Palestine Polytechnic University

Deanship of Graduate Studies and Scientific Research

Master Program of Mathematics

Types of Lattices and Applications of Complete Lattices

By

Maram Khalid Al- Natsheh

Master of Science Thesis

Hebron- Palestine

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Master thesis submitted and accepted, Date …………..

The name and signatures of the examining committee members are as follows:

1. Dr. Nureddin Rabie.	***Head of committee Signature…………..***
2. Dr. Amjad Barham .	***Internal Examiner Signature…………..***
3. Dr. Ali Abdelmohsen.	***External Examiner Signature…………..***

Palestine Polytechnic University

2009

DEDICATION

To my dear parents

To my kind husband Abdel-Muhsen and his family

To my lovely sisters and brothers

To my daughter Lububa

To my son Jawdi

To my teaches and my friends

To everyone hoped me success

And finally,

To the spirit of my grandmother "Raesa".

Maram Al- Natsheh

DECLARATION

I declare that the Master Thesis entitled "*Types of Lattices and Applications of Complete Lattices*" is my own original work, and hereby certify that unless stated, all work contained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgment is made in the text.

Name and Signature: Maram Khalid Al- Natsheh ------------------

Date -------------------------

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ABSTRACT

This thesis aims to develop a better understanding of lattices. It presents lattices, types of lattices and discusses three applications of complete lattices.

Due to the importance of orders and ordered sets to study lattices, this thesis looks first at definitions and some examples of orders and ordered sets, then it defines a lattice in two ways and connects between the two definitions. This thesis then exposes some types of lattices with both important properties to each type and useful theories to determine the type of a given lattices.

Complete lattices, as a type of lattices have several applications in science. This thesis presents three applications of complete lattices .It discusses the impact of retraction operator on a complete lattices , the existence of a fixed point of decreasing functions defined on a complete lattices, and finally this thesis defines the annihilator of a subset and discuss when this annihilator can be an associated prime ideals of complete lattices.

NOTATIONS

In this thesis we used the following notations

$\preccurlyeq$	To denote a relation and read "related to "
$\npreccurlyeq$	To denote a relation and read "not related to "
$\vee$	To denote the join
$\wedge$	To denote the meet
$\cup$	To denote the union
$\cap$	To denote the intersection
$\emptyset$	To denote the empty set
$\mathbb{Z}$	To denote the set of integers
$\mathbb{Z}^{+}$	To denote the set of positive integers
$\mathbb{R}$	To denote the set of real numbers
$\mathbb{N}$	To denote the set of natural numbers
$\mathbb{N}_0$	To denote the set $\mathbb{N} \cup \{0\}$
$\mathbb{Q}$	To denote the set of rational numbers
$\trianglerighteq$	To denote the pointwise order
$\mathbb{P}(X)$	To denote the power set of X
lcd	To denote the least common multiple
gcd	To denote the greatest common divisor
$\subseteq$	To denote the inclusion relation
$\downarrow Q$	To denote the down set of Q
A^u	To denote the set of upper bounds of A

TABLE OF CONTENT

Introduction

The concept of a lattice was introduced by Peirce and Schroder towards the end of the nineteenth century. It derives from pioneering work by Boole on the formalization of propositional logic. The terms idempotent, commutative, associative, and absorption are mostly due to Boole.

The study of lattices became systematic with Birkhoff's first paper in 1933 and his book Lattice Theory (first edition) which appeared in 1940 and was for several decades the bible of lattice theorists.

Over the years the theory of lattices and its many applications has grown considerably. Notable reference works include books by Abbott, Balbes and Dwinger, Crawley and Dilworth, Davey and Priestley Dubreil-Jacotin, Lesieur and Croisot Freese, Jeˇzek and Nation, Ganter and Wille, Hermes, and Maeda, Rutherford Saliˇı, Sikorski and Sz´asz . In recent times the Birkhoff bible has been replaced by that of Gratzer General Lattice Theory.

This thesis consist of four chapters, each one contains basic definitions, examples, figures and important theorems.

In chapter one, we begin with basic definitions needed in this work. In section one we define ordered sets, partial order, total order, quasi order, partially ordered sets and totally ordered sets, each definition supported with examples. In section two we learn how to represent any finite partially ordered set graphically. Special elements within an order such least and greatest, minimal and maximal, upper and lower bounds will be given in section three. Duality and duality principle which is very important in order theory will introduced in section four. In section five we give several maps between ordered sets such as order-preserving, order-reversing, order embedding and order isomorphism. In section six we give many ways to construct new orders. Finally, special subsets of ordered sets such as upper sets, lower sets, ideals, and filters will appear in section seven.

In chapter two, we define lattices in two ways. Lattices as partially ordered sets introduce in section one with definitions and examples of semilattices and bounded lattices. Lattices as algebraic structure introduce in section two again with definition and examples of semilattices and bounded lattices. Connecting lemma and other theorems appear in the same section. Connection between the two definitions of lattices given in section three. Section four define sublattices and include important result that the direct product of two lattices is lattice. We introduce two concept of isomorphism for lattices in section five, important theorems about lattice isomorphism appear in the same section. Finally, important lattice notions as join and meet irreducible element, join and meet prime element, atom element and complement element will be given in section six. Furthermore ideal and its dual notion filter introduce in the same section which is end with important theorem about the set of all ideals of any lattice.

In chapter three we introduce some special types of lattices. Complete lattices which is very important given in section one with several examples and important theorems as Tarski-Knaster fixed point theorem and lattice theoretical fixed point theorem. Modular and semimodular lattices introduced in section two with examples, results and two basic theorems the first one let us know if the lattice is modular or not and the

the second is the isomorphism theorem. Distributive lattices introduced in section three with examples, characteristic properties and two important theorems Birkhoff 1934 and Birkhoff 1933. Section four introduce Complemented lattices with examples and related definitions and remarks. Finally, Boolean lattices introduced in section five which also introduce Boolean algebra, finite Boolean algebra and important theorems about them.

Finally, we end this work with chapter four in which we study three applications about complete lattices, the first one about retraction operator on complete lattices, the second about fixed point and complete lattices, the third one about associated prime ideals of a complete lattice.

Chapter One

Preliminaries

This chapter mainly contains the basic definitions, results, lemmas and theorems to be used later in this thesis.

When humans are asked to express preferences among a set of options, they often report that establishing a ranked list is difficult if not impossible. Instead, they prefer to report a partial order- where comparisons are made between certain pairs of options but not between others. Here we make these observations more concrete by introducing the concept of ordered sets.

1.1. Ordered Sets

A binary relation from a set X to a set Y is a set of pairs (x, y) where x is an element of X and y is an element of Y. When an ordered pair is in the relation $\preccurlyeq$ we write $x \preccurlyeq y$ or $(x, y) \in \preccurlyeq$. Which means that x is related to y in the relation $\preccurlyeq$. When X=Y, we call a relation from X to Y a relation on X.

Orders are special binary relations, three types of orders will be given in this section namely Partial, Total, and Quasi order.

Definition.1.1.1. (Partial order). Suppose that X is a set and that $\preccurlyeq$ is a binary relation on X. Then $\preccurlyeq$ is a **partial order** if it **is reflexive, antisymmetric**, and **transitive**, i.e., for all a, b and c in X, we have that

P_1: $a \preccurlyeq a$. (reflexivity)

P_2: if $a \preccurlyeq b$ and $b \preccurlyeq a$ then $a = b$. (antisymmetry)

P_3: if $a \preccurlyeq b$ and $b \preccurlyeq c$ then $a \preccurlyeq c$. (transitivity)

Definition.1.1.2. (Partially ordered set). A set with a partial order on it is called **a partially ordered set, poset,** or just an ordered set if the intended meaning is clear.

By checking the properties P_1- P_3, one immediately sees that the well-known orders on natural numbers, integers, rational numbers and real numbers are all orders in the above sense.

Remarks.1.1.3.

1.During this thesis we write P to mean $(X, \preccurlyeq)$.

2.The symbol $\preccurlyeq$ is read " related to ", If $x \preccurlyeq y$ but $x \neq y$, one writes $x \prec y$. The relation $x \preccurlyeq y$ is also written $y \succcurlyeq x$.

There are many examples of partially ordered sets. Three of such examples are following.

Example.1.1.4. Let P consists of all the subsets of any set A, (including A itself and the empty set ∅),and define ≼ on P by: A≼B means that A is a subset of B. Then ≼ is a partial order and P is a partially ordered set. (see Figure1 where we listed all subsets and their relations for a set of three elements).

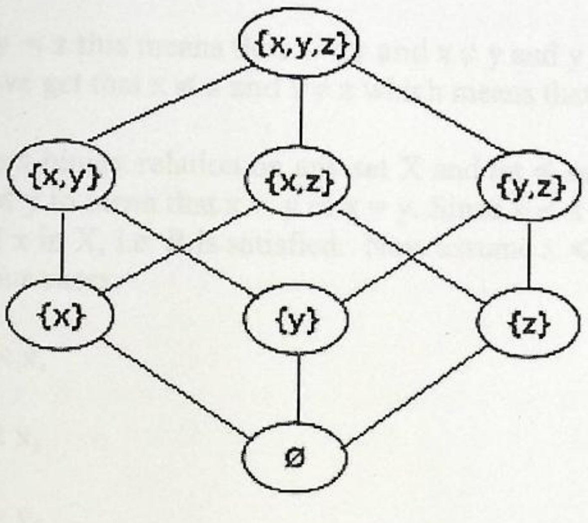


Figure 1 : inclusion order of the set { x ,y, z }

Example.1.1.5. Let P consists of the positive integers, define ≼ on P by : x ≼y means x divides y. Then ≼ is a partial order and P is a partially ordered set. (see Figure 2 in which we listed all divisors and their relation of the number 60).

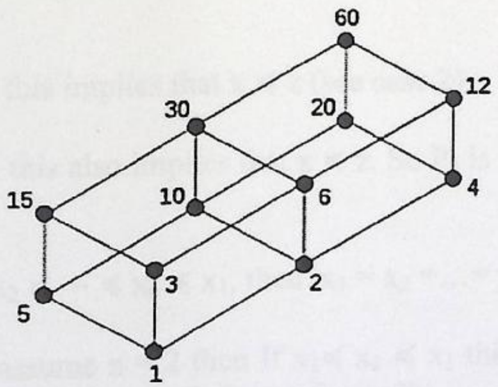


Figure 2: set of divisors of 60 ordered by divisibility

Example.1.1.6. Let **P** consists of all real single- valued functions defined on [-1, 1], and define $\preccurlyeq$ on P by : $g \preccurlyeq f$ means $g(x) \preccurlyeq f(x)\ \forall\, x \in [-1, 1]$. Then $\preccurlyeq$ is a partial order and P is a partially ordered set.

Lemma.1.1.7. In any poset $x \prec x$ for no x, while $x \prec y$ and $y \prec z$ imply $x \prec z$. Conversely if a binary relation $\preccurlyeq$ satisfies the two preceding conditions, define $x \preccurlyeq y$ to mean that $x \prec y$ or $x = y$, then the relation $\preccurlyeq$ satisfies P_1- P_3.

Proof. By way of contradiction let P be any poset and assume that $x \prec x$ for some x in P then $x \preccurlyeq x$ but $x \neq x$. (see remarks 1.1.3).

Now let $x \prec y$ and $y \prec z$ this means that $x \preccurlyeq y$ and $x \neq y$ and $y \preccurlyeq z$ and $y \neq z$, using transitivity property we get that $x \preccurlyeq z$ and $x \neq z$ which means that $x \prec z$.

Conversely, let $\preccurlyeq$ be a binary relation on any set X and let $\preccurlyeq$ satisfies the above two conditions, define $x \preccurlyeq y$ to mean that $x \prec y$ or $x = y$. Since $x \prec x$ for no x in X, and $x = x$ then $x \preccurlyeq x$ for all x in X, i.e. $P_1$is satisfied. Now assume $x \preccurlyeq y$ and $y \preccurlyeq x$ then we have the following four cases:

Case 1: $x \prec y$ and $y \prec x$,

Case 2: $x = y$ and $y \prec x$,

Case 3: $x \prec y$ and $x = y$,

Case 4: $x=y$ and $y = x$.

The first three cases have a contradiction, only case 4 is true so P_2 is satisfied. Similarly if we assume that $x \preccurlyeq y$ and $y \preccurlyeq z$ we again get another four cases:

Case 1: $x \prec y$ and $y \prec z$, this implies that $x \preccurlyeq z$ (using condition two above).

Case 2: $x = y$ and $y \prec z$, this implies that $x \prec z$ (i.e. $x \neq z$) so $x \preccurlyeq z$ (using the definition of $\preccurlyeq$).

Case 3: $x \prec y$ and $y = z$, this implies that $x \preccurlyeq z$ (see case 2)

Case 4: $x = y$ and $y = z$. this also implies that $x \preccurlyeq z$. So P_3 is satisfied. This completes the proof. □

Lemma.1.1.8. If $x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq x_1$, then $x_1 = x_2 = \ldots = x_n$ (Antcircularity).

Proof. By induction, assume $n = 2$ then If $x_1 \preccurlyeq x_2 \preccurlyeq x_1$ this means that $x_1 \preccurlyeq x_2$ and $x_2 \preccurlyeq x_1$ so by property $\mathbf{P_2}$ $x_1 = x_2 = x_1$.(it is true for n=2).
Now, assume it is true for (n=k-1) i.e.

if $x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_{k-1} \preccurlyeq x_1$, then $x_1 = x_2 = \ldots = x_{k-1}$. (*)

To prove for k let $x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_{k-1} \preccurlyeq x_k \preccurlyeq x_1$ so $x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_{k-1} \preccurlyeq x_1$ and so by (*) $x_1 = x_2 = \ldots = x_{k-1}$ we get that $x_1 \preccurlyeq x_n \preccurlyeq x_1$ and so $x_1 = x_k$, thus $x_1 = x_2 = \ldots = x_k$. □

Theorem.1.1.9. Any subset S of a partially ordered set P is itself a partially ordered set under the same inclusion relation.

Proof: Let S be any subset of a partially ordered set P with the same relation $\preccurlyeq$ of P. Let x, y belongs to **S**, so x, y belongs to P . So $\mathbf{P_1}$, $\mathbf{P_2}$, $\mathbf{P_3}$ in definition 1.1.1 are satisfied by $\preccurlyeq$ in P. So they are satisfied in **S.** □

Another important order which we will define is total order.

Definition.1.1.10 (Total order). A binary relation $\preccurlyeq$ over a set X **is a total** order if and only if it is partial order and for any pair of elements a and b in X a is related to b or b is related to a (or both). This is

P_4: for any a and b in X, either a $\preccurlyeq$ b or b $\preccurlyeq$ a. (Totality)

Example.1.1.11. " less than or equal to" is a total relation over the set of real numbers, because for any two numbers either the first is less than or equal to the second, or the second is less than or equal to the first. Total relations are sometimes said to have comparability.

Definition.1.1.12. (Totally ordered set). A partially ordered set which satisfies $\mathbf{P_4}$ is said to be **totally ordered,** and is called **a chain**, i.e. every distinct pair of elements are comparable in P.

We call a partially ordered set an "antichain" if every distinct pair of elements is incomparable.

Example.1.1.13. The set $\mathbb{Z}^+$ of positive integers is chain under the relation $\leq$. But $\mathbb{Z}^+$ is a partially ordered set which is not a chain under the partial ordering of Example 1.1.5

Example.1.1.14. Let R = $\mathbb{Z}$ which is a ring, n$\mathbb{Z}$ is the form of its ideals.
The set { n $\mathbb{Z}$, n$\in$ $\mathbb{N}$} is a partially ordered set with inclusion relation, But it is not a chain.

Example.1.1.15. The set of prime numbers partially ordered by divisibility is an antichain.

Definition.1.1.16. (Quasi order). A binary relation $\preccurlyeq$ on a set X which is irreflexive, and transitive but not necessarily antisymmetric is called **quasi order.**

Example.1.1.17. The less-than relation on the set of integers $\mathbb{Z}$ is a quasi order.

Example.1.1.18. The proper subset relation on the power set of a set is also a quasi order.

1.2 Diagrams.

In any hierarchy, it is important to know when one man is another's immediate superior. The notation of immediate superior can be defined abstractly in any partially ordered set, as follows.

Definition. 1.2.1. By " **a cover b**" in a poset P, it is meant that $a \succ b$ but that $a \succ x \succ b$ is not satisfied by any $x \in P$. This leads to a graphical representation of any finite partially ordered set X

- Small circles are drawn to represent the elements of P so that a is higher than b whenever $a \succ b$.
- A segment is then drawn from a to b whenever a covers b.

Any figure so obtained is called " diagram" of P (this diagram called Hasse-diagram)

Definition.1.2.2. Poset Diagram (Hasse Diagram): A graph representing a poset but with only immediate predecessor edges, and the edges are oriented up from x to y when $x \prec y$. Examples are drawn in Figure 3a-3e.

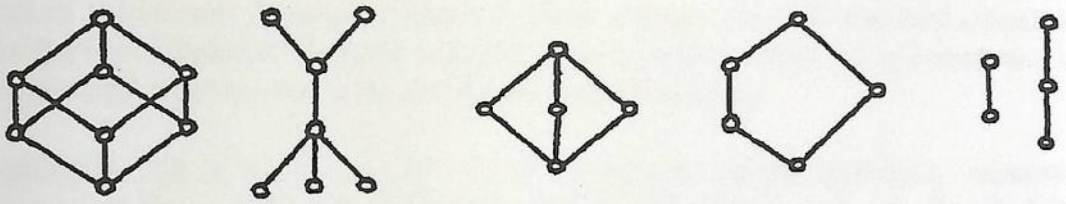


Fig.3a Fig.3b Fig.3c Fig.3d Fig.3e

Figure 3: Examples of Hasse diagrams

1.3. Special Elements Within An Order

In a partially ordered set there are some elements that play a special role. In this section we will define them.

Definition.1.3.1. (Least and greatest elements). By a **least element** of P we mean an element a of P such that $a \preccurlyeq x$ for all x in P. By a **greatest element** of P we mean an element b of P such that $x \preccurlyeq b$ for all x in P.

The least and greatest elements of the whole partially ordered set play a special role and are also called bottom and top or zero (0) and unit (1), respectively. The latter notation of 0 and 1 is only used when no confusion is happen, i.e. when we are not talking about partial orders of numbers that already contain elements 0 and 1. Bottom and top are often represented by the symbols $\perp$ and $\top$, respectively.

In P = ($\mathbb{P}$(X), $\subseteq$), we have $\perp = \emptyset$ and $\top$= X. A finite chain always has bottom and top elements, but an infinite chain need not have.

Example.1.3.2. The chain $\mathbb{N}$ has bottom element 1, but no top, while the chain $\mathbb{Z}$ of integers have neither bottom nor top. Bottom and top do not exist in any antichain with more than one element.

Least and greatest elements may fail to exist, we can see this in the following example.

Example.1.3.3. Consider the divisibility relation | on the set {2,3,4,5,6}. This set has neither top nor bottom, the elements 2, 3, and 5 do not have any elements below them, while 4, 5, and 6 have no other elements above.

Definition.1.3.4. (Minimal and maximal elements). **A minimal element** of a partially ordered set P is an element a such that $x \prec a$ for no x in P. And **a maximal element** of P is an element b such that $b \prec x$ for no x in P.

Greatest elements of a partially ordered subset must not be confused with maximal elements of such a set. A partially ordered subset can have several maximal elements without having a greatest element. Clearly a least element must be minimal and a greatest element must be maximal, but the converse is not true.

Example1.3.5. let X = { 2 , 3 , 4 , 9, 16 } where $\preccurlyeq$ is the divisibility relation. Then P is given by { (2, 2) , (3 , 3) (4, 4) , (9 , 9),(16, 16), (2 , 4), (3 , 9), (2, 16), (4, 16) }.It is clear that 9 and 16 are maximal elements but not greatest elements since $2 \not\preccurlyeq 9$ and $3 \not\preccurlyeq 16$.

Definition.1.3.6. (Upper and lower bounds). Given a subset S of some partially ordered set P, an **upper bound** of S is an element b of P that is above all elements of S. Formally, this means that $s \preccurlyeq b$, for all s in S. **Lower bounds** are defined by inverting the order.

A least upper bound (l.u.b) is an upper bound which is related to every other upper bound , this concept is also called **supremum** or **join** and is denoted **sup (S)** or $\bigvee S$.

A greatest lower bound (g.l.b) is a lower bound that every other lower bound related to it , this concept also called **infimum** or **meet** and is denoted **inf (S)** or $\bigwedge$**S.**

Example.1.3.7. consider the relation | (divides) on natural numbers. The least upper bound of two numbers is the smallest number that is divided by both of them, i.e. the least common multiple of the numbers. Greatest lower bounds in turn are given by the greatest common divisor.

Like upper bounds may fail to exist, this can be seen in the following example.

Example.1.3.8. $\mathbb{Z}$ in $\mathbb{R}$ has no upper bound, and if we let the relation "$\preccurlyeq$" on {a, b, c, d} be given by $a \preccurlyeq c$, $a \preccurlyeq d$, $b \preccurlyeq c$, $b \preccurlyeq d$. The set {a, b} has upper bounds c and d, but no least upper bound.

Zorn's lemma.1.3.9. Every partially ordered set in which every chain (i.e. totally ordered subset) has an upper bound contains at least one maximal element.

Notation.1.3.5. In this thesis the following notations will appear: $a \vee b$ (read as" a join b") will be written in place of sup {a,b} when it exists and $a \wedge b$ (read as "a meet b") in place of inf {a,b}when it exists. Similarly $\bigvee S$ (the "join of S") and $\bigwedge S$ (the " meet of S") are used instead of sup S and inf S when these exist.

1.4. Duality, And Duality Principle

In the previous section we see that a concept can be defined by just inverting the ordering in a former definition. This is the case for **"least"** and **"greatest"**, for **"minimal"** and **"maximal"**, for **"upper bound"** and **"lower bound",** and so on. This is a general situation in order theory: A given order can be inverted by just exchanging its direction. This yields the so-called **dual, inverse**, or **opposite order**

Every order theoretic definition has its **dual**: it is the notion one obtains by applying the definition to the inverse order. Since the symmetry of all concepts, this operation preserves the theorems of partial orders. For a given mathematical result, one can just invert the order and replace all definitions by their duals and one obtains another correct theorem. This fact is important and useful, since one obtains two theorems for the price of one and reduce the work.

Every partially ordered set P gives rise to a dual (or opposite) partially ordered set which is often denoted by P^{op} or P^{d}. It is easy to see that P^{d} diagram can be obtained by turn the Hasse diagram for P upside down, and this will give a partially ordered set. (see Figure 4)

P **P^d**

Figure 4 : Diagrams of a poset P and P^d

This dual order $\mathbf{P^{op}}$ is defined to be the set with the inverse order, i.e. $x \preccurlyeq y$ holds in $\mathbf{P^{op}}$ if and only if $y \preccurlyeq x$ holds in **P**. Familiar examples of dual partial orders include

- the subset and superset relations $\subset$ and $\supset$ on any collection of sets,

- the divides and multiple relations on the integers.

The importance of this simple definition come from the fact that each and every definition and theorem of order theory can readily be transferred to the dual order. Formally, this is captured by the **Duality Principle** for ordered sets:

"If a given statement is valid for all partially ordered sets, then its dual statement, obtained by inverting the direction of all order relations and by dualizing all order theoretic definitions involved, is also valid for all partially ordered sets."

As an example, take the statement: " If sup H exists then it is unique". We get as its dual " If inf H exists then it is unique". The dual of "(P, $\preccurlyeq$) has a zero" is "(P, $\succcurlyeq$) has a unit".

If a statement or definition is equivalent to its dual then it is said to be **self-dual.**

Theorem.1.4.1. Any finite subset X of a partially ordered set has maximal and minimal members.

Proof : let X consist of { $x_1, x_2, \ldots, x_n$ }. Define $m_1 = x_1$, and $m_k = x_k$ if $x_k \prec m_{k-1}$, and $m_k = m_{k-1}$ otherwise. Then m_n will be minimal, Dually X will have a maximal element. □

Corollary.1.4.2. In chains the notation of minimal and least (maximal and greatest) element of a subset are effectively equivalent. Hence any finite chain has a least and greatest element.

Proof: If $x \prec a$ for no x in X (i.e. a is minimal) , then by $\mathbf{P_4}$ $a \preccurlyeq x$ for all x in X.□

1.5 Maps Between Ordered Sets

Definition. 1.5.1. A function f from an ordered set P to an ordered set Q is said to be

1- **" order- preserving"** or **" monotone"** if it satisfies :

$x \preccurlyeq y$ in P implies $f(x) \preccurlyeq f(y)$ in Q for all x,y in X **(1)**

2- The converse of this implication leads to functions that are **order-reflecting,** if it satisfies

$f(x) \preccurlyeq f(y)$ implies $x \preccurlyeq y$ **(2)**

3- It is may also be **order-reversing** ,

if $x \preccurlyeq y$ implies $f(y) \preccurlyeq f(x)$. **(3)**

4- **An order-embedding** (and we write f: $P \hookrightarrow Q$) if

$\forall x_1, x_2 \in P$: $x_1 \preccurlyeq x_2$ if and only if $f(x_1) \preccurlyeq f(x_2)$ in Q. **(4)**

It is important to understand the difference between the notation "order- preserving" and "order – embedding" map.

Definition.1.5.2. By an **order isomorphism** from an ordered set P to an ordered set Q we shall mean an order-preserving bijection $f : P \to Q$ whose inverse $f^{-1} : Q \to P$ is also order-preserving.

An isomorphism from an ordered set P to itself is called an **automorphism**

Two ordered sets are called **isomorphic** if and only if there exist an isomorphism between them. The fact that ordered sets P and Q are isomorphic denoting by $P \cong Q$.

Theorem.1.5.3. (Theorem 1.10 in [8]). Ordered sets P and Q are isomorphic if and only if there is a surjective mapping $f : P \to Q$ such that

$$x \preccurlyeq y \Leftrightarrow f(x) \preccurlyeq f(y).$$

Proof. The necessity is clear. Suppose conversely that such a surjective mapping f exists. Then f is also injective; for if $f(x) = f(y)$ then from $f(x) \preccurlyeq f(y)$ we obtain $x \preccurlyeq y$, and from $f(x) \succcurlyeq f(y)$ we obtain $x \succcurlyeq y$, so that $x = y$. Hence f is a bijection. Clearly, f is monotone; and so also is f^{-1}since $x \preccurlyeq y$ can be written $f[f^{-1}(x)] \preccurlyeq f[f^{-1}(y)]$ which gives $f^{-1}(x) \preccurlyeq f^{-1}(y)$. □

Remarks.1.5.4.

(1) Let $f: P \to Q$ and $g: Q \to R$ be order preserving maps. Then the composite map $g \circ f$, given by $(g \circ f)(x) = g(f(x))$ for $x \in P$, is order preserving.

(2) Let $f: P \hookrightarrow Q$ and let f(P) defined as $\{f(x) : x \in P\}$be the image of f. Then $f(P) \cong P$.

(3) An order embedding is a one-to-one map.

It is easily shown that any finite partially ordered set is defined up to isomorphism by its diagram; $a > b$ if and only if sequence $x_0, x_1, \ldots, x_n$ exist such that $a = x_0$, $b = x_n$ and x_{i-1} covers x_i for $i = 1,2,\ldots, n$. Graphically, this means that one can move from a to b downward along a broken line.

The isomorphism or non isomorphism of a partially ordered sets having few elements can be tested most simple by inspecting their diagrams. Any isomorphism must be one to one between lowest elements, between elements just above lowest elements, and so on, corresponding elements must be covered by equal numbers of different elements, etc; with a little imagination, it does not take long to complete the test.

We shall say that P and Q are dually isomorphic if $P \cong Q^d$ or equivalently, $Q \cong P^d$. In the particular case where $P \cong P^d$ we say that P is self-dual.

Many important ordered sets are self-dual (i.e. anti-isomorphism with themselves).The power set of a set ordered by inclusion is self-dual since the correspondence which carries each subset into its complement is one to one and inverts inclusion. Similarly the set of all linear subspaces of n-dimensional Euclidean space which contains the origin is self dual since the correspondence carrying each subspace into its orthogonal complement is one to one and invert inclusion.

Example.1.5.5. Let Sub$\mathbb{Z}$ be the set of subgroups of the additive abelian group $\mathbb{Z}$ and order Sub$\mathbb{Z}$ by set inclusion. Then $(\mathbb{N}_0, |)$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, is dually isomorphic to (Sub$\mathbb{Z}$,$\subseteq$) under the assignment $n \mapsto n\,\mathbb{Z}$. In fact, since every subgroup of $\mathbb{Z}$ is of the form $n\,\mathbb{Z}$ for some $n \in \mathbb{N}$, this assignment is surjective. Also, we have $n\,\mathbb{Z} \subseteq m\,\mathbb{Z}$ if and only if $m|n$. Note that we include 0 in $\mathbb{N}_0$. Then 0 is the top element of $(\mathbb{N}_0, |)$ and corresponds to the trivial subgroup $\{0\}$. The result therefore follows by Theorem 1.5.3.

Example.1.5.6 The negation function $f : (\mathbb{R}, \leq) \rightarrow (\mathbb{R}, \geq)$ such that $f(x) = -x$ is an order- isomorphism since $x \leq y$ if and only if $-x \geq -y$.

Example.1.5.7 Let $f : (\mathbb{N}_0, \leq) \rightarrow (\mathbb{N}, \leq)$, define f as $f(n) = n+1$,then f is an order-isomorphism.

Example.1.5.8. Figure 5 shows some maps between ordered sets. The map φ_1 is not ordered– preserving. Each of φ_2 to φ_5 is ordered– preserving but not order embedding. The map φ_6 is order embedding but not an order-isomorphism.

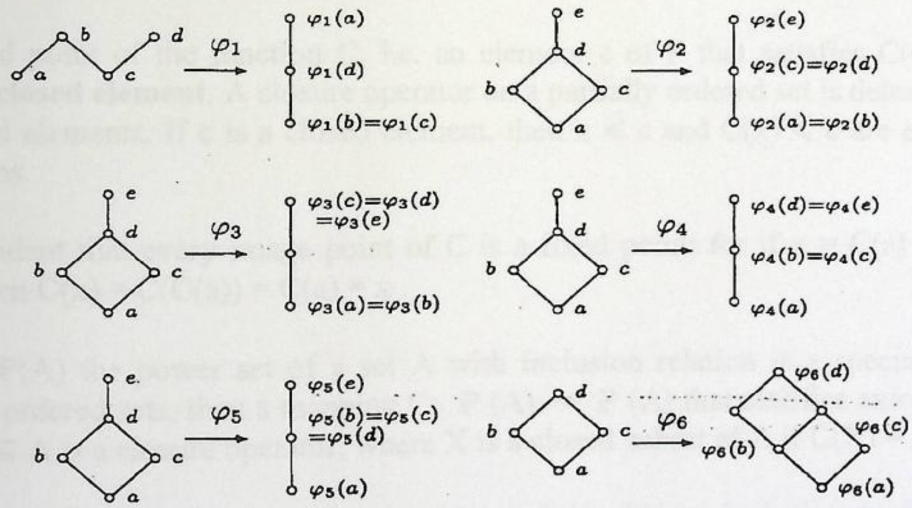


Figure 5: maps between ordered sets

Definition.1.5.9. A function $f: P \rightarrow Q$ from an ordered set P to an ordered set Q is "**antitone**" (or **order-inverting**) if and only if

$$\mathbf{x \leqslant y \Rightarrow f(y) \leqslant f(x),} \tag{5}$$

$$\mathbf{f(x) \leqslant f(y) \Rightarrow y \leqslant x.} \tag{6}$$

Example.1.5.10. If E is a non-empty set and $A \subseteq E$ then $f_A : \mathbb{P}(E) \rightarrow \mathbb{P}(E)$ given by $f_A(X) = A \cap X$ is monotone.

If X^* is the complement of X in E then the assignment $X \rightarrow X^*$ defines an antitone mapping on $\mathbb{P}(E)$.

Now, we define another special type of self-maps on a partially ordered sets which are **closure operators**.

Definition.1.5.11. (Closure Operator). A function $C: P \rightarrow P$ from a partial order P to itself is called a **closure operator** if it satisfies the following axioms for all elements x, y in P.

C_1: $x \preccurlyeq C(x)$ (extensive)

C_2: $C^2(x) = C(C(x)) = C(x)$ (idempotent)

C_3: $x \preccurlyeq y$ implies $C(x) \preccurlyeq C(y)$ (isotone).

If the first condition is changed to $C(x) \preccurlyeq x$, then C is called a dual closure map on P.

Example.1.5.12. The least integer function from the real numbers to the real numbers, which assigns to every real x the smallest integer not smaller than x, is a closure operator. The rounding function [.] is an example of a dual closure map.

A fixed point of the function C, i.e. an element c of P that satisfies $C(c) = c$, is called a **closed element**. A closure operator on a partially ordered set is determined by its closed elements. If c is a closed element, then $x \preccurlyeq c$ and $C(x) \preccurlyeq c$ are equivalent conditions.

It is evident that every image point of C is a fixed point: for if $x = C(a)$ for some $a \in P$, then $C(x) = C(C(a)) = C(a) = x$.

Since $\mathbb{P}(A)$ the power set of a set A with inclusion relation is a special kind of partially ordered sets, then a mapping $C: \mathbb{P}(A) \rightarrow \mathbb{P}(A)$ that satisfies axioms C_1-C_3 for $X, Y \subseteq A$ is a closure operator, where X is a closed subset of A if $C(X) = X$.

The partially ordered set of closed subsets of A with set inclusion as the partial ordering is denoted by $\mathbf{L_C}$.

1.6 Constructing New Orders

There are many ways to construct orders from a given orders. The dual order is one example. Another important construction is the **Cartesian product** of two partially ordered sets.

Definition.1.6.1. (Cartesian product). The **Cartesian product** of two sets X and Y, denoted X × Y, is the set of all possible ordered pairs whose first component is a member of X and whose second component is a member of Y,

$$X \times Y = \{(x, y): x \in X \text{ and } y \in Y\};$$

taken together with the product order on pairs of elements. The ordering is defined by $(a, x) \preccurlyeq (b, y)$ if (and only if) $a \preccurlyeq b$ and $x \preccurlyeq y$. (Notice carefully that there are three distinct meanings for the relation symbol $\preccurlyeq$ in this definition).

Informally, a product X × Y of finite partially ordered sets is drawn by replacing each point of a diagram of X by a copy of a diagram for Y, and connecting the corresponding. See Figure 6.

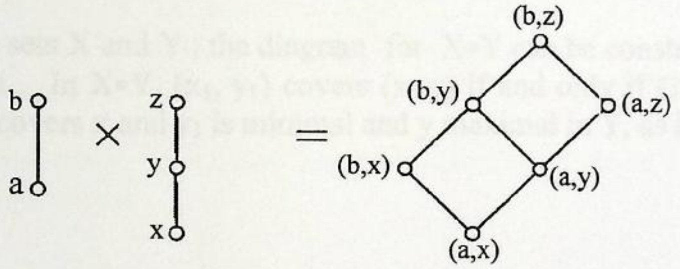


Figure 6

The **disjoint union** of two partially ordered sets is another typical example of order construction.

Definition.1.6.2. Suppose that P and Q are (disjoint) ordered sets. The **disjoint union** P∪Q of P and Q **is** the ordered set formed by defining x, y in P and $x \preccurlyeq y$ in P or x , y in Q and $x \preccurlyeq y$ in Q.

Definition.1.6.3. (Cardinal sum and Cardinal product). Let X and Y be two sets , each with a relation $\preccurlyeq$. By **cardinal sum** X + Y of X and Y, we mean the set of all elements of X and Y, where $\preccurlyeq$ keeps its meaning within X and Y, $y \preccurlyeq x$ [$x \in X$, $y \in Y$] never holds, and X and Y are considered as disjoint. Graphically, the addition of two partially ordered sets amounts simply to laying their diagrams side-by-side.

By **cardinal product** XY, we mean the set of all couples (x,y) [$x \in X$, $y \in Y$], where $(x_1,y_1) \preccurlyeq (x,y)$ means $x_1 \preccurlyeq x$ in X and $y_1 \preccurlyeq y$ in Y.

Definition.1.6.4. (Ordinal sum and Ordinal product). Let X and Y be two sets , each with a relation $\preccurlyeq$. By **ordinal sum** $X \oplus Y$ of X and Y, we mean the set of all elements of X and Y, where $x \preccurlyeq x_1$ in X and $y \preccurlyeq y_1$ in Y preserves their original meaning , and $x \preccurlyeq y$ for all $x \in X$ and $y \in Y$.

By **ordinal product** $X \circ Y$, we mean the set of all couples (x,y) where $(x_1,y_1) \prec (x,y)$ means that either $x_1 \prec x$ or $x_1 = x$ and $y_1 \prec y$.

For finite posets, the diagram of $X \oplus Y$ can be constructed by laying the diagram of Y above the diagram of X and drawing lines from all minimal elements of Y to all maximal elements of X, as in Figure 7.

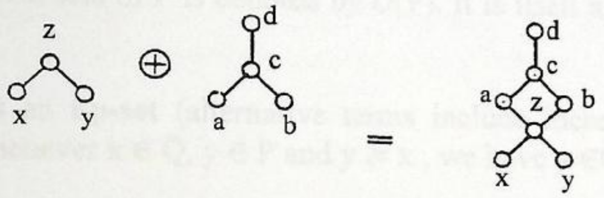


Figure 7

Similarly, for finite sets X and Y , the diagram for $X \circ Y$ can be constructing by using the following result . In $X \circ Y$, (x_1, y_1) covers (x, y) if and only if (1) $x = x_1$ and y_1 covers y, or (2) x_1 covers x and y_1 is minimal and y maximal in Y, as in Figure 8.

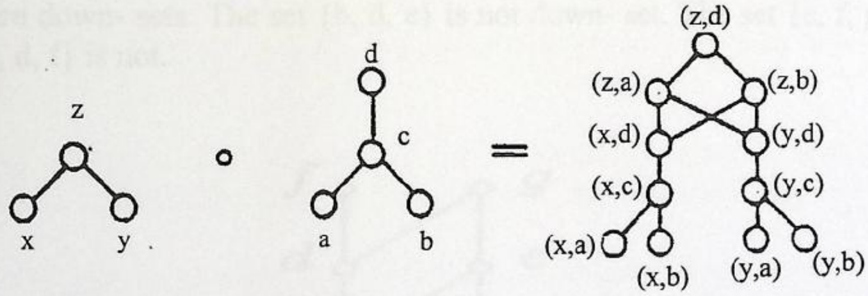


Figure 8

1.7. Subsets Of Ordered Sets

In an ordered set, one can define many types of special subsets based on the given order. A simple examples are **down- sets** and **up – sets.**

Definition.1.7.1. (down- sets and up – sets).
Let P be an ordered set and $Q \subseteq P$.

(1) Q is a **down-set** (alternative terms include decreasing set and order ideal) if, whenever $x \in Q$, $y \in P$ and $y \preccurlyeq x$, we have $y \in Q$.

$$\downarrow Q = \{ y \in P : (\exists x \in Q)\ y \preccurlyeq x \} \quad \text{and} \quad \downarrow x = \{ y \in P : y \preccurlyeq x \}.$$

The family of all down sets of P is denoted by $\mathcal{O}(P)$. It is itself an ordered set , under the inclusion order.

(2) Dually, Q is an **up-set** (alternative terms include increasing set and order filter) if , whenever $x \in Q$, $y \in P$ and $y \succcurlyeq x$, we have $y \in Q$.

$$\uparrow Q = \{ y \in P : (\exists x \in Q)\ y \succcurlyeq x \} \quad \text{and} \quad \uparrow x = \{ y \in P : y \succcurlyeq x \}.$$

It is easily checked that $\downarrow Q$ is the smallest down – set containing Q and that Q is down set if and only if $Q = \downarrow Q$, and dually for $\uparrow Q$. Clearly $\downarrow\{x\} = \downarrow x$, and dually. Down- sets (up – sets) of the form $\downarrow x$ ($\uparrow x$) are called **principle.**

Example.1.7.2. In the chain $\mathbb{Q}^+$ of positive rationales the set $\{q \in \mathbb{Q}^+ : q^2 \leq 2\}$ is a down-set that is not principal.

Example.1.7.3. Consider the ordered set in Figure 9. The sets {c},{a, b, c, d, e} and {a, b, d, f} are down- sets. The set {b, d, e} is not down- set. The set {e, f, g} is up-set, but {a, b, d, f} is not.

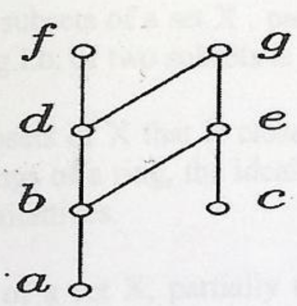


Figure 9

More complicated down- subsets are **ideals,** which have the additional property that each two of their elements have an upper bound within the ideal. Their duals are given by **filters.** (more details in chapter two).

Chapter Two

Lattices

The general theory of partially ordered sets is based on a single undefined relation. That of lattices is also based indirectly on this relation, but directly on two dual binary operations which are analogous in many ways to ordinary addition and multiplication. It is this analogy which makes lattice theory a branch of algebra.

There are two standard ways of defining lattices - one based on the notion of order, and the other puts them on the same (algebraic) footing as groups or rings. In this chapter we will introduce the two definitions illustrated with examples and connect between them, then some basic theorems and lemmas about lattices will be given.

2.1 Lattices As Partially Ordered Sets.

Two kinds of partially ordered sets will be introduced: semilattices and lattices.

Definition.2.1.1.(Semilattices). A join- semilattice is a partially ordered set in which every two elements a and b have a least upper bound $a \vee b$. Replacing " least upper bound " with "greatest lower bound " results in the dual concept of **a meet-semilattice** i.e. **a meet-semilattice** is a partially ordered set in which every two elements a and b have a greatest lower bound $a \wedge b$.

A join-semilattice is bounded if it has a least element. Dually, a meet-semilattice is bounded if it has a greatest element.

Definition.2.1.2. (Subsemilattice). By a **meet-subsemilattice** of a **meet-semilattice** L we mean a nonempty subset E of L that is closed under the meet operation, in the sense that if $x, y \in E$ then $x \wedge y \in E$. A **join-subsemilattice** of a **join-semilattice** is defined dually.

Example.2.1.3. The set of all subsets of a set X , partially ordered by inclusion, is a **meet-semilattice**, in which the g.l.b. of two subsets is their intersections.

This extends to any set of subsets of X that is closed under intersections: thus, the subgroups of a group, the subrings of a ring, the ideals of a ring, the submodules of a module, all constitute meet- semilattices.

Again, the set of all subsets of a set X, partially ordered by inclusion, is a join-semilattice, in which the l.u.b. of two subsets is their union.

The subgroups of a group constitute a join-semilattice, in which the supremum of two subgroups is the subgroup generated by their union, the ideals of a ring and the submodules of a module also constitute a join-semilattices, in which the supremum of two ideals or submodules is their sum.

Example.2.1.4. Every chain is a meet-semilattice in which $x \wedge y = \min\{x, y\}$.

Example. 2.1.5. ($\mathbb{N}_0$; |) is a meet-semilattice in which $m \wedge n = \gcd(m, n)$.

Example.2.1.6. ($\mathbb{N}_0$; |) is a join-semilattice in which $m \vee n = \mathrm{lcm}(m, n)$.

Let us now define lattices.

Definition.2.1.7. (Lattices). **A lattice** is a partially ordered set that is both a join-semilattice and a meet-semilattice. Equivalently, a partially ordered set L is **a lattice** if and only if for every a, b in L both sup {a , b} and inf {a , b} exist (in L).

To make the definition of a lattice less arbitrary, we note that an equivalent definition is the following:

A partially ordered set L is a lattice if and only if inf H and sup H exists , for any finite non empty subset H of L.

Note. It is enough to prove that the first definition implies the second. So let L satisfy the first definition and let $H \subseteq L$ be non empty and finite.

If $H = \{a\}$, then $\inf H = \sup H = a$ follows from the reflexivity of $\preccurlyeq$ and the definitions of inf and sup.

Now let H = { a, b, c}. To show that inf H exists, set d= inf {a ,b}, e = inf { c ,d} we claim that e = inf H.

First of all $d \preccurlyeq a$, $d \preccurlyeq b$ and $e \preccurlyeq c$, $e \preccurlyeq d$; therefore (by transitivity) $e \preccurlyeq x$ for all $x \in H$. Secondly, if f is a lower bound of H, then $f \preccurlyeq a$, $f \preccurlyeq b$ and thus $f \preccurlyeq d$, also $f \preccurlyeq c$, therefore $f \preccurlyeq e$, since e = inf{c,d}. Thus e is the infimum of H.

Now if $H = \{ a_0,\ldots, a_{n-1}\}$, $n \geq 1$, the inf $\{\ldots, \inf \{ a_0, a_1\}\ldots,a_{n-1}\}$ is the infimum of H, by an inductive proof whose steps are similar to those in the preceding paragraph. By duality, we conclude that sup X exists.

Example.2.1.8. Every totally ordered set is a lattice.

If, say, $a \preccurlyeq b$, then inf {a , b} = a and sup {a , b} = b; thus $\mathbb{N}$,$\mathbb{Q}$ and $\mathbb{R}$, with their usual order relations, are lattices.

Example.2.1.9. If X is a set, and L is a set of subsets of X that is closed under intersections and contains X , then L , partially ordered by inclusion, is a lattice.
For if we let A, B $\in$ L . Then $A \cap B \in L$ is the g.l.b. of A and B and the l.u.b of A and B is the intersection of all $C \in L$ that contain $A \cup B$ (including X),which belongs to L by the hypothesis.

Example.2.1.10. As a special case of example 2.1.9, the set of subgroups of a group G ordered by set inclusion form a lattice. In this lattice, the join of two subgroups is the subgroup generated by their union, and the meet of two subgroups is their intersection.

Example.2.1.11. Let J be the set of positive integers, and let $a \preccurlyeq b$ mean " a divides b" it can be seen that this partially ordered set is a lattice with $\sup\{a,b\} = \text{lcm}(a,b)$ and $\inf\{a,b\} = \gcd(a,b)$.

Clearly the dual of any lattice is again a lattice, with meets and joins interchanged. Hence there is a **duality principle** for lattices :A theorem that holds in every lattice remains true when the order relation is reversed.

Definition.2.1.12. (Bounded Lattice). A partially ordered set is **a bounded lattice** if and only if every finite set of elements (including the empty set) has a join and a meet.

Here, the join of an empty set of elements is defined to be the least element $\vee\emptyset = 0$, and the meet of the empty set is defined to be the greatest element $\wedge\emptyset = 1$.

This convention is consistent with the associativity and commutativity of meet and join: the join of a union of finite sets is equal to the join of the joins of the sets, and dually, the meet of a union of finite sets is equal to the meet of the meets of the sets, i.e., for finite subsets A and B of a partially ordered set L,

$$\vee(A \cup B) = (\vee A) \vee (\vee B)$$

and

$$\wedge(A \cup B) = (\wedge A) \wedge (\wedge B)$$

hold.

Taking B to be the empty set,

$$\vee(A \cup \emptyset) = (\vee A) \vee (\vee\emptyset) = (\vee A) \vee 0 = \vee A$$

and

$$\wedge(A \cup \emptyset) = (\wedge A) \wedge (\wedge\emptyset) = (\wedge A) \wedge 1 = \wedge A$$

which is consistent with the fact that $A \cup \emptyset = A$

Any lattice can be converted into a bounded lattice by adding a greatest and least element, and every finite lattice is bounded, by taking the join of all elements, denoted by

$$\vee A = a_1 \vee \ldots \vee a_n,$$

and the meet of all elements denoted by

$$\wedge A = a_1 \wedge \ldots \wedge a_n,$$

where $A = \{a_1, a_2, \ldots, a_n\}$.

Remarks.2.1.13.

1- A lattice need not have 1 or 0, so $\inf\emptyset$ and $\sup\emptyset$ may not exits: the set of real numbers which is a lattice with $a \vee b$ is the grater and $a \wedge b$ is the smaller has no top nor bottom.

2- If a and b are elements of a partially ordered set such that $b \preccurlyeq a$, then $a \wedge b = b$.

proof (2): $b \preccurlyeq a \Longrightarrow b \preccurlyeq a \wedge b$, But $a \wedge b \preccurlyeq b \Longrightarrow b = a \wedge b$. □

Theorem.2.1.14. (Theorem 2.7 in [8]). Let L be a lattice and let $f : L \rightarrow L$ be a closure. Then Im f is a lattice in which the lattice operations are given by

$$\inf\{a, b\} = a \wedge b, \quad \sup\{a, b\} = f(a \vee b).$$

Proof. Recall that for a closure map f on L we have $\operatorname{Im} f = \{x \in L\colon x = f(x)\}$. If $a, b \in \operatorname{Im} f$ and since f is isotone with $f \succcurlyeq id_L$(identity map on L) , then we have

$$f(a) \wedge f(b) = a \wedge b \preccurlyeq f(a \wedge b) \preccurlyeq f(a) \wedge f(b)$$

the resulting equality gives $a \wedge b \in \operatorname{Im} f$. It follows that Im f is a $\wedge$-subsemilattice of L. As for the supremum in Im f of $a, b \in \operatorname{Im} f$, we observe first that

$$a \vee b \preccurlyeq f(a \vee b)$$

and so $f(a \vee b) \in \operatorname{Im} f$ is an upper bound of $\{a, b\}$.

Suppose now that $c = f(c) \in \operatorname{Im} f$ is any upper bound of $\{a,b\}$ in Im f. Then from $a \vee b \preccurlyeq c$ we obtain $f(a \vee b) \preccurlyeq f(c) = c$. Thus, in the subset Im f, the upper bound $f(a \vee b)$ is less than or equal to every upper bound of $\{a, b\}$. Consequently, $\sup\{a, b\}$exists in Im f and is $f(a \vee b)$. □

Example.2.1.15 Consider the lattice L in Figure 10. Let $f : L \rightarrow L$ be given by

$$f(t) = \begin{cases} 1 & \text{if } t = z; \\ t & \text{otherwise.} \end{cases}$$

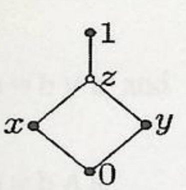


Figure 10

It is readily seen that f is a closure with $\operatorname{Im} f = \{0, x, y, 1\}$. In the corresponding lattice (the elements of which are denoted by •) we have $\sup\{x, y\} = f(x \vee y) = f(z)$ $=1$.

2.2 Lattices As Algebraic Structures.

It is possible to give an equivalent definition of a lattice without any specific mention of any type of ordering. In this setting, lattice can be considered more like the other members of the family of algebraic systems.

Definition.2.2.1. (Semilattice). A **join-semilattice** is an algebraic structure $(L, \vee)$ consisting of a set L with the binary operation $\vee$, such that for all members a, b, c of L, the following identities hold:

Idempotency : $a \vee a = a$

Commutativity: $a \vee b = b \vee a$

Associativity : $a \vee (b \vee c) = (a \vee b) \vee c$

If $\wedge$, denoting meet, replaces $\vee$ in the definition just given, a **meet-semilattice** results.

A meet-semilattice $(L, \wedge)$ is bounded if L includes the identity element 1 such that for all x in L,

$$x \wedge 1 = x.$$

Dually, $(L, \vee)$ is bounded join-semilattice if L includes the zero element such that for all x in L,

$$x \vee 0 = x.$$

Definition.2.2.2. An algebraic structure $(L,\vee,\wedge)$ consisting of a non empty set L and two binary operations $\vee$ and $\wedge$ on L is called **a lattice** if the following axiomatic identities hold for all elements a, b, c of L

L_1: Idempotent laws $a \vee a = a$ and $a \wedge a = a$

L_2: Commutative laws $a \vee b = b \vee a$, and

$a \wedge b = b \wedge a.$

L_3: Associative laws $a \vee (b \vee c) = (a \vee b) \vee c$, and

$a \wedge (b \wedge c) = (a \wedge b) \wedge c$

L_4: Absorptions laws $a \wedge (a \vee b) = a$, and $a \vee (a \wedge b) = a$.

Example.2.2.3. Let L be the set of propositions, let $\vee$ denote the connective "or" and $\wedge$ denote the connective "and". Then L_1 to L_4 are well-known properties from propositional logic.

Example.2.2.4. Let L be the set of natural numbers, let $\vee$ denote the least common multiple and $\wedge$ denote the greatest common divisor. Then properties $\mathbf{L_1}$ to $\mathbf{L_4}$ are easily veritable.

Example.2.2.5. Let L be the set of normal subgroups of any group G, define $H \wedge K$ to be the usual intersection $H \cap K$, and $H \vee K = HK = \{hk : h \in H, k \in K\}$. Then properties $\mathbf{L_1}$ to $\mathbf{L_4}$ are satisfied.

Let L be a lattice, it may happen that $(L,\preccurlyeq)$ has a top and bottom elements as defined in (1.3.1), when thinking of L as $(L,\vee,\wedge)$, it is appropriate to view these elements from a more algebraic stand point, we say L has a one if there exist $1 \in L$ such that $a = a \wedge 1$ for all $a \in L$. dually, L is said to have a zero if there exists $0 \in L$ such that $a = a \vee 0$ for all $a \in L$. A lattice $(L,\vee,\wedge)$ has a one if and only if $(L,\preccurlyeq)$ has a top element. A dual statement holds for 0 and bottom.

Definition.2.2.6. A **bounded lattice** is an algebraic structure of the form $(L,\vee,\wedge,1,0)$ such that $(L,\vee,\wedge)$ is a lattice, 0 (the lattice's bottom) is the identity element for the join operation $\vee$, and 1 (the lattice's top) is the identity element for the meet operation $\wedge$. A finite lattice is automatically bounded, with $1 = \bigvee L$ and $0 = \bigwedge L$.

Example.2.2.7. Let $X = \{a, b, c\}$ and let $L = (L, \vee, \wedge, 0, 1)$ be the power set of X. Then $L = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X \}$is bounded lattice with $0 = \emptyset$ and $1 = X$, $\vee$ is the union: for $A , B \in L$, $A \vee B = A \cup B$,and $\wedge$ is the intersection: for $A , B \in L$, $A \wedge B = A \cap B$.

Example.2.2.8. Note that $(\mathbb{N}_0, \mathrm{lcm}, \gcd)$ is bounded with 1=0 and 0=1, while $(\mathbb{N}_0, \min, \max)$ is not bounded (0 is bottom, there is no top).

Example.2.2.9. For every infinite set E let $P_f(E)$ be the set of finite subsets of E. Then $(P_f(E);\cap,\cup,\subseteq)$ is a lattice with no top element.

Lemma.2.2.10. (2.8 The Connecting Lemma in [14]). Let L be a lattice and let $a,b \in L$. Then the following are equivalent

1. $a \preccurlyeq b$;
2. $a \vee b = b$;
3. $a \wedge b = a$.

Proof: It is easy to show that (1) implies (2) and (3) by the definition of $\vee$ and $\wedge$. Now assume (2) i.e. $a \vee b = b$, then b is an upper bound for $\{a, b\}$ where $a \preccurlyeq b$. Thus (1) holds similarly (3) implies (1). □

Theorem.2.2.11.(Theorem 7.21 in [24]) If a, b, and c are arbitrary elements of a lattice. The following equality holds.

1- $a \vee b = b \vee a$, $a \wedge b = b \wedge a$.

2- $(a \vee b) \vee c = a \vee (b \vee c)$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

3- $a \vee a = a$, $a \wedge a = a$.

4- $(a \vee b) \wedge a = a$, $(a \wedge b) \vee a = a$.

Proof: Since there are only " language" differences between the l.u.b of a and b, and the l.u.b of b and a, it is clear that $a \vee b = b \vee a$.

A similar reflection shows that $a \wedge b = b \wedge a$, so (1) is proved. In order to establish (2), first note that :$(a \vee b) \preccurlyeq (a \vee b) \vee c$ and $c \preccurlyeq (a \vee b) \vee c$, so that $a \preccurlyeq (a \vee b) \vee c$, $b \preccurlyeq (a \vee b) \vee c$ and $c \preccurlyeq (a \vee b) \vee c$. Also if x is any element of the lattice such that $a \preccurlyeq x$, $b \preccurlyeq x$ and $c \preccurlyeq x$. then $a \vee b \preccurlyeq x$, $(a \vee b) \vee c \preccurlyeq x$ it follows that $(a \vee b) \vee c$ is a l. u. b of a, b and c. A similar argument shows that $a \vee (b \vee a)$ is the l.u.b of a,b and c . So $(a \vee b) \vee c = a \vee (b \vee c)$.

Similar argument shows that $a \wedge (b \wedge c)$ is g.l.b. of a, b and c and $(a \wedge b) \wedge c$ is a g.l.b of a, b and c. Hence $(a \wedge b) \wedge c = a \wedge (b \wedge c)$. Now the definition of l.u.b requires $a \vee a = a$ and the definition of g.l.b requires $a \wedge a = a$, so (3) is proved.

Finally, since $a \preccurlyeq a \vee b$, we have $(a \vee b) \wedge a = a$, and since $a \wedge b \preccurlyeq a$ it follows $(a \wedge b) \vee a = a$. This proved (4) and completes the proof of the theorem. □

Lemma.2.2.12. In any lattice, the operations of join and meet are **isotone:**

$$\textbf{If } y \preccurlyeq z, \textbf{ then } x \wedge y \preccurlyeq x \wedge z \textbf{ and } x \vee y \preccurlyeq x \vee z. \tag{7}$$

Proof: By L_1- L_4 in definition 2.2.2 and the connecting lemma, $y \preccurlyeq z$ implies $x \wedge y = (x \wedge x) \wedge (y \wedge z) = (x \wedge y) \wedge (x \wedge z)$ whence $x \wedge y \preccurlyeq x \wedge z$.
The second inequality can be proved dually (Duality principle). □

Lemma.2.2.13. In any lattice we have the **distributive inequalities (8)** and **(9)** :

$$(x \wedge y) \vee (x \wedge z) \preccurlyeq x \wedge (y \vee z). \tag{8}$$

$$(x \vee y) \wedge (x \vee z) \preccurlyeq x \vee (y \wedge z). \tag{9}$$

Proof. clearly $x \wedge y \preccurlyeq x$ and $x \wedge y \preccurlyeq y \preccurlyeq y \vee z$; hence $x \wedge y \preccurlyeq x \wedge (y \vee z)$. Also $x \wedge z \preccurlyeq x$, $x \wedge z \preccurlyeq z \preccurlyeq y \vee z$; whence $x \wedge z \preccurlyeq x \wedge (y \vee z)$. That is $x \wedge (y \vee z)$ is an upper bound of $x \wedge y$ and $x \wedge z$, from which **(8)** follows. The distributive inequality **(9)** follows from **(8)** by duality. □

Lemma.2.2.14. The elements of any lattice satisfy the **modular inequality (10)** :

$$x \preccurlyeq z \textbf{ implies } x \vee (y \wedge z) \preccurlyeq (x \vee y) \wedge z. \tag{10}$$

Proof : $x \preccurlyeq x \vee y$ and $x \preccurlyeq z$. Hence $x \preccurlyeq (x \vee y) \wedge z$. Also $y \wedge z \preccurlyeq y \preccurlyeq x \vee y$ and $y \wedge z \preccurlyeq z$. Therefore $y \wedge z \preccurlyeq (x \vee y) \wedge z$. Whence $x \vee (y \wedge z) \preccurlyeq (x \vee y) \wedge z$. □

Lemma.2.2.15.(4.2 Lemma in [14]). Let L be a lattice. Then the following are equivalent:

1. $(\forall a, b, c \in L)\ c \preccurlyeq a \Longrightarrow a \wedge (b \vee c) = (a \wedge b) \vee c$,

2. $(\forall a, b, c \in L)\ c \preccurlyeq a \Longrightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

3. $(\forall p, q, r \in L)\ p \wedge (q \vee (p \wedge r)) = (p \wedge q) \vee (p \wedge r)$.

Proof. The Connecting Lemma gives the equivalence of (1) and (2). To prove that (3) implies (2), assume that $c \preccurlyeq a$ and apply (3) with $p = a$, $q = b$ and $r = c$. Conversely, assume (2) holds and that p, q and r are any elements of L. We may put $a = p$, $b = q$ and $c = p \wedge r$ in (2), and this gives (3). □

The following theorem shows that properties $\mathbf{L_1}$ to $\mathbf{L_4}$ are not only characteristic of any lattice, but they are enough to define this type of system.

Theorem.2.2.16. (Theorem 7.22 in [24]). Let L be a set in which are defined two binary operations $\vee$ and $\wedge$, and which possess the identities listed in theorem 2.2.11. It is then possible to define a partial ordering $\preccurlyeq$ in L such that L is a lattice with $\vee$ and $\wedge$ are l.u.b and g.l.b respectively.

Proof: we must define a partial ordering $\preccurlyeq$ in L and show that

(1) $a \preccurlyeq b$ and $b \preccurlyeq a$ if and only if $a = b$.

(2) if $c \preccurlyeq b$ and $b \preccurlyeq a$, then $c \preccurlyeq a$.

We define the partial ordering as follows , $b \preccurlyeq a$ if and only if $a \vee b = a$. The two required properties of this partial ordering can now be derived.

Suppose that $b \preccurlyeq a$ and $a \preccurlyeq b$. Then $a \vee b = a$ and $b \vee a = b$, so that by (1) in theorem 2.2.11 it follows that $a = b$. Conversely, if $a=b$, by (3) in theorem 2.2.11 we have $a \vee b = a$ and so $b \preccurlyeq a$. In like manner we can show that $a \preccurlyeq b$ if $a = b$, thus completing the verification of (1).

Now suppose that $c \preccurlyeq b$ and $b \preccurlyeq a$. Then $b \vee c = b$ and $b \vee a = a$, so that $a \vee c = (a \vee b) \vee c = a \vee (b \vee c) = a \vee b = a$, Hence $c \preccurlyeq a$, and so (2) is established.

There remains only to show that $\vee$ and $\wedge$ play the respective roles of least upper bound and greatest lower bound.

Since $a \wedge (b \vee a) = a$, by 4, we have $a \preccurlyeq a \vee b$: and a similar argument leads to $b \preccurlyeq a \vee b$. Now let c be any element of L such that $a \preccurlyeq c$ and $b \preccurlyeq c$. Then $a \vee c = c$ and $b \vee c = c$, and hence $(a \vee b) \vee c = a \vee (b \vee c) = a \vee c = c$. Thus $a \vee b \preccurlyeq c$ and so $a \vee b$ is the least upper bound of a and b. In like manner we can show that $a \wedge b$ is the greatest lower bound of a and b, thus completing the proof of the theorem. □

2.3 Connection Between The Two Definitions.

An order- theoretic lattice gives rise to the two binary operations $\vee$ and $\wedge$. Since the commutative, associative and absorptions laws can easily be verified for these operations, they make $(L, \vee, \wedge)$ into a lattice in algebraic sense. The ordering can be recovered from the algebraic structure because $a \preccurlyeq b$ holds if and only if $a = a \wedge b$.

The converse is also true. Given an algebraically defined lattice $(L, \vee, \wedge)$ one can define a partial order $\preccurlyeq$ on L by setting:

$$a \preccurlyeq b \text{ if and only if } a \wedge b = a, \text{ or}$$
$$a \preccurlyeq b \text{ if and only if } a \vee b = b,$$

for all elements a and b from L. The laws of absorption ensure that both definitions are equivalent.

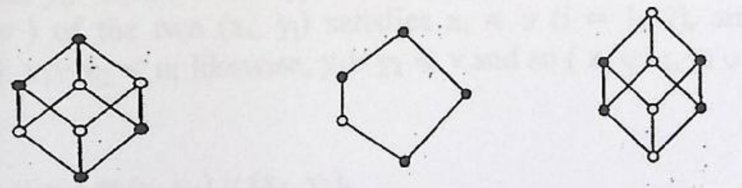
2.4 Sublattices And Products.

It is usually the case that an algebraic system has subsystems of the same kind. Lattices are no exception to this, and it is customary to define a subset of lattice L as a sublattice.

Definition.2.4.1. **A sublattice** of a lattice L is a subset $M \neq \emptyset$ of L that is closed under infimums and supremums.
Equivalently, $M \subseteq L$ is a sublattice of L if and only if $x, y \in M$ implies $x \wedge y \in M$ and $x \vee y \in M$, where $\vee$ and $\wedge$ are the lattice operations of L. If M is a sublattice of L then for a, b in M we will of course have $a \preccurlyeq b$ in M if and only if $a \preccurlyeq b$ in L.

Example.2.4.2. Any one- element subset of a lattice is a sublattice more generally, any non- empty chain in a lattice is a sublattice. (In fact, when testing that a non-empty subset M is a sublattice, it is sufficient to consider non- comparable elements).

Example.2.4.3. In the diagrams in Figure (11) the shaded elements in lattices (i) and (ii) form sublattices, while those in (iii) do not.



(i) (ii) (iii)

Figure 11

It is interesting to note that given a lattice L one can often find subset which as posets (using the same order relation) are lattices, but which do not qualify as sublattices as the operations $\vee$ and $\wedge$ do not agree with those of the original lattice L.

Example.2.4.4. In Figure 12 note that P = {a, d, c, e} as a partially ordered set is in fact a lattice, but P is not a sublattice of the lattice {a, b, c, d, e}.

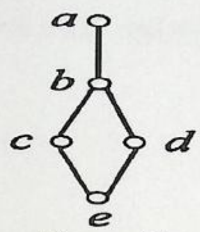


Figure 12

Example.2.4.5. The lattice L of subsets of a group G with $H \vee K = H \cup K$ and $H \wedge K = H \cap K$, along with the lattice M of subgroups of G. Since every subgroup of G is a subset of G, it is clear that $M \subseteq L$. However if H and K are arbitrary subgroups of G, the subset $H \vee K = H \cup K$ is not in general a subgroup of G, and so it not a member of M. Hence M is not a sublattice of L. While M is a lattice with $H \vee K$ to be the subgroup generated by H and K, and same definition of $H \wedge K$ as for L.

New lattices can be constructed from given ones by informing direct products. This is analogous to processes of forming direct products of groups and direct sum of rings.

Definition.2.4.6. (Products). Let L and K be lattices. Define $\vee$ and $\wedge$ coordinates wise on L×K, as follows:

$$(x_1,y_1) \vee (x_2,y_2) = (x_1 \vee x_2, y_1 \vee y_2)$$

$$(x_1,y_1) \wedge (x_2,y_2) = (x_1 \wedge x_2, y_1 \wedge y_2).$$

Theorem.2.4.7.(Theorem 7 in [7]). The direct product LM of any two lattices is a lattice.

Proof: For any two elements (x_i, y_i) in LM (i = 1, 2), the element $(x_1 \vee x_2, y_1 \vee y_2)$ contains both of (x_i, y_i) hence is an upper bound for the pair. Moreover every other upper bound (u, v) of the two (x_i, y_i) satisfies $x_i \preccurlyeq u$ (i = 1, 2), and hence (by definition of l.u.b) $x_1 \vee x_2 \preccurlyeq u$; likewise, $y_1 \vee y_2 \preccurlyeq v$ and so $(x_1 \vee x_2, y_1 \vee y_2) \preccurlyeq (u, v)$. This shows that

$$(x_1 \vee x_2, y_1 \vee y_2) = (x_1,y_1) \vee (x_2, y_2), \tag{11}$$

whence the latter exists. Dually,

$$(x_1 \wedge x_2, y_1 \wedge y_2) = (x_1, y_1) \wedge (x_2, y_2) \tag{12}$$

which proves that L is a lattice. □

2.5 Isomorphic lattices

An isomorphism between two partially ordered sets P and Q was defined as a one to one correspondence which preserved order, so that.

$$x \preccurlyeq y \text{ in P if and only if } f(x) \preccurlyeq f(y) \text{ in Q.} \tag{13}$$

Such a correspondence must preserve joins and meets, whenever they exist, so that if P and Q are lattices.

$$f(x \wedge y) = f(x) \wedge f(y). \tag{14}$$

$$f(x \vee y) = f(x) \vee f(y). \tag{15}$$

Now look closely at several correspondence f: $L \longrightarrow L^*$ between lattices, they may satisfy. $x \preccurlyeq y$ implies $f(x) \preccurlyeq f(y)$, but neither (14) nor (15), they may satisfy (14) but not (15), or (15) but not (14), or they may satisfy both.

Such correspondences are called **monotones(isotones), meet-homomorphisms, join-homomorphisms, and lattice homomorphisms,** respectively.

Remark.2.5.1. For bounded lattices L and K it is appropriate to consider homomorphisms f: $L \longrightarrow K$ such that $f(0) = 0$ and $f(1) = 1$. Such maps are called **{0, 1}-homomorphisms**

Example.2.5.2. Let $L = L^* = (\mathbb{Z}, \vee, \wedge)$, where $a \vee b$ is the grater, and $a \wedge b$ is the smaller. Now f: $L \longrightarrow L^*$ such that $f(a) = a+1$ is monotone, and satisfying (14) and (15) so it is a lattice homomorphism.

Example.2.5.3 In Figure 5, note that each of $\varphi_2 - \varphi_6$ is an order preserving map(monotone) from one lattice to another. The maps φ_2 and φ_3 are homomorphisms, the remainder are not. Neither join nor meet is preserved by φ_4. The map φ_5 preserves joins but does not preserve all meets; φ_6 is meet - preserving but does not preserve all joins.

Lemma.2.5.4. Any meet-homomorphism, join- homomorphism, and(lattice)-homomorphism, are all monotone.

This statement will be proved for meet-homomorphism.

Proof. Let $f : L_1 \longrightarrow L_2$ be a meet homomorphism , then $f(a \wedge b) = f(a) \wedge f(b)$, for all a, b in L_1, and if $a, b \in L_1$ with $a \preccurlyeq b$ in L_1, then $a = a \wedge b$; thus $f(a) = f(a \wedge b) = f(a) \wedge f(b)$, so $f(a) \preccurlyeq f(b)$.
(similar argument for join and lattice homomorphisms). □

Note that the converse of the above lemma does not hold, nor is there any connection between join- and meet- homomorphisms.

Example.2.5.5. Figure 13 shows three maps of four-elements lattice L into the three element chain C_3.

The map of figure (i) is monotone but is neither a meet nor a join-homomorphism. The map of figure (ii) is a join-homomorphism but is not a meet-homomorphism, thus not a homomorphism. The map of figure (iii) is a homomorphism.

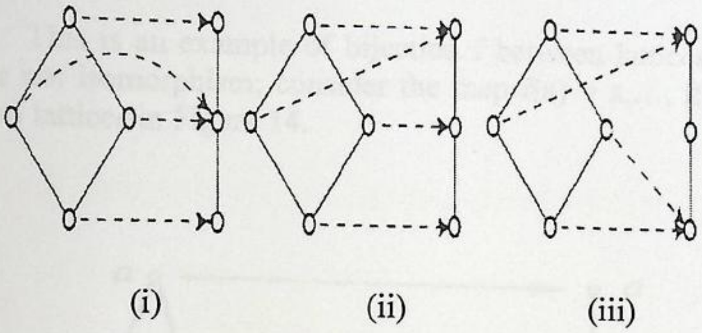


Figure 13

Definition.2.5.6. (Isomorphism For Lattices).
We can introduce two concepts of isomorphism for lattices.

- Two lattices $L_1 = (X,\preccurlyeq)$ and $L_2 = (Y,\preccurlyeq)$ are **isomorphic** (in symbol $L_1 \cong L_2$) if there is a bijection map f (one to one and onto) from L_1 to L_2 such that for every a,b in L_1 we have

$$a \preccurlyeq b \text{ in } L_1 \text{ if and only if } f(a) \preccurlyeq f(b) \text{ in } L_2 .$$

- Two lattices $L_1 = (X, \vee, \wedge)$ and $L_2 = (Y, \vee, \wedge)$ are **isomorphic** (in symbol $L_1 \cong L_2$) if there is a bijection map f (one to one and onto) from L_1 to L_2 such that for every a, b in L_1 the following two equations hold:

$$f(a \vee b) = f(a) \vee f(b),$$
and $f(a \wedge b) = f(a) \wedge f(b)$. Such f is called **an isomorphism.**

It is useful to note that if f is an isomorphism from L_1 to L_2 then f^{-1} is an isomorphism from L_2 to L_1, and if g is an isomorphism from L_2 to L_3 then $g \circ f$ is an isomorphism from L_1 to L_3.

Definition.2.5.7. A lattice L_1 can be **embedded** into a lattice L_2 if there is a sublattice of L_2 isomorphic to L_1, in this case we also say L_2 contains a copy of L_1.

Theorem.2.5.8. (Theorem 2.3 in [10]). Two lattices L_1 and L_2 are isomorphic if and only if there is a bijection f from L_1 to L_2 such that both f and f^{-1} are order-preserving.

Proof. If f is an isomorphism from L_1 to L_2 and $a \preccurlyeq b$ holds in L_1 then $a = a \wedge b$, so $f(a) = f(a \wedge b) = f(a) \wedge f(b)$, hence $f(a) \preccurlyeq f(b)$, and thus f is order-preserving. As f^{-1} is an isomorphism, it is also order-preserving.

Conversely, let f be a bijection from L_1 to L_2 such that both f and $f^{-1}$are order preserving . For a, b in L_1 we have $a \preccurlyeq a \vee b$ and $b \preccurlyeq a \vee b$, so $f(a) \preccurlyeq f(a \vee b)$ and $f(b) \preccurlyeq f(a \vee b)$, hence $f(a) \vee f(b) \preccurlyeq f(a \vee b)$.
Furthermore, if $f(a) \vee f(b) \preccurlyeq u$ then $f(a) \preccurlyeq u$ and $f(b) \preccurlyeq u$, hence $a \preccurlyeq f^{-1}(u)$ and $b \preccurlyeq f^{-1}(u)$, so $a \vee b \preccurlyeq f^{-1}(u)$, and thus $f(a \vee b) \preccurlyeq u$.
This implies that $f(a) \vee f(b) = f(a \vee b)$. Similarly, it can be argued that $f(a) \wedge f(b) = f(a \wedge b)$. □

Example.2.5.9. This is an example of bijection f between lattices which is order-preserving but is not isomorphism; consider the map f(a) = a,…, f(d) = d where L_1 and L_2 are the two lattices in Figure 14.

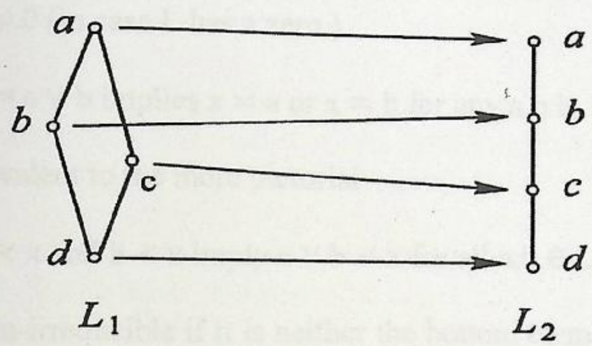


Figure 14

The possible demarcation dispute between order-isomorphism and lattice isomorphism does not arise, as 2.5.10(2) below shows.

Theorem.2.5.10.(2.19 Proposition in [14]). Let L and K be lattices and $f: L \longrightarrow K$ be a map.

(1) The following are equivalent:

(a) f is order-preserving,
(b) $(\forall\, a, b \in L)\ f(a \vee b) \succcurlyeq f(a) \vee f(b)$,
(c) $(\forall\, a, b \in L)\ f(a \wedge b) \preccurlyeq f(a) \wedge f(b)$.

In particular, if f is a homomorphism, then f is order preserving.

(2) f is a lattice isomorphism if and only if it is an order-isomorphism.

Proof. Part (1) is an easy consequence of the Connecting Lemma (2.2.10).
Consider (2). Assume that f is a lattice isomorphism. Then, by the Connecting Lemma,

$$a \preccurlyeq b \Leftrightarrow a \vee b = b \Leftrightarrow f(a \vee b) = f(b) \Leftrightarrow f(a) \vee f(b) = f(b) \Leftrightarrow f(a) \preccurlyeq f(b),$$

whence f is an order-embedding and so is an order- isomorphism.

Conversely, assume that f is an order-isomorphism. Then f is bijective (see 1.5.2). By (1) and duality, to show that f is a lattice isomorphism it suffices to show that $f(a) \vee f(b) \geqslant f(a \vee b)$ $\forall$ a,b$\in$ L. Since f is surjective, there exists c $\in$ L such that $f(a) \vee f(b) = f(c)$. Then $f(a) \leqslant f(c)$ and $f(b) \leqslant f(c)$.
Since f is an order-embedding, it follows that $a \leqslant c$ and $b \leqslant c$, whence $a \vee b \leqslant c$. Because f is order-preserving, $f(a \vee b) \leqslant f(c) = f(a) \vee f(b)$, as required. □

2.6 Important lattice- theoretic notions

In the following, let L be a lattice. We define some order-theoretic notions that are of particular importance in lattice theory.

Definition.2.6.1. An element x of L is called **join- irreducible** if

1- $x \neq 0$ (in case L has a zero)

2- $x = a \vee b$ implies $x = a$ or $x = b$ for any a,b in L.

Condition (2) is equivalent to the more pictorial

3- $a \prec x$ and $b \prec x$ imply $a \vee b \prec x$ for all a,b $\in$ L.

Equivalently, x is join-irreducible if it is neither the bottom element of the lattice (the join of zero elements) nor the join of any two smaller elements. For instance, in the lattice of divisors of 120, there is no pair of elements whose join is 4, so 4 is join-irreducible.
A **meet irreducible** element is defined dully.

We denote the set of join-irreducible elements of L by $\mathcal{J}(L)$ and the set of meet irreducible elements by $\mathcal{M}(L)$.

Example.2.6.2. In a lattice $\mathbb{P}(A)$ of the power set of a set A with inclusion relation the join-irreducible elements are exactly the singleton sets, $\{x\}$, for x $\in$ A.

Example.2.6.3. In a chain, all the elements except the bottom one are join irreducible. Dually, all the elements except the top one are a meet-irreducible. Thus if L is an n-element chain, then $\mathcal{J}(L)$ and $\mathcal{M}(L)$ are an (n-1) - element chains.

Example.2.6.4 In a finite lattice L, an element is join-irreducible if and only if it has exactly one lower cover. Figure 15 gives two examples. The join irreducible elements are shaded.

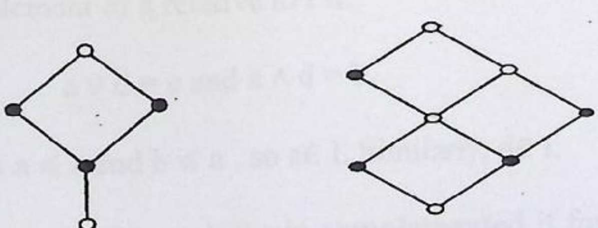


Figure 15

Definition.2.6.5. An element x of L is called **join prime** if

1- $x \neq 0$ (in case L has a zero)

2- $x \preccurlyeq a \vee b$ implies $x \preccurlyeq a$ or $x \preccurlyeq b$.

Again, this can be dualized to yield **meet prime**. Any join-prime element is also join irreducible, and any meet-prime element is also meet irreducible.

Definition.2.6.6 An element x of L is an **atom**, if L has a 0, $0 \prec x$, and there exists no element y of L such that $0 \prec y \prec x$. L is **atomic**, if for every nonzero element x of L there exists an atom a of L such that $a \preccurlyeq x$.

L is atomistic, if every element of L is a supremum of atoms, that is, for all a, b in L such that $a \not\preccurlyeq b$, there exists an atom x of L such that $x \preccurlyeq a$ and $x \not\preccurlyeq b$.

Example.2.6.7. In example (2.6.2). All singleton subsets {x} of A are atoms in L. And L with usual intersection and union as the lattice operations meet and join is atomistic: every subset B of A is the union of all the singleton subsets of B .

Example.2.6.8 In the lattice of example (2.1.11), any prime number p is an atom. This lattice is atomic. But it is not atomistic: 36 is not a join of 2 and 3 this is just a counter example.

Now we want to define two important elements in a lattice which are named complement element and relatively complement element, but before we need this definition.

Definition.2.6.9. ([a,b]). Given any two elements a, b ∈ A with $a \preccurlyeq b$, we denote by [a,b] the interval with the endpoints a and b, that is, the set of all elements x ∈ A for which $a \preccurlyeq x \preccurlyeq b$, in symbols,

$$[a,b] = E_x[x \in A \text{ and } a \preccurlyeq x \preccurlyeq b\,].$$

Definition.2.6.10. (Complement and relatively complement). Let L be a bounded lattice with greatest element 1 and least element 0. Two elements a and b of L are **complements** of each other if and only if

$$a \vee b = 1 \text{ and } a \wedge b = 0.$$

Let L be a lattice, a an element of L, and I = [b,c] an interval in L. An element d∈ L is said to be a complement of a relative to I if:

$$a \vee d = c \text{ and } a \wedge d = b.$$

It is easy to see that $a \preccurlyeq c$ and $b \preccurlyeq a$, so a∈ I. Similarly, d∈ I.

An element a∈ L is said to be **relatively complemented** if for every interval I in L with a∈ I, it has a complement relative to I.

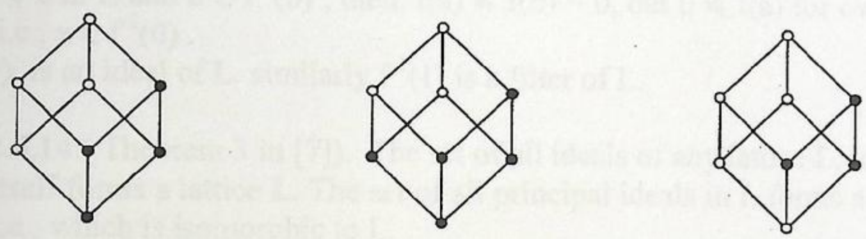
Example.2.6.11. In a lattice of subsets of a set A with $1 = A$ and $0 = \emptyset$, we define complement of any subset of A as the collection of all elements of A which are not in the subset.

Other important notions in lattice theory are **ideal** and its dual notion **filter**. Both terms describe special subsets of a lattice (or of any partially ordered set in general).

Definitions.2.6.12. (Ideals And Filters)
Let L be a lattice, A non- empty subset J of L is called an **Ideal** if

1- $a, b \in J$ implies $a \vee b \in J$.

2- $a \in L$, $b \in J$ and $a \preccurlyeq b$ imply $a \in J$ (see figure 16 for illustrations.)



shaded elements an ideal | shaded elements not an ideal | shaded elements not an ideal

Figure 16

Clearly, every ideal K of a lattice L is a sublattice, since $a \wedge b \preccurlyeq a$ for any $a, b \in L$, every lattice L is an ideal of itself, and every intersection of ideals of L is an ideal of L.

A dual ideal is called a **filter.** Specifically, a non- empty subset G of L is called **a filter** if :

1 - $a, b \in G$ implies $a \wedge b \in G$,

2 - $a \in L$, $b \in G$ and $a \succcurlyeq b$ imply $a \in G$.

The set of all ideals of L by denoted **Id(L)** and the set of all filters denoted by **F(L)**, and carries the usual inclusion order.

Given an element a of a lattice L, the set L(a) of all elements $x \preccurlyeq a$ is evidently an ideal, it is called **a principal ideal** for L.

A prime ideal of a lattice L is an ideal $J \neq \emptyset$, such that $x \wedge y \in J$ implies $x \in J$ or $y \in J$.

An **order ideal** of a lattice is a subset I of L such that $x \preccurlyeq y$, $y \in I$ implies $x \in I$. Order ideals have been called a variety of other names.

An ideal or filter is called **proper** if it does not coincide with L. It is very easy to show that an ideal J of a lattice with 1 is proper if and only if $1 \notin J$, and dually a filter G of a lattice with 0 is proper if and only if $0 \notin G$.

Example.2.6.13. Let L and K be bounded lattices and $f{:}L \longrightarrow K$ a $\{0, 1\}$-homomorphism. Then $f^{-1}(0)$ is an ideal and $f^{-1}(1)$ is a filter in L.

Proof. Let $f^{-1}(0) = \{ x \in L, f(x) = 0\}$, note that $f^{-1}(0)$ is not empty, $0 \in f^{-1}(0)$.
Assume $a,b \in f^{-1}(0)$ then $f^{-1}(a) = 0$, and $f^{-1}(b) = 0$.
Now $f(a \vee b) = f(a) \vee f(b)$
$= 0 \vee 0$
$= 0.$
So, $a \vee b \in f^{-1}(0)$.

Now, let $a \preccurlyeq b$ in L and $b \in f^{-1}(0)$, then, $f(a) \preccurlyeq f(b) = 0$, but $0 \preccurlyeq f(a)$ for every a in L so $f(a) = 0$ i.e., $a \in f^{-1}(0)$.
Hence $f^{-1}(0)$ is an ideal of L. similarly $f^{-1}(1)$ is a filter of L.

Theorem.2.6.14.(Theorem 3 in [7]). The set of all ideals of any lattice L, ordered by inclusion, itself forms a lattice $\mathbb{L}$. The set of all principal ideals in L forms a sublattice of this lattice , which is isomorphic to L.

Proof. Given any two ideals J and K of L, they have a common element since if $a \in J$ and $b \in K$, then $a \wedge b \in J \wedge K$. Thus we can take $J \wedge K$ as the set –intersection of J and K ; this is clearly an ideal.

Again, any ideal which contains J and K must contain the set M of all elements x such that $x \preccurlyeq a \vee b$ for some $a \in J$, $b \in K$.
But the set M is an ideal: if $x \in M$ and $y \preccurlyeq x \preccurlyeq a \vee b$, then $y \preccurlyeq a \vee b$ by P_3; and if $\{x, y\} \subset M$, then since $x \preccurlyeq a \vee b$ and $y \preccurlyeq a_1 \vee b_1$ for some $a, a_1 \in J$ and $b, b_1 \in K$,

$$x \vee y \preccurlyeq (a \vee b) \vee (a_1 \vee b_1) = (a \vee a_1) \vee (b \vee b_1),$$

where $a \vee a_1 \in J$ and $b \vee b_1 \in K$ since J and K are ideals. Hence M= sup {J, K}is the set of all ideals of L.

If J and K are principal ideal of L with generators a and b, then $J \vee K$ and $J \wedge K$ are principal ideals generated by $a \vee b$ and $a \wedge b$, respectively. The principal ideals thus form a sublattice of $\mathbb{L}$ which is isomorphic with L. □

Chapter Three

Types Of Lattices

It is usually possible to enlarge the set of postulates of an algebraic system, and thereby obtain a special system with properties not characteristic of the general system. In this chapter we will consider some special kinds of lattices namely complete, modular, distributive, complemented, and Boolean lattices.

3.1 Complete lattices.

Definition.3.1.1. A partially ordered set P is **complete** if for every subset A of P both sup A and inf A exist (in P).

The elements sup A and inf A will be denoted by $\bigvee A$ and $\bigwedge A$, respectively. All complete partially ordered sets are lattices, and a lattice L which is complete as a partially ordered set is **a complete lattice**

In particular, a complete lattice L has a least element 0 (such that $0 \preccurlyeq x$ for all x in L),which is the g.l.b. of L (and the l.u.b. of the empty subset of L), and it is has a greatest element 1 (such that $x \preccurlyeq 1$ for all x in L), which is the l.u.b. of L (and the g.l.b. of the empty subset of L).

Theorem.3.1.2. (Theorem 4.2 in [10]). Let P be a partially ordered set such that $\bigwedge A$ exists for every subset A, or such that $\bigvee A$ exists for every subset A. Then P is a complete lattice.

Proof. Suppose $\bigwedge A$ exists for every $A \subseteq P$. Then letting A^u be the set of upper bounds of A in P, it is clear that $\bigwedge A^u$ is indeed $\bigvee A$. The other half of the theorem is proved similarly. □

Any finite lattice is complete, and so is any lattice whose chains are finite. So is any cardinal product of complete lattices, and so are the ordinal sum and product of any two complete lattices

The opposite of a complete lattice is a complete lattice. Hence there is a duality principle for complete lattices: a theorem that holds in every complete lattice remains true when the order relation is reversed.

Definition.3.1.3. A sublattice M of a complete lattice L is called **a complete sublattice** of L if for every subset A of M the elements $\bigvee A$ and $\bigwedge A$, as defined in L, are actually in M.

Example.3.1.4. The power set of a given set, ordered by inclusion is a complete lattice, for, in this case, the supremum of any subset of the lattice is given by the set-theoretic union of the elements of the subset and so is an element of the lattice, while the infimum of any subset of the lattice is given by the set- theoretic intersection of the elements of the subset and so is an element of the lattice.

Example.3.1.5. The non-negative integers, ordered by divisibility is a complete lattice. The least element in this lattice is the number 1, since it divides any other number. The greatest element is 0, because it can be divided by any other number. The supremum of finite sets is given by the least common multiple and the infimum by the greatest common divisor. For infinite sets, the supremum will always be 0 while the infimum can well be greater than 1. For example, the set of all even numbers has 2 as the greatest common divisor.

Example.3.1.6. The subgroups of any given group under inclusion is a complete lattice. (While the infimum here is the usual set-theoretic intersection, the supremum of a set of subgroups is the subgroup generated by the set-theoretic union of the subgroups, not the set-theoretic union itself). If e is the identity of G, then the trivial group {e} is the minimum subgroup of G, while the maximum subgroup is the group G itself.

Example.3.1.7. The submodules of a module, ordered by inclusion is a complete lattice. The supremum is given by the sum of submodules and the infimum by the intersection.

Example.3.1.8. Every finite lattice is complete. On the other hand not all lattices are complete, for example the lattices $\mathbb{N},\mathbb{Q},\mathbb{R}$ with the usual relation ($\leq$) are not complete.

A complete lattice may, of course, have sublattices which are incomplete (for example, consider the reals as a sublattice of the extended reals). It is also possible for a sublattice of a complete lattice to be complete, but the sups and infs of the sublattice not to agree with those of the original lattice (for example look at the sublattice of the extended reals consisting of those numbers whose absolute value is less than one together with the numbers −2; +2).

Theorem.3.1.9. (Theorem 5.2 in [10]). Let C be a closure operator (see def^n 1.5.11) on a set A: Then L_C is a complete lattice with

$$\bigwedge_{i\in I} C(A_i) = \bigcap_{i\in I} C(A_i)$$

and

$$\bigvee_{i\in I} C(A_i) = C\Big(\bigcup_{i\in I} A_i\Big).$$

Proof. Let $(A_i)_{i\in I}$ be an indexed family of closed subsets of A. From

$$\bigcap_{i\in I} A_i \subseteq A_i,$$

for each i; we have

$$C\Big(\bigcap_{i\in I} A_i\Big) \subseteq C(A_i) = A_i,$$

so

$$C\Big(\bigcap_{i\in I} A_i\Big) \subseteq \bigcap_{i\in I} A_i\,,$$

hence

$$C\Big(\bigcap_{i\in I} A_i\Big) = \bigcap_{i\in I} A_i;$$

so $\bigcap_{i\in I} A_i$ is in L_C. Then, if one notes that A itself is in L_C; it follows that L_C is a complete lattice. The verification of the formulas for the V's and Λ's of families of closed sets is straightforward. □

Interestingly enough, the converse of this theorem is also true, which shows that the lattices L_C arising from closure operators provide typical examples of complete lattices.

Remarks.3.1.10.

1- Homomorphisms of complete lattices: The traditional homomorphisms between complete lattices are the complete homomorphisms (or complete lattice homomorphisms). These are characterized as functions that preserve all joins and all meets. Explicitly, this means that a function $f: L \to M$ between two complete lattices L and M is a complete homomorphism if

$$f(\bigwedge A) = \bigwedge \{ f(a) : a \in A\} \text{ and}$$

$$f(\bigvee A) = \bigvee \{ f(a) : a \in A\} ,$$

for all subsets A of L.

Such functions are automatically monotonic, but the condition of being a complete homomorphism is in fact much more specific. For this reason, it can be useful to consider weaker notions of homomorphisms, that are only required to preserve all meets or all joins, which are indeed in equivalent conditions. This notion may be considered as a homomorphism of complete meet-semilattices or complete join semilattices, respectively.

2-Representation: There are various mathematical concepts that can be used to represent complete lattices. One means of doing so is the Dedekind-MacNeille completion. When this completion is applied to a partially ordered set that already is a complete lattice, then the result is a complete lattice of sets which is isomorphic to the original one. Thus we immediately find that every complete lattice is isomorphic to a complete lattice of sets.

Another representation is obtained by noting that the image of any closure operator on a complete lattice is again a complete lattice (called its closure system). Since the identity function is a closure operator too, this shows that the complete lattices are exactly the images of closure operators on complete lattices.

Besides the previous representation results, there are some other statements that can be made about complete lattices, or that take a particularly simple form in this case. An example is the **Knaster-Tarski theorem**, named after Bronisław Knaster and Alfred Tarski, which we will state and prove after give this definition.

Definition.3.1.11. The **least fixed point** of f is the least element x such that $f(x) = x$, or, equivalently, such that $f(x) \preccurlyeq x$, the dual holds for the **greatest fixed point**, the greatest element x such that $f(x) = x$.

Since complete lattices cannot be empty, the theorem in particular guarantees the existence of at least one fixed point of f, and even the existence of a least (or greatest) fixed point. In many practical cases, this is the most important implication of the theorem. Let's state the theorem:

Theorem.3.1.12. (The Knaster-Tarski Fixed point Theorem in [14]).
Let L be a complete lattice and $f : L \longrightarrow L$ an order preserving map. Then

$$\alpha := \bigvee \{ x \in L : x \preccurlyeq f(x)\}$$

is a fixed point of f. Further, α is the greatest fixed point of f . Dually, f has a least fixed point, given by $\bigwedge\{ x \in L : f(x) \preccurlyeq x\}$

Proof. Let $H = \{ x \in L : x \preccurlyeq f(x)\}$. For all $x \in H$ we have $x \preccurlyeq \alpha$, so $x \preccurlyeq f(x) \preccurlyeq f(\alpha)$. Thus $f(\alpha) \in H^u$, whence $\alpha \preccurlyeq f(\alpha)$. We now use this inequality to prove the reverse one and thereby complete the proof that α is a fixed point.
Since f is order preserving, $f(\alpha) \preccurlyeq f(f(\alpha))$. This says that $f(\alpha) \in H$, so $f(\alpha) \preccurlyeq \alpha$. If β is any fixed point of f then $\beta \in H$, so $\beta \preccurlyeq \alpha$. □

The Knaster-Tarski theorem can be restate to say that : **The set of fixed points of f is itself a complete lattice.**

Theorem.3.1.13. (Lattice theoretical Fixedpoint Theorem).[31]

Let (1) $L = (A, \preccurlyeq)$ be A complete lattice,

(2) f be an order- preserving map (increasing function) on A to A,

(3) P be the set of all fixedpoints of f.

Then the set P is not empty and the system $(P, \preccurlyeq)$ is a complete lattice; in particular we have

$$\bigvee P = \bigvee E_x [f(x) \succcurlyeq x] \in P$$

And

$$\bigwedge P = \bigwedge E_x [f(x) \preccurlyeq x] \in P.$$

Proof. Let

(1) $u = \bigvee E_x [f(x) \succcurlyeq x] \in P$

We clearly have $x \preccurlyeq u$ for every element x with $f(x) \succcurlyeq x$; hence, the function f being order- preserving (or increasing),

$$f (x) \preccurlyeq f(u) \text{ and } x \preccurlyeq f(u) .$$

By (1) we conclude that

(2) $u \preccurlyeq f(u)$.

Therefore

$$f(u) \preccurlyeq f(f(u)),$$

so that f(u) belongs to the set $E_x[f(x) \succcurlyeq x]$; consequently, by (1),

(3) $f(u) \preccurlyeq u.$

Formulas (2) and (3) imply that u is a fixedpoint of f; hence we conclude by (1) that u is the join of all fixedpoints of f, so that

(4) $\bigvee P = \bigvee E_x[f(x) \succcurlyeq x] \in P$

Consider the dual lattice $L^d = (A, \succcurlyeq)$, L^d like L, is complete, and f is again an increasing function in L^d. The join of any elements in L^d obviously coincides with the meet of these elements in L. Hence, by applying to L^d the result established for L in (4), we conclude that

(5) $\bigwedge P = \bigwedge E_x[f(x) \preccurlyeq x] \in P.$

Now let Y be any subset of P. The system

$$\mathcal{B} = ([\bigvee Y, 1], \preccurlyeq)$$

is a complete lattice. For any $x \in Y$ we have $x \preccurlyeq \bigvee Y$ and hence

$$x = f(x) \preccurlyeq f(\bigvee Y);$$

therefore $\bigvee Y \preccurlyeq f(\bigvee Y)$. Consequently, $\bigvee Y \preccurlyeq z$ implies

$$\bigvee Y \preccurlyeq f(\bigvee Y) \prec f(z).$$

Thus, by restricting the domain of f to the interval $[\bigvee Y, 1]$, we obtain an increasing function f ' on $[\bigvee Y, 1]$ to $[\bigvee Y, 1]$ By applying formula (5) established above to the lattice $\mathcal{B}$ and to the function f ', we conclude that the greatest lower bound v of all fixedpoints of f ' is itself a fixedpoint of f '. Obviously, v is a fixedpoint of f, and in fact the least fixedpoint of f which is an upper bound of all elements of Y; in other words, v is the least upper bound of Y in the system $(P, \preccurlyeq)$.

Hence, by passing to the dual lattices L^d and $\mathcal{B}^d$, we see that there exists also a greatest lower bound of Y in $(P, \preccurlyeq)$. Since Y is an arbitrary subset of P, we finally conclude that

(6) the system $(P, \preccurlyeq)$ is a complete lattice .

In view of (4) - (6), the proof has been completed. □

The next lemma is useful for showing that certain subsets of a complete lattices are themselves complete lattices.

Now another important kind of lattices will be introduced.

3.2 Modular Lattices.

Definition.3.2.1. A **modular lattice** is any lattice which satisfies the modular law

$$\mathbf{L_5}: \; x \preccurlyeq z \text{ implies } x \vee (y \wedge z) \approx (x \vee y) \wedge z.$$

The modular law is obviously equivalent (for lattices) to the identity

$$(x \wedge y) \vee (x \wedge z) \approx x \wedge (y \vee (x \wedge z)) \text{ for all } x, y, z \in L .$$

since $a \preccurlyeq b$ holds if and only if $a = a \wedge b$. Also it is not difficult to see that every lattice satisfies

$$x \preccurlyeq z \text{ implies } x \vee (y \wedge z) \preccurlyeq (x \vee y) \wedge z,$$

so to verify the modular law it suffices to check the implication

$$x \preccurlyeq z \text{ implies } (x \vee y) \wedge z \preccurlyeq x \vee (y \wedge z).$$

for all $x, y, z \in L$.

The opposite of a modular lattice is a modular lattice. Hence there is a duality principle for modular lattices: a theorem that holds in every modular lattice remains true when the order relation is reversed.

Example.3.2.2. The lattice of submodules of any module is modular.

Proof. We saw that the submodules of a module constitute a lattice, in which $A \wedge B = A \cap B$ and $A \vee B = A + B$. Let A, B, C be submodules such that $A \subseteq C$. Then $A + (B \cap C) \subseteq (A + B) \cap C$ (this holds in every lattice).
Conversely, if $x \in (A + B) \cap C$, then $x = a + b \in C$ for some $a \in A \subseteq C$ and $b \in B$, whence $b = x - a \in C$ and $x = a + b \in A + (B \cap C)$. □

Example.3.2.3. The lattice of normal subgroups of any group G is modular

Proof. We saw that the normal subgroups of a group G constitute a lattice, in which $A \wedge B = A \cap B$ and $A \vee B = AB$,with $\subseteq$ order.
Let A,B,C be normal subgroups of G, with $C \subseteq A$. Take $x \in A \wedge (B \vee C)$, so $x \in A$ and $x = k\,n$, for some $k \in B$, and $n \in C$. Then $k = xn^{-1} \in A$ since $C \subseteq A$ and A is a subgroup. This proves that $x \in (A \wedge B) \vee C$. Hence $A \wedge (B \vee C) \subseteq (A \wedge B) \vee C$. □

Example.3.2.4. Every chain is a modular lattice.

Example.3.2.5. The following lattice which called M_3 (shown in Fig 17) is a modular lattice , this lattice is normal subgroup of V_4 Hence by the above example it is modular.

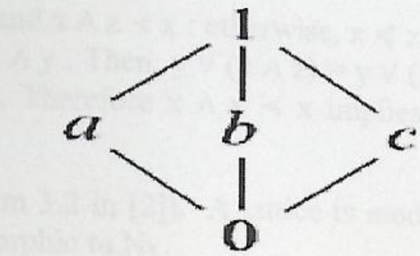


Figure 17: M_3

The smallest non-modular lattice is the "pentagon" lattice N_5 consisting of five elements 0,1,a,b,c such that $0 \prec b \prec c \prec 1$, $0 \prec a \prec 1$, and a is not comparable to c or to b. For this lattice $b \vee (a \wedge c) = b \vee 0 = b$, and $(b \vee a) \wedge c = 1 \wedge c = c$.
Every non-modular lattice contains a copy of N_5 as a sublattice

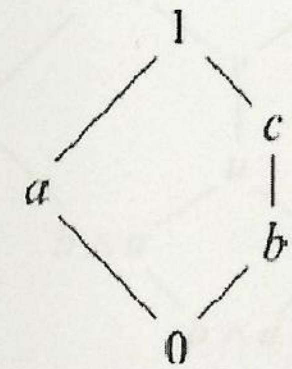


Figure 18: N_5

Remark.3.2.6.
New lattices can be manufactured by forming sublattices , products and homomorphic images. Modularity preserved by these constructions , as follows.

1. If L is a modular lattice , then every sublattice of L is modular .
2. If L and K are modular lattices ,then $L \times K$ is modular .
3. If L is modular and K is the image of L under a homomorphism , then K is modular.

Here (1) is immediate and (2) holds because $\vee$ and $\wedge$ are defined coordinatewise on products. For (3) we use the fact that a join- and a meet- preserving map preserves any lattice identity, then invoke (1) $\Leftrightarrow$ (3) in lemma(2.2.15). □

Theorem.3.2.7. In a modular lattice, $x \wedge y \prec x$ if and only if $y \prec x \vee y$.

Proof. Suppose that $x \wedge y \prec x$ but $y \nprec x \vee y$. Then $x \wedge y \prec x$, $x \npreceq y$, $y \prec x \vee y$, and $y \prec z \prec x \vee y$ for some z . Then $x \vee y \preccurlyeq x \vee z \preccurlyeq x \vee y$ and $x \vee z = x \vee y$.

Also $x \wedge y \preccurlyeq x \wedge z \preccurlyeq x$, and $x \wedge z \prec x$: otherwise, $x \preccurlyeq z$ and $x \vee z = z \prec x \vee y$ .Hence $x \wedge z \prec x$, and $x \wedge z = x \wedge y$. Then $y \vee (x \wedge z) = y \vee (x \wedge y) = y \prec z = (y \vee x) \wedge z$, contradicting modularity. Therefore $x \wedge y \prec x$ implies $y \prec x \vee y$. The converse implication is dual. □

Theorem.3.2.8. (Theorem 3.2 in [2]). A lattice is modular if and only if it contains no sublattice that is isomorphic to N_5 .

Proof. A sublattice of a modular lattice is modular and not isomorphic to N_5. Conversely, a lattice L that is not modular contains elements a, b, c such that $b \preccurlyeq c$ and $u = b \vee (a \wedge c) \prec (b \vee a) \wedge c = v$. Then $v \preccurlyeq b \vee a$, $a \preccurlyeq u \prec v \preccurlyeq c$,and $b \vee a \preccurlyeq b \vee u \preccurlyeq b \vee v \preccurlyeq b \vee a$, so that $b \vee u = b \vee v = b \vee a$. Similarly, $b \wedge c \preccurlyeq u$, $v \preccurlyeq b \wedge c$, and $b \wedge c \preccurlyeq b \wedge u \preccurlyeq b \wedge v \preccurlyeq b \wedge c$, so that $b \wedge u = b \wedge v = b \wedge c$. Thus b, u , v , $b \wedge u = b \wedge v$, and $b \vee u = b \vee v$ constitute a sublattice of L . We show that these five elements are distinct, so that our sublattice is isomorphic to N_5 :

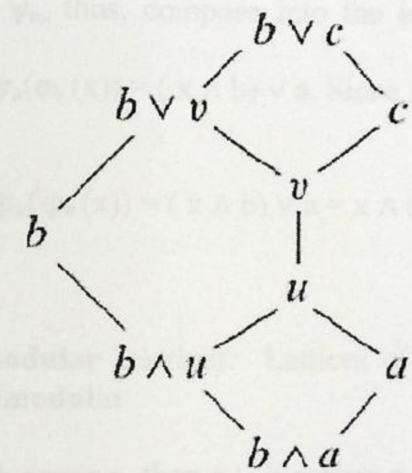


Figure 19

Already $u \prec v$. Moreover, $b \not\preccurlyeq c$: otherwise, $b = b \wedge c \preccurlyeq u \prec v$ and $u = b \vee u = b \vee v = v$; and $a \not\preccurlyeq b$: otherwise, $b = b \vee a \succcurlyeq v \succ u$ and $u = b \wedge u = b \wedge v = v$.
Hence $v \prec b \vee v$ (otherwise, $b \preccurlyeq v \preccurlyeq c$), $b \wedge u \prec u$ (otherwise, $a \preccurlyeq u \preccurlyeq b$), $b \wedge u \prec b$, (otherwise, $b = b \wedge u = b \wedge c \preccurlyeq c$), $b \prec b \vee v$ (otherwise, $b = b \vee v = b \vee a \preccurlyeq a$). □

Theorem.3.2.9. (The Isomorphism Theorem). (Theorem 2 in [19]).
Let L be a modular lattice and let a, b ∈ L. Then

$$\varphi_b : x \mapsto x \wedge b, \quad x \in [a, a \vee b],$$

is an isomorphism of $[a, a \vee b]$ and $[a \wedge b, b]$. The inverse isomorphism is

$$\psi_a : y \mapsto y \vee a, \quad y \in [a \wedge b, b].$$

(See Figure 20.)

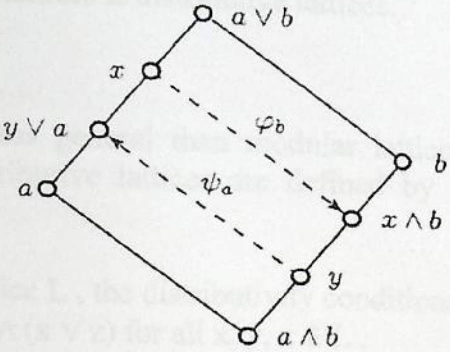


Figure 20

Proof. It is sufficient to show that $\psi_a(\varphi_b(x)) = x$, for $x \in [a, a \vee b]$. Indeed if this is true then by duality $\varphi_b(\psi_a(y)) = y$, for all $y \in [a \wedge b, b]$.
The isotone maps φ_b and ψ_a, thus, compose into the identity maps, then they are isomorphism, as claimed.
So let $x \in [a, a \vee b]$. Thus $\psi_a(\varphi_b(x)) = (x \wedge b) \vee a$. Since $x \in [a, a \vee b]$, we have $a \preccurlyeq b$, and so modularity applies:

$$\psi_a(\varphi_b(x)) = (x \wedge b) \vee a = x \wedge (b \vee a) = x,$$

because $x \preccurlyeq a \vee b$ □

Definition.3.2.10. (Semimodular Lattice). Lattices of finite length which satisfy **(16)** or **(17)** are called **semimodular**

If $a \neq b$ and both a and b cover c, then $a \vee b$ covers a as well as b. (16)
If $a \neq b$ and c covers both a and b, then a and b both cover $a \wedge b$. (17)

A lattice which satisfy **(16)** is called upper semimodular, and one which satisfy **(17)** is called lower semimodular.

Example.3.2.11. The smallest lattice which is semimodular but not modular is show in (Figure 21) since $d \preccurlyeq a$ but $a \wedge (c \vee d) \neq (a \wedge c) \vee d$.

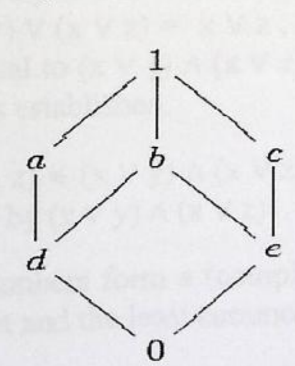


Figure 21

Another important type of lattices is distributive lattices.

3.3 Distributive Lattices.

Distributive lattices are less general than modular lattices but still include some important examples. Distributive lattices are defined by the following equivalent properties.

Proposition.3.3.1 In a lattice L , the distributivity conditions
(1) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in L$,
(2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$,
are equivalent, and imply modularity.

Proof. Assume (1). Then $x \leqslant z$ implies
$$\begin{aligned} x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \\ &= (x \vee y) \wedge z. \end{aligned}$$
Hence L is modular.

Then $x \wedge z \leqslant x$ yields
$$\begin{aligned} (x \wedge z) \vee (y \wedge x) &= x \wedge (z \vee (y \wedge x)) \\ &= x \wedge (z \vee y) \wedge (z \vee x) \\ &= x \wedge (z \vee y). \end{aligned}$$
and (2) holds.

Dually, (2) implies (1) . □

Definition.3.3.2. A lattice is **distributive** when it satisfies the equivalent conditions in proposition 3.3.1

The dual of a distributive lattice is a distributive lattice. Hence there is duality principle for distributive lattices: a theorem that holds in every distributive lattice remains true when the order relation is reversed.

Theorem.3.3.3. Every distributive lattice is modular.
Proof. Just use (1) in proposition 3.3.1, noting that $a \vee b = b$ whenever $a \leqslant b$. □

Example.3.3.4. Every totally ordered set is a distributive lattice with max as join and min as meet.

Proof. We will show that $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. We may suppose $y \leqslant z$ (If not, $z \leqslant y$ and we may switch y and z). Recall that $y \leqslant z$ is equivalent to $y \vee z = z$. Hence $y \leqslant z$ implies $(x \vee y) \vee (x \vee z) = x \vee z$, i.e., $x \vee y \leqslant x \vee z$, so the right hand side of the equation is equal to $(x \vee y) \wedge (x \vee z) = x \vee y$. On the left hand side we have $y \wedge z = y$, so equality is established.

Note that the relation $x \vee (y \wedge z) \leqslant (x \vee y) \wedge (x \vee z)$ is true in all lattices, as both x and $(y \wedge z)$ are bounded above by $(x \vee y) \wedge (x \vee z)$.

Example.3.3.5. The natural numbers form a (complete) distributive lattice with the greatest common divisor as meet and the least common multiple as join.

Example.3.3.6. Given a positive integer n, the set of all positive divisors of n forms a distributive lattice, again with the greatest common divisor as meet and the least common multiple as join.

Example.3.3.7. The lattice $\mathbb{P}(X)$ of all subsets of a set is distributive. So is every sublattice of $\mathbb{P}(X)$.

However, while many lattices are distributive , there are also many important ones which are not .For example the lattice M_3 (shown in Fig 17 page 40) is not distributive since $a \vee (b \wedge c) = a$ but $(a \vee b) \wedge (a \vee c) = 1$. In fact, M_3 and N_5 are the quintessential nondistributive lattices.

Remarks.3.3.8.
1- New lattices can be manufactured by forming sublattices , products and homomorphic images. Distributivity preserved by these constructions , as follows.

1. If L is a distributive lattice , then every sublattice of L is distributive.
2. If L and K are distributive lattices ,then $L \times K$ is distributive .
3. If L is distributive and K is the image of L under a homomorphism , then K is distributive.

Here (1) is immediate and (2) holds because $\vee$ and $\wedge$ are defined coordinatewise on products. For (3) we use the fact that a join- and a meet- preserving map preserves any lattice identity. □

2-Characteristic Properties: Various equivalent formulations to the above definition exist. For example, L is distributive if and only if the following holds for all elements x, y, z in L:

$$(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x).$$

Similarly, L is distributive if and only if

$$x \wedge z = y \wedge z \text{ and } x \vee z = y \vee z \text{ always imply } x = y.$$

Another popular characterization is obtained from two well-known prototypical non-distributive lattices: M_3, the "**diamond**", and N_5, the "**pentagon**", shown with their Hasse diagrams in Fig 17and Fig 18. A lattice is distributive if and only if none of its sublattices is isomorphic to M_3 or N_5 where a sublattice is a subset that is closed under the meet and join operations of the original lattice. This will be prove in the following theorem.

Theorem.3.3.19. (Birkhoff [1934]). (Theorem 4.2 in [2]). A lattice is distributive if and only if it contains no sublattice that is isomorphic to M_3 or to N_5.

Proof. A sublattice of a distributive lattice is distributive and is not, therefore, isomorphic to M_3 or N_5 .

Conversely, assume that the lattice L is not distributive. We may assume that L is modular: otherwise, L contains a sublattice that is isomorphic to N_5 , by 3.2.8, and the theorem is proved. Since L is not distributive, $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$ for some $a, b, c \in L$. Let
$u = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$ and $v = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$.
Then $u \preccurlyeq v$, since $a \wedge b \preccurlyeq a \vee b$, etc.
Let,

$$x = u \vee (a \wedge v) = (u \vee a) \wedge v,$$

$$y = u \vee (b \wedge v) = (u \vee b) \wedge v,$$
$$z = u \vee (c \wedge v) = (u \vee c) \wedge v.$$

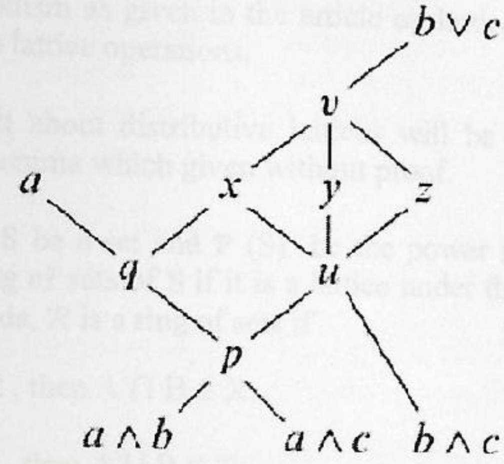


Figure 22

Since L is modular and $a \wedge v = a \wedge (b \vee c)$, $b \vee u = b \vee (c \wedge a)$,

$$\begin{aligned} x \wedge y &= (u \vee (a \wedge v)) \wedge (u \vee (b \wedge v)) \\ &= u \vee (a \wedge v) \wedge (u \vee b) \wedge v)) \\ &= u \vee ((a \wedge v) \wedge (u \vee b)) \\ &= u \vee (a \wedge (b \vee c)) \wedge (b \vee (c \wedge a)) \\ &= u \vee (((a \wedge (b \vee c)) \wedge b) \vee (c \wedge a))) \\ &= u \vee (a \wedge b) \vee (c \wedge a) = u. \end{aligned}$$

Permuting a, b, and c yields $y \wedge z = z \wedge x = u$. Dually, $x \vee y = y \vee z = z \vee x = v$. Thus $\{u, v, x, y, z\}$ is a sublattice of L . We show that u, v, x, y, z are distinct, so that $\{u, v, x, y, z\} \cong M_3$:
Since $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$, but $a \wedge b \preccurlyeq a \wedge (b \vee c)$ and $a \wedge c \preccurlyeq a \wedge (b \vee c)$, we have
$p = (a \wedge b) \vee (a \wedge c) \prec a \wedge (b \vee c) = q$.

Now, $a \wedge v = a \wedge (b \vee c) = q$ and modularity yields

$$\begin{aligned} u \wedge a &= (((a \wedge b) \vee (a \wedge c)) \vee (b \wedge c)) \wedge a \\ &= ((a \wedge b) \vee (a \wedge c)) \vee ((b \wedge c) \wedge a) = (a \wedge b) \vee (a \wedge c) = p. \end{aligned}$$

Therefore $u \prec v$. Hence x, y, z are distinct (if, say, $x = y$, then $u = x \wedge y = x \vee y = v$) and distinct from u and v (if, say, $x = u$, then $y = x \vee y = v$ and $z = x \vee z = v = y$). □

An example of distributive lattice is order ideals which defined in chapter two.

Proposition.3.3.10. The order ideals of a partially ordered set S , partially ordered by inclusion, constitute a distributive lattice Id (S).

Proof. First, S is an order ideal of itself, and every intersection of order ideals of S is an order ideal of S. Id (S) is a complete lattice, in which infimums are intersections.

Moreover, every union of order ideals of S is an order ideal of S , so that supremums in Id (S) are unions. Hence Id (S) is a sublattice of $\mathbb{P}$ (S) and is distributive. □

Definition.3.3.11. (Homomorphism). A **homomorphism** of distributive lattices is just a lattice homomorphism as given in the article on lattices, i.e. a function that is compatible with the two lattice operations.

A very important result about distributive lattices will be given after this needed definitions and needed lemma which given without proof.

Definition.3.3.12. Let S be a set and $\mathbb{P}$ (S) be the power set of S . A subset $\mathcal{R}$ of $\mathbb{P}$ (S) is said to be **a ring of sets** of S if it is a lattice under the intersection and union operations. In other words, $\mathcal{R}$ is a ring of sets if

- for any A , B$\in \mathcal{R}$, then $A \cap B \in \mathcal{R}$,
- for any A, B$\in \mathcal{R}$, then $A \cup B \in \mathcal{R}$.

Example.3.3.13. The subset {{a}, {a,b}} of the $\mathbb{P}$ ({a,b}) is a ring of sets since $\{a\} \cup \{a,b\} = \{a,b\}$ and $\{a\} \cap \{a,b\} = \{a\}$.

Lemma.3.3.14. Let L be a distributive lattice and a,b $\in$L with $a \neq b$. Then there is a prime ideal containing one or the other.

Theorem.3.3.15. (Birkhoff [1933]). (Theorem 5.7 in [8]). Every distributive lattice is isomorphic to ring of sets.

Proof. For every $x \in L$ let $\mathcal{F}(x)$ denote the set of prime filters of L that contain x. Then we have

$$F \in \mathcal{F}(x \wedge y) \Leftrightarrow x \wedge y \in F \Leftrightarrow x, y \in F \Leftrightarrow F \in \mathcal{F}(x) \cap \mathcal{F}(y);$$

$$F \in \mathcal{F}(x \vee y) \Leftrightarrow x \vee y \in F \Leftrightarrow x \in F \text{ or } y \in F \Leftrightarrow F \in \mathcal{F}(x) \cup \mathcal{F}(y).$$

Thus $\mathcal{F}(x \wedge y) = \mathcal{F}(x) \cap \mathcal{F}(y)$ and $\mathcal{F}(x \vee y) = \mathcal{F}(x) \cup \mathcal{F}(y)$. Consequently $\mathcal{F} = \{ \mathcal{F}(x) \; ; x \in L\}$ is a ring of sets and the mapping described by $x \longmapsto \mathcal{F}(x)$ is a lattice homomorphism of L onto $\mathcal{F}$. By the dual of lemma 3.3.14 this mapping is injective , so we conclude that L ≅ F. □

Now complemented lattices will be defined.

3.4 Complemented lattices

Definition.3.4.1. A lattice L is called **complemented** if all its elements have complements. And it is called **relatively complement** if every element of L is relatively complemented. Equivalently, L is relatively complemented if and only if each of its interval is a complemented lattice. **A complemented modular lattice** is a complemented lattice L that is modular.

Remark.3.4.2 A relatively complemented lattice is complemented if it is bounded. Conversely, a complemented lattice is relatively complemented if it is modular.

In general an element may have more than one complement. However, in a bounded distributive lattice every element will have at most one complement.

Definition.3.4.3. A complemented lattice such that every element has a unique complement is said to be **uniquely complemented.**

If a is an element of a uniquely complemented lattice, a* denotes its (unique) complement one can think of a as a unary operator on the lattice. One of the first consequences is $a^{**} = a$. To see this, we have that $a \vee a^*=1$, $a \wedge a^*= 0$, as well as $a^{**} \vee a^* = 1$, $a^{**} \wedge a^*= 0$. So $a = a^{**}$, since they are both complements of a*.

Below are some additional (and non-trivial) properties of a uniquely complemented lattice:

- There exists a uniquely complemented lattice that is not distributive.
- A uniquely complemented lattice L is distributive if at least one of the following is satisfied:

1. *, as an operator on L , is order reversing;
2. $(a \vee b)^* = a^* \wedge b^*$;
3. $(a \wedge b)^* = a^* \vee b^*$;
4. (von Neumann) L is a modular lattice;
5. (Birkhoff-Ward) L is an atomic lattice.

- (Dilworth) every lattice can be embedded in a uniquely complemented lattice.

The above statements proof can be found in [8].

Example.3.4.4. The lattice of subsets of a set A is **uniquely complemented.** For we identify the whole set A with 1 and the empty set with 0, and then define the complement of any subset of A as the collection of all elements of A which are not in the subset.

Example.3.4.5. Let M be the lattice of all subspaces of Euclidean n-space E_n . Then M is **modular** (by example 3.2.3), and **complemented (so it is complemented modular lattice**) since the orthogonal complement $S^{\perp}$ of any subspace S satisfies $S \cap S^{\perp} = 0$, and $S + S^{\perp} = E_n$.

Definition.3.4.6. let L be a complemented lattice and denote M the set of complements of elements of L . M is clearly a subposet of L , with $\leqslant$ inherited from L. For each $a \in L$, let $M_a \subseteq M$ be the set of complements of a .
L is said to be **orthocomplemented** if there is a function $\perp: L \longrightarrow M$,called an orthocomplementation, whose image is written $a^{\perp}$ for any $a \in L$, such that

1. $a^{\perp} \in M_a$,
2. $(a^{\perp})^{\perp} = a$, and
3. $\perp$ is order-reversing; that is, for any $a,b \in L$, $a \leqslant b$ implies $b^{\perp} \leqslant a^{\perp}$.

The element $a^{\perp}$ is called an orthocomplement of a .

Example.3.4.7. Look at the Hasse diagrams of the two finite complemented lattices below, the one on the right is orthocomplemented (there exist three orthocomplementation), while the one on the left is not.

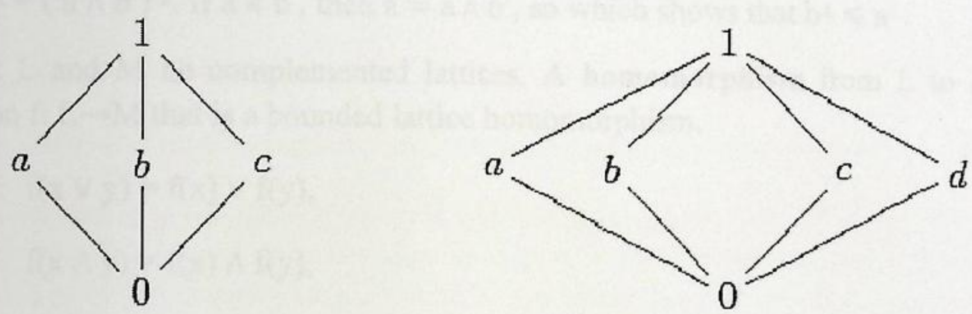


Figure 23

Remarks.3.4.8

1. From the first condition above, we see that an orthocomplementation $\perp$ is a bijection. It is one-to-one: if $a^{\perp} = b^{\perp}$, then $a = (a^{\perp})^{\perp} = (b^{\perp})^{\perp} = b$. And it is onto: if we pick $a \in M \subseteq L$, then $(a^{\perp})^{\perp} = a$. As a result, M=L, every element of L is an orthocomplement. Furthermore, we have $0^{\perp}=1$ and $1^{\perp} = 0$.

2. Let L^d be the dual lattice of L (a lattice having the same underlying set, but with meet and join operations switched). Then any orthocomplementation $\perp$ can be viewed as a lattice isomorphism between L and L^d .

3. From the above conditions, it follows that elements of L satisfy the de Morgan's laws: for a, $b \in L$, we have

$$a^{\perp} \vee b^{\perp} = (a \wedge b)^{\perp}$$

$$a^{\perp} \wedge b^{\perp} = (a \vee b)^{\perp}$$

To derive the first equation, first note $a \leqslant a \vee b$. Then

$(a \vee b)^{\perp} \leqslant a^{\perp}$. Similarly,

$(a \vee b)^{\perp} \leqslant b^{\perp}$. So $(a \vee b)^{\perp} \leqslant a^{\perp} \wedge b^{\perp}$.

For the other inequality, we start with $a^{\perp} \wedge b^{\perp} \leqslant a^{\perp}$. Then

$$a \preccurlyeq (a^{\perp} \wedge b^{\perp})^{\perp}.$$

Similarly,

$$b \preccurlyeq (a^{\perp} \wedge b^{\perp})^{\perp}.$$

Therefore,

$$a \vee b \preccurlyeq (a^{\perp} \wedge b^{\perp})^{\perp},$$

which implies that

$$a^{\perp} \wedge b^{\perp} \preccurlyeq (a \vee b)^{\perp}.$$

4. Conversely, any of two equations in the previous remark can replace the third condition in the definition above. For example, suppose we have the first equation $a^{\perp} \vee b^{\perp} = (a \wedge b)^{\perp}$. If $a \preccurlyeq b$, then $a = a \wedge b$, so which shows that $b^{\perp} \preccurlyeq a^{\perp}$.

5. Let L and M be complemented lattices. A **homomorphism** from L to M is a function $f: L \to M$ that is a bounded lattice homomorphism.

$$f(x \vee y) = f(x) \vee f(y),$$

$$f(x \wedge y) = f(x) \wedge f(y),$$

$$f(0) = 0,$$

and

$$f(1) = 1.$$

3.5 Boolean Lattices

Boolean lattices generalize the lattice of subsets of a set. They were introduced by Boole [1847] for use in mathematical logic, as formal algebraic systems in which the properties of infimums, supremums, and complements match those of conjunctions, disjunctions, and negations. Boolean lattices are still in use today, as a source of models of set theory, and in the design of electronic logic circuits.[2]

Proposition.3.5.1 In a distributive lattice with a least element and a greatest element:

(1) an element has at most one complement,
(2) if a^* is the complement of a and b^* is the complement of b, then $a^* \vee b^*$ is the complement of $a \wedge b$, and $a^* \wedge b^*$ is the complement of $a \vee b$.

Proof. (1). If b and c are complements of a, then

$$b = b \wedge (a \vee c) = (b \wedge a) \vee (b \wedge c) = b \wedge c \preccurlyeq c$$

exchanging b and c then yields $c \preccurlyeq b$.

(2). By distributivity,

$$(a \wedge b) \wedge (a^* \vee b^*) = (a \wedge b \wedge a^*) \vee (a \wedge b \wedge b^*) = 0 \vee 0 = 0$$

Dually,

$(a^* \vee b^*) \vee (a \wedge b) = 1$. Hence $a^* \vee b^*$ is a complement of $a \wedge b$. Dually, $a^* \wedge b^*$ is a complement of $a \vee b$. □

Definition.3.5.2. A **Boolean lattice** B is a distributive lattice in which for each element $x \in B$ there exists a complement $x^* \in B$ such that

$$x \wedge x^* = 0$$

$$x \vee x^* = 1$$

$$(x^*)^* = x$$

$$(x \wedge y)^* = x^* \vee y^*$$

$$(x \vee y)^* = x^* \wedge y^*$$

In other words a **Boolean lattice**, is a complemented distributive lattice.

The opposite of a Boolean lattice L is a Boolean lattice; in fact, $L^{op} \cong L$, by proposition 3.5.1. Hence there is a duality principle for Boolean lattices: a theorem that holds in every Boolean lattice remains true when the order relation is reversed.

Definition.3.5.3. (Boolean Algebra). A Boolean lattices was defined to be a special kind of distributive lattice. In such a lattice it is often more natural to regard the distinguished elements 0 and 1 and the operation * as an integral part of the structure. Accordingly, a **Boolean algebra** is defined to be the structure $(B; \vee, \wedge, *, 0, 1)$ such that:

(1) $(B; \vee, \wedge)$ is distributive lattice,

(2) $a \vee 0 = a$ and $a \wedge 1 = a$ for all $a \in B$,

(3) $a \vee a^* = 1$ and $a \wedge a^* = 0$.

Example.3.5.4. The set of all subsets of a set , with the usual compositions of intersection , union, and complementation is Boolean lattice (algebra).

Another important example will be given after define the square-free integer.

Definition.3.5.5. The positive integer n is **square-free** if and only if in the prime factorization of n, no prime number occurs more than once.

Another formulation: n is **square-free** if and only if in every factorization $n = ab$, the factors a and b are coprime. For example, 10 is square-free but 18 is not, as it is divisible by $9 = 3^2$. The smallest square-free numbers are 1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 37, 38, 39…

Example.3.5.6. For any natural number n, the set of all positive divisors of n forms a distributive lattice if we write $a \preccurlyeq b$ for $a \mid b$. This lattice is a Boolean algebra if and only if n is square-free. The smallest element 0 of this Boolean algebra is the natural number 1, the largest element 1 of this Boolean algebra is the natural number n.

Definition.3.5.7. **A Boolean sublattice** of a Boolean lattice L is a sublattice S such that $0 \in S$, $1 \in S$, and $x \in S$ implies $x^* \in S$. A Boolean sublattice of L is a Boolean lattice in its own right.

Definition.3.5.8. A map $f: B \to C$ is **Boolean homomorphism** if f is a lattice homomorphism which also preserves 0, 1 and * (that is $f(0) = 0$, $f(1) = 1$ and $f(a^*) = (f(a))^*$ for all $a \in B$).

Theorem.3.5.9. A set B is a Boolean algebra if and only if there exist binary operations $\vee$ and $\wedge$ on B satisfying the following axioms.

1. $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ for $a, b \in B$.

2. $a \vee (b \vee c) = (a \vee b) \vee c$ and $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ for $a, b, c \in B$.

3. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for $a, b, c \in B$.

4. There exist elements 1 and 0 such that $a \vee 0 = a$ and $a \wedge 1 = a$ for all $a \in B$.

5. For every $a \in B$ there exists an $a^* \in B$ such that $a \vee a^* = 1$ and $a \wedge a^* = 0$.

Proof. Let B be a set satisfying (1) - (5) in the theorem. One of the idempotent laws is satisfied since

$$\begin{aligned} a &= a \vee 0 \\ &= a \vee (a \wedge a^*) \\ &= (a \vee a) \wedge (a \vee a^*) \\ &= (a \vee a) \wedge 1 \\ &= a \vee a \end{aligned}$$

Observe that

$$1 \vee b = (1 \vee b) \wedge 1 = (1 \wedge 1) \vee (b \wedge 1) = 1 \vee 1 = 1$$

Consequently, the first of the two absorption laws holds, since

$$\begin{aligned} a \vee (a \wedge b) &= (a \wedge 1) \vee (a \wedge b) \\ &= a \wedge (1 \vee b) \\ &= a \wedge 1 \\ &= a \end{aligned}$$

The other idempotent and absorption laws are proven similarly. Since B also satisfies (1)-(2), the conditions of Theorem 2.2.11 are met; therefore, B must be a lattice. Condition (3) tells us that B is a distributive lattice. For $a \in B$, $0 \vee a = a$, hence, $0 \preccurlyeq a$ and 0 is the smallest element in B.

To show that 1 is the largest element in B, we will first show that $a \vee b = b$ is equivalent to $a \wedge b = a$. Since $a \vee 1 = a$ for all $a \in B$, using the absorption laws we can determine that

$$a \vee 1 = (a \wedge 1) \vee 1 = 1 \vee (1 \wedge a) = 1$$

or $a \preccurlyeq 1$ for all a in B. Finally, since we know that B is complemented by (5), B must be a Boolean algebra.

Conversely, suppose that B is a Boolean algebra. Let 1 and 0 be the greatest and least elements in B, respectively. If we define $a \vee b$ and $a \wedge b$ as least upper and greatest lower bounds of {a, b}, then B is a Boolean algebra by Theorem 2.2.16, Proposition 3.3.1, and our hypothesis. □

Many other identities hold in Boolean algebras. Some of these identities are listed in the following theorem.

Theorem.3.5.10. Let B be a Boolean algebra. Then

1. $a \vee 1 = 1$ and $a \wedge 0 = 0$ for all $a \in B$.

2. If $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$ for a; b; $c \in B$, then $b = c$.

3. If $a \vee b = 1$ and $a \wedge b = 0$, then $b = a^*$.

4. $(a^*)^* = a$ for all $a \in B$.

5. $1^* = 0$ and $0^* = 1$.

6. $(a \vee b)^* = a^* \wedge b^*$ and $(a \wedge b)^* = a^* \vee b^*$ (De Morgan's Laws).

Definition.3.5.11. (Finite Boolean Algebras). A Boolean algebra is a finite Boolean algebra if it contains a finite number of elements as a set.

Finite Boolean algebras are particularly nice since we can classify them up to isomorphism. Let B and C be Boolean algebras. A bijective Boolean homomorphism map $f: B \rightarrow C$ is an isomorphism of Boolean algebras.

We will show that any finite Boolean algebra is isomorphic to the Boolean algebra obtained by taking the power set of some finite set X. We need some lemmas and recall the definition of an atom element.

Recall. Let B be a finite Boolean algebra. An element $a \in B$ is an atom of B if $a \neq 0$ and $a \wedge b = a$ for all $b \in B$. Equivalently, a is an atom of B if there is no nonzero $b \in B$ distinct from a such that $0 \prec b \prec a$.

Lemma.3.5.12. Let B be a finite Boolean algebra. If b is a nonzero element of B, then there is an atom a in B such that $a \preccurlyeq b$.

Proof. If b is an atom, let a = b. Otherwise, choose an element b_1, not equal to 0 or b, such that $b_1 \preccurlyeq b$. We are guaranteed that this is possible since b is not an atom. If b_1 is an atom, then we are done. If not, choose b_2, not equal to 0 or b_1, such that $b_2 \preccurlyeq b_1$. Again, if b_2 is an atom, let $a = b_2$. Continuing this process, we can obtain a chain

$$0 \preccurlyeq \ldots \preccurlyeq b_3 \preccurlyeq b_2 \preccurlyeq b_1 \preccurlyeq b.$$

Since B is a finite Boolean algebra, this chain must be finite. That is, for some k, b_k is an atom. Let $a = b_k$. □

Lemma.3.5.13 Let a and b be atoms in a finite Boolean algebra B such that $a \neq b$. Then $a \wedge b = 0$.

Proof. Since $a \wedge b$ is the greatest lower bound of a and b, we know that $a \wedge b \preccurlyeq a$. Hence, either $a \wedge b = a$ or $a \wedge b = 0$ However, if $a \wedge b = a$, then either $a \preccurlyeq b$ or $a = 0$. In either case we have a contradiction because a and b are both atoms; therefore, $a \wedge b = 0$. □

Lemma.3.5.14. Let B be a Boolean algebra and $a, b \in B$. The following statements are equivalent.

1. $a \preccurlyeq b$.

2. $a \wedge b^* = 0$.

3. $a^* \vee b = 1$.

Proof. (1) ⇒ (2). If $a \preccurlyeq b$, then $a \vee b = b$. Therefore,

$$\begin{aligned} a \wedge b^* &= a \wedge (a \vee b)^* \\ &= a \wedge (a^* \wedge b^*) \\ &= (a \wedge a^*) \wedge b^* \\ &= 0 \wedge b^* \\ &= 0. \end{aligned}$$

(2) ⇒(3). If $a \wedge b^* = 0$, then $a^* \vee b = (a \wedge b^*)^* = 0^* = 1$.

(3) ⇒ (1). If $a^* \vee b = 1$, then

$$\begin{aligned} a &= a \wedge (a^* \vee b) \\ &= (a \wedge a^*) \vee (a \wedge b) \\ &= 0 \vee (a \wedge b) \\ &= a \wedge b \end{aligned}$$

Thus, $a \preccurlyeq b$. □

Lemma.3.5.15. Let B be a Boolean algebra and b and c be elements in B such that $b \not\preccurlyeq c$. Then there exists an atom $a \in B$ such that $a \preccurlyeq b$ and $a \neq c$.

Proof. By Lemma 3.5.14, $b \wedge c^* \neq 0$. Hence, there exists an atom a such that $a \preccurlyeq b \wedge c^*$. Consequently, $a \preccurlyeq b$ and $a \not\preccurlyeq c$. □

Lemma.3.5.16. Let $b \in B$ and $a_1, \ldots, a_n$ be the atoms of B such that $a_i \preccurlyeq b$. Then

$$b = a_1 \vee \ldots \vee a_n.$$

Furthermore, if $a, a_1, \ldots, a_n$ are atoms of B such that:

$a \preccurlyeq b$, $a_i \preccurlyeq b$, and $b = a \vee a_1 \vee \ldots \vee a_n$, then

$a = a_i$ for some $i = 1, \ldots, n$.

Proof. Let $b_1 = a_1 \vee \ldots \vee a_n$. Since $a_i \preccurlyeq b$ for each i, we know that $b_1 \preccurlyeq b$. If we can show that $b \preccurlyeq b_1$, then the lemma is true by antisymmetry. Assume $b \not\preccurlyeq b_1$. Then there exists an atom a such that $a \preccurlyeq b$ and $a \not\preccurlyeq b_1$. Since a is an atom and $a \preccurlyeq b$, we can deduce that $a = a_i$ for some a_i. However, this is impossible since $a \preccurlyeq b_1$. Therefore, $b \preccurlyeq b_1$.

Now suppose that $b = a_1 \vee \ldots \vee a_n$.

If a is an atom less than b, then

$$a = a \wedge b = a \wedge (a_1 \vee \ldots \vee a_n) = (a \wedge a_1) \vee \ldots \vee (a \wedge a_n).$$

But each term is 0 or a with $a \wedge a_i$ occurring for only one a_i. Hence, by Lemma 3.5.13 $a = a_i$ for some i. □

Theorem.3.5.17. (Theorem 17.12 in [21]). Let B be a finite Boolean algebra. Then there exists a set X such that B is isomorphic to $\mathbb{P}(X)$.

Proof. We will show that B is isomorphic to $\mathbb{P}(X)$,where X is the set of atoms of B. Let $a \in B$. By Lemma 3.5.16, we can write a uniquely as

$$a = a_1 \vee \ldots \vee a_n \quad \text{for} \quad a_1, \ldots, a_n \in X.$$

Consequently, we can define a map $f : B \rightarrow \mathbb{P}(X)$ by

$$f(a) = f(a_1 \vee \ldots \vee a_n) = \{ a_1, \ldots, a_n \}.$$

Clearly, f is onto.

Now let $a = a_1 \vee \ldots \vee a_n$ and $b = b_1 \vee \ldots \vee b_m$ be elements in B, where each a_i and each b_i is an atom. If $f(a) = f(b)$, then $\{ a_1, \ldots, a_n \} = \{ b_1, \ldots, b_m \}$. and $a = b$. Consequently, f is injective.

The join of a and b is preserved by f since

$$f(a \vee b) = f(a_1 \vee \ldots \vee a_n \vee b_1 \vee \ldots \vee b_m).$$

$$= \{ a_1, \ldots, a_n, b_1, \ldots, b_m \}.$$

$$= \{ a_1, \ldots, a_n \} \cup \{ b_1, \ldots, b_m \}.$$

$= f(a_1 \vee \ldots \vee a_n) \cup f(b_1 \vee \ldots \vee b_m)$

$= f(a) \cup f(b).$

Similarly, $f(a \wedge b) = f(a) \cap f(b)$. □

Chapter Four

Applications of Complete Lattices

In this chapter three applications of complete lattices will be discussed. Each one contains important and basic definitions, propositions, and theorems. First one talk about retraction on a complete lattices [11], second one talk about fixed point property of decreasing functions on a complete lattices [23], finally third one talk about associated prime ideals of a complete lattices [4].

4.1. <u>All Retraction Operators On A Complete Lattice Form A Complete Lattice.</u>

The main topic studied in this section is the following problem raised by H Crapo in 1982: [11]

Crapo's Problem : If P is a complete lattice , is Retr(P) a complete lattice?

An affirmative answer to Crapo's Problem will be given after some needed definitions

Definition 4.1.1. (Retraction Operator). By **a retraction operator** on a partially ordered set P we mean an order- preserving map of P to itself which is idempotent (i.e., $f^2 = f$). The set of all retraction operators on P is denoted by Retr(P).

Definition 4.1.2. (Pointwise Order). For any two maps f and g of P to itself , define $f \trianglelefteq g$ if and only if $f(x) \preccurlyeq g(x)$ for any $x \in P$. This $\trianglelefteq$ is an order between maps and is called **pointwise order.**

According to the above definitions (Retr(P) , $\trianglelefteq$) forms a partially ordered set.

In the proof of the following theorem Tarski's fixed point theorem on closed intervals of a complete lattice L will be used [31].

Apparently , any closed interval of a complete lattice L is still a complete lattice (here , when a subset X of (L, $\preccurlyeq$) is considered as a poset , its order is the induced order) .

If an order- preserving map f on L maps a closed interval [a, b] into itself ($f(x) \in [a, b]$ for any $x \in [a, b]$), this f can be considered as an order- preserving map on [a, b], and then f has a smallest fixed point x in [a, b], i.e., for any fixed point y of f, $x \preccurlyeq y$ whenever $y \in [a, b]$.

Now the following theorem which answers Crapo's question will be proved.

Theorem.4.1.3.[10]. (P, $\preccurlyeq$) is a complete lattice if and only if (Retr(P), $\trianglelefteq$) is a complete lattice.

Proof . We have to deal with two partially ordered sets (P, $\preccurlyeq$) and (Retr(P), $\trianglelefteq$). To simplify notations, we shall use the same order- theoretic terminology, such as per bounds and supremum. This will not make any confusion, since we can distinguish their meaning according to underlying sets (P or Retr(P)) to which the subsets belong.

First, we show the necessity.

Let 0 and 1 be the smallest and greatest element of $(P, \preccurlyeq)$ respectively. Obviously, the map taking 0 as the value at every $x \in P$ is a retraction operator and is the smallest element of (Retr(P), $\trianglelefteq$). So, for the completeness of (Retr(P), $\trianglelefteq$) it will suffice to show that every nonempty subset F of Retr(P) has the supremum.

First, let us define a map $: P \to P$ as follows. For $x \in P$,

$$i(x) = \sup \{f(x): f \in F\}.$$

(1) is an order- preserving map.

Suppose x and $y \in P$ and $x \preccurlyeq y$. Since for every $f \in F$, f is order preserving, hence $f(x) \preccurlyeq f(y)$. By the definition of i, $f(y) \preccurlyeq i(y)$. This shows that i (y) is an upper bound of the subset $\{f(x): f \in F\}$. Then $i(x) = \sup \{f(x): f \in F\} \preccurlyeq i(y)$.

(2) $f \trianglelefteq i$ for every $f \in F$.

The conclusion is coming immediately from the definition of .

(3) $\trianglelefteq i^2$.

Note that for each $x \in P$,

$$i(x) = \sup \{ f(x): f \in F\} \text{ and } i^2(x) = \sup \{ f(i(x)): f \in F\}.$$

For any $f \in F$, by the definition of retraction operators and (2),

$$f(x) = f^2(x) = f(f(x)) \preccurlyeq f(i(x)) \preccurlyeq i(i(x)) = i^2(x).$$

Therefore, $i^2(x)$ is an upper bound of the subset $\{f(x): f \in F\}$. Hence $i(x) \preccurlyeq i^2(x)$.

But this map i may not be idempotent, we should adapt it to a retraction operator g on P. For any $x \in P$, (3) guarantees that i maps the closed interval $[i(x), 1]$ into itself because

$$i(x) \preccurlyeq i^2(x) = i(i(x)) \preccurlyeq i(y) \preccurlyeq 1$$

for any $y \in [i(x), 1]$, according to (1) and (3). By Tarski's fixed point theorem, i has a smallest fixed point in $[i(x), 1]$ which is defined to be g(x).

(4) g is an order preserving map .

Suppose $x, y \in P$ and $x \preccurlyeq y$. (1) guarantees $i(x) \preccurlyeq i(y)$, g(y) is a fixed point of i in $[i(y), 1]$, hence is a fixed point of i in $[i(x), 1]$. And g(x) is the smallest fixed point of i in $[i(x), 1]$, hence $g(x) \preccurlyeq g(y)$.

(5) g is idempotent , therefore is a retraction operator .

By the definition of g, for any $x \in P$, $g^2(x)$ is the smallest fixed point of i in the closed interval $[i(g(x)), 1]$ and $g(x) = i(g(x))$ is just a fixed point of i in the closed interval . So, $g^2(x) = g(x)$, namely g is idempotent.

(6) g is an upper bound of F.

By the definition of g, $i \trianglelefteq g$. Then (6) is an immediate consequence of (2).

(7) g is the supremum of F.

Let $h \in Retr(P)$ be an upper bound of F. We shall show $g \trianglelefteq h$, i.e. ,$g(x) \preccurlyeq h(x)$ for each $x \in P$.

It is easy to see $i \trianglelefteq h$ (h upper bound of F), so $i(x) \preccurlyeq h(x)$. For any $y \in [i(x) , h(x)]$, we have by (1), (3), $i \trianglelefteq h$ and the definition of retraction operators , that

$$i(x) \preccurlyeq i^2(x) = (i(x)) \preccurlyeq i(y) \preccurlyeq i(h(x)) \preccurlyeq h(h(x) = h^2(x) = h(x).$$

This shows that i maps the closed interval $[i(x) , h(x)]$ into itself . Again , by Tarski's fixed point theorem, i has a fixed point z in $[i(x) , h(x)]$. This z is a fixed point of i in $[i(x), 1]$ too. Since g(x) is the smallest fixed point of i in $[i(x), 1]$,

$$g(x) \preccurlyeq z \preccurlyeq h(x).$$

Hence $g \trianglelefteq h$. This completes the proof of the necessity.

In the following we show the sufficiency.

For each $a \in P$ define a map $f_a : P \rightarrow P$ as follows. For each $x \in P$, $f_a (x) = a$. It is always a retraction operator on P.

By completeness of $(Retr(P), \trianglelefteq)$, we have

$$f = \sup \{f_a: a \in A\}$$

exists for an arbitrary non empty subset A of P.

(8) There exist a $b \in P$ such that $f = f_b$.

It is enough to show that f takes the same value at every $x \in P$. Let $y , z \in P$, $c = f(y)$ and $d = f(z)$. For any $a \in A$ it follows from $f_a \trianglelefteq f$ that

$$a = f_a(y) \preccurlyeq f(y) = c.$$

So, $f_a \trianglelefteq f_c$, i.e., f_c is an upper bound $\{f_a: a \in A\}$. Then $f \trianglelefteq f_c$ since f is the supremum of $\{f_a: a \in A\}$. Therefore

$$d = f(z) \preccurlyeq f_c(z) = c \qquad (*)$$

Now, since $f_a \trianglelefteq f$ for any $a \in A$ it follows that

$$a = f_a(z) \preccurlyeq f(z) = d.$$

So, $f_a \trianglelefteq f_d$, i.e., f_d is an upper bound $\{f_a: a \in A\}$. Then $f \trianglelefteq f_d$ since f is the supremum of $\{f_a: a \in A\}$. Therefore

$$c = f(y) \preccurlyeq f_d(y) = d \qquad (**)$$

From (*) and (**) we have $f(y) = f(z)$ for any $y , z \in P$.

(9) b is the supremum of A .

For any a $\in$A, take x $\in$P. Then by the definition of f

$$a = f_a(x) \preccurlyeq f(x) = f_b(x) = b.$$

This shows that b is an upper bound of A . Let c be an upper bound of A. Then f_c is an upper bound of $\{f_a \colon a \in A\}$. $f_b \trianglelefteq f_c$ by (8) and the definition of f . Taking x $\in$P we have

$$b = f_b(x) \preccurlyeq f_c(x) = c.$$

Which shows that b is the supremum of A.

Similarly, to show that the infimum of A exists we may assume (by completeness of (Retr(P), $\trianglelefteq$), that $f = \inf \{f_a \colon a \in A\}$ and do the same steps.

So (P, $\preccurlyeq$) is a complete lattice.□

4.2 Fixed Points And Complete Lattices.

Tarski proved in 1955 that every complete lattice has the fixedpoint property[31]. Later, Davis proved the converse that "every lattice with the fixed point property is complete" [13]. For a chain complete ordered set, there is the well known Abian-Brown fixed point result. As a consequence of the Abian- Brown result, every chain complete ordered set with a smallest element has the fixed point property[1]. Here a fixed point theorems are given for(order- reversing) decreasing functions.[23]

Recall:

1. A function from a poset $(P,\preccurlyeq)$ into itself is (order- reversing) decreasing if whenever $x \preccurlyeq y$, then $f(y) \preccurlyeq f(x)$.

2. A mapping from a poset $(P,\preccurlyeq)$ into itself has a fixed point if there exists an $x \in P$ such that $f(x) = x$.

3. The poset $(P,\preccurlyeq)$ has the fixed point property if every increasing mapping of $(P,\preccurlyeq)$ into itself has a fixed point.

Definition.4.2.1. (Chain-complete). Let P be an ordered set. Then P is called chain-complete if and only if each non empty subchain $C \subseteq P$ has a supremum and an infimum.

Now, without the poset $(P,\preccurlyeq)$ being a lattice, we have the well known Abian/ Brown result which says: "Let $(P,\preccurlyeq)$ be a chain-complete poset. Let f be an increasing mapping from $(P,\preccurlyeq)$ into itself such that for some $a \in P$, $a \preccurlyeq f(a)$. Then the mapping f has a fixed point"[1].

Example.4.2.2. Any complete poset is a chain-complete.

Exampe.4.2.3. The set of all linearly independent subsets of a vector space V, ordered by inclusion is a chain-complete.

Before proving some theorems, some needed notes will be given.

Definition.4.2.4. The mapping f from a poset P into itself has a **fixed apex** u if there exists a v in P such that $f(v) = u$ and $f(u) = v$.

Example.4.2.5. let $X = \{1, 2\}$, and let f be a map from $(\mathbb{P}(X), \subseteq)$ to itself such that the image of any subset of X is its complement, then $\{1\}$ and $\{2\}$ are both fixed apex since $f(\{1\}) = \{2\}$ and $f(\{2\}) = \{1\}$.

Definition.4.2.6. let A be a set and let $\preccurlyeq$ be a binary relation on A. Then a subset B of A is said to be **cofinal** if it satisfies the following condition:

For every $a \in A$, there exists some $b \in B$ such that $a \preccurlyeq b$.

Example.4.2.7. $\mathbb{N}$ is cofinal in $\mathbb{R}$ since for every real number r there exists natural number n such that $r \leq n$.

In the following, the set P will denote a partially ordered set where

(4) $P^* = P \times \{1,-1\}$

We define an ordering on P* as follows:

(5) If $a \preccurlyeq b$, define $(a,-1) \preccurlyeq (b,-1)$ and $(b, 1) \preccurlyeq (a, 1)$.

(6) For all c, d $\in$P, define $(c,-1) \preccurlyeq (d, 1)$.

In view of 5 and 6, the set P* is clearly a partially ordered set. We will denote an element of the form (x, i) by x* where $i \in \{1,-1\}$. Also, we denote π_1 as the projection from $P \times \{1,-1\}$ onto P.

(7) We say a non empty subset D of $(P^*,\preccurlyeq)$ has the **bounded property** if the set D is bounded and if every non empty subset of $\pi_1(D)$ has a supremum and an infimum in $(P,\preccurlyeq)$.

Theorem.4.2.8. [23] . The lattice $(P,\preccurlyeq)$ is complete is equivalent to the following statement *" every decreasing function from the partially ordered set $(P^*,\preccurlyeq)$ into itself has a fixed point where the function f and the partially ordered set $(P^*,\preccurlyeq)$ have the following properties: there exists a nonempty linearly ordered subset M of the lattice $(P^*,\preccurlyeq)$ such that $f(M) \subseteq M$ and*

(i) if $f(b^*) \preccurlyeq a^* \preccurlyeq b^* \preccurlyeq f(a^*)$ holds in M, then there exists an element $x^* \in M$ such that $a^* \prec f(x^*) \prec b^*$ and $a^* \prec x^* \prec b^*$

(ii) if D is a nonempty bounded subset of M with the bounded property, then sup(D) exists in M." (*)

Proof. Assume that the lattice $(P,\preccurlyeq)$ is complete. Let f be a decreasing mapping of P* into itself. Let M be a nonempty linearly ordered subset of P* such that $f(M) \subseteq M$. Since the set M is not empty, let $a^* \in M$.

Without loss of generality, we assume that $a^* \preccurlyeq f(a^*)$. But then by (1). we have

(8). $f(f(a^*)) = f^2(a^*) \preccurlyeq f(a^*)$.

Let the sets A and B be defined as follows

(9). $A = \{ x^* \in M : x^* \preccurlyeq f(x^*)\}$ and $B = \{ x^* \in M : f(x^*) \preccurlyeq x^*\}$

In view of our assumption $a^* \preccurlyeq f(a^*)$, we see that $a^* \in A$ and from (8) it follows that $f(a) \in B$. Thus, A and B are nonempty. We observe by (1) and by (9) that the following can be readily verified

(10). if $x^* \in A$, then $f(x^*) \in B$, $f^2(x^*) \in A$ and if $y^* \in B$, then $f(y^*) \in A$ and $f^2(y^*) \in B$. Thus, $f(A) \subseteq B$ and $f(B) \subseteq A$.

Since M is a linearly ordered set, we have
(11) . M = AUB.

We show that

(12). $x^* \preccurlyeq y^*$ for $x^* \in A$ and every $y^* \in B$.

For if $y^* \prec x^*$, by 10. we have $f(y^*) \preccurlyeq y^* \prec x^* \preccurlyeq f(x^*)$. However, since f is decreasing function we must also have $f(x^*) \preccurlyeq f(y^*)$. Thus, we arrive at a contradiction and hence $x^* \preccurlyeq y^*$.

Since the partially ordered set $(P,\preccurlyeq)$ is complete for all nonempty subsets D of A, we have

(13). $\sup(\pi_1(D))$ and $\inf(\pi_1(D))$ exist in the lattice $(P,\preccurlyeq)$.

Thus in view of (ii) and since A is a bounded subset of P^*, we have that

(14). $e^* = \sup A$ exists in the subset M of P^*.

Since M is a linearly ordered subset of $(P^*,\preccurlyeq)$, we have that either

$e^* \preccurlyeq f(e^*)$ or $f(e^*) \prec e^*$.

If $e^* \preccurlyeq f(e^*)$, we have by (10). that $f(e^*) \in B$, and $f^2(e^*) \in A$. Since e^* is the least upper bound of A, we have that $f^2(e^*) \preccurlyeq e^*$. Hence,

$f^2(e^*) \preccurlyeq e^* \prec f(e^*) \preccurlyeq f(e^*)$.

By hypothesis of the Theorem, there exists $x^* \in M$ such that:

(15). $e^* \prec x^* \prec f(e^*)$ and $e^* \prec f(x^*) \prec f(e^*)$.

From 11. it follows that $x^* \in A$ or $x^* \in B$. If $x^* \in A$, then $x^* \preccurlyeq e^*$ by 14. Which contradicts 15.

If $x^* \in B$, then $f(x^*) \in A$ and thus in view of 14, $f(x^*) \preccurlyeq e^*$ which also contradicts 15.

Therefore, e^* is not $\prec f(e^*)$. The second case $f(e^*) \prec e^*$ is similar to the previous case. Thus, the two cases cannot happen. Therefore, e^* must be a fixed point as desired.

Conversely, assume that *(*)* holds and that the lattice $(P,\preccurlyeq)$ is not complete. Thus, there exists a maximal linearly ordered subset M_1 of P which is not complete. Thus, there exists a subset L of M_1 such that sup L or inf L does not exist in the partially ordered set $(P,\preccurlyeq)$. Without loss of generality, assume that sup L does not exist in $(P,\preccurlyeq)$. Clearly, the set L does not have a maximum. Thus, there exists an infinite subset L_1 of L such that

(16). $L_1 = \{x_0 < x_1 < \ldots < x_i < \ldots\}$

where $i \in \beta$ and β is a limit ordinal where L_1 is cofinal in L.

Let $L_0 = \{x \in M_1 : x \preccurlyeq x_\alpha \text{ for some } \alpha \prec \beta\}$.
Then consider the partially ordered set P^* and let $M = L_0 \times \{1,-1\}$.

Define a mapping f from P^* into itself (where k = 1 or -1) by $f(x, k) = (x,-k)$.Clearly, the mapping f is decreasing on P^* and $f(M) \subseteq M$.

In order to show (i) is true, assume that

(17). $f(b^*) \preccurlyeq a^* \prec b^* \preccurlyeq f(a^*)$ holds in M

In order for (17) to hold, we must have $a^* = (a,-1)$, $b^* = (b, 1)$ and $a = b$. In view of (16), there exists $x_\alpha \in L_1$ such that $(x_\alpha,-1)$and $f(x_\alpha,-1) = (x_\alpha,1)$ are between $(a,-1)$ and $(b, 1)$. Thus part (i) is true.

In order to show part (ii), let D be a non empty bounded subset of M having the bounded property.

We show that sup D exists in M. If there exists an element of the form $(x, 1) \in D$, then $e = \inf(\pi_1(\{(x, 1):(x, 1)) \in D\})$ exists in $(P,\preccurlyeq)$.

Since M_1 is a maximal linearly ordered subset of $(P,\preccurlyeq)$, we have that $e \in M_1$. Thus, $e \in L_0$. Then $(e, 1) = \sup D \in M$. If no element of the form $(x, 1)$ is in D, then since $e = \sup \pi_1(D)$ exists in the lattice $(P,\preccurlyeq)$ and M_1 is a maximal linearly ordered subset of P we have that $e \in M_1$.Thus $e \in L_0$.

Thus, clearly the supremum of D in M is $(e,-1)$. In either case we have that sup D exists in M. Thus, the mapping f satisfies the hypothesis of *(*)* but the mapping f does not have a fixed point. □

From the above proof, we have the corollary given below. The following corollary gives a sufficient condition for a decreasing function to have a fixed point

Corollary.4.2.9. Let $(P,\preccurlyeq)$ be a nonempty linearly ordered partially ordered set in which every nonempty bounded above subset of P has a least upper bound and let f be a decreasing function from $(P,\preccurlyeq)$ into itself. Assume also that for every element.

(18). $a, b \in P$ such that $f(b) \preccurlyeq a \prec b \preccurlyeq f(a)$ then $a \prec x \prec b$ and $a \prec f(x) \prec b$ for some $x \in P$.

Then the mapping f has a fixed point.

Below some remarks concerning the previous corollary.

Example.4.2.10. Any continuous function on the set of real numbers $\mathbb{R}$ satisfies (18) of the previous Corollary. Thus, since the set $\mathbb{R}$ has the property that every nonempty set which is bounded above has a least upper bound, then every decreasing continuous function from $\mathbb{R}$ into $\mathbb{R}$ has a fixed point.

Example.4.2.11. The Corollary 4.2.9 is not true if (18) is deleted. For example let $P = \{a, b\}$ where $a \preccurlyeq b$. Let $f : P \to P$ be defined by $f(a) = b$ and $f(b) = a$. The mapping f is decreasing on P and the poset $\{a, b\}$ is complete, but f does not have a fixed point.

Example.4.2.12. To see in the above Corollary that (18) is not sufficient let $f: [0, 2] \to [0, 2]$ be defined by $f(x) = 2$ if $0 \preccurlyeq x \preccurlyeq 1$; $f(x) = 0.5$ if $x \in [1, 2]-\{1+1/n : n \in N\}$ and $f(1 + 1/n) = 1 + 1/(2n)$, $1 \leq n$. Clearly, the mapping f satisfies (18). but does not have a fixed point.(take $a = 0.9$, $b = 1.9$, $x = 1.5$).

Example.4.2.13. In (18) of the previous corollary, $a \prec f(x) \prec b$ cannot be replaced by

(19). $a \prec f(x) \preccurlyeq b$

For let f map the closed interval [0, 1] into itself be defined by $f(1) = 0$ and $f(x) = 1$, if $0 \preccurlyeq x \prec 1$. The mapping satisfies 19 and the other hypotheses of the theorem, but the mapping f does not have a fixed point.(take $a = 0$, $b = 1$, $x = 0.5$)

Theorem.4.2.14. [23] Let $(P,\preccurlyeq)$ be a complete partially ordered set. Then every decreasing function from the partially ordered set $(P^*,\preccurlyeq)$ into itself such that for some a^* in P^*, $a^* \preccurlyeq f^2(a^*)$ has a fixed apex or fixed point.

Proof. Assume the partially ordered set $(P,\preccurlyeq)$ is complete. Let f be a mapping of the lattice $(P^*,\preccurlyeq)$ into itself. Since P is a complete lattice, the poset $(P^*,\preccurlyeq)$ is a complete lattice.

We consider the mapping $F = f^2$. Then the mapping F is an increasing mapping. By Tarski's Fixed Point Result, the mapping F has a fixed point and thus the mapping f has a fixed point or a fixed apex.□

From the Abian-Brown Fixed Point result, we have the following corollary:

Corollary.4.2.15. Let $(P,\preccurlyeq)$ be a nonempty linearly ordered partially ordered set in which every nonempty bounded subset has a least upper bound and let f be a decreasing mapping from P into itself. Assume that there exists an element $a \in M$ such that $a \preccurlyeq f^2(a) \preccurlyeq f(a)$. Then the mapping f has a fixed apex or a fixed point.

4.3 Associated Prime Ideals of a Complete Lattice

This section present an application of a complete distributive lattices, it is shown that Annihilator of any subset of a complete chain is an associated prime [4].

Annihilator of a subset of a complete distributive lattice , annihilator primes and associated primes of a lattice will be defined and illustrative examples will be given in the beginning of this section.

Definition.4.3.1. Let L be a complete lattice, which is also distributive. Let A be a non empty subset of L. **Annihilator** of A is denoted by **Ann(A)**, and is defined as:

$$\mathbf{Ann(A)} = \{J \in L, \text{ such that } a \wedge J = 0, \text{ for all } a \in A\}.$$

Example.4.3.2. Let L = {1, 3, 9, 27}. L is a complete, distributive lattice with respect to divisibility. Let A = {3, 9}. Then Ann(A) = {1}.
Let K = {1, 2, 3, 4, 6, 8, 12, 24}. K is a complete, distributive lattice with respect to divisibility, and let B = {3}. Then Ann(B) = {1, 2, 4, 8}.

Example.4.3.3. Let X = {a, b, c}. Now $\mathbb{P}$ (X), the power set of X is a complete distributive lattice with respect to set inclusion .
Let A = { {a}, {a, b}}.
Then Ann(A) = {∅ ,{c}} since {c} ∩ {a} = ∅ and {c} ∩ {a,b} = ∅ which is the zero element in $\mathbb{P}$ (X) with respect to inclusion.

Proposition.4.3.4. Let L be a complete lattice, which is also distributive. Let A ⊆ B be two non empty subsets of L. Then Ann(B) ⊇ Ann(A).

Proof. Let J ∈ Ann(B). Then b ∧ J = 0, for all b ∈ B. Now let a ∈ A. Then a ∈ B, and therefore a ∧ J = 0. So J ∈ Ann(A). Hence Ann(B) ⊇ Ann(A).□

Definition.4.3.5. Let L be a complete lattice, which is also distributive. A subset B ⊆ L is said to be **faithful** if Ann(B) = {0}.

Example.4.3.6. Let L = {1, 2, 3, 6, 10, 15, 30} with respect to divisibility.
Then B = {1, 3, 6} is a faithful subset of L since Ann(B) = {1}which is the zero element with respect to divisibility here. But C = {1, 3} is not faithful, as Ann(C) = {1, 2, 10}.

Proposition.4.3.7. Let L be a complete lattice, which is also distributive. Let A be a non empty subset of L. Then Ann(A) is an ideal of L.

Proof. Ann(A) ≠ ∅ , as 0 ∈ Ann(A).

Now for any r, s ∈ Ann(A)

r ∧ a = 0 and s ∧ a = 0 for all a ∈ A.

Now for all $a \in A$,

$$(r \vee s) \wedge a = (r \wedge a) \vee (s \wedge a) = 0 \vee 0 = 0.$$

Therefore,

$$(r \vee s) \in \mathrm{Ann}(A).$$

Now let $r \in \mathrm{Ann}(A)$, and $J \in L$. We have

$$r \wedge a = 0 \text{ for all } a \in A,$$

and

$$(r \wedge J) \wedge a = (J \wedge r) \wedge a = J \wedge (r \wedge a) = J \wedge 0 = 0.$$

Therefore,

$$(r \wedge J) \in \mathrm{Ann}(A).$$

Hence Ann(A) is an ideal of L.□

Definition.4.3.8. Let L be a complete lattice, which is also distributive. We say that L is a **fully faithful** lattice if Ann(A) = 0, where A is any non empty subset of L.

Example.4.3.9. In Example (4.3.2) above, L is fully faithful.

Proposition.4.3.10. Every chain, which is also complete, is a fully faithful lattice.

Proof. Let L be a chain and $\{0\} \neq A$ be a non empty subset of L. Let $x \in \mathrm{Ann}(A)$. Then $x \wedge a = 0$ for all $a \in A$.
Now L is a chain, therefore $x \preccurlyeq a$ for at least one $0 \neq a \in A$, or $a \preccurlyeq x$ for all $a \in A$.

If $a \preccurlyeq x$ for all $a \in A$, then

$$x \wedge a = a,$$

which implies that $a = 0$ for all $a \in A$, a contradiction. So $x \preccurlyeq a$ for at least one $0 \neq a \in A$, and therefore,

$$x \wedge a = x. \text{ Thus } x = 0.$$

Therefore Ann(A) = 0. Hence L is a fully faithful lattice.□

Note: Recall that an ideal P of a lattice L is called a prime ideal if $P \neq L$, and if for any $a, b \in L$ with $(a \wedge b) \in P$, we have $a \in A$ or $b \in A$.

Definition.4.3.11. Let L be a lattice. An ideal P is said to be a **minimal prime ideal** if P is a prime ideal and does not contain properly a prime ideal.

Example.4.3.12. Consider $L = \{1, 2, 5, 10\}$. L is a lattice with respect to divisibility. $P = \{2\}$ is a minimal prime ideal. $S = \{1, 2\}$ is a prime ideal, but is not a minimal prime ideal, as $\{2\} \subset \{1, 2\}$, and $\{2\}$ is a prime ideal.

Definition.4.3.13. Let L be a complete distributive lattice. An ideal A of L is called an **annihilator prime** if A is a prime ideal and is also annihilator of some non empty subset $B \neq \{0\}$ of L.

Example.4.3.14. Let L = {1, 2, 5, 10}. L is complete distributive lattice with respect to divisibility, A = {1, 2} is a prime ideal. Now let M = {5}. Then A = Ann(M). Therefore A is an annihilator prime.
Let B = {1}, and C = {2, 5}. Then Ann(C) = B. Now B is not a prime ideal, as $2 \wedge 5 = 1 \in P$, but $2 \notin B$ and $5 \notin B$. Therefore B is not an annihilator prime.

Example.4.3.15. In Example (4.3.3) above, Ann(A) is not a prime as

$$\{b, c\} \cap \{a, c\} \in \mathrm{Ann}(A) = \{\{c\}\},$$

but

$$\{b, c\} \notin \mathrm{Ann}(A)$$

and

$$\{a, c\} \notin \mathrm{Ann}(A).$$

Definition.4.3.16. Let L be a complete distributive lattice. An ideal A of L is called an **associated prime** if A is a prime ideal and is annihilator of some non empty subset $\{0\} \neq B$ of L. Furthermore, A is also the annihilator of any non empty subset C of B.

It is clear that an associated prime is an annihilator prime, but every annihilator prime need not be an associated prime.

Example.4.3.17. Let L = {1, 3, 5, 7, 35} with respect to divisibility.
Let B = {3, 7}, and C = {7}. Then Ann(B) = {1, 5}, and Ann(C) = {1, 3, 5}.
Now {1, 5} is a prime ideal of L, so it is an annihilator prime, but not an associated prime.

Example.4.3.18. Let L = {1, 2, 4, 5, 10, 20}. With respect to divisibility, L is a complete distributive lattice. Let A = {1, 5}, and B = {2, 4}. Then A = Ann(B), and A = Ann(C) for any subset C of B. More over A is a prime ideal. Therefore A is an associated prime ideal.

Theorem.4.3.19. In a chain, which is also complete, annihilator of any nonzero subset is an associated prime ideal.

Proof. Let L be a chain, which is also complete. Let $\{0\} \neq B$ be a non empty subset of L.
Let Ann(B) = A. Now A is an ideal by Proposition (4.3.7).

We will show that A is a prime ideal.

Let a, b ∈ L be such that $(a \wedge b) \in A$. Now L is a chain, therefore $b \preccurlyeq a$, or $a \preccurlyeq b$.

If $a \preccurlyeq b$, then $a \wedge b = a$, and so $a \in A$.
If $b \preccurlyeq a$, then $a \wedge b = b$, and so $b \in A$.

Therefore A is a prime ideal.

Let $\{0\} \neq C \subseteq B$ be a non empty subset of B. Then by Proposition (4.3.4), $\mathrm{Ann}(B) \subseteq \mathrm{Ann}(C)$.

We will show that A = Ann(C).

Suppose not, and $0 \neq b \in \mathrm{Ann}(C)$, $b \notin A$.

Now $b \wedge c = 0$, for all $c \in C$. Now L is a chain, therefore $b \preccurlyeq c$ for at least one $c \in C$, or $c \preccurlyeq b$ for all $c \in C$.

If $b \preccurlyeq c$ for at least one $c \in C$. Then $b = b \wedge c = 0$, a contradiction.

If $c \preccurlyeq b$ for all $c \in C$, then $c = b \wedge c = 0$. Then $C = \{0\}$, again a contradiction.

So our supposition must be wrong. Therefore $A = \mathrm{Ann}(C)$, and hence A is an associated prime ideal of L.□

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