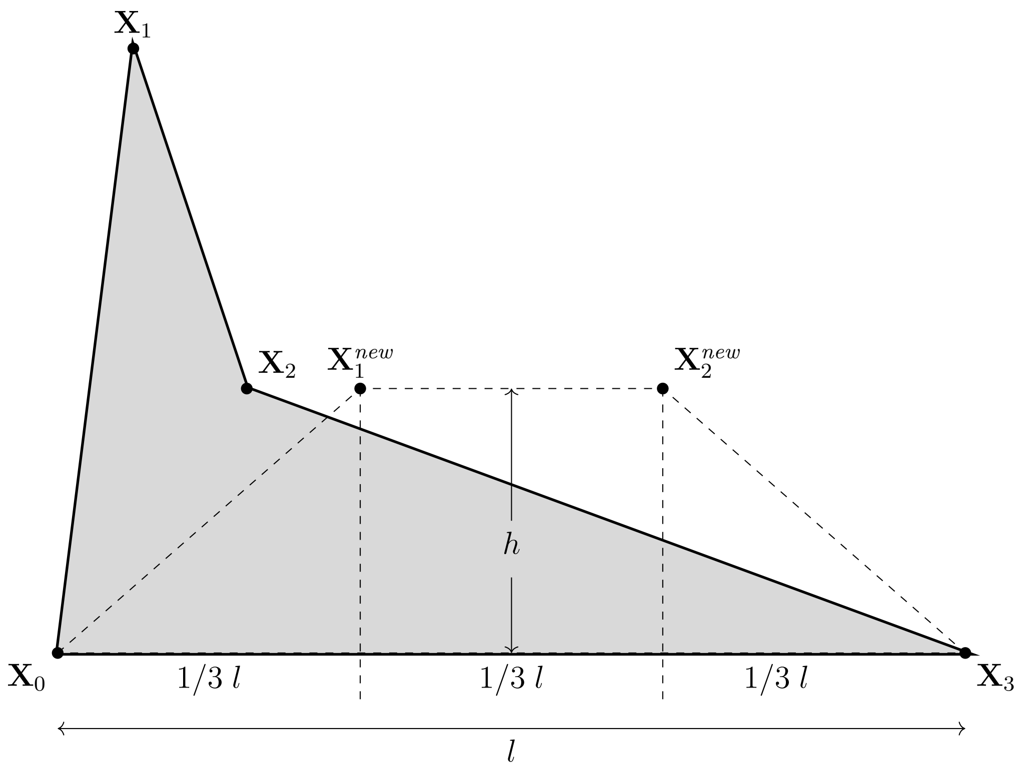


Figure 5.3: **Smoothing the nodes $\mathbf{x}_1, \mathbf{x}_2$ by edge relaxation method.**

therefore,

$$h = \frac{3A_{quad}^{prev}}{2l} \tag{5.4}$$

and with the unit normal vector $\hat{n} = \frac{(x_3 - x_0)^{\perp}}{\|(x_3 - x_0)^{\perp}\|}$, thus

$$\begin{aligned} \mathbf{x}_1^{\text{new}} &= \mathbf{x}_0 + \frac{1}{3}(\mathbf{x}_3 - \mathbf{x}_0) + h\hat{n} \\ \mathbf{x}_2^{\text{new}} &= \mathbf{x}_0 + \frac{2}{3}(\mathbf{x}_3 - \mathbf{x}_0) + h\hat{n} \end{aligned} \tag{5.5}$$

Algorithm 6 The edge relaxation method for area conserving smoothing of a closed plane curve

for $i = 0$ to $n-1$ **do**
  Perform the smoothing operation on the neighbourhood $\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{x}_{i+3}$ (i.e. Relax the node $\overline{\mathbf{x}_{i+1}\mathbf{x}_{i+2}}$)
  Compute the unit normal vector $\hat{n} = \frac{(x_{i+3}-x_i)^\perp}{\|(x_{i+3}-x_i)^\perp\|}$
  Compute the perpendicular to the edge

- $(x_{i+2} - x_i)^\perp = (-(x_{i+2} - x_i)_y, (x_{i+2} - x_i)_x)$
- $(x_{i+3} - x_i)^\perp = (-(x_{i+3} - x_i)_y, (x_{i+3} - x_i)_x)$

  Compute local area A^{prev}_{quad} as follows:
    $A^{prev}_{quad} = \frac{1}{2}(x_{i+3} - x_i)^\perp \cdot (x_{i+2} - x_i) + \frac{1}{2}(x_{i+2} - x_i)^\perp \cdot (x_{i+1} - x_i)$
  $h = \frac{3}{2}\frac{A^{prev}_{quad}}{\|\mathbf{x}_{i+3}-\mathbf{x}_i\|}$
  $\mathbf{x}^{\text{new}}_{i+1} = \mathbf{x}_i + \frac{1}{3}(\mathbf{x}_{i+3} - \mathbf{x}_i) + h\hat{n}$
  $\mathbf{x}^{\text{new}}_{i+2} = \mathbf{x}_i + \frac{2}{3}(\mathbf{x}_{i+3} - \mathbf{x}_i) + h\hat{n}$
end for

Lemma 5.1.1. Let $\mathbf{a} = (a_x, a_y)$, and $\mathbf{b} = (b_x, b_y)$, where $a_x, a_y, b_x, b_y \in \mathbb{R}$, and $\mathbf{a}^\perp = (-a_y, a_x)$ and $\mathbf{b}^\perp = (-b_y, b_x)$, then $(\mathbf{a}+\mathbf{b})^\perp = (\mathbf{a})^\perp + (\mathbf{b})^\perp$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$.

Proof.

$$\begin{aligned}
(\mathbf{a}+\mathbf{b}) \cdot (\mathbf{a}^\perp + \mathbf{b}^\perp) &= (\mathbf{a}_x + \mathbf{b}_x, \mathbf{a}_y + \mathbf{b}_y) \cdot (-\mathbf{a}_y, \mathbf{a}_x) + (-\mathbf{b}_y, \mathbf{b}_x) \\
&= (\mathbf{a}_x + \mathbf{b}_x, \mathbf{a}_y + \mathbf{b}_y) \cdot (-\mathbf{a}_y, \mathbf{a}_x) + (\mathbf{a}_x + \mathbf{b}_x, \mathbf{a}_y + \mathbf{b}_y) \cdot (-\mathbf{b}_y, \mathbf{b}_x) \\
&= -\mathbf{a}_x\mathbf{a}_y + -\mathbf{b}_x\mathbf{a}_y + \mathbf{a}_x\mathbf{a}_y + \mathbf{b}_y\mathbf{a}_x \\
&- \mathbf{a}_x\mathbf{b}_y + -\mathbf{b}_x\mathbf{b}_y + \mathbf{b}_x\mathbf{a}_y + \mathbf{b}_y\mathbf{b}_x \\
&= 0
\end{aligned}$$

Hence, $(\mathbf{a}+\mathbf{b})^\perp = (\mathbf{a})^\perp + (\mathbf{b})^\perp$. □

Remark. The previous lemma is basic part for proving the next claim.

Claim: Algorithm 5 is an area preserving algorithm.

Proof:

$$
\begin{aligned}
A_{quad}^{new} &= \frac{1}{2}(x_{i+3}-x_i)^{\perp}\cdot(x_{i+2}^{new}-x_i)+\frac{1}{2}(x_{i+2}^{new}-x_i)^{\perp}\cdot(x_{i+1}^{new}-x_i)\\
&= \frac{1}{2}(x_{i+3}-x_i)^{\perp}\cdot(\frac{2}{3}(\mathbf{x}_{i+3}-\mathbf{x}_i)+h\hat{n})+\frac{1}{2}(\frac{2}{3}(\mathbf{x}_{i+3}-\mathbf{x}_i)+h\hat{n})^{\perp}\cdot(\frac{1}{3}(\mathbf{x}_{i+3}-\mathbf{x}_i)+h\hat{n})\\
&= \frac{1}{2}(x_{i+3}-x_i)^{\perp}\cdot(h\hat{n})+(\frac{1}{3}(\mathbf{x}_{i+3}-\mathbf{x}_i)+\frac{h\hat{n}}{2})^{\perp}\cdot(\frac{1}{3}(\mathbf{x}_{i+3}-\mathbf{x}_i)+\frac{h\hat{n}}{2}+\frac{h\hat{n}}{2})\\
&= \frac{1}{2}(x_{i+3}-x_i)^{\perp}\cdot(h\hat{n})+(\frac{1}{3}(\mathbf{x}_{i+3}-\mathbf{x}_i)+\frac{h\hat{n}}{2})^{\perp}\cdot(\frac{1}{3}(\mathbf{x}_{i+3}-\mathbf{x}_i)+\frac{h\hat{n}}{2})\\
&+(\frac{1}{3}(\mathbf{x}_{i+3}-\mathbf{x}_i)+\frac{h\hat{n}}{2})^{\perp}\cdot(\frac{h\hat{n}}{2})\\
&= \frac{1}{2}(x_{i+3}-x_i)^{\perp}\cdot(h\hat{n})+(\frac{1}{3}(\mathbf{x}_{i+3}-\mathbf{x}_i+\frac{h\hat{n}}{2})^{\perp}\cdot(\frac{h\hat{n}}{2})\\
&= \frac{1}{2}(x_{i+3}-x_i)^{\perp}\cdot(h\hat{n})+(\frac{1}{3}(\mathbf{x}_{i+3}-\mathbf{x}_i))^{\perp}\cdot(\frac{h\hat{n}}{2})+(\frac{h\hat{n}}{2})^{\perp}\cdot(\frac{h\hat{n}}{2}) \quad \textbf{using lemma5.1.1}\\
&= \frac{1}{2}(x_{i+3}-x_i)^{\perp}\cdot(h\hat{n})+\frac{1}{6}(\mathbf{x}_{i+3}-\mathbf{x}_i)^{\perp}\cdot(h\hat{n})\\
&= \frac{2}{3}(x_{i+3}-x_i)^{\perp}\cdot(h\hat{n})\\
&= \frac{2h}{3}(x_{i+3}-x_i)^{\perp}\cdot\frac{(x_{i+3}-x_i)^{\perp}}{\|(x_{i+3}-x_i)^{\perp}\|}\\
&= \frac{2h}{3}\|(x_{i+3}-x_i)^{\perp}\|\\
&= \frac{2h}{3}\|(x_{i+3}-x_i)\|\\
&= \frac{2}{3}\frac{3A_{quad}^{prev}}{2\|\mathbf{x}_{i+3}-\mathbf{x}_i\|}\|(x_{i+3}-x_i)\|\\
&= A_{quad}^{prev}
\end{aligned}
$$

■

5.2 Area Conserving Smoothing of Planar Polygons in 3D

Based on the previous section, in this section , we take the smoothing process into another level which represented by area conserving smoothing of planar polygon in 3D using the smoothing operation in the previous section with the mapping between 2D and 3D realms ψ including the generation of orthonormal basis steps and other needed steps for smoothing process.
The smoothing operation in 3D for any object ϑ with vertices $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_{n-1}$ can be achieved by two main steps as follows:

I Applying the map ψ in section 1.7 to map the nodes from $\mathbb{R}^3$ to $\mathbb{R}^2$.

II Smooth the shape in 2D and use the inverse map ψ^{-1} to map the nodes to $\mathbb{R}^3$.

We will clarify the above steps.

- For step **I**.
  The mapping is based on finding the orthonormal basis, so we need to find the orthonormal basis to map every node $\mathbf{x}_i$,$i = 0, 1, 2, \cdots, n-1$.
  The orthonormal basis for every node is computed as follows:

$$\begin{aligned}
\hat{n} &= \frac{(\mathbf{x}_1 - \mathbf{x}_0) \times (\mathbf{x}_2 - \mathbf{x}_0)}{\|(\mathbf{x}_1 - \mathbf{x}_0) \times (\mathbf{x}_2 - \mathbf{x}_0)\|} \\
\hat{u_1} &= \frac{\mathbf{x}_1 - \mathbf{x}_0}{\|\mathbf{x}_1 - \mathbf{x}_0\|} \\
\hat{u_2} &= \frac{\hat{n} \times \hat{u_1}}{\|\hat{n} \times \hat{u_1}\|}
\end{aligned}$$

  thus, every node $\mathbf{x}_i, i = 0, 1, 2, \cdots, n-1$ can be expressed as

$$\begin{aligned}
\mathbf{x}_i &= (\mathbf{x}_i.\hat{n})\hat{n} + (\mathbf{x}_i.\hat{u_1})\hat{u_1} + (\mathbf{x}_i.\hat{u_2})\hat{u_2} \\
&= (\mathbf{p}_0.\hat{n})\hat{n} + (\mathbf{x}_i.\hat{u_1})\hat{u_1} + (\mathbf{x}_i.\hat{u_2})\hat{u_2}
\end{aligned} \tag{5.6}$$

  As the equation of the plane that contained the shape is: $(\mathbf{x}_i - \mathbf{p}_0).\hat{n} = 0$.
  Apply the map ψ on the node $\mathbf{x}_i, i = 0, 1, 2, \cdots, n-1$ from $\mathbb{R}^3$ to $\mathbb{R}^2$.

$$\begin{aligned}
\psi(\mathbf{x}_i) &= \psi(\mathbf{x}_i.\hat{n}, \mathbf{x}_i.\hat{u_1}, \mathbf{x}_i.\hat{u_2}) \\
&= (\xi_i, \eta_i) \\
&= \mathbf{x}_i^*
\end{aligned}$$

  where $\mathbf{x}_i^*$ represents the mapped node $\mathbf{x}_i$ from $\mathbb{R}^3$ to $\mathbb{R}^2$.

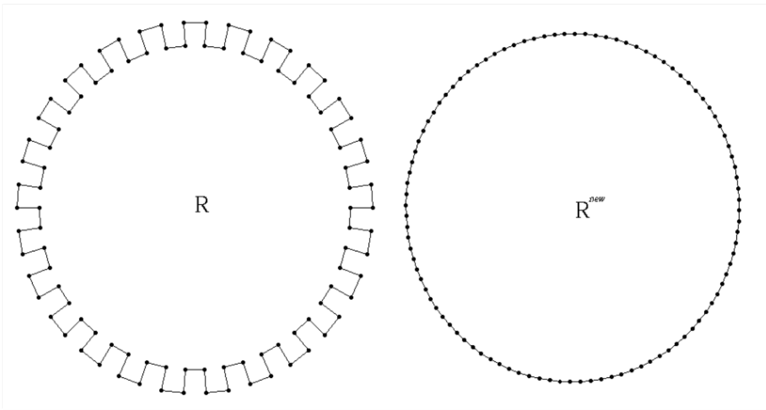
- For step **II**.

  This step includes the following steps to apply to the mapped shape: $\left\{\mathbf{x}_k^*\right\}_{k=0}^{n-1}$:

  1. specifies if we need tangential smoothing (i.e., a single node (vertex) $\mathbf{x}_i$) only or we need normal smoothing (i.e., two consecutive nodes (an edge) $\mathbf{x}_{i+1}\mathbf{x}_{i+2}$), where $i = 0, 1, 2, \cdots, n-1$ and $\mathbf{x}_n = \mathbf{x}_0$ as the shape is closed.
  2. For tangential smoothing, we need to apply the single node relaxation method and apply algorithm 1, but if we need both smoothings, then we need the edge relaxation method and apply algorithm 2.
  3. Map every vertex in the achieved smoothed shape $\mathbf{x}_0^{*\text{new}}\mathbf{x}_1^{*\text{new}}\mathbf{x}_2^{*\text{new}}\cdots\mathbf{x}_{n-1}^{*\text{new}}$ from $\mathbb{R}^2$ to $\mathbb{R}^3$ by the inverse map ψ^{-1} as follows:

$$
\begin{aligned}
\psi^{-1}(\mathbf{x}_i^{*\text{new}}) &= \psi^{-1}(\xi_i^{\text{new}}, \eta_i^{\text{new}}) \\
&= (\mathbf{x}_i^{\text{new}}.\hat{n})\hat{n} + (\mathbf{x}_i^{\text{new}}.\hat{u_1})\hat{u_1} + (\mathbf{x}_i^{\text{new}}.\hat{u_2})\hat{u_2} \\
&= (\mathbf{p}_0.\hat{n})\hat{n} + (\mathbf{x}_i^{\text{new}}.\hat{u_1})\hat{u_1} + (\mathbf{x}_i^{\text{new}}.\hat{u_2})\hat{u_2} \\
&= \mathbf{x}_i^{\text{new}}
\end{aligned}
\tag{5.7}
$$

Remark. • The following figures explain the area-conservative smoothing process in 2D and 3D.



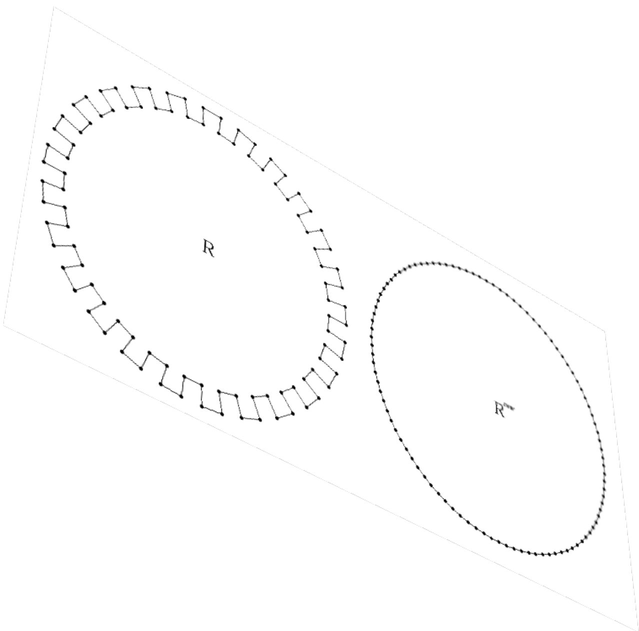


Figure 5.4: Area-conservative smoothing in 3D

- The following figures explain the curve smoothing process in 2D and 3D.

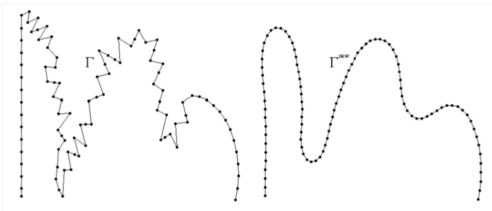


Figure 5.5: Curve smoothing in 2D

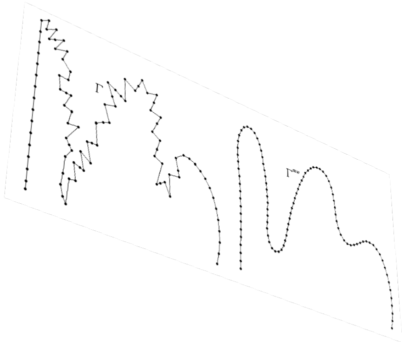


Figure 5.6: Curve smoothing in 3D

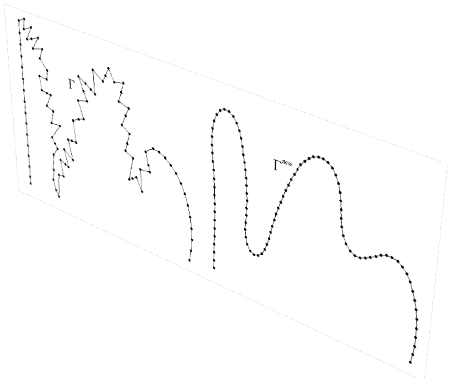


Figure 5.7: Curve smoothing in 3D in another plane

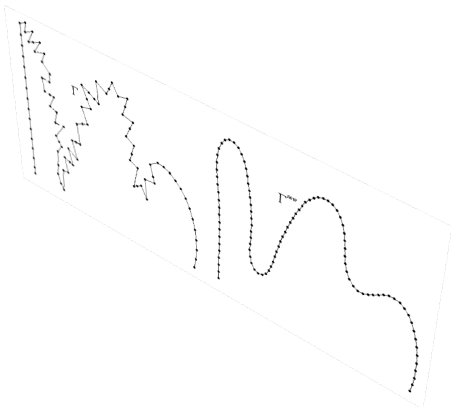


Figure 5.8: Curve smoothing in 3D in another plane

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