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Master of Mathematics

Interval Fractional Calculus

Submitted by:

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Areen Ali Robin Shweiki

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Dedication

To my parents,
To my husband Safwan.
To my second family.
To my son Hisham and my daughter Ameera.
To my brothers and sisters.
To my friends .

Areen Ali Shweiki

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Abstract

In this thesis, we give the definitions of fractional derivatives, fractional integral, and the concept of the fuzzy sets. In particular, we use these definitions in addition to the generalization of the Hukuhara differences for the closed intervals on the real line to develop the theory of inerval-valued fractional calculus and fractional differential equations with fractional order. Several examples are presented.

DEDICATION

This thesis is dedicated to:
The sake of Allah, my Creator and my Master,
My great supervisor Dr.Mohammad Adam, who encourage and support me,
My external committee member,
My parents, the reason of what I become today.

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Contents

Chapter 1

Fractional Calculus and Interval Analysis

1.1 Frational Calculus

Fractional calculus is a branch of mathematics which is as old as calculus [12]. It deals with studies several different possibilities of defining real number orders or complex number order of the differentiation operator.

It's story can be traced back to the end of the 17^{th} century, the time when Newton and Leibniz developed the foundation of differential and integral calculus. It was started as a trial to understand the question of whether the meaning of a derivative to an integer order n could be extended when n is not an integer. This question was first raised by L'Hopital in $30^{th}, 1695$. In a letter to Leibniz, he posed a question about $\frac{\mathrm{d}^n}{\mathrm{d}x^n}$, Leibniz notation for the n^{th} derivative of the linear function $f(x) = x$. L'hopital curiously asked what the result would be if $n = \frac{1}{2}$. Leibniz responded that it would be " an apparent paradox, from which one day useful consequences will be drawn"[26].

In the following, the discussion of the subject of fractional calculus caught the attention of other great mathematicians, many of whom directly or indirectly contributed to it's development. They included Euler, Laplace, Fourier, Riemann, Liouville and many others [19].

Over the years, many mathematicians, using their own notation and approachs, have found various definitions that fit the idea of non-integer order integral or derivative such that Riemann Liouville, Grunwald-Letnikov, and Caputo derivatives.

In this chapter, we present several definitions of fractional derivetives and integrals such as Riemann Liouville, Grunwald-Letnikov, Caputo derivatives, and conformable fractional derivative and the fractional integral.
In the next chapter, we will present these definition using interval valued functions in addition to present the fractional differential equation.

1.1.1 Riemann Liouville Fractional Derivative

In this section, we present the necessary definitions and theorems that are related to the Riemann Liouville fractional derivative and integral.

Riemann Liouville definition of fractional integral can be motivated by the generalization of Gamma formula which defined as a generalization of the factorial for all real numbers as follows

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \exp\{-t\}\, dx.$$

In the following we denote by $C([a,b],\mathbb{R})$ the set of all continuous functions in the interval of real numbers $[a,b]$.

Definition 1.1. *[3] Let $f(x) \in C([a,b],\mathbb{R})$. The integrals*

$${}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (x-t)^{\alpha-1} f(t)\, dx,\ t > a,$$

and

$${}_xI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t-x)^{\alpha-1} f(t)\, dx,\ t < b,$$

where $\alpha > 0$, are called, respectively the left and right Riemann-Liouville fractional integrals of order α.

In this thesis, we will consider only the left Riemann-Liouville fractional integral, and denoted by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x)\, dx,$$

Proposition 1.1. *[3] If $f(t) \in C([a,b],\mathbb{R})$, $\alpha_1, \alpha_2 \geq 0$, then*

$$I_a^{\alpha_1} I_a^{\alpha_2} f(t) = I_a^{\alpha_1+\alpha_2} f(t) \tag{1.1}$$

Proof.

$$\begin{aligned} I_a^{\alpha_1} I_a^{\alpha_2} f(t) &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^t \int_\tau^t (t-x)^{\alpha_1-1} (x-\tau)^{\alpha_2-1} f(\tau)\, dx\, d\tau \\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^t f(\tau) \left[\int_\tau^t (t-\tau)^{\alpha_1-1} (x-\tau)^{\alpha_2-1}\, dx \right] d\tau, \end{aligned}$$

the substitution $x = \tau + s(t-\tau)$, yields

$$\begin{aligned} I_a^{\alpha_1} I_a^{\alpha_2} f(t) &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^t f(\tau) \int_0^1 [(t-\tau)(1-s)]^{\alpha_1-1}[s(t-\tau)]^{\alpha_2-1}(t-\tau)\, ds\, d\tau \\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^t f(\tau)(t-\tau)^{\alpha_1+\alpha_2-1} \int_0^1 (1-s)^{\alpha-1} s^{n-1}\, ds\, d\tau \\ &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^t f(\tau)(t-\tau)^{\alpha_1+\alpha_2-1} \beta(\alpha_1,\alpha_2)\, d\tau, \end{aligned}$$

where $\beta(\alpha_1, \alpha_2)$ is the *Beta function*, which is defined as the following [10]

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \quad x, y > 0. \tag{1.2}$$

Also, the Beta function has a close relationship to the Gamma function, where

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.3}$$

Hence,

$$\begin{aligned} I_a^{\alpha_1} I_a^{\alpha_2} f(t) &= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_a^t f(\tau)(x-\tau)^{\alpha_1+\alpha_2-1} \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}\, d\tau \\ &= \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t (t-\tau)^{\alpha_1+\alpha_2-1} f(\tau)\, d\tau \\ &= I^{\alpha_1+\alpha_2} f(t). \end{aligned}$$

□

Definition 1.2. *[28] The Riemann-Liouville fractional derivative of order α for a function $f(t) \in C^1([a,b],\mathbb{R}); b > 0$ is given by:*

$$\begin{aligned} D_a^{\alpha} f(t) &= \frac{d^n}{dt^n} I_a^{1-\alpha} f(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-x)^{n-\alpha-1} f(x)\, dx, \end{aligned}$$

for every $t \in [a,b]$ and $n-1 < \alpha < n$, where $n > 0$ is an integer.

The following Example illustrates the previous two definitions

Example 1.1. *Let $f(t) = C$ where C is an arbitrary constant in the interval $[0,2]$. Then*

- *The Riemann-Liouville Fractional Integral*

$$\begin{aligned} I_0^\alpha f(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1} C\,dx \\ &= \frac{C}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}\,dx \\ &= \frac{-C}{\Gamma(\alpha)}\left[\frac{(t-x)^\alpha}{\alpha}\right|_0^t \\ &= \frac{C}{\alpha\Gamma(\alpha)}t^\alpha \\ &= \frac{C}{\Gamma(\alpha+1)}t^\alpha. \end{aligned}$$

- *The Riemann-Liouville Fractional Derivative*

$$\begin{aligned} D_0^\alpha f(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t (t-x)^{n-\alpha-1} C\,dx \\ &= \frac{C}{\Gamma(1-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(-\frac{(t-x)^{n-\alpha}}{n-\alpha}\right|_0^t \\ &= \frac{C}{(n-\alpha)\Gamma(1-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(t^{n-\alpha}\right). \end{aligned}$$

after n-times derivatives, we get that

$$D_0^\alpha f(t) = \frac{C}{(n-\alpha)\Gamma(1-\alpha)}(n-\alpha)(n-\alpha-1)\dots(-\alpha)t^{-\alpha}.$$

Definition 1.3. *[12] For order α and $f \in (C[a,b],\mathbb{R})$, $D_a^\alpha f(t) = I_a^{-\alpha} f(t)$.*

The following is the composition rules and the relation between the Riemann Liouville Fractional Derivative and Integral which can be derived from definitions

Theorem 1.1. *Composition Rule* *[12]*
Let $f \in C([a,b],\mathbb{R})$. Then

1. $D_a^{\alpha_1} D_a^{\alpha_2} f = D_a^{\alpha_1+\alpha_2} f, \quad \alpha_1, \alpha_2 > 0.$
2. $D_a^{\alpha_1} I_a^{\alpha_2} f = \begin{cases} D^{\alpha_1-\alpha_2} f & \text{if } \alpha_1 > \alpha_2, \\ I^{\alpha_2-\alpha_1} f & \text{if } \alpha_1 < \alpha_2. \end{cases}$

Proof. Using Proposition 1.1. and Definition 1.3., we get that

1.

$$\begin{aligned} D_a^{\alpha_1} D_a^{\alpha_2} f &= I_a^{-\alpha_1} I_a^{-\alpha_2} f, \\ &= I_a^{-(\alpha_1+\alpha_2)} f, \\ &= D_a^{(\alpha_1+\alpha_2)} f. \end{aligned}$$

2.

$$\begin{aligned} D_a^{\alpha_1} I_a^{\alpha_2} f &= I^{-\alpha_1} I_a^{\alpha_2}, \\ &= I^{\alpha_2-\alpha_1}, \text{ if } \alpha_2 > \alpha_1, \\ &= D^{\alpha_1-\alpha_2}, \text{ if } \alpha_1 > \alpha_2. \end{aligned}$$

□

Theorem 1.2. *[3, 18]* ***The Relation between Riemann-Loiuville Fractional Integral and Derivative***
Let $\alpha > 0$. Then for every $f \in (C[a,b], \mathbb{R})$:

1. $D_a^\alpha I_a^\alpha f(t) = f(t)$.
2. $I_a^\alpha D_a^\alpha f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\alpha-k-1}}{\Gamma(\alpha-1)} I_a^{1-\alpha} f(t)$.

Example 1.2. *In this example, we verify that*

$$I_a^\alpha (x-a)^p = \frac{\Gamma(p+1)}{\Gamma(p+\alpha+1)} (x-a)^{\alpha+p},$$

for some $p > -1$ and $\alpha > 0$.
By definition

$$I_a^\alpha (x-a)^p = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} (t-a)^p \, dt$$

Now by substituting $t = a + s(x-a)$, we get

$$\begin{aligned} I_a^\alpha (x-a)^p &= \frac{1}{\Gamma(\alpha)} \int_0^1 (x-a)^\alpha (x-a)^p (1-s)^{\alpha-1} s^p \, ds \\ &= \frac{(x-a)^{\alpha+p}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^p \, ds \\ &= \frac{(x-a)^{\alpha+p}}{\Gamma(\alpha)} \beta(p+1, \alpha), \qquad \textit{by } (1.2). \\ &= \frac{(x-a)^{\alpha+p}}{\Gamma(\alpha)} \frac{\Gamma(p+1) \cdot \Gamma(\alpha)}{\Gamma(\alpha+p+1)}, \qquad \textit{by } (1.3) \\ &= \frac{\Gamma(p+1)}{\Gamma(p+\alpha+1)} (x-a)^{\alpha+p}. \end{aligned}$$

1.1.2 Caputo Fractional Derivative

In this section we present a definition of fractional derivative which is called the *Caputo Fractional Definition* and it's relationship to the Riemann-Liouville Fractional derivative. The *Caputo definition of fractional derivative* can be written as [30]

$$ {}_a^C D_x^\alpha = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}\, dt, \;\; n-1 < \alpha < n. $$

and

$$ {}_a^C D_x^n f(x) = f^{(n)}(x), $$

for $n \in \mathbb{N}$.

Remark 1.1.1. *[12]* ${}_a^C D_x^p K = 0$ *if* K *is constant.*

The relationship between Riemman Liouville fractional derivative and the Caputo derivative can be obtained using the composition Rule which was derived before as follows [39]. For $n-1 < \alpha < n, n \in \mathbb{N}$:

$$ \begin{aligned} {}_a^{RL} D_x^\alpha f(x) &= {}^{RL}D^n \left({}^{RL}D^{\alpha-n} f(x) \right) \\ &= {}^{RL}D^n \left({}^{RL}D^{\alpha-n} (D^n f^{(n)}(x)) \right) \\ &= {}^{RL}D^n \left(I^{n-\alpha} \left(I^n f^{(n)} + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(x-a)^k}{k!} \right) \right) \\ &= I^{n-\alpha} f^{(n)}(x) + \sum_{k=0}^{n-1} f^{(k)}(a) \frac{(x-a)^{k-a}}{\Gamma(k+1-\alpha)}. \end{aligned} $$

Hence

$$ {}_a^{RL} D_x^\alpha f(x) = {}_a^C D_x^\alpha f(x) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}. $$

Thus under the homogeneous initial conditions, the Riemann-Liouville and the Caputo fractional derivatives are equivalent. i.e.,

$$ {}_a^{RL} D_x^\alpha f(x) = {}_a^C D_x^\alpha f(x) \iff f^{(k)}(a) = 0, \qquad 0 \le k \le n-1. $$

1.1.3 Grünwald-Letnikov Fractional Derivative

In this section, we introduce another definition of fractional derivative, which is called the *Grünwald-Letnikov Fractional Derivative* and its implementation.

Let f be a continuous function. Then the ordinary derivatives are defined in terms of the so-called *backward differences* as follows

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h\to 0}\frac{f(x)-f(x-h)}{h},\\
\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} &= \lim_{h\to 0}\frac{f(x)-2f(x-h)+f(x-2h)}{h},\\
\frac{\mathrm{d}^3 f}{\mathrm{d}x^3} &= \lim_{h\to 0}\frac{f(x)-3f(x-h)+3f(x-2h)-f(x-3h)}{h}.
\end{aligned}$$

If we continue in this manner, then we can write a general formula for the n^{th}-derivative of a function $f(x)$. Indeed, for $n \in \mathbb{N}, f \in (C^j[a,b],\mathbb{R})$ and $j > n$, then

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} = \lim_{h\to 0}\frac{1}{h^n}\sum_{j=0}^{n}\left[(-1)^j\binom{n}{j}f(x-jh)\right], \tag{1.4}$$

where the relation (1.4) expresses a linear combination of function values of $f(x)$ in variable x, and the binomial coefficient with alternating signs for positive values of n [30]

$$\begin{aligned}
\binom{n}{j} &= \frac{n!}{j!(n-j)!}\\
&= \frac{n(n-1)\dots(n-j+1)}{j!}.
\end{aligned}$$

In the case of negative value of n, we define

$$\begin{aligned}
\binom{-n}{j} &= \frac{-n(-n-1)(-n-2)\dots(-n-j+1)}{j!}\\
&= (-1)^j\frac{n(n+1)\dots(n+j-1)}{j!}\\
&= (-1)^j\begin{bmatrix} n\\ j\end{bmatrix}.
\end{aligned}$$

so, replacing n by $-n$ in (1.4), we get

$$\begin{aligned}
\frac{\mathrm{d}^{-n} f}{\mathrm{d}x^{-n}} &= f^{(-n)}(x)\\
&= \lim_{h\to 0}\frac{1}{h^n}\sum_{j=0}^{n}(-1)^j\begin{bmatrix} n\\ j\end{bmatrix} f(x-jh),
\end{aligned}$$

where n is a positive integer number.

Remark 1.1.2. *[30]*

For the binomial coefficients calculations, we can use the relation between Euler's Gamma function and fractorial, defined as

$$\binom{n}{j} = \frac{n!}{j!(n-j)!} = \frac{\Gamma(n+1)}{\Gamma(j+1)\Gamma(n-j+1)},$$

and for $\binom{n}{0} := 1$.

Grünwald-Letnikov fractional derivative is the generalization of the n-derivative function which is given by (1.4). The idea behind it is that h should approaches 0 as n approaches ∞, and since $\binom{n}{j} := 0$ for $n < j, n \in \mathbb{N}$, and assume that h take only the values

$$h_N = \frac{x-a}{N}, N = 1, 2, \ldots.$$

Hence

$$\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} = \lim_{h \to 0} h^{-n} \sum_{j=0}^{\infty} (-1)^j \binom{n}{j} f(x - jh),$$

since N goes to ∞ when h_N goes to 0, then the following definition is derived.

Definition 1.4. *If* $n > 0, f \in C^n[a,b]$ *and* $a \le x \le b$ *and* $n < \alpha < n+1$*, then*

$$D_a^{\alpha} f(x) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{N} (-1)^j \binom{\alpha}{j} f(x - jh)$$

is called the Grünwald-Letnikov derivative of order α of $f(x)$.

Implementation of Grünwald-Letnikov Fractional Order Derivative

The most contents of this subsection are from [38].
In this subsection we present the The Grünwald-Letnikov definition of the fractional order derivative which defined as

$$D_a^{\alpha} f(x) = \lim_{h \to 0} \sum_{j=o}^{L} w_j^{(\alpha)} f(x - jh),$$

where $w_j^{(\alpha)}$ are the binomial coefficient calculated recursively as following:

$$w_0^{(\alpha)} = 1, w_j^{(\alpha)} = \left(1 - \frac{\alpha+1}{j}\right) w_{j-1}^{(\alpha)}, j = 1, 2, 3, \ldots,$$

where h is the step size and L is the window size.
In order to implement this operator, an approximate version of length L is given by:

$$D^{\alpha}_{x-L} f(x) = \frac{1}{h^{\alpha}} \sum_{j=0}^{(\alpha)} w_j^{(\alpha)} f(x - jh),$$

According to the short memory principle, the error in calculating this approximated derivative is bounded by:

$$\Delta x = \left| D^{\alpha}_{a} f(x) - D^{\alpha}_{x-L} f(x) \right| \leq \frac{M L^{-\alpha}}{|\Gamma(1-\alpha)|},$$

where $(a+L < x < b)$ and $|f(x)| < M$ when $a < x < b$. So, it can be concluded that the error is reduced by increasing the window size and the magnitude of the w_j^{α} decreases with increasing of j.
The above procedure can be applied to solve the general form of a fractional order differential equation which is given as following

$$D^{q_1} = P(x, t).$$

To simulate this system based on Grünwald-Letnikov definition, the following set of equations are used:

$$x_{t_k} = P(x(t_{k-1}), t_k) h^{q_1} - \sum_{j=1}^{m} w^{q_1}_{j=1} x(t_{k-j}),$$

where $m = L$ for the approximated window variation of the Grünwald-Letnikov operator and $m = k$ when the entire state memory is used in calculation.

Grünwald-Letnikov equation consists of two parts, the first is the binomial coefficients:

$$w_0^{(\alpha)} = 1, w_j^{(\alpha)} = \left(1 - \frac{\alpha + 1}{j}\right) w_{j-1}^{(\alpha)}, j = 1, 2, 3, \ldots, \tag{1.5}$$

the second part is the dot product of the row and column vectors presented by

$$D^{\alpha}_{x-L} f(x) = \frac{1}{h^{\alpha}} \sum_{j=0}^{(\alpha)} w_j^{(\alpha)} f(x - jh),$$

The following system represented the row and column vector dot products output:

$$\begin{aligned}
&w_0x_0\\
&w_0x_1 + w_1x_0\\
&w_0x_2 + w_1x_1 + w_2x_0\\
&\vdots\\
&w_0x_n + w_1x_{n-1} + \ldots + w_nx_0.
\end{aligned}$$

The above procedure can be extended to solve order system of three differential equations, the following example illustrates an application of the Grünwald-Letnikov system of equations:

Example 1.3. ***(Liu System Implementation)****:*
The fractional order Liu system is given by:

$$\begin{aligned}
D^{q_1} &= -ax - ey^2\\
D^{q_2} &= by - kxz\\
D^{q_3} &= -cz + mxy.
\end{aligned}$$

A numerical solution of the Liu system can be represented as follows

$$\begin{aligned}
x_{t_k} &= (-ax(t_{k-1}) - ey^2(t_{k-1}))h^{q_1} - \sum_{j=1}^{m} w_{j=1}^{q_1} x(t_{k-j})\\
y_{t_k} &= (by(t_{k-1}) - kx(t_{k-1})z(t_{k-1}))h^{q_2} - \sum_{j=1}^{m} w_{j=1}^{q_2} y(t_{k-j})\\
z_{t_k} &= (-cz(t_{k-1}) + mx(t_{k-1})y(t_{k-1}))h^{q_3} - \sum_{j=1}^{m} w_{j=1}^{q_3} z(t_{k-j}).
\end{aligned}$$

where q_1, q_2, q_3 *are the fractional orders.*

1.1.4 Conformable Fractional Derivative

In this section, we present the definition of the Conformable Fractional and it's proprieties.
In 2014, the authors in [17] defined a new simple fractional derivative called *"The Conformable Fractional Derivative"* depending just on the basic limit definition of the derivative.

Definition 1.5. *[17] Given a function $f : [0, \infty) \to \mathbf{R}$. Then the* The Conformable Fractional Derivative *of f of order α is defied by*

$$T^{(\alpha)}(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon},$$

for all $t > 0, \alpha \in (0, 1)$. So, if the Conformable fractional derivative of f of order α exist, then we simply say f is α-diffrentiable.

If f is α-differentiable in some intervals $(0, a), a > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exist, then define

$$f^{(\alpha)}(0) := \lim_{t \to 0^+} f^{(\alpha)}(t).$$

Note that, this definition coincides with the classical definition of Riemann-Liouville and Caputo Fractional Derivative on polynomials, i.e., up to constant multiple.
As a consequence of the above definition, we can easily show that T_α satisfies all the properties in the following theorem.

Theorem 1.3. *[17, 11] Let $\alpha \in (0, 1]$ and f, g be α-differentiable. Then*

1. Linearity
$$T^{(\alpha)}(af + bg) = aT(\alpha)(f) + bT^{(\alpha)}(g).$$

2. Leibniz Rule
$$T^{(\alpha)}(fg) = [T^{(\alpha)}(f)]g + f[T^{(\alpha)}(g)].$$

3. $T^{(\alpha)}(t^p) = pt^{p-\alpha}$, *for all $p \in \mathbb{R}$.*

4. $T^{(\alpha)}(\lambda) = 0$, *for all constant functions $f(t) = \lambda$.*

5. $T^{(\alpha)}(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.

6. *If f is differentiable, then $T^{(\alpha)}(f)(t) = t^{1-\alpha} \frac{\mathrm{d}f}{\mathrm{d}t}$.*

As a consequence of the previous theorem, the following are Conformable Fractional Derivative of some certain functions.

(1) $T^{(\alpha)}(e^{ct}) = ct^{1-\alpha} e^{ct}, c \in \mathbb{R}$.

(2) $T^{(\alpha)}(\cos(bt)) = -bt^{1-\alpha} \sin(bt)$.

(3) $T^{(\alpha)}(\sin(bt)) = bt^{1-\alpha}\cos(bt)$.

(4) $T^{(\alpha)}\left(\frac{1}{\alpha}t^{\alpha}\right) = 1$.

such that

1. $T_\alpha\left(\sin\left(\frac{1}{\alpha}t^\alpha\right)\right) = cos\frac{1}{\alpha}t^\alpha$.
2. $T_\alpha\left(\cos\left(\frac{1}{\alpha}t^\alpha\right)\right) = -sin\frac{1}{\alpha}t^\alpha$.
3. $T_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$.

Remark 1.1.3. *A function f could be α- differentiable but not differentiable.*

For example; take $f(t) = 2\sqrt{t}$, then $T_{\frac{1}{2}}f(0) = \lim_{t\to 0^+} T_{\frac{1}{2}}f(t) = 1$, where $T_{\frac{1}{2}}f(t) = 1$, for $t > 0$. But $T_1 f(t)$ does not exist.

Fractional Integral

When it comes to integration, the most important class of functions to define the integral is the space of continuous functions. particularly, define the fractional integral on polynomials, using the Weistrass theorem [7].and let $J_\alpha(f(t))$ denote the fractional integral of a continuous function $f(t)$.
Let $\alpha \in (0,\infty)$. Define $J_\alpha(t^p) = \frac{t^{p+\alpha}}{p+\alpha}$ for any $p \in \mathbf{R}$, and $\alpha \neq -p$, then

1. If $f(t) = \sum_{k=0}^{n} b_k t^k$, then define

$$J_\alpha(f) = \sum_{k=0}^{n} b_k J_\alpha(t^k) = \sum_{k=0}^{n} b_k \frac{t^{k+\alpha}}{k+\alpha}.$$

2. If $f(t) = \sum_{k=0}^{n} b_k t^k$, where the series is uniformly convergent, then define

equation*J$_\alpha(f) = \sum_{k=0}^{\infty} b_k \frac{t^{k+\alpha}}{k+\alpha}$.

Note that, J_α is linear on it is domain, and if $\alpha = 1$, then J_α is the usual integral.
The following definition for the α-fractional integral of a function $f \in C[0,\infty), a \geq 0$.

Definition 1.6. *[17]*

$$I_\alpha^a f(t) = I_1^a(t^{\alpha-1}f)(t) = \int_a^t \frac{f(s)}{s^{1-\alpha}}\, ds,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1)$.

For example, $I^0_{\frac{1}{2}}(\sqrt{t}sint) = \int_0^t \sin x\, dx = 1 - \cos t$.
The following theorems illustrates some results of the Conformable Fractional Integral.

Theorem 1.4. *$T_\alpha I^a_\alpha f(t) = f(t)$, for $t \geq a$, where f is any continuous function in the domain of I_α.*

Proof. Since f is continuous, then $I^a_\alpha f(t)$ is differentiable. Hence,

$$\begin{aligned} T_\alpha(I^a_\alpha f)(t) &= t^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t} I^a_\alpha f(t) \\ &= t^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t \frac{f(s)}{s^{1-\alpha}}\, ds \\ &= t^{1-\alpha}\frac{f(t)}{t^{1-\alpha}} \\ &= f(t). \end{aligned}$$

□

Theorem 1.5. *[33] Let f be α- differentiable. Then*

$$I^a_\alpha T_\alpha f(t) = f(t) - f(a).$$

Proof. Using the definition of the usual Riemann improper integral and the Conformable fractional derivative, we get that

$$\begin{aligned} I^a_\alpha T_\alpha f(t) &= \int_a^t \frac{T_\alpha f(s)}{s^{1-\alpha}}\, ds \\ &= \int_a^t \frac{s^{1-\alpha} f'(s)}{s^{1-\alpha}}\, ds \\ &= f(t) - f(a). \end{aligned}$$

□

1.2 Interval Analysis

Since not all numbers can be represented exactly with finite number of digits such that irrational number, so the result of each calculation may contain some errors, then we need a new arithmetic form to use with mathematical calculations which works with an interval $[a, b]$ that defines the range of values that x can have instead of working with an uncertain single real number.
This treatment is typically limited to real intervals, i.e., quantities in the form

$$[a, b] = \left\{x \in \mathbb{R} | a \leq x \leq b\right\},$$

where $a = -\infty$ and $b = \infty$ are allowed [27].
This interval would be an unbounded interval; with both infinite or would be the extended real number line, addition to other types of intervals open such that (a, b),or half-open such that $(a, b]$, and $[a, b)$ appear through mathematics.

1.2.1 Basic Terms and Concepts:

We will adopt the convenient of denoting intervals and their endpoints by capital letters. The following are some definitions corresponding to intervals concepts:

1. **End Point Notation**
   The left and right endpoint of an interval X will be denoted by $\underline{X}$ and $\overline{X}$, respectively. Thus,
   $$X = [\underline{X}, \overline{X}].$$

2. **Interval Equality**
   Two intervals X and Y are said to be *equal* if the corresponding endpoints are equal:
   $$X = Y \text{ if } \underline{X} = \underline{Y} \text{ and } \overline{X} = \overline{Y}.$$

3. **Intersection Interval**
   - *The intersection of two intervals X and Y* is empty if either $\overline{Y} < \underline{X}$ or $\overline{X} < \underline{Y}$, and we write
     $$X \cap Y = \emptyset$$
   - *Interval intersection* is defined as follows
     $$\begin{aligned} X \cap Y &= \{z : z \in X \text{ and } z \in Y\} \\ &= [\max\{\underline{X}, \underline{Y}\}, \min\left\{\overline{X}, \overline{Y}\right\}]. \end{aligned}$$
   - Intersection plays a key role in interval analysis. If we have two intervals containing a result of interest, then the intersection which may be narrower, also contains the result.

4. **Union Interval and Interval Hull**

- *The union of two intervals* X *and* Y can be defined as

$$\begin{aligned} X \cup Y &= \{z : z \in X or z \in Y\} \\ &= [\min\{\underline{X}, \underline{Y}\}, \max\left\{\overline{X}, \overline{Y}\right\}]. \end{aligned}$$

- In general, the union of two intervals may not be an interval, such that $[0, 2]$, and $[4, 5]$ are two intervals, but there union is not an intervals, since they have a nonempty intersection, where ϕ is not an interval.

- *The interval hull* of two intervals defined by

$$X \underline{\cup} Y = [\min\{\underline{X}, \underline{Y}\}, \max\left\{\overline{X}, \overline{Y}\right\}],$$

is always an interval and it is used in interval computations.In general;

$$X \cup Y \subseteq X \underline{\cup} Y.$$

5. **Width, Absolute Value, and Midpoints**

- The *width (length)* of an interval X is defined and denoted by

$$w(X) = \overline{X} - \underline{X},$$

- *The absolute values (Magnitude)* of X, denoted by $|X|$ is the maximum of the absolute values of it's endpoints

$$|X| = \max\left\{|\underline{X}|, |\overline{X}|\right\},$$

where $|x| \leq |X|$ for all $x \in X$.

- *The midpoint* of X is given by

$$m(X) = \frac{1}{2}\left\{\underline{X} + \overline{X}\right\}.$$

The following example illustrate the previous definitions of the interval concepts:

Example 1.4. *Let* $X = [0, 3]$ *and* $Y = [-2, 2]$. *Then:*

- *The intersection and union of* X *and* Y *are*

$$\begin{aligned} X \cap Y &= [0, 2], \\ X \cup Y &= [-2, 3]. \end{aligned}$$

- *The width of X and Y respectively*

$$\begin{aligned} w(X) &= 3, \\ w(Y) &= 4. \end{aligned}$$

- *The absolute value of X and Y is*

$$\begin{aligned} |X| &= 3, \\ |Y| &= 2. \end{aligned}$$

- *The midpoint of X is $m(X) = \frac{3}{2}$ and $m(Y) = 0$.*

1.2.2 Order Relations for Intervals

We say that

$$X < Y \text{ if } \overline{X} < \underline{Y},$$

where the relation $<$ is transitive such that let A, B and C three intervals where

$$A < B \text{ and } B < C \text{ then } A < C,$$

Also; we call the interval X is positive if $x > 0$ for all $x \in X$ or negative if $x < 0$ for all $x \in X$.
Another transitive order relation for intervals is the set inclusion:

$$X \subseteq Y \text{ if and only if } \underline{Y} \leq \underline{X} \text{ and } \overline{X} \leq \overline{Y}.$$

1.2.3 Arithmetic Operations on Intervals

We define arithmetic operations and functions on intervals in such a way that the result of the calculation is a new interval that is guaranteed to contain the true range of the function.
Let $\mathcal{K}$ indicate the set of all nonempty compact intervals of the real line $\mathbb{R}$, and let X and Y be two intervals such that $x \in X = [\underline{X}, \overline{X}]$ and $y \in Y = [\underline{Y}, \overline{Y}]$, then using the intervals proprieties we have the following;

1. *The Minkowski Addition* of two intervals X and Y is the set[22]

$$\begin{aligned} X + Y &= \{x + y : x \in X, y \in Y\} \\ &= [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}]. \end{aligned}$$

2. *The difference* of two intervals X and Y is the set

$$\begin{aligned} X - Y &= \{x - y : x \in X, y \in Y\} \\ &= [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}]. \end{aligned}$$

3. *The scalar multiplication* is defined by [22, 33], let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$, $\lambda_1, \lambda_2 \geq 0$, and λ_3, λ_4 are both have the same sign (i.e. either both of them are positive or negative), then it holds that

   (a)

$$\begin{aligned} \lambda X &= \lambda[\underline{X}, \overline{X}] \\ &= \begin{cases} [\lambda\underline{X}, \lambda\overline{X}], & \text{if} \quad \lambda > 0, \\ 0, & \text{if} \quad \lambda = 0, \\ [\lambda\overline{X}, \lambda\underline{X}], & \text{if} \quad \lambda < 0. \end{cases} \end{aligned}$$

   respectively.

   (b)

$$\begin{aligned} \lambda_1(\lambda_2 X) &= (\lambda_1\lambda_2)X, \\ (\lambda_3 + \lambda_4)X &= \lambda_3 X + \lambda_4 X. \end{aligned}$$

   (c) If $\lambda = -1$, then the scalar multiplication gives the opposite of X

$$\begin{aligned} -X &= (-1)X \\ &= (-1)[\underline{X}, \overline{X}] \\ &= [-\underline{X}, -\overline{X}]. \end{aligned}$$

   (d) In general; $(X) + (-X) \neq 0$; that is the opposite of X is not the inverse of X with respect to Minkowski addition. Minowski difference is $X - Y = X + (-Y) = [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}]$ with respect to the above operations.

4. The product of X and Y is given by

$$\begin{aligned} X \cdot Y &= \{xy : x \in X, y \in Y\} \\ &= [\min\{S\}, \max\{S\}], \end{aligned}$$

   where $S = \left\{\underline{XY}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{XY}\right\}$, we sometimes write $X \cdot Y$ more briefly as XY.

5. The quotient $\frac{X}{Y}$ is defined as

$$\frac{X}{Y} = X \cdot \left(\frac{1}{Y}\right),$$

where

$$\begin{aligned} \frac{1}{Y} &= \left\{y : \frac{1}{y} \in Y\right\} \\ &= \left[\frac{1}{\overline{Y}}, \frac{1}{\underline{Y}}\right], \end{aligned}$$

this assumes $0 \notin Y$.

6. Given that, an interval X can be written as

$$X = m(X) + \frac{1}{2}w(X)[-1, 1]$$

The following example illustrates the interval arithmetic operations:

Example 1.5. *let X=[-2,-1] and Y=[-2,4] then:*

- $X + Y = [-4, 3]$,
- $X - Y = [0, -5]$,
- *To find $X \cdot Y$, we need to find first $S = \{-4, 2, -8, 4\}$, so $X \cdot Y = [\min S, \max S] =$* $[-8, 4]$,
- $\frac{Y}{X} = [1, -4]$

1.2.4 Interval Vectors and Matrices

By an n-dimensional interval vector, we mean an ordered $n-$tuple of intervals,

$$(X_1, X_2, \ldots, X_n).$$

We will also denote interval vectors as capital letters such as X.

Example 1.6. *A three dimensional interval vector*

$$\begin{aligned} X &= (X_1, X_2, X_3) \\ &= ([\underline{X_1}, \overline{X_1}], [\underline{X_2}, \overline{X_2}], [\underline{X_3}, \overline{X_3}]), \end{aligned}$$

can be represented in the $x_1x_2x_3-$plane ; it is the set of all points (x_1, x_2, x_3) such that

$$\underline{X_1} \leq x_1 \leq \underline{X_1},$$
$$\underline{X_2} \leq x_1 \leq \underline{X_2},$$
$$\underline{X_3} \leq x_1 \leq \underline{X_3}.$$

Let $X = (X_1, X_2, \ldots, X_n)$ and $Y = (Y_1, Y_2, \ldots, Y_n)$ be two interval vectors. Then we have the following proprieties:

1. *The intersection of two interval vectors* is empty if the intersection of any of their corresponding components is empty, i.e.,

   $$\text{If } X_i \cap Y_i = \emptyset \text{ for some } i \in \{1, 2, \ldots, n\}$$

   then $X \cap Y = \emptyset$. Otherwise,

   $$X \cap Y = (X_1 \cap Y_1, X_2 \cap Y_2, \ldots, X_n \cap Y_n),$$

   which is again an interval vector.

2. $X \subseteq Y$ if $X_i \subseteq Y_i$ for all $i = 1, \ldots, n$.

3. *The width(length)* of an interval vector X is the largest of the width of any of it is component intervals

   $$w(X) = \max_{i \in \{1,2,\ldots,n\}} w(X_i),$$

4. *The midpoint* of an interval vector X is

   $$m(X) = (m(X_1), m(X_2), \ldots, m(X_n)),$$

5. *The norm* of an interval vector X is

   $$||X|| = \max_{i \in \{1,2,\ldots,n\}} |X_i|,$$

   this serves as a generalization of absolute value.

Chapter 2

Interval Fractional Analysis

The study of fractional calculus starts from the beginning of 1695s, and continued to benifit from it's development forms in various fields such as electrochemistry and radiology. Also, the fractional differential equation and its applications have been widely used in various fields such, as science and engineering.
The interval arithmetic provides a possibility to measure uncertainties of uncertain variables regarding the lack of knowledge of the complex information of the system, where the interval-valued arithmetic and interval- valued differential equations are the particular cases of the set-valued analysis and set differential equations, respectively [33].
Recently, the theory of fuzzy calculus and fuzzy differential equations have become one of the most important subjects in the mathematical analysis area [39], where the connection between the fuzzy analysis and the interval analysis introduced as an attempt to handle interval uncertainty that appears in many mathematical models of some deterministic real world phenomena [22].
The concept of Hukuhara derivative of a set-valued mapping is rigorously combined with the theoretical foundation of the initial differential equations and the fuzzy differential equation, where we can use this mapping to permit them to achieve the solutions of initial differential equations with diminishing diameter of solutions values [33].

2.1 Derivatives and Integrals of Interval-Valued Functions

In this section, we present some recent and basic notions on the integral and differential calculus for interval-valued functions. In addition to that, some essential theorems for interval spaces and interval functions are introduced.

Generalized Hukuhara Differences:

The generalized Hukuhara difference (or *gH-difference* for short) of two intervals $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}] \in \mathcal{K}$ is defined as follows [25, 37]

$$A\,\theta_g B = [\min\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\}, \max\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\}],$$

which can be written as follows

$$A\,\theta_g B = \begin{cases} [\underline{a} - \underline{b}, \overline{a} - \overline{b}] & \text{if } w(A) \geq w(B), \\ [\overline{a} - \overline{b}, \underline{a} - \underline{b}] & \text{if } w(B) > w(A). \end{cases}$$

If $A, B \in \mathcal{K}$ and $w(A) \geq w(B)$, then the gH-difference $A\,\theta_g B$ will be denoted by $A\theta_g B$ and it is called *Hukuhara Difference*or (H^+- *difference* for short) of A and B. Note that $A\,\theta_g B \neq A + (-1)B$. If $w(A) \leq w(B)$, then the gH- Differences$A\,\theta_g B$ which denoted by $A \boxminus B$[24] and it is called *the second Hukuhara difference* (H^-*-difference*) [24].

Note that the second Hukuhara difference is equivalent to

$$A \boxminus B = A\,\theta_g\,B = -(B\,\theta\,A), \quad \text{if} \quad w(A) < w(B).$$

We recall that

1. $w(-A) = w(A)$.
2. $w(A + B) = w(A) + w(B)$.
3. $w(A\,\theta_g\,B) = |w(A) - w(B)|$.

If $A = [\underline{a}, \overline{a}]$, then norm of A is given by

$$\|A\| := \max\left\{|\underline{a}|, |\overline{a}|\right\}.$$

A metric structure on $\mathcal{K}$ is given by the Housedorff-Pompeiu distance $\mathcal{H} : \mathcal{K} \times \mathcal{K} \to [0, \infty)$ is defined by $\mathcal{H}(A, B) = \max\left\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\right\}$ for $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ [22]. It is known that $(\mathcal{K}, \mathcal{H})$ is a complete, separable and locally compact metric space [13].
Let $A, B \in \mathcal{K}$. Then we have [33, 39]

- $\mathcal{H}(A, B) = \|A\theta_g B\|$.
- If there exist an interval $C \in \mathcal{K}$ such that $A = B + C$, then we call C the Hukuhara difference of A and B.

Definition 2.1. *[24] Let $X, Y \in \mathcal{K}$. Then $X \preceq Y (Y \preceq X)$ if and only if $\underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$ ($\underline{X} \geq \underline{Y}$ and $\overline{X} \geq \overline{Y}$). We call $\preceq$ the partial ordering on $\mathcal{K}$.*

In following lemma, we will show some interesting properties on the partial ordering $\preceq$.

Lemma 2.1. *[39] Suppose $X, Y, Z, W \in \mathcal{K}$ and $c \in \mathbb{R}^+$. Then*

1. *$X = Y$ iff $X \preceq Y$ and $Y \preceq X$.*
2. *If $X \preceq Y$, then $X + Z \preceq Y + Z$.*
3. *If $X \preceq Y$, then $cX \preceq cY$.*
4. *$X \preceq Y$, then $(-1)Y \preceq (-1)X$.*
5. *If $X\theta Y$ exists, then $X \preceq Y$ iff $Y\theta X \preceq 0$.*
6. *If the Hukuhara difference $X\theta Z$, and $X\theta Y$ exist, then $Z \preceq Y \iff X\theta Y \preceq X\theta Z$.*
7. *If $X \preceq Y \preceq Z$, then $\mathcal{H}(X, Y) \leq \mathcal{H}(X, Z)$ and $\mathcal{H}(Y, Z) \leq \mathcal{H}(X, Z)$.*

On the other hand, for $k \in \mathbb{N}$. We say that the sequence $(X_k), k \in \mathbb{N}, X_k \in \mathcal{K}$ is nondecreasing (nonincreasing), if $X_k \preceq X_{k+1} (X_k \succeq X_{k+1})$ for all $k \in \mathbb{N}$. Consider the interval functions $X, Y : [a, b] \to \mathcal{K}$, then the partial ordering $\preceq$ can be extended to the space of interval functions as follows [24]

$$X \preceq Y \longleftrightarrow \underline{X}(t) \leq \underline{Y}(t) \text{ and } \overline{X}(t) \leq \overline{Y}(t),$$

for all $t \in [a, b]$.
If $F : [a, b] \to \mathcal{K}$ is an interval-valued function such that $F(t) = [\underline{f}(t), \overline{f}(t)]$, then $\lim_{t\to t_0} F(t)$ exist if and only if $\lim_{t\to t_0} \underline{f}(t)$ and $\lim_{t\to t_0} \overline{f}(t)$ exist as finite number.
In this case, we have[22]

$$\lim_{t\to t_0} F(t) = [\lim_{t\to t_0} \underline{f}(t), \lim_{t\to t_0} \overline{f}(t)].$$

In particular, F is continuous if and only if $\underline{f}$ and $\overline{f}$ are continuous. If $F, G : [a, b] \to \mathcal{K}$ are two interval-valued function, then we define the interval-valued function $F\theta_g G : [a, b] \to \mathcal{K}$ by $(F\theta_g G)(t) = F(t)\theta_g G(t)$ for all $t \in [a, b]$. If there exist $\lim_{t\to t_0} F(t) = A$ and $\lim_{t\to t_0} G(t) = B$, then $\lim_{t\to t_0} (F\,\theta_g\,G)(t)$ exist, and [22]

$$\lim_{t\to t_0} (F\,\theta_g\,G)(t) = A\,\theta_g\,B.$$

In particular, If $F, G : [a,b]$ into $\mathcal{K}$ are continuous functions, then the interval function $(F\theta_g G)$ is a continues interval-valued function.
Let $C([a,b],\mathcal{K})$ denote the set of continuous interval-valued function from $[a,b] \to \mathcal{K}$. Then $C([a,b],\mathcal{K})$ is a complete normed space with respect to the norm [22]

$$\|F\|_c := \sup_{a\leq t\leq b} \|F(t)\|.$$

We say that an interval-valued function $F : [a,b] \to \mathcal{K}$ is w-increasing (w-decreasing) on $[a,b]$ if the real function $t \to w_F(t) := w(F(t))$ is increasing (decreasing) on $[a,b]$, and we say that F is w-monotone on $[a,b]$ [22, 24].

Lemma 2.2. *[24] Let $X : [a,b] \to \mathcal{K}$ be a w-monotone interval-valued function and $A \in \mathcal{K}$ and let $Y : [a,b] \to \mathcal{K}$, be the interval-valued function defined by $Y(t) = A\,\theta_g\,X(t), t \in [a,b]$. Then*

1. *If $w(X(t)) \leq w(A)$ for all $t \in [a,b]$, then Y and X are differently w-monotone (that is, one is w-increasing and the other is w-decreasing) on $[a,b]$.*
2. *If $(w(X(t)) \geq w(A)$ for all $t \in [a,b]$, then Y and X are equally w-monotone (that is, both are w-increasing or both w-decreasing) on $[a,b]$.*

Generalized Hukuhara Derivative:

Definition 2.2. *[25, 37] Let $F : [a,b] \to \mathcal{K}$ be an interval-valued function and let $t_0 \in [a,b]$. We define $F'(t_0) \in \mathcal{K}$ (provided it exists) as follows*

$$F'(t_0) = \lim_{h\to 0} \frac{F(t_0+h)\,\theta_g\,F(t_0)}{h} \in \mathcal{K}.$$

We call $F'(t_0)$ the generalized Hukuhara derivative *(*gH-derivative *for short) of F at t_0.*

Also we define the *left gH-derivative* $F'_-(t_0) \in \mathcal{K}$ (provided it exists) as

$$F'_-(t_0) = \lim_{h\to 0^-} \frac{F(t_0+h)\,\theta_g\,F(t_0)}{h},$$

and the *right gH-derivative* $F'_+(t_0) \in \mathcal{K}$ (provided it exists) as

$$F'_+(t_0) = \lim_{h\to 0^+} \frac{F(t_0+h)\,\theta_g\,F(t_0)}{h}.$$

We say that F is *generalized Hukuhara differentiable* (*gH-differentiable* for short) on $[a,b]$ if $F'(t) \in \mathcal{K}$ exist for all $t \in [a,b]$. At the endpoints of [a,b] we consider only the one sided gH-derivative [22].
The following are some properties which illustrate the behaviour of the gH-derivative

Proposition 2.1. *[25, 37] Let $F : [a,b] \to \mathcal{K}$ be such that $F(t) = [\underline{f}(t), \overline{f}(t)], t \in [a,b]$. If the real-valued functions $\underline{f}$ and $\overline{f}$ are differentiable at $t \in [a,b]$, then F is gH-differentiable at $t \in [a,b]$ and*

$$F'(t) = [\min\left\{\underline{f}'(t), \overline{f}'(t)\right\}, \max\left\{\underline{f}'(t), \overline{f}'(t)\right\}].$$

Note that the gH-differentiablity of F does not imply the differentiability of $\underline{f}$ and $\overline{f}$ [25, 9]. For example, $F(t) = [t, \sqrt{t}] : [0,1] \longrightarrow \mathcal{K}$, is gH- differentiable but $\underline{f} = \sqrt{t}$ is not differntiable at $t = 0$.

Proposition 2.2. *[25] Let $F : [a,b] \to \mathcal{K}$ be such that $F(t) = [\underline{f}(t), \overline{f}(t)], t \in [a,b]$. If F is w-monotone and gH-differentiable on $[a,b]$, then $\underline{f}'$ and $\overline{f}'$ exist for all $t \in [a,b]$. Moreover, we have that:*

1. *$F'(t) = [\underline{f}'(t), \overline{f}'(t)]$ for all $t \in [a,b]$, if F is w-increasing.*
2. *$F'(t) = [\overline{f}'(t), \underline{f}'(t)]$ for all $t \in [a,b]$, if F is w-decreasing.*

Note that if $F : [a,b] \to \mathcal{K}$ is gH-differentiable and w-monotone on $[a,b]$, then $w(F(t))$ is differentiable on $[a,b]$ and for all $t \in [a,b]$ we have [24]

$$w(F'(t)) = \frac{\mathrm{d}}{\mathrm{dt}}(\overline{f}(t) - \underline{f}(t)) = \frac{\mathrm{d}}{\mathrm{dt}}(w(F(t))),$$

if F is w-increasing on $[a,b]$, and

$$w(F'(t)) = \frac{\mathrm{d}}{\mathrm{dt}}(\underline{f}(t) - \overline{f}(t)) = \frac{\mathrm{d}}{\mathrm{dt}}(-w(F(t))),$$

if F is w-decreasing on $[a,b]$. The follows is an example that illustrates the previous propositions.

Example 2.1. *Consider the interval-valued function $F : [0,1] \to \mathcal{K}$ given by*

$$F(t) = [t^2 - 2t, t^2]$$

Since $w(F(t)) = 2t$, it follows that F is w-increasing on $[0,1]$. Because $\underline{f}(t) = t^2 - 2t$ and $\overline{f}(t) = t^2$ are differentiable on $[0,1]$, then by Proposition 2.1, we obtain that:

$$F'(t) = [2t - 2, 2t], \;\; t \in [a,b].$$

If we consider the interval-valued function $G : [2t-3, |t^2-1|]$, then G is w-decreasing on $[0,1]$ and w-increasing on $[1,2]$. Also $\underline{g}$ and $\overline{g}$ are differentiable on $[0,2]/\{1\}$, where [22]

$$\frac{\mathrm{d}^-}{\mathrm{d}t}\underline{g}(1) = \frac{\mathrm{d}^+}{\mathrm{d}t}\underline{g}(1) = 2,$$

and

$$\frac{\mathrm{d}^-}{\mathrm{d}t}\overline{g}(1) = -2 \quad \text{and} \quad \frac{\mathrm{d}^+}{\mathrm{d}t}\overline{g}(1) = 2.$$

It follows that $G'_-(1) = [-2,2]$, $G'_+(1) = 2$, then G is gH-differentiable on $[0,2]/\{1\}$ and

$$G'(t) = \begin{cases} [-2t, 2] & \text{if } t \in [0,1), \\ [2, 2t] & \text{if } t \in (1,2]. \end{cases}$$

Proposition 2.3. *[25] Let $F : [a,b] \to \mathcal{K}$ be w-monotone and gH-differentiable on $[a,b]$. The following are true:*

1. *For all $F \in \mathcal{K}$ and for all $\lambda \in \mathbb{R}$, the interval-valued functions $F + \lambda$, $F\,\theta_g\lambda$ and λF are the gH-differentiable on $[a,b]$, and*

$$\begin{aligned} (F+\lambda)' &= F', \\ (F\,\theta_g\,\lambda)' &= F', \\ (\lambda F)' &= \lambda F'. \end{aligned}$$

2. *If F and G are equally w-monotone, then*

$$\begin{aligned} (F+G)' &= F' + G' \quad \textit{and} \\ (F\,\theta_g\,G)' &= F'\,\theta_g\,G'. \end{aligned}$$

3. *If F and G are differently w-monotone, then*

$$\begin{aligned} (F+G)' &= F'\,\theta_g\,(-G') \quad \textit{and} \\ (F\,\theta_g\,G)' &= F' + (-G'). \end{aligned}$$

The Lebesgue integral for interval-valued function is a special case of the Lebesgue integral for the set-valued mapping [6].

Let $F : [a,b] \to \mathcal{K}$ be an interval valued function such that $F(t) = [\underline{f}(t), \overline{f}(t)]$, where $\underline{f}$

and $\overline{f}$ are measurable and Lebesgue integrable on $[a,b]$. Then we define $\int_a^b F(t)\,dt$ by [22]

$$\int_a^b F(t)\,dt = \left[\int_a^b \underline{f}(t)\,dt, \int_a^b \overline{f}(t)\,dt\right],$$

and we say that F is *Lebesgue integrable* on $[a,b]$.

An interval-valued function $F : [a,b] \to\in \mathcal{K}$ is called a *step set valued function* if there exists a partition $\{J_k : k = 1,2,\ldots,n\}$ of disjoint Lebesgue measurable subset in $[a,b]$. i.e,

$$\bigcup_{k\in\mathbb{N}}^{n} J_k = [a,b],$$

such that F is constant on each set $J_k, k \in \mathbb{N}$ [22].

In the following, we present a definition which we will need it in the coming formulas

Definition 2.3. *[31]**The Space** $L^\infty(\Omega)$*

Let Ω be any set, and a function f that is measurable on Ω is said to be essentially bounded *on Ω, if there is a constant K such that $|f(x)| \leq K$ a.e on Ω. The greatest lower bound of such constants K is called the* essential supremum *of $|f|$ on Ω, and is denoted by*

$$ess\sup_{x\in\Omega} |f(x)|.$$

We denote by $L^\infty(\Omega)$ the vector space of all functions f, that are essentially bounded on Ω, functions being once again identified if they are equal a.e. on Ω. Then $||f||_\infty$ can be defined by

$$||f||_\infty = ess\sup_{x\in\Omega} |f(x)|.$$

An interval-valued function $F : [a,b] \to \mathcal{K}$ is called measurable if it is almost everywhere in $[a,b]$ a point wise-limit of the sequence $F_m : [a,b] \to \mathcal{K}, m \geq 1$ of simple interval-valued functions such that

$$\lim_{m\to\infty} \mathcal{H}(F_m(t), F(t)) = 0.$$

for a.e $t \in [a,b]$[22].

It is clear that an interval-valued function $F : [a,b] \to \mathcal{K}$ is measurable if and only if $\underline{f}$ and $\overline{f}$ are measurable. In addition, it is clear that $F : [a,b] \to \mathcal{K}$ is integrable on $[a,b]$ if and only if F is measurable and the real function $t \mapsto ||F(t)||$ is Lebesgue integrable on $[a,b]$ [6, 5].

For $1 \leq p \leq \infty$, let $L^p([a,b])$ be the set of all interval-valued functions $F : [a,b] \to \mathcal{K}$

such that the real function $t \mapsto ||F(t)||$ belongs to $L^p([a,b])$. Then $L^p([a,b])$ is a complete metric space with respect to the metric $\mathcal{H}_p$ defined by $\mathcal{H}_p(F,G) := ||F\,\theta_g\,G||_p$, where

$$||F||_p := \begin{cases} (\int_a^b ||F||^p\,dt)^{\frac{1}{p}} & 1 \leq p < \infty, \\ ess\,\sup_{t\in[a,b]} |F(t)| & p = \infty. \end{cases}$$

An interval-valued function $F : [a,b] \to \mathcal{K}$ is said to be *absolutely continuous* if for all $\xi > 0$, there exist $\delta > 0$ such that for each family $\big\{(s_k, t_k) : k = 1, 2, \ldots, n\big\}$ of disjoint open intervals in $[a,b]$ with

$$\sum_{k=1}^{n}(t_k - s_k) < \delta,$$

we have

$$\sum_{k=1}^{n} \mathcal{H}(F(t_k) - F(s_k)) < \xi.$$

Let $AC([a,b],\mathcal{K})$ denote the set of all absolutely continuous interval function from $[a,b]$ to $\mathcal{K}$ [22].

Proposition 2.4. *[22] An interval-valued function $F : [a,b] \to \mathcal{K}$ is absolutely continuous if and only if $\underline{f}$ and $\overline{f}$ are both absolutely continuous.*

Proposition 2.5. *[22, 25] Let $F : [a,b] \to \mathcal{K}$ be Lebesgue integrable on $[a,b]$. Then the interval-valued function $G : [a,b] \to \mathcal{K}$ defined by*

$$G(t) := \int_a^t F(s)\,ds, \qquad t \in [a,b]. \tag{2.1}$$

Then we have the following

1. *G is absolutely continuous and $G'(t) = F(t)$, for a.e. $t \in [a,b]$.*

2. *If F is continuous on $[a,b]$, then G is continonusly gH-differentiable on $[a,b]$ and $G'(t) = F(t)$, for all $t \in [a,b]$.*

Proposition 2.6. *[22, 25] If $F \in AC([a,b],\mathcal{K})$, then F is gH-differentiable for a.e. on $[a,b]$ and $F' \in L^1([a,b],\mathcal{K})$.*
Moreover, if F is w-monotone on $[a,b]$, then

$$F(t)\,\theta_g F(a) = \int_a^t F'(s)\,ds \tag{2.2}$$

for all $t \in [a,b]$.

Also, if we assume that if F is w-increasing on $[a,b]$, then (2.2) is equivalent to

$$F(t) = F(a) + \int_a^t F'(s)\,ds,$$

and if F is w-decreasing on $[a,b]$, then (2.2) is equivalent to

$$F(t) = F(a)\theta_g(-1)\int_a^t F'\,ds,$$

for all $t \in [a,b]$][24].
We note that the relation (2.2) is not true if F is not w-monotone on $[a,b]$ [22].

Example 2.2. *If $F : [0,1] \to \mathcal{K}$ is the interval valued function given by*

$$F(t) = [-2t, 1-t^2],$$

since $w(F(t)) = -(t^2 - 2t - 1)$, then F is w-increasing on $[0,1]$, and since $\underline{f}$, and $\overline{f}$ are differentiable on $[0,1]$. Then by Proposition 2.2., we obtain that

$$F'(t) = [-2, -2t], \quad t \in [0,1].$$

Now, we have that

$$\begin{aligned} \int_0^t F'(s)\,ds &= \int_0^t [-2,-2t]\,ds \\ &= [-2t, -t^2], \end{aligned}$$

and

$$F(t)\ \theta\ F(0) = [2t, t^2 - 2] \neq \int_0^t F'(s)\,ds$$

for all $t \in [0,1]$. Therefore, (2.2) is not true for each $t \in [0,1]$.

2.2 Fractional Derivatives and Integrals of Interval-Valued Functions

In this section we present definitions and some properties of the Riemann-Liouville, Conformable, and Caputo fractional derivatives and integrals of interval-valued functions.

2.2.1 Riemann-Liouville Fractional Integral of Interval-Valued Functions

We recall that if $f \in L^1[a,b]$, then the Riemann-Liouville fractional integral $I^{\alpha}_{a^+}f$ of order $\alpha > 0$ is defined by

$$(I^{\alpha}_{a^+}f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

for a.e. $t \in [a,b]$.

Definition 2.4. *For a given real valued function $f \in L^1[a,b]$ and $\alpha > 0$, we define $f_\alpha : [a,b] \to \mathbb{R}$ by $f_\alpha(t) := (I^{\alpha}_{a^+}f)(t), t \in [a,b]$, and for $\alpha \in (0,1]$, we define $f_{1-\alpha} : [a,b] \to \mathbb{R}$ by $f_{1-\alpha}(t) := (I^{1-\alpha}_{a^+}f)(t), t \in [a,b]$.*

In the following, we present some properties of the Riemann-Liouville fractional integral [24, 14, 34].

1. If $f \in AC[a,b]$, then $f_\alpha \in AC[a,b]$ and
$$\frac{\mathrm{d}}{\mathrm{d}t} f_\alpha(t) = (I^{\alpha}_{a^+} \frac{\mathrm{d}}{\mathrm{d}t} f)(t) + \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} f(a),$$
for a.e. $t \in [a,b]$.

2. For any $\alpha \in (0,1]$, we have that $f_\alpha(t) \in AC[a,b]$ if and only if $f_{1-\alpha}(t) \in AC[a,b]$. Moreover, in this case
$$\frac{\mathrm{d}}{\mathrm{d}t} f_{1-\alpha}(t) = (I^{1-\alpha}_{a^+} \frac{\mathrm{d}}{\mathrm{d}t} f)(t) + \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a),$$
for a.e. $t \in [a,b]$.

3. If $f \in L^\infty[a,b]$, then $f_\alpha(t) \in C[a,b]$ and $f_\alpha(a) = 0$.

Definition 2.5. *[24] Let $F \in L^p([a,b],\mathcal{K}), 1 \leq p \leq \infty$. Then the* interval-valued Riemann-Liouville fractional integral of order $\alpha > 0$ *of the interval-valued function F is defined for a.e. $t \in [a,b]$ by*

$$(\mathcal{J}^{\alpha}_{a^+}F)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} F(s)\, ds.$$

If $F = [\underline{f}, \overline{f}] \in L^1([a,b],\mathcal{K})$ and $\alpha > 0$, it is obvious that

$$(\mathcal{J}^{\alpha}_{a^+}F)(t) = [I^{\alpha}_{a^+}\underline{f}(t), I^{\alpha}_{a^+}\overline{f}(t)],$$

for all $t \in [a,b]$.

In the following, we will discuss the cases of increasing and decreasing of the interval-valued Riemann-Liouville fractional integral.

Lemma 2.3. *[22]*
Let $\Phi \in L^1[a,b]$ be a positive and increasing real function on $[a,b]$ and let $\alpha \in (0,1]$. Then the real function $\phi(t) := \int_a^t (t-s)^{-\alpha}\Phi(s)\,ds$ is also increasing on [a,b].

Remark 2.2.1. *[22]*
From the previous lemma it follows that if $F \in L^p([a,b],\mathcal{K})$ $(1 \leq p \leq \infty)$ is w-increasing on $[a,b]$, then the interval-valued functions $(\mathcal{J}_{a^+}^{\alpha}F)(t)$ and $(\mathcal{J}_{a^+}^{1-\alpha}F)(t)$ are w-increasing on $[a,b]$ if $\alpha \in (0,1)$. If $\Phi \in L^1[a,b]$ is positive and decreasing on $[a,b]$, then the real function $\phi(t)$ is not decreasing on $[a,b]$, in general.
For example, if $\phi : [0,2] \to [0,2]$ is given by $\phi(t) = 1-t$. Then the function

$$\begin{aligned} \phi(t) &= \int_0^t (t-s)^{-\alpha}\phi(s)\,ds \\ &= \int_0^t (t-s)^{-\alpha}(1-s)\,ds \end{aligned}$$

let $t-s=x$, and substitute it in the integral. Then we get

$$\phi(t) = \frac{t^{1-\alpha}}{1-\alpha}\left(1-\frac{t}{\alpha-2}\right),$$

is increasing on $[0,2-\alpha]$ and decreasing on $[2-\alpha,2]$ for $\alpha \in (0,1)$.

In the following theorems, we present the operations on the interval-valued Riemann-Liouville fractional integral.

Theorem 2.1. *[24]*
If $F,G \in L^p([a,b],\mathcal{K}),(p \in [1,\infty))$ and $\alpha,\beta > 0$, then for a.e. $t \in [a,b]$ we have that

1. $w\left((\mathcal{J}_{a^+}^{\alpha}F)(t)\right) = I_{a^+}^{\alpha}w(F(t))$.
2. $\left(\mathcal{J}_{a^+}^{\alpha}((\mathcal{J}_{a^+}^{\beta}F)(t))\right)(t) = (\mathcal{J}_{a^+}^{\alpha+\beta}F)(t)$.
3. $\left((\mathcal{J}_{a^+}^{\alpha}(cF))\right)(t) = c(\mathcal{J}_{a^+}^{\alpha}F)(t)$ *for each* $c \in \mathbb{R}^+$.
4. $\left(\mathcal{J}_{a^+}^{\alpha}(F+G)\right)(t) = (\mathcal{J}_{a^+}^{\alpha}F)(t) + (\mathcal{J}_{a^+}^{\alpha}G)(t)$.

Theorem 2.2. *[22] If $F,G \in L^1([a,b],\mathcal{K})$ and $\alpha > 0$, then*

$$\mathcal{J}_{a^+}^{\alpha}F(t)\,\theta_g\,\mathcal{J}_{a^+}^{\alpha}G(t) \subseteq \mathcal{J}_{a^+}^{\alpha}(F\,\theta_g\,G)(t), \tag{2.3}$$

for all $t \in [a,b]$.
Moreover, if the difference $w(F(t)) - w(G(t))$ *has a constant sign on* $[a,b]$, *then*

$$\mathcal{J}_{a^+}^{\alpha} F(t)\, \theta_g\, \mathcal{J}_{a^+}^{\alpha} G(t) = \mathcal{J}_{a^+}^{\alpha} (F\, \theta_g\, G)(t), \tag{2.4}$$

for all $t \in [a,b]$.

In the following example, we show that the inclusion in Theorem 2.2 is strict.

Example 2.3. *[22] Let* $F, G : [0,3] \to \mathcal{K}$ *be given by* $F(t) = [0,t]$ *and* $G(t) = [-2,0]$, *respectively. First, we remark that* $w(F(t)) - w(G(t))$ *has not a constant sign on* $[0,3]$. *so we have*

$$\begin{aligned}
\mathcal{J}_{0^+}^{\frac{1}{2}} F(t) &= \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-s)^{-\frac{1}{2}} [0,s]\, ds \\
&= \frac{1}{\sqrt{\pi}} \left[0, \int_0^t (t-s)^{\frac{-1}{2}} s\, ds \right] \\
&= \left[0, \frac{4}{3\sqrt{\pi}} t^{\frac{3}{2}} \right], \quad t \in [0,3]. \\
\mathcal{J}_{0^+}^{\frac{1}{2}} G(t) &= \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-s)^{-\frac{1}{2}} [-2,0]\, ds \\
&= \frac{1}{\sqrt{\pi}} \left[-2 \int_0^t (t-s)^{\frac{-1}{2}}\, ds, 0 \right] \\
&= \left[\frac{4}{\sqrt{\pi}} t^{\frac{1}{2}}, 0 \right], \quad t \in [0,3].
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{J}_{0^+}^{\frac{1}{2}} F(t) \theta_g \mathcal{J}_{0^+}^{\frac{1}{2}} G(t) &= [\min \left\{ \frac{4}{\sqrt{\pi}} t^{\frac{1}{2}}, \frac{4}{3\sqrt{\pi}} t^{\frac{3}{2}} \right\}, \max \left\{ \frac{4}{\sqrt{\pi}} t^{\frac{1}{2}}, \frac{4}{3\sqrt{\pi}} t^{\frac{3}{2}} \right\}] \\
&= \left[\frac{4}{\sqrt{\pi}} t^{\frac{1}{2}}, \frac{4}{3\sqrt{\pi}} t^{\frac{3}{2}} \right] \\
&= \frac{4}{3\sqrt{\pi}} [t^{\frac{3}{2}}, 3t^{\frac{1}{2}}],
\end{aligned}$$

for all $t \in [0,3]$.
On the other hand, we have that

$$\begin{aligned}
(F\theta_g G)(t) &= [\min \{2,t\}, \max \{2,t\}] \\
&= \begin{cases} [t,2] & \textit{if} \quad t \in [0,2], \\ [2,t] & \textit{if} \quad t \in (2,3], \end{cases}
\end{aligned}$$

for $t \in [0,2]$, the difference $w(F(t)) - w(G(t)) = t - 2$ has a constant sign on $[0,2]$, and we have

$$\begin{aligned}
\mathcal{J}_{0^+}^{\frac{1}{2}}(F\theta_g G)(t) &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}}[s,2]\,ds \\
&= \frac{1}{\sqrt{\pi}}\left[\int_0^t s(t-s)^{\frac{1}{2}}\,ds, 2\int_0^t (t-s)^{\frac{1}{2}}\,ds\right] \\
&= \frac{4}{3\sqrt{\pi}}[t^{\frac{3}{2}}, 3t^{\frac{1}{2}}],
\end{aligned}$$

that is

$$\mathcal{J}_{0^+}^{\frac{1}{2}}(F\theta_g G)(t) = \mathcal{J}_{0^+}^{\frac{1}{2}}F(t)\theta_g \mathcal{J}_{0^+}^{\frac{1}{2}}G(t).$$

But for $t \in (2,3]$, we have

$$\begin{aligned}
\mathcal{J}_{0^+}^{\frac{1}{2}}(F\theta_g G)(t) &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^2 (t-s)^{-\frac{1}{2}}[s,2]\,ds + \frac{1}{\Gamma(\frac{1}{2})}\int_2^t (t-s)^{-\frac{1}{2}}[2,s]\,ds \\
&= \frac{4}{3\sqrt{\pi}}\left[t^{\frac{3}{2}} - (t-2)^{\frac{3}{2}}, 3t^{\frac{1}{2}} + (t-2)^{\frac{3}{2}}\right] \\
&\supset [t^{\frac{3}{2}}, 3t^{\frac{1}{2}}],
\end{aligned}$$

that is $\mathcal{J}_{0^+}^{\frac{1}{2}}(F\theta_g G)(t) \supset \mathcal{J}_{0^+}^{\frac{1}{2}}F(t)\theta_g \mathcal{J}_{0^+}^{\frac{1}{2}}G(t)$ for $t \in (2,3]$. It follows that the inclusion in (2.3) *is strict on $[0,3]$.*

Theorem 2.3. *[22, 34] The interval-valued Riemann-Liouville fractional integral of order $\alpha > 0$ is a bounded operator from $L^p([a,b],\mathcal{K})$ into $L^p([a,b],\mathcal{K})$ where $p \in [1,\infty)$,i.e.,*

$$||\mathcal{J}_{a^+}^{\alpha}F||_p \leq \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}||F||_p.$$

Moreover, if $\alpha \in (0,1)$ and $1 < p < \frac{1}{\alpha}$, then $\mathcal{J}_{a^+}^{\alpha}$ is a bounded operator from $L^p([a,b],\mathcal{K})$ into $L^q([a,b],\mathcal{K})$ where $q = \frac{p}{(1-\alpha p)}$.

2.2.2 Riemann-Liouville Fractional Derivative of Interval-Valued Functions

First, from Section 1.1 we recall the Riemann-Liouville fractional derivative of order $\alpha \in (0,1]$ for a real function $f \in C([a,b],\mathbb{R})$ is defined for a.e. $t \in [a,b]$ by

$$D_{a^+}^{\alpha}f(t) = \frac{\mathrm{d}}{\mathrm{d}t}I_{a^+}^{1-\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t (t-\tau)^{-\alpha}f(\tau)\,d\tau.$$

In particular, when $\alpha = 1$, then $D^1_{a+} f(t) = f'(t)$ for a.e. $t \in [a, b]$. For a given interval-valued function $F = [\underline{f}, \overline{f}] \in L^1([a, b], \mathcal{K})$ and $\alpha \in (0, 1]$, we define the interval-valued function $F_{1-\alpha} : [a, b] \to \mathcal{K}$ by

$$F_{1-\alpha}(t) = (\mathcal{J}^{1-\alpha}_{a^+} F)(t) := \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} F(s)\, ds,$$

for a.e. $t \in [a, b]$.

If the gH-derivative $(F_{1-\alpha})'$ exist for a.e. $t \in [a, b]$, then $(F_{1-\alpha})'$ is called the *interval-valued Riemann-Liouville fractional derivative (or Riemann-Liouville gH-fractional derivative) of order* $\alpha \in (0, 1]$. The Riemann-Liouville gH-fractional derivative of F will be denoted by $\mathcal{D}^\alpha_{a+} F$. Therefore

$$(\mathcal{D}^\alpha_{a+} F)(t) = (\mathcal{J}^{1-\alpha}_{a^+} F)'(t),$$

for a.e. $t \in [a, b]$.

In particular, when $\alpha = 1$ and $F \in AC([a, b], \mathcal{K})$, then $(\mathcal{D}^1_{a+} F)(t) = F'(t)$ for a.e. $t \in [a, b]$.

In the following, we present some theorems and lemmas which illustrate the Interval-Valued Riemann-Liouville Fractional Derivative with the interval function's increasing or decreasing.

Theorem 2.4. *[22] Let* $F = [\underline{f}, \overline{f}] \in AC([a, b], \mathcal{K})$. *Then*

1. $F_{1-\alpha} \in AC([a, b], \mathcal{K})$ *and*

$$\mathcal{D}^\alpha_{a+} f(t) = \left[\min\left\{(D^\alpha_{a+}\underline{f})(t), (D^\alpha_{a+}\overline{f})(t)\right\}, \max\left\{(D^\alpha_{a+}\underline{f})(t), (D^\alpha_{a+}\overline{f})(t)\right\}\right],$$

for a.e. $t \in [a, b]$.

2. *If either* F *is w-increasing on* $[a, b]$ *or* F *is w-decreasing and* $F_{1-\alpha}$ *is w-increasing on* $[a, b]$, *then*

$$(\mathcal{D}^\alpha_{a+} F)(t) = [(D^\alpha_{a+}\underline{f})(t), (D^\alpha_{a+}\overline{f})(t)]$$

for a.e. $t \in [a, b]$.

3. *If* $F_{1-\alpha}$ *is w-decreasing on* $[a, b]$. *Then*

$$(\mathcal{D}^\alpha_{a+} F)(t) = [(D^\alpha_{a+}\overline{f})(t), (D^\alpha_{a+}\underline{f})(t)]$$

In the following, we present a lemma that will be used to prove the coming theorem.

Lemma 2.4. *[18] If* $0 < \alpha < 1$*, then*

$$(I^{\alpha}_{a^+} D^{\alpha}_{a^+} f)(t) = f(t) - \frac{f_{1-\alpha}(a)}{\Gamma(\alpha)} (t-a)^{\alpha-1}.$$

Moreover, if $\alpha = n \in \mathbb{N}$*. Then the following equality holds*

$$(I^{n}_{a^+} D^{n}_{a^+} f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^{k}.$$

Theorem 2.5. *[22] Let* $F \in L^1([a,b], \mathcal{K})$ *be such that* $F_{1-\alpha} \in AC([a,b], \mathcal{K})$*. Then*

1. *If* $\frac{\mathrm{d}}{\mathrm{dt}} w(F_{1-\alpha}) \geq 0$ *for a.e.* $t \in [a,b]$*, then*

$$w(F(t)) \geq \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} w(F_{1-\alpha}(a)),$$

   for a.e. $t \in [a,b]$*.*

2. *If* $\frac{\mathrm{d}}{\mathrm{dt}} w(F_{1-\alpha}) \leq 0$ *for a.e.* $t \in [a,b]$*, then*

$$w(F(t)) \leq \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} w(F_{1-\alpha}(a)),$$

   for a.e. $t \in [a,b]$*.*

Proof. (1)
Using lemma 2.4., we get

$$I^{\alpha}_{a^+} D^{\alpha}_{a^+} w(F(t)) = w(F(t)) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} w(F_{1-\alpha}(a)) \tag{2.5}$$

for a.e. $t \in [a,b]$. Now since

$$\begin{aligned} I^{\alpha}_{a^+} D^{\alpha}_{a^+} w(F(t)) &= I^{\alpha}_{a^+} \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha}_{a^+} w(F(t)) \\ &= I^{\alpha}_{a^+} \frac{\mathrm{d}}{\mathrm{d}t} w(F_{1-\alpha}(a)) \end{aligned}$$

for a.e. $t \in [a,b]$, it follows that

$$w(F(t)) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} w(F_{1-\alpha}(a)) = I^{\alpha-1}_{a^+} \frac{\mathrm{d}}{\mathrm{d}t} w(F_{1-\alpha}(t)),$$

for a.e. $t \in [a,b]$.
Noting that $I^{\alpha}_{a^+} \frac{\mathrm{d}}{\mathrm{d}t} w(F_{1-\alpha}(t)) \geq 0$ for a.e. $t \in [a,b]$ if $\frac{\mathrm{d}}{\mathrm{d}t} w(F_{1-\alpha}(t)) \geq 0$ for a.e. $t \in [a,b]$, so we are done. Similarly, we can prove (2). □

Lemma 2.5. *[24] Let $F = [\underline{f}, \overline{f}] \in AC([a,b], \mathcal{K})$. Then*

1. *If either F is w-increasing on $[a,b]$ or F is w-decreasing and $F_{1-\alpha}$ is w-increasing on $[a,b]$, then*
$$(\mathcal{D}^{\alpha}_{a^+}F)(t) = (\mathcal{J}^{1-\alpha}_{a^+}F')(t) + \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}F(a),$$
for a.e. $t \in [a,b]$.

2. *If F and $F_{1-\alpha}$ are w-decreasing on $[a,b]$, then*
$$(\mathcal{D}^{\alpha}_{a^+}F)(t) = (\mathcal{J}^{1-\alpha}_{a^+}F')(t)\theta_g \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}(-F(a)),$$
for a.e. $t \in [a,b]$.

The following theorem is a direct consequence of Proposition 2.3.

Theorem 2.6. *[24] Let $F, G \in AC([a,b], \mathcal{K})$ be w-monotone on $[a,b]$ and let $\alpha \in (0,1]$*

a) If $F_{1-\alpha}$ and $G_{1-\alpha}$ are equally w-monotone on $[a,b]$, then
$$(\mathcal{D}^{\alpha}_{a^+}(F+G))(t) = (\mathcal{D}^{\alpha}_{a^+}(F))(t) + (\mathcal{D}^{\alpha}_{a^+}(G))(t),$$
for a.e. $t \in [a,b]$, and
$$(\mathcal{D}^{\alpha}_{a^+}(F \,\theta_g\, G))(t) = (\mathcal{D}^{\alpha}_{a^+}(F))(t) \,\theta_g\, (\mathcal{D}^{\alpha}_{a^+}(G))(t),$$
for a.e. $t \in [a,b]$.

b) If $F_{1-\alpha}$ and $G_{1-\alpha}$ are differently w-monotone on $[a,b]$, i.e., if one of them is w-increasing on $[a,b]$, then the other is w-deacreasing on $[a,b]$, then
$$(\mathcal{D}^{\alpha}_{a^+}(F+G))(t) = (\mathcal{D}^{\alpha}_{a^+}(F))(t)\theta_g(-(\mathcal{D}^{\alpha}_{a^+}(G)))(t),$$
for a.e. $t \in [a,b]$, and
$$(\mathcal{D}^{\alpha}_{a^+}(F\theta_g G))(t) = (\mathcal{D}^{\alpha}_{a^+}(F))(t) + (-(\mathcal{D}^{\alpha}_{a^+}(G)))(t),$$
for a.e. $t \in [a,b]$.

The following propositions represent the relation between interval-valued Riemann-liouville fractional integral and derivative.

Proposition 2.7. *[22] If $F \in L^P([a,b],\mathcal{K})$ $(1 \le p \le \infty)$, then*

$$\mathcal{D}^{\alpha}_{a^+}\mathcal{J}^{\alpha}_{a^+}F(t) = F(t),$$

for a.e. $t \in [a,b]$.

Proof. From previous definitions and theorems, we have

$$\begin{aligned}
\mathcal{D}^{\alpha}_{a^+}\mathcal{J}^{\alpha}_{a^+}F(t) &= (\mathcal{J}^{1-\alpha}_{a^+}\mathcal{J}^{\alpha}_{a^+}F)'(t) \\
&= (\mathcal{J}^{1}_{a^+}F)'(t) \\
&= (\int_a^t F(s)\,ds)' \\
&= F(t).
\end{aligned}$$

for a.e. $t \in [a,b]$. □

Proposition 2.8. *[22] Let $F \in L^p([a,b],\mathcal{K})$ $(1 \le p \le \infty)$ be such that $F_{1-\alpha} \in AC([a,b],\mathcal{K})$. If there exists an interval-valued function $G \in L^p([a,b],\mathcal{K})$ with $F = \mathcal{J}^{\alpha}_{a^+}G$. Then*

$$\mathcal{J}^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}F(t) = F(t),$$

for a.e. $t \in [a,b]$.

Proof. Indeed, we have that

$$\begin{aligned}
\mathcal{J}^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}F(t) &= \mathcal{J}^{\alpha}_{a^+}(\mathcal{J}^{1-\alpha}_{a^+}F)'(t) \\
&= \mathcal{J}^{\alpha}_{a^+}(\mathcal{J}^{1-\alpha}_{a^+}\mathcal{J}^{\alpha}_{a^+}G)' \quad \text{Using the proof of the previous proposition.} \\
&= F(t). \quad \text{Given from the statement of the proposition.}
\end{aligned}$$

□

Proposition 2.9. *[22, 24] Let $F \in L^1([a,b],\mathcal{K})$ be such that $F_{1-\alpha} \in AC([a,b],\mathcal{K})$. If either $\frac{\mathrm{d}}{\mathrm{dt}}w(F_{1-\alpha}(t)) \ge 0$ for a.e. $t \in [a,b]$ or $\frac{\mathrm{d}}{\mathrm{dt}}w(F_{1-\alpha}(t)) \le 0$ for a.e. $t \in [a,b]$, then the gH-Difference $F(t)\,\theta_g\,\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}F_{1-\alpha}(a)$ exists for a.e. $t \in [a,b]$, and*

$$\mathcal{J}^{\alpha}_{a^+}\mathcal{D}^{\alpha}_{a^+}F(t) = F(t)\,\theta_g\,\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}F_{1-\alpha}(a), \tag{2.6}$$

for a.e. $t \in [a,b]$.

Proof. From Theorem 2.5 and its proof, we get that

1. The existence of the difference

$$F(t)\theta_g \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} F_{1-\alpha}(a),$$

for a.e. $t \in [a,b]$.

2.

$$I^{\alpha}_{a^+} D^{\alpha}_{a^+} F(t) = F(t) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} F_{1-\alpha}(a),$$

for a.e. $t \in [a,b]$.

3. If $\frac{\mathrm{d}}{\mathrm{d}t} w(F_{1-\alpha}(t)) \geq 0$ for a.e. $t \in [a,b]$, then $F_{1-\alpha}$ is w-increasing on $[a,b]$. Thus

$$\begin{aligned}
\mathcal{J}^{\alpha}_{a^+} \mathcal{D}^{\alpha}_{a^+} F(t) &= \mathcal{J}^{\alpha}_{a^+} (F_{1-\alpha})'(t) \\
&= \mathcal{J}^{\alpha}_{a^+} [D^{\alpha}_{a^+} \underline{f}, D^{\alpha}_{a^+} \overline{f}] \\
&= [(I^{\alpha}_{a^+} D^{\alpha}_{a^+} \underline{f})(t), (I^{\alpha}_{a^+} D^{\alpha}_{a^+} \overline{f})(t)] \\
&= [\underline{f}(t) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} \underline{f}_{1-\alpha}(a), \overline{f}(t) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} \overline{f}_{1-\alpha(a)]} \\
&= F(t)\theta_g \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} F_{1-\alpha}(a),
\end{aligned}$$

for a.e. $t \in [a,b]$. By a similar reasoning we obtain (2.6) if $\frac{\mathrm{d}}{\mathrm{d}t} w(F_{1-\alpha}(t)) \leq 0$ for a.e. $t \in [a,b]$.

□

Remark 2.2.2. *[22] Under the conditions of Proposition 2.7, the relation* (2.6) *can be written as*

$$F(t) = \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} F_{1-\alpha}(a) + \mathcal{J}^{\alpha}_{a^+} \mathcal{D}^{\alpha}_{a^+} F(t)$$

for a.e. $t \in [a,b]$*, if* $\frac{\mathrm{d}}{\mathrm{d}t} w(F_{1-\alpha}(t)) \geq 0$ *for a.e.* $t \in [a,b]$*, and as*

$$F(t) = \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} F_{1-\alpha}(a)\theta_g(-\mathcal{J}^{\alpha}_{a^+} \mathcal{D}^{\alpha}_{a^+} F(t))$$

for a.e. $t \in [a,b]$*, if* $\frac{\mathrm{d}}{\mathrm{d}t} w(F_{1-\alpha}(t)) \leq 0$ *for a.e.* $t \in [a,b]$

The following example illustrates the previous proposition.

Example 2.4. *[22] Let us consider the interval-valued function $F : [0,1] \to \mathcal{K}$ given by $F(t) = [t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ if $t \in (0,1]$ and $F(0) = [0,1]$. Then $F \in L^1([0,1],\mathcal{K})$, F is w-decreasing on $(0,1]$, and*

$$F_{1-\frac{1}{2}}(t) = \mathcal{J}_{0^+}^{1-\frac{1}{2}} F(t) = \frac{1}{\sqrt{\pi}} \left[\frac{\pi}{2}t, \pi\right],$$

for all $t \in [0,1]$.
It follows that $F_{1-\frac{1}{2}} \in AC([0,1],\mathcal{K})$ and $\frac{\mathrm{d}}{\mathrm{d}t} w(F_{1-\alpha}(t)) \leq 0$ for all $t \in [0,1]$. Since the interval-valued function $F_{1-\frac{1}{2}}$ is w-decreasing on $[0,1]$, then we have that

$$\mathcal{D}_{0^+}^{\frac{1}{2}} F(t) = F'_{1-\frac{1}{2}}(t) = [0, \frac{\sqrt{\pi}}{2}],$$

for all $t \in [0,1]$, and so $\mathcal{J}_{0^+}^{\frac{1}{2}} \mathcal{D}_{0^+}^{\frac{1}{2}} F(t) = [0, t^{\frac{1}{2}}]$, for all $t \in [0,1]$.
On the other hand,

$$F(t)\theta_g \frac{t^{\frac{1}{2}-1}}{\Gamma(\frac{1}{2})} F_{1-\frac{1}{2}}(0) = [t^{\frac{1}{2}}, t^{-\frac{1}{2}}]\theta_g[0, t^{-\frac{1}{2}}] = [0, t^{\frac{1}{2}}].$$

that is $\mathcal{J}_{0^+}^{\frac{1}{2}} \mathcal{D}_{0^+}^{\frac{1}{2}} F(t) = F(t)\theta_g \frac{t^{\frac{1}{2}-1}}{\Gamma(\frac{1}{2})} F_{1-\frac{1}{2}}(0)$ for all $t \in [0,1]$.

2.2.3 Conformable Fractional Derivative of Interval-Valued Functions

In this subsection, we introduce and study the conformable fractional derivative which developed under interval arithmetic. The most content of this subsection is from [7, 33].

Definition 2.6. *Let $\mathcal{T} : (a,b) \to \mathbb{K}$ and $t \in (a,b)$, we say that $\mathcal{T}$ is* generalized conformable fractional differential *at t, if there exists $\mathcal{T}_\alpha(t) \in \mathbb{R}$ such that*

$$\mathcal{T}_{gH}^{(\alpha)}(t) = \lim_{\epsilon \to 0} \frac{\mathcal{T}(t + \epsilon t^{1-\alpha})\theta_g \mathcal{T}(t)}{\epsilon},$$

then $\mathcal{T}$ is called α-differentiable at $t \in (a,b)$.

In the following theorem, we discuss the value of $\mathcal{T}$ according to the function behaviour of monotonicity .

Theorem 2.7. *Let $\mathcal{T}(t) = [\mathcal{T}_1(t), \mathcal{T}_2(t)]$ be α-differentiable and w-monotone on (a,b). Then for every $t \in (a,b)$, the derivatives $\mathcal{T}_1(t)$ and $\mathcal{T}_2(t)$ exist and*

1. *$\mathcal{T}^{(\alpha)}(t) = [\mathcal{T}_1^{(\alpha)}(t), \mathcal{T}_2^{(\alpha)}(t)]$, if $\mathcal{T}$ is w-increasing.*

2. *$\mathcal{T}^{(\alpha)}(t) = [\mathcal{T}_2^{(\alpha)}(t), \mathcal{T}_1^{(\alpha)}(t)]$, if $\mathcal{T}$ is w-decreasing.*

Proof. Using the generalized conformable fractional derivative, we have for the case of w-increasing

$$\begin{aligned}
\mathcal{T}^{(\alpha)}(t) &= \lim_{\epsilon\to 0}\frac{\mathcal{T}(t+t^{1-\alpha})\theta_g\mathcal{T}(t)}{\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{[\mathcal{T}_1^{(\alpha)}(t+\epsilon t^{1-\alpha}),\mathcal{T}_2^{(\alpha)}(t+\epsilon t^{1-\alpha})]\theta_g[\mathcal{T}_1^{(\alpha)}(t),\mathcal{T}_2^{(\alpha)}(t)]}{\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{[\mathcal{T}_1^{(\alpha)}(t+\epsilon t^{1-\alpha})-\mathcal{T}_1^{(\alpha)}(t),\mathcal{T}_2^{(\alpha)}(t+\epsilon t^{1-\alpha})-\mathcal{T}_2^{(\alpha)}(t)]}{\epsilon}\\
&= [\mathcal{T}_1^{(\alpha)}(t),\mathcal{T}_2^{(\alpha)}(t)].
\end{aligned}$$

Similarly, we can get the proof of the case of w-decreasing. □

In the following theorem, we state the relation between Conformable fractional derivative and Conformable fractional integral of interval-valued functions.

Theorem 2.8. *[33] Let F be α-differentiable and w-monotone, then*

$$I_\alpha\mathcal{T}^{(\alpha)}F(t)=F(t)\theta_gF(a)$$

for a.e. $t\in[a,b]$, where I_α is the fractional integral.

Proof. Similarly to the proof of Theorem 1.4 and using the definition of integrability, differentiability and Theorem 2.7, the proof is straightforward. □

2.2.4 Caputo Fractional Derivative of Interval-Valued Functions

First, from Section 1.1.3, we get that if $f\in C[a,b]$, then the Caputo fractional derivative denoted by ${}^cD^\alpha_{a^+}f$ of order α is defined for a.e. $t\in[a,b]$ by

$${}^C_aD^\alpha_x=\frac{1}{\Gamma(n-\alpha)}\int_a^x\frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}\,dt,\ \ n-1<\alpha<n.$$

Let $F\in L^1([a,b],\mathcal{K})$ such that the Riemann-Liouville fractional derivative $\mathcal{D}^\alpha_{a^+}F$ exists a.e. on $[a,b]$, for $\alpha\in(0,1]$. In this case we will define the *interval valued Caputo fractional derivative* $({}^c\mathcal{D}^\alpha_{a^+}F)(t)$ *of order* $\alpha\in(0,1]$ *of* F by

$$({}^c\mathcal{D}^\alpha_{a^+}F)(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t(t-s)^{-\alpha}F'(s)\,ds,$$

for a.e. $t\in[a,b]$. Certainly, $({}^c\mathcal{D}^\alpha_{a^+}F)(t)=\mathcal{J}^{1-\alpha}_{a^+}F'(t)$ for a.e. $t\in[a,b]$ where $({}^c\mathcal{D}^\alpha_{a^+}F)(t)$ is called the *Interval-Valued Caputo Fractional Derivative (or Caputo gH-Fractional*

Derivative) of order $\alpha \in (0,1)$ [22].
In the following Proposition and Remark, we show how we can find the value of the interval-valued Caputo fractional derivative with respect to its from the function increasing or decreasing behaviour.

Proposition 2.10. *[23] Let $F \in AC([a,b],\mathcal{K})$ with $F(t) = [\underline{f}, \overline{f}]$. Then*

$$({}^{c}D^{\alpha}_{a^+}F)(t) \supseteq [\min\left\{{}^{c}\mathcal{D}^{\alpha}_{a^+}\underline{f}, {}^{c}\mathcal{D}^{\alpha}_{a^+}\overline{f}\right\}, \max\left\{{}^{c}\mathcal{D}^{\alpha}_{a^+}\underline{f}, {}^{c}D^{\alpha}_{a^+}\overline{f}\right\}], \tag{2.7}$$

for a.e. $t \in [a,b]$.

Now, the following remark illustrates the condition where the inequality (2.7) is hold.

Remark 2.2.3. *[22] If $F(t) = [\underline{f}, \overline{f}] \in AC([a,b],\mathcal{K})$ and $\alpha \in (0,1)$, then it is obvious that*

$$({}^{c}\mathcal{D}^{\alpha}_{a^+}F)(t) = [\min\left\{{}^{c}\mathcal{D}^{\alpha}_{a^+}\underline{f}, {}^{c}\mathcal{D}^{\alpha}_{a^+}\overline{f}\right\}, \max\left\{{}^{c}\mathcal{D}^{\alpha}_{a^+}\underline{f}, {}^{c}\mathcal{D}^{\alpha}_{a^+}\overline{f}\right\}],$$

for a.e. $t \in [a,b]$. If F is w-monotone, then

i . ${}^{c}\mathcal{D}^{\alpha}_{a^+}F(t) = [{}^{c}D^{\alpha}_{a^+}\underline{f}, {}^{c}D^{\alpha}_{a^+}\overline{f}]$ for a.e. $t \in [a,b]$, if F is w-increasing.

ii . ${}^{c}\mathcal{D}^{\alpha}_{a^+}F(t) = [{}^{c}D^{\alpha}_{a^+}\overline{f}, {}^{c}D^{\alpha}_{a^+}\underline{f}]$ for a.e. $t \in [a,b]$, if F is w-decreasing.

In the following theorems, we will discuss the operation between interval function via interval-valued Caputo fractional derivative.

Theorem 2.9. *[22] Let $F, G \in AC([a,b],\mathcal{K})$ be w-monotone on $[a,b]$ and let $\alpha \in (0,1)$*

1. *If F and G are equally w-monotone on $[a,b]$. Then*

$$ {}^{c}\mathcal{D}^{\alpha}_{a^+}(F+G)(t) = ({}^{c}\mathcal{D}^{\alpha}_{a^+}F)(t) + ({}^{c}\mathcal{D}^{\alpha}_{a^+}G)(t),$$

for a.e. $t \in [a,b]$, and

$$ {}^{c}\mathcal{D}^{\alpha}_{a^+}(F\theta_g G)(t) \supseteq ({}^{c}\mathcal{D}^{\alpha}_{a^+}F)(t)\theta_g({}^{c}\mathcal{D}^{\alpha}_{a^+}G)(t),$$

for a.e. $t \in [a,b]$. Moreover, if the difference $\big(w(F'(t)) - w(G'(t))\big)$ has a constant sign for a.e. $t \in [a,b]$, then

$$ {}^{c}\mathcal{D}^{\alpha}_{a^+}(F\theta_g G)(t) = ({}^{c}\mathcal{D}^{\alpha}_{a^+}F)(t)\theta_g({}^{c}\mathcal{D}^{\alpha}_{a^+}G)(t),$$

for a.e. $t \in [a,b]$.

2. If F and G are differently w-monotone on $[a,b]$, then

$$^{c}\mathcal{D}^{\alpha}_{a^+}(F\,\theta_g\,G)(t) = (^{c}\mathcal{D}^{\alpha}_{a^+}F)(t) + (-^{c}\mathcal{D}^{\alpha}_{a^+}G)(t),$$

for a.e. $t \in [a,b]$, and

$$^{c}\mathcal{D}^{\alpha}_{a^+}(F+G)(t) \supseteq (^{c}\mathcal{D}^{\alpha}_{a^+}F)(t)\theta_g(-^{c}\mathcal{D}^{\alpha}_{a^+}G)(t),$$

for a.e. $t \in [a,b]$. Moreover, if the difference $\big(w(F'(t)) - w(G'(t))\big)$ has a constant sign for a.e. $t \in [a,b]$, then

$$^{c}\mathcal{D}^{\alpha}_{a^+}(F\theta_g G)(t) = (^{c}\mathcal{D}^{\alpha}_{a^+}F)(t)\theta_g(-^{c}\mathcal{D}^{\alpha}_{a^+}G)(t),$$

for a.e. $t \in [a,b]$.

The following example illustrates that relations in the Theorem 2.9 can be false if the assumptions are not satisfied.

Example 2.5. *[22] Consider the interval-valued functions $F, G : [0,2] \to \mathcal{K}$, given by $F(t) = [0, -t^2+2t]$ and $G(t) = [0, 2t^2-4t+3]$, respectively. We have that $w_F(t) = -t^2+2t$ and $w_G(t) = 2t^2 - 4t + 3$ for all $t \in [0,2]$. It follows that F is w-increasing on $[0,1]$ and w-decreasing on $[1,2]$, and G is w-decreasing on $[0,1]$ and w-increasing on $[1,2]$. Then we have that*

$$\begin{aligned} (F+G)(t) &= [0, t^2 - 2t + 3], \\ (F\theta_g G)(t) &= [-3t^2 + 6t - 3, 0], \end{aligned}$$

for all $t \in [0,2]$*. Also, it is easy to check that*

$$F'(t) = \begin{cases} [0, 2-2t], & t \in [0,1), \\ \{0\}, & t = 1, \\ [2-2t, 0], & t \in (1,2], \end{cases}$$

$$G'(t) = \begin{cases} [4t-4, 0], & t \in [0,1), \\ \{0\}, & t = 1, \\ [0, 4t-4], & t \in (1,2], \end{cases}$$

$$(F+G)'(t) = \begin{cases} [2-2t, 0], & t \in [0,1), \\ \{0\}, & t = 1, \\ [0, 2t-2], & t \in (1,2], \end{cases}$$

and

$$(F\theta_g G)'(t) = \begin{cases} [0, 6-6t], & t \in [0,1), \\ 0, & t = 1, \\ [6-6t, 0], & t \in (1,2]. \end{cases}$$

We see that $w(F'(t)) - w(G'(t))$ *has a constant sign on each interval* $[0,1]$ *and* $[1,2]$*, but it does not have a constant sign on the interval* $[0,2]$*. For all* $t \in [0,1]$*, we obtain that*

$$\begin{aligned} {}^c\mathcal{D}^{\frac{1}{2}}_{0^+} F(t) &= \frac{4}{\sqrt{\pi}} \left[0, t^{\frac{1}{2}} - \frac{2}{3} t^{\frac{3}{2}}\right], \\ {}^c\mathcal{D}^{\frac{1}{2}}_{0^+} G(t) &= \frac{8}{\sqrt{\pi}} \left[-t^{\frac{1}{2}} + \frac{2}{3} t^{\frac{3}{2}}, 0\right], \\ {}^c\mathcal{D}^{\frac{1}{2}}_{0^+} (F+G)(t) &= \frac{4}{\sqrt{\pi}} \left[-t^{\frac{1}{2}} + \frac{2}{3} t^{\frac{3}{2}}, 0\right], \\ {}^c\mathcal{D}^{\frac{1}{2}}_{0^+} (F\theta_g G)(t) &= \frac{12}{\sqrt{\pi}} \left[-t^{\frac{1}{2}} + \frac{2}{3} t^{\frac{3}{2}}, 0\right], \end{aligned}$$

and

$$\begin{aligned} {}^c\mathcal{D}^{\frac{1}{2}}_{0^+} F(t) + {}^c\mathcal{D}^{\frac{1}{2}}_{0^+} G(t) &= \frac{4}{\sqrt{\pi}} \left[-t^{\frac{1}{2}} + \frac{2}{3} t^{\frac{3}{2}}, t^{\frac{1}{2}} - \frac{2}{3} t^{\frac{3}{2}}\right] \\ &\neq {}^c\mathcal{D}^{\frac{1}{2}}_{0^+} (F+G)(t), \\ {}^c\mathcal{D}^{\frac{1}{2}}_{0^+} F(t) \theta_g {}^c\mathcal{D}^{\frac{1}{2}}_{0^+} G(t) &= \frac{4}{\sqrt{\pi}} \left[t^{\frac{1}{2}} - \frac{2}{3} t^{\frac{3}{2}}, 2t^{\frac{1}{2}} - \frac{4}{3} t^{\frac{3}{2}}\right] \\ &\nsubseteq {}^c\mathcal{D}^{\frac{1}{2}}_{0^+} (F\theta_g G)(t). \end{aligned}$$

Also,

$$ {}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}}(F\theta_g G)(t) \not\subseteq {}^{c}\mathcal{D}_{0^+}^{\alpha}F(t)\theta_g {}^{c}\mathcal{D}_{0^+}^{\alpha}G(t), $$

but

$$ \begin{aligned} {}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}}F(t) + (-{}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}})G(t) &= \frac{12}{\sqrt{\pi}}\left[-t^{\frac{1}{2}} + \frac{2}{3}t^{\frac{3}{2}}, 0\right] \\ &= {}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}}(F\theta_g G)(t), \end{aligned} $$

and

$$ \begin{aligned} {}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}}F(t)\theta_g(-{}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}})G(t) &= \frac{4}{\sqrt{\pi}}\left[-t^{\frac{1}{2}} + \frac{2}{3}t^{\frac{3}{2}}, 0\right] \\ &= {}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}}(F+G)(t). \end{aligned} $$

Similarly, using the same procedure we obtain that for $t \in (1, 2]$

$$ \begin{aligned} {}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}}F(t) + (-{}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}})G(t) &= \frac{8}{3\sqrt{\pi}}\left[3(t-1)^{\frac{3}{2}} + (t-\frac{3}{2})t^{\frac{1}{2}}, 3(t-1)^{\frac{3}{2}} + (\frac{3}{2}-t)t^{\frac{1}{2}}\right] \\ &\neq {}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}}(F\theta_g G)(t), \end{aligned} $$

and

$$ \begin{aligned} {}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}}F(t)\theta_g(-{}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}})G(t) &= \frac{8}{\sqrt{\pi}}\left[(t-1)^{\frac{3}{2}}, (t-1)^{\frac{3}{2}} + (\frac{3}{2}-t)t^{\frac{1}{2}}\right] \\ &\neq {}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}}(F+G)(t). \end{aligned} $$

But

$$ {}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}}F(t)\theta_g(-{}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}})G(t) \supset {}^{c}\mathcal{D}_{0^+}^{\frac{1}{2}}(F+G)(t) \quad \textit{for } t \in (\frac{3}{2}, 2]. $$

The next theorem gives an equivalent formula of the interval-value Caputo fractional derivative. But before that, we will present a lemma we need it in the proof of the theorem.

Lemma 2.6. *[34] For a real-valued function* $\phi \in AC[a, b]$*, we have*

$$ D_{a^+}^{\alpha}\phi(t) = {}^{c}\mathcal{D}_{a^+}^{\alpha}\phi(t) + \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}\phi(a), $$

for a.e. $t \in [a, b]$.

Theorem 2.10. *[22, 24] Let* $F = [\underline{f}, \overline{f}] \in AC([a, b], \mathcal{K})$*. Then the following Properties are then true*

1. *If either F is w-increasing on $[a,b]$ or F is w-decreasing and $F_{1-\alpha}$ is w-increasing on $[a,b]$, then*
$$^c\mathcal{D}^{\alpha}_{a^+}F(t) = \mathcal{D}^{\alpha}_{a^+}F(t)\theta_g \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}F(a), \tag{2.8}$$
for a.e. $t \in [a,b]$.

2. *If both F and $F_{1-\alpha}$ are w-decreasing on $[a,b]$, then*
$$\mathcal{D}^{\alpha}_{a^+}F(t) = {}^c\mathcal{D}^{\alpha}_{a^+}F(t)\theta_g \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}(-F(a)), \tag{2.9}$$
for a.e. $t \in [a,b]$.

Proof. If F is w-increasing, then $F_{1-\alpha}$ is w-increasing on $[a,b]$ and we have

$$\begin{aligned}
{}^c\mathcal{D}^{\alpha}_{a^+}F(t) + \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}F(a) &= [{}^cD^{\alpha}_{a^+}\underline{f}(t), {}^cD^{\alpha}_{a^+}\overline{f}(t)] + \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}[\underline{f}(a), \overline{f}(a)] \\
&= [{}^cD^{\alpha}_{a^+}\underline{f}(t) + \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}\underline{f}(a), {}^cD^{\alpha}_{a^+}\overline{f}(t) + \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}\overline{f}(a)] \\
&= [D^{\alpha}_{a^+}\underline{f}(t), D^{\alpha}_{a^+}\overline{f}(t)] \\
&= \mathcal{D}^{\alpha}_{a^+}F(t),
\end{aligned}$$

that is (2.8) is true for a.e. $t \in [a,b]$.
If F is w-decreasing and $F_{1-\alpha}$ is w-increasing on $[a,b]$, then

$$\begin{aligned}
{}^c\mathcal{D}^{\alpha}_{a^+}F(t) + (-\mathcal{D}^{\alpha}_{a^+}F(t)) &= [{}^cD^{\alpha}_{a^+}\underline{f}(t), {}^cD^{\alpha}_{a^+}\overline{f}(t)] + [D^{\alpha}_{a^+}\underline{f}(t), D^{\alpha}_{a^+}\overline{f}(t)] \\
&= \left[-\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}\overline{f}(a), \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}\underline{f}(a)\right] \\
&= \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}(-F(a)).
\end{aligned}$$

It follows that

$$\begin{aligned}
{}^c\mathcal{D}^{\alpha}_{a^+}F(t) &= \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}(-F(a))\theta_g(-\mathcal{D}^{\alpha}_{a^+}F(t))) \\
&= \mathcal{D}^{\alpha}_{a^+}F(t))\theta_g \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}(F(a)),
\end{aligned}$$

for a.e. $t \in [a,b]$.
Using the same procedures, we get (2.9) if both F and $F_{1-\alpha}$ are w-decreasing on $[a,b]$. □

In the following there are some properties which illustrate the relationships between the interval-valued Riemann-Liouville fractional integral and Caputo fractional derivative.

Proposition 2.11. *[22] If $F \in AC([a,b],\mathcal{K})$ is a w-monotone interval-valued function and $\alpha \in (0,1]$, then*

$$\mathcal{J}^{\alpha}_{a^+}\,{}^{c}\mathcal{D}^{\alpha}_{a^+}F(t) = F(t)\theta_g F(a), \tag{2.10}$$

for a.e $t \in [a,b]$.

Proof. By Proposition 2.5, we have that

$$\begin{aligned}
\mathcal{J}^{\alpha}_{a^+}\,{}^{c}\mathcal{D}^{\alpha}_{a^+}F(t) &= \mathcal{J}^{\alpha}_{a^+}\mathcal{J}^{1-\alpha}_{a^+}F(t) \\
&= \mathcal{J}^{1}_{a^+}F'(t) \\
&= \int_a^t F'(s)\,ds \\
&= F(t)\theta_g F(a),
\end{aligned}$$

for a.e. $t \in [a,b]$. □

Remark 2.2.4. *[22] The relation* (2.10) *can be written as*

$$F(t) = F(a) + \mathcal{J}^{\alpha}_{a^+}\,{}^{c}\mathcal{D}^{\alpha}_{a^+}F(t),$$

if F is w-increasing on $[a,b]$, and as

$$F(t) = F(a)\theta_g(-1)\mathcal{J}^{\alpha}_{a^+}\,{}^{c}\mathcal{D}^{\alpha}_{a^+}F(t),$$

if F is w-decreasing on $[a,b]$. Also, we remark that the equality (2.10) *can be false if F is not monotone on $[a,b]$.*
Indeed, for interval-valued function $F(t) = [0, -t^2 + 2t], t \in (1,2]$, we have that

$${}^{c}\mathcal{D}^{\frac{1}{2}}_{0^+}F(t) = \frac{8}{3\sqrt{\pi}}\left[(t-1)^{\frac{3}{2}}, (t-1)^{\frac{3}{2}} + (\frac{3}{2} - t)t^{\frac{1}{2}}\right],$$

for all $t \in (1,2]$. Hence we obtain that

$$\begin{aligned}
\mathcal{J}^{\frac{1}{2}}_{0^+}\,{}^{c}\mathcal{D}^{\frac{1}{2}}_{0^+}F(t) &= [-2t^2 + 4t + 2, t^2 + 2t + 2] \\
&\neq F(t)\theta_g F(0).
\end{aligned}$$

Proposition 2.12. *[22] Let $F \in L^{\infty}([a,b],\mathcal{K})$ be such that either F is w-increasing on $[a,b]$, or F is w-decreasing on $[a,b]$ and $F_{\alpha}(t) := \mathcal{J}^{\alpha}_{a^+}F(t)$ is w-increasing on $[a,b]$, then*

$${}^{c}\mathcal{D}^{\alpha}_{a^+}\mathcal{J}^{\alpha}_{a^+}F(t) = F(t), \tag{2.11}$$

for a.e. $t \in [a,b]$.

Proof. It is known that for a real-valued function $\phi \in L^{\infty}[a,b]$, we have that ${}^{c}D^{\alpha}_{a+} I^{\alpha}_{a+}\phi(t) = \phi(t)$ for a.e. $t \in [a,b]$[1]. If F is w-increasing on $[a,b]$, then from Lemma 2.3 it follows that F_{α} is also w-increasing on $[a,b]$. Therefore in both cases, F_{α} is w-increasing on $[a,b]$ and we have

$$\begin{aligned} {}^{c}\mathcal{D}^{\alpha}_{a+}\mathcal{J}^{\alpha}_{a+}F(t) &= {}^{c}\mathcal{D}^{\alpha}_{a+}[I^{\alpha}_{a+}\underline{f}, I^{\alpha}_{a+}\overline{f}](t) \\ &= [{}^{c}D^{\alpha}_{a+}I^{\alpha}_{a+}\underline{f}, {}^{c}D^{\alpha}_{a+}I^{\alpha}_{a+}\overline{f}](t) \\ &= [\underline{f}, \overline{f}](t) \\ &= F(t), \end{aligned}$$

for a.e. $t \in [a,b]$. □

The equality (2.11) can be false if F is not w-monotone on $[a,b]$ [22]. Now, the following example illustrates the Proposition 2.12.

Example 2.6. *[22] Consider the interval-valued function $F : [0,1] \to \mathcal{K}$, given by $F(t) = [t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ if $t \in (0,1]$, and $F(0) = [0,1]$. Then $F \in L^{1}([0,1],\mathcal{K})$, but $F \notin L^{\infty}([0,1],\mathcal{K})$. we have that*

$$F_{\frac{1}{2}}(t) = \mathcal{J}^{\frac{1}{2}}_{0+}F(t) = \frac{1}{\sqrt{\pi}}\left[\frac{\pi}{2}t, \pi\right],$$

for all $t \in [0,1]$.

It follows that $F_{\frac{1}{2}} \in AC([0,1],\mathcal{K})$ and $\frac{\mathrm{d}}{\mathrm{d}t}w(F_{\frac{1}{2}}) \leq 0$ for all $t \in [0,1]$. Since the interval-valued function $F_{\frac{1}{2}} = \mathcal{J}^{\frac{1}{2}}_{0+}F$ is w-decreasing on $[0,1]$, we have that

$${}^{c}\mathcal{D}^{\alpha}_{0+}F_{\frac{1}{2}} = F'_{\frac{1}{2}}(t) = [0, \frac{\sqrt{\pi}}{2}], \tag{2.12}$$

for all $t \in [0,1]$, and so ${}^{c}\mathcal{D}^{\frac{1}{2}}_{0+}\mathcal{J}^{\frac{1}{2}}_{0+}F(t) = [0, t^{\frac{1}{2}}] \neq F(t)$, for all $t \in [a,b]$.
The following theorem show the interval-valued differential equations with there initial conditions.

Theorem 2.11. *[24] Let H be an interval-valued function such that $H(t) \in C^{1}([a,b],\mathcal{K})$, and let $(\mathcal{J}^{\alpha}_{0+}H)(t)$ be w-increasing on $[a,b]$. Then there is a w-monotone unique solution $X \in C([a,b],\mathcal{K})$ of the initial-value problem*

$$\begin{cases} ({}^{c}\mathbf{D}^{\alpha}_{a+}X)(t) = H(t), \\ X(a) = X_0 \in \mathcal{K}, \end{cases} \tag{2.13}$$

given by

$$X(t)\theta_g X_0 = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}H(s)\,ds, \tag{2.14}$$

[1] see Lemma 2.21 in Kilbas et al. [18]

Proof. From (2.14) we get that $X(a) = X_0$ and $X \in C([a,b],\mathcal{K})$, and for $t \in (a,b]$ and $H \in C([a,b],\mathcal{K})$, we get that

$$\begin{aligned}
(\mathcal{D}^{\alpha}_{a^+}[X(t)\theta_g X(a)]) &= (\mathcal{J}^{1-\alpha}_{a^+}[X(t)\theta_g X(a)])' \\
&= (\mathcal{J}^{1-\alpha}_{a^+}\mathcal{J}^{\alpha}_{a^+}H)'(t) \\
&= H(t).
\end{aligned}$$

This infers that X is a solution of (2.13). Then the proof is completed. □

If $X \in C([a,b],\mathcal{K})$ is such that $w(X(t)) \geq w(X_0)$ for all $t \in [a,b]$, then (2.14) can be written as

$$X(t) = X_0 + \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} H(s)\,ds, \quad t \in [a,b].$$

Otherwise, if $w(X(t)) \leq w(X_0)$ for all $t \in [a,b]$, then (2.14) can be written as

$$X(t) = X_0\theta(-1)\int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} H(s)\,ds, \quad t \in [a,b].$$

If $X \in C([a,b],\mathcal{K})$ is an interval-valued function such that $X_a(t) := X(t)\theta_g X(a), t \in [a,b]$, satisfies (2.14), then X_a is called the *Condensed solution* of (2.14). Moreover, if $X \in C([a,T],\mathcal{K})$ may produce two solutions of (2.14): a w-increasing solution $X\uparrow(t) = X(a) + X_a(t), t \in [a,b]$, if X is w-increasing on $[a,b]$, and a w-decreasing solution $X\downarrow(t) = X(a)\theta(-X_a(t)), t \in [a,b]$, if X is w-decreasing on $[a,b]$.
The following Example illustrates the interval-valued differential equation and its solution.

Example 2.7. *[24] Let us consider the following initial-value problem*

$$\begin{cases} ({}^{c}\mathcal{D}^{\frac{1}{2}}_{0^+}X)(t) = [t,1], \quad t \in [0,1], \\ X(0) = [0,1], \end{cases} \tag{2.15}$$

and its associated integral equation

$$X(t)\theta_g X(0) = (\mathcal{J}^{\frac{1}{2}}_{0^+}F)(t), \quad t \in [0,1], \tag{2.16}$$

where $F(t) := [t,1], t \in [0,1]$. We have that

$$\begin{aligned}
(\mathcal{J}^{\frac{1}{2}}_{0^+}F)(t) &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}}[s,1]\,ds \\
&= \left[\frac{4}{3\sqrt{\pi}}t^{\frac{3}{2}}, \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}}\right], \quad t \in [t_0,1],
\end{aligned}$$

where $t_0 = 0$. *If we put* $Y_0(t) := (\mathcal{J}_{0^+}^{\frac{1}{2}} F)(t)$, $t \in [t_0, 1]$. *Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} w(Y_0(t)) = \frac{t^{-\frac{1}{2}}}{\sqrt{\pi}}(1-2t), t \in (t_0, 1].$$

It follows that Y_0 *is* w*-increasing on* $[t_0, \frac{1}{2}]$ *and* w*-decreasing on* $[\frac{1}{2}, 1]$. *Then the condensed solution* X_0 *of the integral equation* (2.16), *namely*

$$\begin{aligned}\mathcal{X}_0(t) &= X_0(t)\theta_g X_0(t_0),\\ &= \left[\frac{4}{3\sqrt{\pi}} t^{\frac{3}{2}}, \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}}\right], \quad t \in [t_0, 1]\end{aligned}$$

produces a w*-monotone solution* X_0 *of* (2.15) *only on the interval* $[t_0, \frac{1}{2}]$. *We obtain the* w*-increasing solution*

$$\begin{aligned}X_0 \uparrow (t) &:= X(0) + (\mathcal{J}_{t_0^+}^{\frac{1}{2}})X(t),\\ &= [0,1] + \left[\frac{4}{3\sqrt{\pi}} t^{\frac{3}{2}}, \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}}\right],\\ &= \left[\frac{4}{3\sqrt{\pi}} t^{\frac{3}{2}}, 1 + \frac{2}{\sqrt{\pi}} t^{\frac{1}{2}}\right], \quad t \in [t_0, t_1],\end{aligned}$$

and the w*-decreasing solution*

$$\begin{aligned}X_0 \downarrow (t) &:= [0,1]\theta(-(\mathcal{J}_{0^+}^{\frac{1}{2}} X)(t))\\ &= [0,1]\theta\left[-\frac{2}{\sqrt{\pi}} t^{\frac{1}{2}}, -\frac{4}{3\sqrt{\pi}} t^{\frac{3}{2}}\right]\\ &= \left[\frac{2}{\sqrt{\pi}} t^{\frac{1}{2}}, 1 + \frac{4}{3\sqrt{\pi}} t^{\frac{3}{2}}\right], \quad t \in [t_0, t_1].\end{aligned}$$

The solution $\mathcal{X}_0$ *can be extended to the right of the point* t_1 *up to a point* $t_2 \in (t_1, 1]$ *such that* $(\mathcal{J}_{0^+}^{\frac{1}{2}} F)(t)$ *is* w*-increasing on* $[t_1, t_2]$. *The extension of* $\mathcal{X}_0$ *up to* t_2 *is an interval-valued function* $X_1 : [t_0, t_2] \to \mathcal{K}$ *such that* $\mathcal{X}_1(t) = \mathcal{X}_0(t)$ *for* $t \in [t_0, t_1]$ *and* X_1 *is the solution of the following initial-value problem*

$$\begin{cases}({}^c\mathcal{D}_{0^+}^{\frac{1}{2}} X)(t) = [t, 1], \quad t \in [t_1, 1],\\ X(t_1) = \mathcal{X}_0(t_1).\end{cases} \tag{2.17}$$

where $\mathcal{X}_0(t_1) = X_0(t_1)\theta_g X_0(t_0) = (\mathcal{J}_{0^+}^{\frac{1}{2}} F)(t_1)$; *that is,*

$$X(t_1) = X_0 \uparrow (t_1) = X(t_0) + (\mathcal{J}_{0^+}^{\frac{1}{2}} F)(t_1),$$

if X_0 *is* w*-increasing and*

$$X(t_1) = X_0 \downarrow (t_1) = X_0(t_0)\theta(-(\mathcal{J}_{0^+}^{\frac{1}{2}} F)(t_1)),$$

if X_0 *is w-decreasing. Here,* $X_0(t_0) = [0,1]$ *and* $X(t_1) = X_0 \uparrow (t_1) = \left[\frac{1}{3}\sqrt{\frac{2}{\pi}}, 1 + \sqrt{\frac{2}{\pi}}\right]$, *if* X *is w-increasing, and* $X(t_1) = X_0 \downarrow (t_1) = \left[\sqrt{\frac{2}{\pi}}, 1 + \frac{1}{3}\sqrt{\frac{2}{\pi}}\right]$ *if* X *is w-decreasing.*
The integral equation associated to (2.17) *is*

$$X(t)\theta_g X(t_1) = (\mathcal{J}_{t_1^+}^{\frac{1}{2}})F(t), \quad t \in [t_1, 1]. \tag{2.18}$$

Next, we have that

$$\begin{aligned}(\mathcal{J}_{t_1^+}^{\frac{1}{2}}F)(t) &= \int_{t_1}^{t} \frac{(t-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}[s,1]\,ds \\ &= \left[\frac{4t+2t_1}{3\sqrt{\pi}}\sqrt{t-t_1}, \frac{2}{\sqrt{\pi}}\sqrt{t-t_1}\right], \quad t \in [t_1, 1].\end{aligned}$$

If we put $Y_1(t) := (\mathcal{J}_{t_1^+}^{\frac{1}{2}})F(t), \quad t \in [t_1, 1]$. *Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} w(Y_1(t)) = \frac{1}{2\sqrt{\pi(t-t_1)}}(1 + t_1 - 2t), \quad t \in (t_1, 1],$$

it follows that Y_1 *is w-increasing on* $[t_1, t_2]$, *and w-decreasing on* $[t_2, 1]$, *where* $t_2 = \frac{1}{2}(1 + t_1) = \frac{3}{4}$.
Then the condensed solution X_1 *of the integral equation* (2.18) *is*

$$\mathcal{X}_1(t) := X_1(t)\theta_g X_1(t_1) = \left[\frac{4t+2t_1}{3\sqrt{\pi}}\sqrt{t-t_1}, \frac{2}{\pi}\sqrt{t-t_1}\right], \quad t \in [t_1, t_2],$$

and it is produces four w-monotone solutions for initial value problem (2.17), *namely*

$$\begin{aligned}X_1 \uparrow\uparrow (t) &:= X_0 \uparrow (t_1) + (\mathcal{J}_{t_1^+}^{\frac{1}{2}})F(t), \\ &= \left[\frac{1}{3}\sqrt{\frac{2}{\pi}}, 1 + \sqrt{\frac{2}{\pi}}\right] + \left[\frac{4t+2t_1}{3\sqrt{\pi}}\sqrt{t-t_1}, \frac{2}{\pi}\sqrt{t-t_1}\right], \\ &= \left[\frac{1}{3}\sqrt{\frac{2}{\pi}} + \frac{4t+2t_1}{3\sqrt{\pi}}\sqrt{t-t_1}, 1 + \sqrt{\frac{2}{\pi}} + \frac{2}{\pi}\sqrt{t-t_1}\right], \quad t \in [t_1, t_2],\end{aligned}$$

and

$$\begin{aligned}X_1 \downarrow\uparrow (t) &:= X_1 \downarrow (t) + ((\mathcal{J}_{t_1^+}^{\frac{1}{2}})F(t)) \\ &= \left[\sqrt{\frac{2}{\pi}}, 1 + \frac{1}{3}\sqrt{\frac{2}{\pi}}\right] + \left[\frac{4t+2t_1}{3\sqrt{\pi}}\sqrt{t-t_1}, \frac{2}{\sqrt{\pi}}\sqrt{t-t_1}\right] \\ &= \left[\sqrt{\frac{2}{\pi}} + \frac{4t+2t_1}{3\sqrt{\pi}}\sqrt{t-t_1}, 1 + \frac{1}{3}\sqrt{\frac{2}{\pi}} + \frac{2}{\sqrt{\pi}}\sqrt{t-t_1}\right],\end{aligned}$$

for $t \in [t_1, t_2]$, and two w-decreasing solutions

$$\begin{aligned} X_1 \uparrow\downarrow (t) &:= X_1 \uparrow (t) \theta(-((\mathcal{J}^{\frac{1}{2}}_{t_1^+})F(t))) \\ &= \left[\frac{1}{3}\sqrt{\frac{2}{\pi}}, 1 + \sqrt{\frac{2}{\pi}}\right] \theta \left[-\frac{2}{\pi}\sqrt{t-t_1}, -\frac{4t+2t_1}{3\sqrt{\pi}}\sqrt{t-t_1}\right] \\ &= \left[\frac{1}{3}\sqrt{\frac{2}{\pi}} + \frac{2}{\sqrt{\pi}}\sqrt{t-t_1}, 1 + \sqrt{\frac{2}{\pi}} + -\frac{4t+2t_1}{3\sqrt{\pi}}\sqrt{t-t_1}\right], \quad t \in [t_1, t_2], \end{aligned}$$

and

$$\begin{aligned} X_1 \downarrow\downarrow (t) &:= X_1 \downarrow (t) \theta(-((\mathcal{J}^{\frac{1}{2}}_{t_1^+})F(t))) \\ &= \left[\sqrt{\frac{2}{\pi}}, 1 + \frac{1}{3}\sqrt{\frac{2}{\pi}}\right] \theta \left[-\frac{2}{\sqrt{\pi}} + \sqrt{t-t_1}, -\frac{4t+2t_1}{3\sqrt{\pi}}\sqrt{t-t_1}\right] \\ &= \left[\sqrt{\frac{2}{\pi}} + \frac{2}{\sqrt{\pi}}\sqrt{t-t_1}, 1 + \frac{1}{3}\sqrt{\frac{2}{\pi}} + \frac{4t+2t_1}{3\sqrt{\pi}}\sqrt{t-t_1}\right], t \in [t_1, t_2]. \end{aligned}$$

In fact, $X_1 \uparrow\uparrow (t)$ is the w-increasing solution and $X_1 \uparrow\downarrow (t)$ is the w-decreasing solution of (2.17)*, if we use the initial condition $X(t_1) = X_0 \uparrow (t_1)$. Similarly, $X_1 \downarrow\uparrow (t)$ is the w-increasing solution and $X_1 \downarrow\downarrow (t)$ is the w-decreasing solution of* (2.17) *if we use the initial condition $X(t_1) = X_0 \downarrow\uparrow (t_1)$.*
Now, one can check that the interval-valued function $X \uparrow : [0,1] \to \mathcal{K}$ given by

$$X \uparrow (t) = \begin{cases} X_0 \uparrow (t), & t \in [t_0, t_1], \\ X_1 \uparrow\uparrow (t), & t \in [t_1, t_2], \end{cases}$$

is a w-increasing solution of (2.15) *on $[t_0, t_2]$. Similar, the interval-valued function $X \downarrow :$ $[0,1] \to \mathcal{K}$ given by*

$$X \downarrow (t) = \begin{cases} X_0 \downarrow (t), & t \in [t_0, t_1], \\ X_1 \downarrow\downarrow (t), & t \in [t_1, t_2], \end{cases}$$

is a w-increasing solution of (2.15) *on $[t_0, t_2]$. By mathematical induction we can show that for any X_0 can be extended to the right of the point t_n up to the point $t_{n+1} \in (t_n, 1]$ such that $((\mathcal{J}^{\frac{1}{2}}_{t_1^+})F(t))$ is w-increasing on $[t_n, t_{n+1}]$, where $t_{n+1} = \frac{1}{2}(1+t_n), n \geq 0$; that is $t_n = 1 - (\frac{1}{2})^n, n \geq 0$. Indeed, suppose that $\mathcal{X}_0$ was extended up to the point t_n such that*

$$\begin{aligned} \mathcal{X}_{n-1}(t) &= X_{n-1}(t)\theta_g X_{n-1}(t_{n-1}) \\ &= \left[\frac{4t+2t_{n-1}}{3\sqrt{\pi}}\sqrt{t-t_{n-1}}, \frac{2}{\sqrt{\pi}}\sqrt{t-t_{n-1}}\right], \end{aligned}$$

where $t \in [t_{n-1}, t_n]$. *The extension of* $\mathcal{X}_0$ *up to* t_{n+1} *is an interval-valued function* $\mathcal{X}_n : [t_0, t_{n+1}] \to \mathcal{K}$ *such that* $\mathcal{X}_n(t) = \mathcal{X}_{n-1}(t)$ *for* $t \in [t_0, t_n]$ *and* X_n *is the solution of the following initial value problem*

$$\begin{cases} ({}^c\mathcal{D}^{\frac{1}{2}}_{t_n^+}X)(t) = [1, t], \quad t \in [t_n, 1], \\ X(t_n) = \mathcal{X}_{n-1}(t_n), \end{cases} \tag{2.19}$$

where $\mathcal{X}_{n-1}(t_n) = X_{n-1}(t_n)\theta X_{n-1}(t_{n-1}) = ((\mathcal{J}^{\frac{1}{2}}_{t_{n-1}^+})F(t))$. *Next,*

$$\begin{aligned} ((\mathcal{J}^{\frac{1}{2}}_{t_{n-1}^+})X(t)) &= \int_{t_{n-1}}^{t} \frac{(t-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}[s, 1]\, ds, \\ &= \left[\frac{4t + 2t_n}{3\sqrt{\pi}}\sqrt{t - t_n}, \frac{2}{\sqrt{\pi}}\sqrt{t - t_n}\right], \end{aligned}$$

where $t \in [t_n, 1]$. *If we put* $Y_n := ((\mathcal{J}^{\frac{1}{2}}_{t_n^+})F(t)), t \in [t_n, 1]$, *then* $\frac{\mathrm{d}}{\mathrm{d}t}w(Y_n(t)) = \frac{1}{2\sqrt{\pi(t-t_n)}}(1 + t_n - 2t), t \in (t_n, 1]$. *It follows that* Y_n *is w-increasing on* $[t_n, t_{n+1}]$ *and w-decreasing on* $[t_{n+1}, 1]$, *where* $t_{n+1} = \frac{1}{2}(1 + t_n)$. *Then the condensed solution* $\mathcal{X}_n$ *of the integral equation associated with* (2.19) *is given by*

$$\begin{aligned} \mathcal{X}_n(t) &:= X_n(t)\theta_g X_n(t_n) \\ &= \left[\frac{4t + 2t_n}{3\sqrt{\pi}}\sqrt{t - t_n}, \frac{2}{\sqrt{\pi}}\sqrt{t - t_n}\right], \quad t \in [t_n, t_{n+1}], \end{aligned}$$

and it produces 2^n *w-increasing solutions and* 2^n *w-decreasing solutions for initial value problem* (2.19).

A reasoning, not so difficult, lead us to establish the extended monotone solutions of (2.19) *on* $[0, 1]$. *We obtain the w-increasing solution* $X \uparrow : [0, 1] \to \mathcal{K}$ *given by*

$$X \uparrow (t) = \begin{cases} X_0 \uparrow (t), & t \in [t_0, t_1], \\ X_n \uparrow^n (t), & t \in [t_n, t_{n+1}], n \geq 1, \end{cases}$$

where

$$\begin{aligned} X_n \uparrow^n (t) &:= \left[a_n + \frac{4t + 2t_n}{3\sqrt{\pi}}\sqrt{t - t_n}, b_n + \frac{2}{\sqrt{\pi}}\sqrt{t - t_n}\right], t \in [t_n, t_{n+1}], n \geq 1, \\ a_n &:= \frac{2}{3\sqrt{\pi}}\sum_{k=1}^{n}\left(3 - \frac{1}{2^{k-2}}\right)\sqrt{\frac{1}{2^k}}, b_n = 1 + \frac{2}{\sqrt{\pi}}\sum_{k=1}^{n}\frac{1}{\sqrt{2^k}}, n \geq 1, \end{aligned}$$

and $\uparrow^n$ *means* $\uparrow\uparrow \ldots \uparrow -n + 1 -$ *times*

Also, the w-decreasing solution $X \downarrow : [0, 1] \to \mathcal{K}$ *is given by*

$$X \downarrow (t) = \begin{cases} X \downarrow_0 (t), & \text{if } t \in [t_0, t_1], \\ X \downarrow_n^n (t), & \text{if } t \in [t_n, t_{n+1}], n \geq 1, \end{cases}$$

where

$$X_n \downarrow^n (t) = \left[c_n + \frac{2}{\sqrt{\pi}}\sqrt{t-t_n}, d_n + \frac{4t+2t_n}{3\sqrt{\pi}}\sqrt{t-t_n}\right], t \in [t_n, t_{n+1}], n \geq 1.$$

$$c_n = \frac{2}{\sqrt{\pi}}\sum_{k=1}^{n}\frac{1}{\sqrt{2^k}}, d_n = 1 + \frac{2}{3\sqrt{\pi}}\sum_{k=1}^{n}\left(3 - \frac{1}{2^{k-2}}\right)\frac{1}{\sqrt{2^k}}.$$

and $\downarrow^n$ *means* $\downarrow\downarrow\ldots\downarrow -n+1-$ *times*

2.3 Fuzzy Fractional Differential Equations

This section consists of subsections: In the first, we introduce some basic notions of fuzzy set and fractional calculus, and in the second, we present several topics such as differential equations of fractional order with uncertainty.

2.3.1 Definitions and Preliminaries

Definition 2.7. *[21] A fuzzy set (class) A in X is characterized by a membership function* $U_A(x)$ *which associated with each point in X or a real number in the interval* $[0, 1]$ *with the value of* $U_A(x)$ *at x representing the " grade of membership" of x in A.*

The space of fuzzy numbers in $\mathbb{R}$ is denoted by E with the following properties [4]

1. u is normal. i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

2. u is fuzzy convex. i.e., for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$, then

$$u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}.$$

3. u is upper semicontinuous.

4. $[u]^0 = cl\{x \in \mathbb{R}; u(x) > 0\}$ is compact.

For $0 < q < 1$, denoted $[u]^q = \{x \in \mathbb{R} : u(x) \geq q\}$, it follows that the q-level set $[u]^q$ is closed interval for each $q \in [0, 1]$. From this characterization of fuzzy numbers, it follows that a fuzzy number u is completely determined by the end point of the interval $[u]^q = [u_1^q, u_2^q]$, then we can define $\hat{0}$ as following

$$\hat{0} = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0, \end{cases}$$

Then we need to define a metric space d on E which is denoted by d_H and defined as

$$d(u,v) = \sup_{0\le q\le 1} d_H([u]^q,[v]^q), \tag{2.20}$$

where d_H is the *Hausedorff metric* which is given by

$$d_H([u]^q,[v]^q) = \max\left\{|v_1^q - u_1^q|, |v_2^q - u_2^q|\right\}. \tag{2.21}$$

Note that the arithmetic operations on E are the same of the arithmetic operations under intervals.
Now; we need to define some spaces related to the fuzzy sets, let $T \subset \mathbb{R}$, then [4]

1. $C(T,E)$ is the space of all *continuous fuzzy functions on* T.

2. $L^1(T,E)$ is the space of *all fuzzy functions* $f : T \to E$ which are *Lebesgue integrable on the bounded interval* T *of* $\mathbb{R}$.

3. $C_r([a,b],E)$ is a complete metric space with respect to the metric

$$h_r(u,v) = \max_{t\in[0,a]} t^r d(u(t),v(t)),$$

and

$$C_r([0,a],E) = \left\{u \in C((0,a],E) : t^r u \in C([0,a],E)\right\}.$$

Definition 2.8. *[4] Let $\alpha > 0$ and $u : (0,a] \to E$ be such that $[u(t)]^q = [u_1^q(t), u_2^q(t)]$ for every $t \in (0,a]$ and $q \in [0,1]$. Suppose that $u_1^q, u_2^q \in C((0,a],E) \cap L^1((0,a),\mathbb{R})$ for each $q \in [0,1]$, and let*

$$A_q = \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} u_1^q(s)\,ds, \int_0^t (t-s)^{\alpha-1} u_2^q(s)\,ds\right],$$

then the family $\left\{A_q : q \in [0,1]\right\}$ is defined a fuzzy number $u \in E$ such that $[u]^q = A_q$.

Let $u : T \to E$ be a fuzzy function such that $[u(t)]^q = [u_1^q(t), u_2^q(t)], t \in T$ and $q \in [0,1]$. Then we have [4, 35]

1. The derivative $u'(t)$ of a fuzzy function u is defined as follows:

$$[u'(t)]^q = [(u_1^q)'(t), (u_2^q)'(t)], \quad q \in [0,1], \tag{2.22}$$

where this equation provided a fuzzy number $u'(t) \in E$.

2. The fuzzy integral is defined by

$$\left[\int_a^b u(t)\,dt\right]^q = \left[\int_a^b u_1^q(t)\,dt, \int_a^b u_2^q(t)\,dt\right], q \in [0,1],$$

provided that $u_1^q(t)$ and $u_2^q(t)$ are Lebesgue integrable, and we get that fuzzy integral is a fuzzy number.

3. We have that

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_a^t u(s)\,ds = u(t),$$

for a.e. on $[a,b]$.

4. If the endpoints functions $(u_1^q)'(t)$ and $(u_2^q)'(t)$ in (2.22) are integrable, then

$$u(t) = u(a) + \int_a^t u'(s)\,ds, t \in [a,b].$$

2.3.2 Fuzzy Fractional Derivative and Fuzzy Fractional Integral

Definition 2.9. *[5] Let $u \in C((0,a],E) \cap L^1((0,a),E)$. Then the* fuzzy fractional integral of order $\alpha > 0$ of u

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,ds, \quad t \in (0,a).$$

and

$$[I^\alpha u(t)]^q = \frac{1}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1} u_1^q(s)\,ds, \int_0^t (t-s)^{\alpha-1} u_2^q(s)\,ds\right], t \in (0,a).$$

For $\alpha = 1$, we obtain $I^1 u(t) = \int_0^t u(s)\,ds$, which is the integral operator.
Let $u, v \in C((0,a],E) \cap L^1((0,a),E)$. Then the fuzzy fractional integral function satisfies the following properties [4]:

1. $I^\alpha(cu)(t) = cI^\alpha u(t)$ for each constant $c \in E$.

2. $I^\alpha(u+v)(t) = I^\alpha u(t) + I^\alpha v(t)$.

3. $I^{\alpha_1} I^{\alpha_2} u(t) = I^{\alpha_1+\alpha_2} u(t)$, where $\alpha_1, \alpha_2 > 0$.

Now; we introduce an example to illustrate the concept of fuzzy fractional integral function.

Example 2.8. *Let $u : [0, a] \to E$ be a constant fuzzy function. Then $u(t) = c \in E$ for $t \in [0, a]$.*
Now, if $[c]^q = [c_1^q, c_2^q]$, then

$$\begin{aligned}[I^\alpha u(t)]^q &= \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} c_1^q\, ds, \int_0^t (t-s)^{\alpha-1} c_2^q\, ds\right] \\ &= \frac{1}{\Gamma(\alpha+1)} t^\alpha [c_1^q, c_2^q] \\ &= \frac{1}{\Gamma(1+\alpha)} t^\alpha [c]^q.\end{aligned}$$

Hence,

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha+1)} t^\alpha u(t).$$

Definition 2.10. *[4] Let $u \in C((0, a], E) \cap L^1((0, a), E)$ be a given function such that $[u(t)]^q = [u_1^q(t), u_2^q(t)]$ for every $t \in (0, a]$ and $q \in [0, 1]$. Then* the fuzzy fractional derivative of order $0 < \alpha < 1$ of u *is given by the following*

$$D^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-\alpha} u(s)\, ds,$$

and

$$[D^\alpha u(t)]^q = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha} u_1^q(s)\, ds, \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha} u_2^q(s)\, ds\right],$$

provided that the equation defines a fuzzy number $D^\alpha u(t) \in E$, where

$$[D^\alpha u(t)]^q = [D^\alpha u_1^q(t), D^\alpha u_2^q(t)],$$

for every $t \in (0, a]$ and $q \in [0, 1]$.

Note that $D^\alpha u(t) = \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha} u(t)$ for $t \in (0, a]$.
In the following proposition some properties of the fuzzy fractional derivative are introduced.

Proposition 2.13. *[5] Let $u, v \in C((0, a], E) \cap L^1((0, a), E)$ be a given fuzzy functions and $0 < \alpha < 1$. Then the fuzzy fractional derivative satisfies the following properties*

1. *$D^\alpha(cu)(t) = cD^\alpha u(t)$, for each constant $c \in E$.*

2. *$D^\alpha(u + v)(t) = D^\alpha u(t) + D^\alpha v(t)$.*

3. *$D^\alpha I^\alpha u(t) = u(t)$.*

The following is an example which illustrates the fuzzy fractional derivative.

Example 2.9. *[5] Let $u : (0, a] \to E$ be a constant fuzzy function, $u(t) = c \in E$ for $t \in (0, a]$.*
If $[c]^q = [c_1^q, c_2^q]$, then

$$\begin{aligned} [D^\alpha u(t)]^q &= \frac{1}{\Gamma(1-q)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha} c_1^q\, ds, \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-s)^{-\alpha} c_2^q\, ds\right] \\ &= \frac{t^{-\alpha}}{\Gamma(1-\alpha)}[c_1^q, c_2^q] \\ &= \frac{t^{-\alpha}}{\Gamma(1-\alpha)}[u(t)]^q. \end{aligned}$$

Hence

$$D^\alpha c = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} c,$$

for every $c \in E$

In the following, we combine two types of differential equations of fractional order and with uncertainty [5].
Let $\alpha \in (0, 1], T > 0$, and E be the set of fuzzy numbers. Consider the differential equation with uncertainty of the type:

$$D^\alpha X(t) = f(t, X(t)), t \in (0, T], \tag{2.23}$$

where $f : [0, T] \times E \to E$ is a contionous Riemann-Liouville fractional derivative of order α which is given by Definition 1.2 of the function $X : (0, T] \to \mathbb{R}$.
In this case, the initial condition is

$$\lim_{t\to 0^+} t^{1-\alpha} X(t) = X_0 \in \mathbb{R}. \tag{2.24}$$

For example, consider the corresponding equation, for $a \in \mathbb{R}$ and $\sigma : [0, T] \to \mathbb{R}$

$$D^\alpha X(t) + aX(t) = \sigma(t),\ t \in (0, T], \tag{2.25}$$

with the initial condition

$$X(0) = X_0.$$

If we apply to both side of (2.25) by I^α, then the solution is given by [18]

$$X(t) = X_0\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\sigma(s)\, ds, \tag{2.26}$$

where $E_{\alpha,\alpha}$ is the classical *Mittage Leffler function* which given by [15]

$$E_{\alpha,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha(k+1))}.$$

If $\alpha = 1$, then

$$X(t) = X_0 \exp\{-at\} + \int_0^t \exp\big\{-a(t-s)\big\}\sigma(s)\, ds.$$

Let E be the set of real numbers and consider the nonlinear differential equation with uncertainty

$$X'(t) = f(t, X(t)), t \in (0, T], \tag{2.27}$$

where $f : [0, T] \times E \to E$ is continuous, and the initial condition

$$X(0) = X_0 \in E. \tag{2.28}$$

For $a > 0$, the solution of the linear problem

$$X'(t) = aX(t) + \sigma(t), t \in (0, T],$$

with the initial condition (2.28) is given by [21]

$$X(t) = \exp\{at\} \left(X_0 + \int_0^t \exp\{-as\}\sigma(s)\, ds \right).$$

Let us consider the fractional differential equation with uncertainty (2.24), where f : $[0, T] \times E \to E$ is continuous, with the initial condition

$$\lim_{t \to 0^+} t^{1-\alpha} X(t) = X_0 \in E.$$

Suppose that $\lambda > 0$ such that f is given by

$$f(t, X) = \lambda X + g(t, X),$$

where $g : [0, T] \times E \to E$ is continuous.
We can write (2.23) as

$$D^{\alpha} X(t) = \lambda X(t) + g(t, X(t)), \quad t \in [0, T]. \tag{2.29}$$

Hence the solution of (2.29) can be derived from (2.26), and it can be written as

$$X(t) = \Gamma(\alpha) t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha}) X_0 + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-s)^{\alpha}) g(s, X(s))\, ds,$$

where $a = -\lambda$ and $\sigma(t) = g(t, X(t))$.

Example 2.10. *[2] Consider the fractional differential equation*

$$D^{\alpha}X(t) = 0, t \in (0,T]$$

with the initial condition

$$\lim_{t\to 0^+} t^{1-\alpha}X(t) = X_0 \in \mathbb{R}.$$

Then the general solution given in [18] is

$$X(t) = ct^{\alpha-1},$$

with $c = X_0 \in \mathbb{R}$.

2.3.3 Fuzzy Fractional Differential Equation

The most content of this part is from [5].
Consider the following fuzzy fractional differential equation

$$D^{\alpha}u = f(t,u) \tag{2.30}$$

where $0 < \alpha < 1$ and $f : [0,a] \times E \to E$ is continuous function on $(0,a] \times E$, with the initial condition

$$\lim_{t\to 0^+} t^{1-\alpha}u(t) = u_0 \in E. \tag{2.31}$$

A fuzzy function $u \in C((0,a],E) \cap L^1((0,a),E)$ is a solution of fuzzy fractional differential equation (2.30) if $D^{\alpha}u$ is continuous on $(0,a]$, and satisfy (2.30). A function $u \in C([0,a],E)$ is called an integral solution for (2.30) if $f(t,u(t)) \in C((0,a],E) \cap L^1((0,a),E)$ and

$$u(t) = ct^{\alpha-1} + I^{\alpha}f(t,u(t)), \tag{2.32}$$

holds a.e. on $[0,a]$.
In the following, are some properties of the solution of fuzzy fractional differential equation are presented.

Proposition 2.14. *If* $f(t,u(t)) \in C((0,a],E) \cap L^1((0,a),E)$. *Then an integral solution of* (2.30) *is also a solution of* (2.30).

Proposition 2.15. *Let* $f : (0,a] \times E \to E$ *be a given fuzzy function. If* $t^{1-\alpha}f(t,u)$ *is continuous on* $[0,1] \times E$ *and there exists* $M > 0$ *such that* $d(t^{1-\alpha}f(t,u),\hat{0}) \le M$ *for each* $t \in (0,a]$ *and* $u \in E$. *Then*

$$u(t) = u_0t^{\alpha-1} + I^{\alpha}f(t,u(t)), \tag{2.33}$$

is a solution for the initial value problem (2.30) *and* (2.31).

Proof. It is clear that $u(t)$ given by (2.33) is an integral solution of (2.30). Then, it's sufficient to show that

$$\lim_{t\to 0^+} d(t^{1-\alpha}u(t), u_0) = u_0,$$

so we have

$$\begin{aligned}
\lim_{t\to 0^+} d(t^{1-\alpha}u(t), u_0) &= \lim_{t\to 0^+} d(t^{1-\alpha}u_0 t^{\alpha-1} + t^{1-\alpha}I^{1-\alpha}f(t,u(t)), u_0) \\
&= \lim_{t\to 0^+} d(u_0 + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(t,u(t))\, ds, u_0) \\
&\leq \lim_{t\to 0^+} \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} d(f(s,u(s)),\hat{0})\, ds \\
&\leq \lim_{t\to 0^+} \frac{Mt^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} s^{1-\alpha}\, ds \\
&= \lim_{t\to 0^+} \frac{M\Gamma(\alpha)t^{\alpha}}{\Gamma(2\alpha)} \\
&= 0.
\end{aligned}$$

□

We conclude with the following theorems which describe the integral equation and the cases where the fuzzy differential equation has a unique solution.

Theorem 2.12. *Let $0 < \alpha < 1$. If $f : E \to E$ satisfies the following:*

$$\begin{cases} f(\hat{0}) = \hat{0}, \\ d(f(u), f(v)) \leq L d(u,v), \end{cases} \tag{2.34}$$

for some $L > 0$ independent of $u, v \in E$. Then for any fixed $c \in E$, the fuzzy fractional integral equation

$$u(t) = ct^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(u(s))\, ds$$

has a unique solution $u \in C_{1-\alpha}([0,a], E)$ for each $a > 0$.

Theorem 2.13. *Let $0 < \alpha < 1$. Assume that $t^\alpha f(t,u)$ is continuous on $[0,1] \times E$, and that*

$$h_0(f(t,u), f(t,v)) \leq Lt^{-\alpha} h_0(u,v), \tag{2.35}$$

for every $u, v \in E$ and $t \in (0,1]$. If $L\Gamma(1-\alpha) < 1$, then the initial value problem (2.30) and (2.31) has a unique integral solution $u \in C([0,1], E)$.

Theorem 2.14. *Let $0 < \alpha < 1$, if $f : E \to E$ satisfies (2.34), then for each $u_0 \in E$, the fuzzy value problem*

$$\begin{cases} D^\alpha u = f(u), \\ \lim_{t\to 0^+} t^{1-\alpha}u(t) = u_0, \end{cases} \tag{2.36}$$

has a unique solution $u \in C_{1-\alpha}([0,a],E)$ for all $a > 0$.

Proof. Assume that

$$u(t) = ct^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(u(s))\, ds,$$

be an integral equation for (2.36), with $c \in E$. Then by Theorem 2.12, this integral equation has a unique solution $u \in C_{1-\alpha}([0,a],E)$. So, it is sufficient to show that

$$\lim_{t\to 0^+} t^{1-\alpha} u(t) = c.$$

Then, we have

$$\begin{aligned}
\lim_{t\to 0^+} d(t^{1-\alpha}u(t), c) &= \lim_{t\to 0^+} d(c + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(u(s))\, ds, \hat{0}) \\
&\leq \lim_{t\to 0^+} \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d(f(u(s)), \hat{0})\, ds \\
&\leq \lim_{t\to 0^+} \frac{Lt^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d(u(s), \hat{0})\, ds \\
&\leq \lim_{t\to 0^+} \frac{Lt^{1-\alpha}}{\Gamma(\alpha)} ||u||_{1-\alpha} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1}\, ds \\
&= \lim_{t\to 0^+} \frac{L\Gamma(\alpha)t^{\alpha}}{\Gamma(2\alpha)} ||u||_{1-\alpha} \\
&= 0.
\end{aligned}$$

□

Bibliography

[1] T. Abdeljawad. On Conformable Calculus. Journal of Computational and Applied Mathematics 279(2015)57-66.

[2] R.P.Agarwal, V.Lakshmikantham. J.J.Nieto. On the concept of solutions for fractional differential equations with uncertainty. Nonlinear Analysis 72(2010)2859-2862.

[3] M. Agnieszka, T. Delfim. Introduction to the Fractional Calculus of Variation. World Scientific Publishing Co.Pte.Ltd (2012).

[4] S.Arshad, V.Lupulescu. On the fractional differential equations with uncertainty. Nonlinear Analysis 74(2011)3685-3693.

[5] Z.Artstein. On the Calculus of closed set valued functions. Indian Univ. Math. J.24(1975)433-441.

[6] R.J.Aumann. Integrals of set valued function. J.Math-Anal.App1.12(1965)1-12.

[7] R.G.Bartle, D.R. Sherbert. Introduction to Real Analysis. John Wiley and Sons. Inc (2000).

[8] Y. Chalco-Cano, A. Rufian-Lizana, H. Roman-Flores, M.D. Jimenez-Gamero. Calulus for Interval-Valued Functions Using Generalized Hukuhara Derivative and Applications. Fuzzy Sets and Systems 219(2013)49-67.

[9] Y.Chalco-Cano, H.Roman-Flores, M.D.Jimenez-Gamero. Generalized derivative and π- derivative for set-valued functions. Inf. Sci 181(2011)2177-2188.

[10] M.A.Chaudhry, A.Qadir, M.Rafique, S.M.Zubair. Extension of Euler's Beta function. Journal of Computational and Applied Mathematics 78(1997)19-32.

[11] W.S. Chung. Fractional Newton mechanics with Conformable Fractional Derivative. Journal of Computational and Applied Mathematics 290(2015)150-158.

[12] V. Daftardar-Gejji. Fractional Calculus: Theory and Application. Narosa Publishing House (2014).

[13] G.Debru. Integration of Corresponndence, in; Proc-fifth Berkeley sympos. On Math. statist. and probability. Vol.II. part I, 1996, pp ,351-372.

[14] G. Gripenbeng, S.-O. Londen, O. Staffans, Volterra Integral and Functional Equations. Cambrige University Press, Cambrige, 1990.

[15] H.J.Haubold, A.M.Mathai, R.K.Saxena. Mittag-Leffler functions and their applications. Journal of Applied mathematics(2011).

[16] V. Hoa, Ngo, V.Lupulescu, and D. O'Regan. Solving Interval-Valued Fractional Initial Value Problems under Caputo gH- Fractional Differentiability. Fuzzy Sets and Systems. 309 (2017): 1-34.

[17] R. Khalil, M. Al-Morani, A. Yosef, M. Sababheh. A New Definition of Fractional Derivative. Journal Computational and Applied Mathematics, 246(2014)65-70.

[18] A.A. Kilbas, H.M. Strtvastava, J.J. Trujello. Theory and Applications of Fractional Differential Equations. Elsevier Science B.V, Amesterdam,2006.

[19] J. Kimeu. Fractional Calculus: Definitions and Applications. Bowling Green, Kentucky. May(2009).

[20] G.J.Klir, B.Yuan. Fuzzy sets, Fuzzy logic and Fuzzy systems: Selected by Lotfi A. Zadeh. World Scientific publishing Co.Pte.Ltd, Singapor(1996).

[21] V.Lakshmikantham, R.N.Mohapatra. Theory of Fuzzy differential equations and applications. Taylor and Francis, London(2003).

[22] V. Lupulescu. Fractional Calculus for interval-valued functions. Fuzzy Sets and Systems 265(2015)63-85.

[23] V. Lupulescu. Hukuhara Differentiability of Interval-Valued Function and Interval Differential Equations on Time Scales. Information Sciences 248(2013)50-67.

[24] V. Lupulescu, N. Van Hoa. Existence of Extremal Solutions to Intarval-Valued Delay Fractional Differential Equations Via Monotone iterative technique. Journal of Intelligent and Fuzzy Systems(2018).

[25] S.Markov. Calculus for interval functions of a real Variables. Computing 22(1979)325-337.

[26] K. Miller, B. Ross. An Introduction to the Fractional Calculus and Fractional Differential Equations. John Wiely and Sons, Inc.(1993).

[27] R.E. Moore, R. Baker Kearfott, M.J. Cloud. Introduction to Interval Analysis. Society for Industrial and Applied Mathematics, Philadelphia, USA(2009).

[28] K. Owolabi. Riemann-Liouville Fractional Derivative and Application to Model Chaotic Differential Equations. Natural Sciences Publishing Co,(2018)99-110.

[29] D.Perrin, J.Pin. Infinite Words (Automata, Semigroups, Logic, and Games. Elsevier Academic Press(2003).

[30] I.Petras. Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation. Springer Heidelberg Dordercht. London, NewYork(2011).

[31] A.S.Poznyak. Advanced Mathematical Tools for Automatic Control Engineers: Deterministic Technique, Volume1. Elsevier Science(2008)

[32] Quang, L.Thanh, N.V. Hoa, N.D.Phu, T.T. Tung and N.Dinh. Existence of extremal solutions for interval-valued functional integro-differential equation. Journal of Intelligent and Fuzzy systems 30, no. 6 (2016): 3495-3512.

[33] S. Salahshour, A. Ahmadian, F. Ismail, D. Baleanu, N. Senu. A New Fractional Derivative for Differential Equation of Fractional Order under Interval Uncertainty. Advances in Mechanical Engineering 2015, Vol.7(12)1-11.

[34] S.G.Samko, A.A. Kilbas, O.I. Marichev. Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers, Switzerland 1993.

[35] S.Seikkala. On the fuzzy initial value problem: Fuzzy sets and systems 24(1987)319-330.

[36] L. Stefanini, B. Bede. A Generalization of Hukuhara differentiability of interval-valued functions and interval-differential equation.Nonlinear Analysis: Theory, Methods and Applications 2009;71: pages 1311-1328.

[37] L.Stefanini, B.Bede. Generalized Hukuhara difference and division for interval and fuzzy arithmetic. Fuzzy sets syst. 161(2010)1564-1584.

[38] M. Tolba, A.Abdal Aty, N. Soliman, L. Said, A. Azar, A. Radwan.FPGA Implementation of Two Fractional Order Choatic systems. Introduction Journal of Electrics and Communications (AEU)78(2017)162-172.

[39] L. Zhang, M. Feng, R. Agarwal, G. Wang. Concept and Application of Interval-Valued Fractional Conformable Calculus. Alexandria Engineering Journal (2022)61,11959-11977.