## RELAXING THE NONSINGULARITY ASSUMPTION FOR INTERVALS OF TOTALLY NONNEGATIVE MATRICES

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5Abstract. Totally nonnegative matrices, i.e., matrices having all their minors nonnegative, and 6 matrix intervals with respect to the checkerboard partial order are considered. It is proven that if the 7 two bound matrices of such a matrix interval are totally nonnegative and satisfy certain conditions, 8 then all matrices from this interval are totally nonnegative and satisfy these conditions, too, hereby 9 relaxing the nonsingularity condition in a former paper [M. Adm, J. Garloff, Intervals of totally nonnegative matrices, Linear Algebra Appl. 439 (2013), pp.3796-3806]. 10

11 Key words. Matrix interval, Checkerboard partial order, Totally nonnegative matrix, Cauchon matrix, Cauchon Algorithm, Descending rank conditions.

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1. Introduction. A real matrix is called *totally nonnegative* if all its minors 14 are nonnegative. Such matrices arise in a variety of ways in mathematics and its 15applications. For background information the reader is referred to the monographs 16 [9], [15]. In [2], the following *interval property* was shown: Consider the checkerboard 17 order which is obtained from the usual entry-wise order on the set of the square real 18 19 matrices of fixed order by reversing the inequality sign for each entry in a checkerboard fashion. If the two bound matrices of an interval with respect to the checkerboard 20 order are nonsingular and totally nonnegative, then all matrices lying between the 21 22 two bound matrices are nonsingular and totally nonnegative, too. The purpose of this paper is to relax the nonsingularity assumption on the two bound matrices and 23to allow rectangular matrices instead of square matrices. For a collection of various 24 classes of matrices which enjoy an interval property see [11]. 25

We mention a closely related problem, viz. given a totally nonnegative matrix, 26 27find for each of its entries the maximum allowable perturbation such that the perturbed matrix remains totally nonnegative. This problem was solved in [3] for the 28tridiagonal totally nonnegative and in [7] for the general totally nonnegative matrices. 29For the *totally positive matrices*, i.e., matrices having all their minors positive (here 30 31 the perturbed matrix has in turn to be totally positive), it was established in [10], see also [9, Section 9.5], for a few specified entries and in [6] for arbitrary entries. The 32 similar problem for a uniform perturbation of all the coefficients of a totally positive 33 matrix was considered in [13, Section 7]. 34

The organization of our paper is as follows. In Section 2, we introduce our notation and give some auxiliary results which we use in the subsequent sections. In Section 3, 36 we recall the condensed form of the Cauchon Algorithm and some of its properties. In 37 38 Section 4, we present our new results on the application of the Cauchon Algorithm,

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<sup>39</sup> and apply them in the last section to the above mentioned interval problem.

## 40 **2.** Notation and auxiliary results.

**2.1. Notation.** We introduce the notation used in our paper. For integers  $n, m, \kappa$ , we denote by S the set  $\{1, \ldots, n-1\} \times \{1, \ldots, m-1\}$ , and by  $Q_{\kappa,n}$  the set of all strictly increasing sequences of  $\kappa$  integers chosen from  $\{1, 2, \ldots, n\}$ . Let A be a real *n*-by-*m* matrix. For  $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_\kappa\} \in Q_{\kappa,n}, \beta = \{\beta_1, \beta_2, \ldots, \beta_\mu\} \in Q_{\mu,m}$ , we denote by  $A[\alpha|\beta]$  the  $\kappa$ -by- $\mu$  submatrix of A contained in the rows indexed by  $\alpha_1, \alpha_2, \ldots, \alpha_\kappa$  and columns indexed by  $\beta_1, \beta_2, \ldots, \beta_\mu$ . We suppress the curly brackets when we enumerate the indices explicitly. A measure of the gaps in an index sequence  $\alpha \in Q_{\kappa,n}$  is the dispersion of  $\alpha$ , denoted by  $d(\alpha)$ , which is defined by  $d(\alpha) := \alpha_\kappa - \alpha_1 - \kappa + 1$ . If  $d(\alpha) = 0$ , we call  $\alpha$  contiguous, if  $d(\alpha) = d(\beta) = 0$ , we call the submatrix  $A[\alpha|\beta]$  contiguous, and in the case  $\kappa = \mu$ , we call the corresponding minor contiguous. For any contiguous  $\kappa$ -by- $\kappa$  submatrix  $A[\alpha|\beta]$  of A, we call the submatrix

$$A[\alpha_1,\ldots,\alpha_{\kappa},\alpha_{\kappa}+1,\ldots,n\mid 1,\ldots,\beta_1-1,\beta_1,\ldots,\beta_{\kappa}]$$

of A having  $A[\alpha|\beta]$  in its upper right corner the *left shadow of*  $A[\alpha|\beta]$ , and, analogously, we call the submatrix

$$A[1,\ldots,\alpha_1-1,\alpha_1,\ldots,\alpha_{\kappa} \mid \beta_1,\ldots,\beta_{\kappa},\beta_{\kappa}+1,\ldots,m]$$

41 having  $A[\alpha|\beta]$  in its lower left corner the *right shadow of*  $A[\alpha|\beta]$ . By  $E_{ij}$  we denote 42 the matrix in  $\mathbb{R}^{n,m}$  which has in position (i, j) a one, while all other entries are 43 zero. A matrix  $A \in \mathbb{R}^{n,m}$  is called *totally nonnegative* (abbreviated TN henceforth) 44 if  $\det A[\alpha|\beta] \ge 0$ , for all  $\alpha, \beta \in Q_{\kappa,n'}, \kappa = 1, 2, \ldots, n'$ , where  $n' := \min\{n, m\}$ . If a 45 totally nonnegative matrix is also nonsingular, we write NsTN. If n = m, we set 46  $A^{\#} := TAT$ , where  $T = (t_{ij})$  is the permutation matrix of order n (antidiagonal 47 matrix) with  $t_{ij} := \delta_{i,n-j+1}, i, j = 1, \ldots, n$ . If A is TN, then  $A^{\#}$  is TN, too, e.g., [9, 48 Theorem 1.4.1 (iii)].

We endow  $\mathbb{R}^{n,m}$  with two partial orders: Firstly, with the usual entry-wise partial order: For  $A = (a_{kj}), B = (b_{kj}) \in \mathbb{R}^{n,m}$ 

$$A \leq B : \Leftrightarrow a_{ij} \leq b_{ij}, i = 1, \dots, n, j = 1, \dots, m.$$

Secondly, with the *checkerboard partial order*, which is defined as follows

$$A \leq B : \Leftrightarrow (-1)^{i+j} a_{ij} \leq (-1)^{i+j} b_{ij}, i = 1, \dots, n, \ j = 1, \dots, m.$$

We denote by  $\mathbb{I}(\mathbb{R}^{n,m})$  the set of all *matrix intervals* of order *n*-by-*m* with respect to the checkboard partial order

$$[A, B] := \{ Z \in \mathbb{R}^{n, m} \mid A \leq^* Z \leq^* B \}.$$

**2.2.** Auxiliary results. In this subsection we list some facts that will be employed in Sections 4 and 5. We will often make use of the following determinantal identity.

LEMMA 1. (Sylvester's Determinantal Identity), see, e.g., [9, pp.29-30] Partition  $A \in \mathbb{R}^{n,n}$ ,  $n \geq 3$ , as follows:

$$A = \begin{pmatrix} c & A_{12} & d \\ A_{21} & A_{22} & A_{23} \\ e & A_{32} & f \end{pmatrix},$$

## $\mathbf{2}$

where  $A_{22} \in \mathbb{R}^{n-2,n-2}$  and c, d, e, f are scalars. Define the submatrices

$$C := \begin{pmatrix} c & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, D := \begin{pmatrix} A_{12} & d \\ A_{22} & A_{23} \end{pmatrix},$$
$$E := \begin{pmatrix} A_{21} & A_{22} \\ e & A_{32} \end{pmatrix}, F := \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & f \end{pmatrix}.$$

Then if det  $A_{22} \neq 0$ , the following relation holds

$$\det A = \frac{\det C \det F - \det D \det E}{\det A_{22}}$$

The following two lemmata provide information on the rank of certain submatrices of TN matrices.

LEMMA 2. [9, Theorem 7.2.8] Suppose that  $A \in \mathbb{R}^{n,m}$  is TN,  $B := A[\alpha \mid \beta]$  is a contiguous, rank deficient submatrix of A, and both  $A[1, \ldots, n \mid \beta]$  and  $A[\alpha \mid 1, \ldots, m]$ have greater rank than B. Then either the left shadow or the right shadow of B has the same rank as B.

LEMMA 3. E.g., [15, Theorem 1.13] All principal minors of an NsTN matrix are positive.

62 Monotonicity properties of the determinant through matrix intervals are given in 63 the next two lemmata.

LEMMA 4. [2, Lemma 3.2] Let  $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ , A be NsTN, and B be TN. Then for any  $Z \in [A, B]$ , the following inequalities hold

$$\det A \le \det Z \le \det B.$$

LEMMA 5. Let  $[A, B] \in \mathbb{I}(\mathbb{R}^{n,n})$ , A and B be TN, and  $A[2, \ldots, n]$  be nonsingular. Then for any  $Z \in [A, B]$ , the following inequalities are true

$$\frac{\det A}{\det A[2,\ldots,n]} \le \frac{\det Z}{\det Z[2,\ldots,n]} \le \frac{\det B}{\det B[2,\ldots,n]}$$

64 Proof. Put  $A_1 := A + \epsilon E_{11}$ ,  $Z_1 := Z + \epsilon E_{11}$ , and  $B_1 := B + \epsilon E_{11}$  for some  $\epsilon > 0$ . 65 Then  $A_1 \leq^* Z_1 \leq^* B_1$ ,  $A_1$  is NsTN since  $A[2, \ldots, n]$  is nonsingular, and  $B_1$  is TN. 66 By [2, Lemma 3.2] 67

68 (1) 
$$\frac{\det A_1}{\det A_1[2,\ldots,n]} \le \frac{\det Z_1}{\det Z_1[2,\ldots,n]} \le \frac{\det B_1}{\det B_1[2,\ldots,n]}$$

By Laplace expansion along the first row of  $A_1$  we obtain det  $A_1 = \det A + \epsilon \det A[2, \ldots, n]$ , with similar expansions of det  $Z_1$ , and det  $B_1$ , which we substitute into (1) to get

$$\frac{\det A}{\det A[2,\ldots,n]} + \epsilon \le \frac{\det Z}{\det Z[2,\ldots,n]} + \epsilon \le \frac{\det B}{\det B[2,\ldots,n]} + \epsilon,$$

69 from which the claim follows.

Finally, we recall a certain type of rank conditions associated with the rank of sets of submatrices of a matrix.

DEFINITION 6. Let  $A \in \mathbb{R}^{n,n}$ . Then A satisfies the descending rank conditions if for all l with  $1 \leq l \leq n-1$ , for all z with  $0 \leq z \leq l-1$ , and for all p with  $l-z \leq p \leq n-z-1$ , the following two sets of inequalities are satisfied

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$$\operatorname{rank} A[p+1,\ldots,p+z+1|1,\ldots,l] \le \operatorname{rank} A[p,\ldots,p+z|1,\ldots,l],$$

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$$\operatorname{rank} A[1, \dots, l|p+1, \dots, p+z+1] < \operatorname{rank} A[1, \dots, l|p, \dots, p+z].$$

# 3. The condensed form of the Cauchon Algorithm and some of its properties.

**3.1. The condensed form of the Cauchon Algorithm.** We recall the definition of Cauchon diagrams and from [4] the condensed form of the Cauchon Algorithm which reduces the complexity of the orginal algorithm [12], [14].

In order to formulate the Cauchon Algorithm we need the following notation. We denote by  $\leq$  and  $\leq_c$  the lexicographic and colexicographic orders, respectively, on  $\mathbb{N}^2$ , i.e.,

 $(g,h) \le (i,j) : \Leftrightarrow (g < i) \text{ or } (g = i \text{ and } h \le j),$  $(g,h) <_c (i,j) : \Leftrightarrow (h < j) \text{ or } (h = j \text{ and } g < i).$ 

B3 DEFINITION 7. An n-by-m Cauchon diagram C is an n-by-m grid consisting of R4  $n \cdot m$  squares colored black and white, where each black square has the property that R5 either every square to its left (in the same row) or every square above it (in the same R6 column) is black.

We denote by  $C_{n,m}$  the set of all *n*-by-*m* Cauchon diagrams. We fix positions in a Cauchon diagram in the following way: For  $C \in C_{n,m}$  and  $i \in \{1, \ldots, n\}, j \in$  $\{1, \ldots, m\}, (i, j) \in C$  if the square in row *i* and column *j* is black. Here we use the usual matrix notation for the (i, j) position in a Cauchon diagram, i.e., the square in the (1, 1) position of the Cauchon diagram is in its top left corner.

DEFINITION 8. Let  $A \in \mathbb{R}^{n,m}$  and let  $C \in \mathcal{C}_{n,m}$ . We say that A is a Cauchon matrix associated with the Cauchon diagram C if for all (i, j),  $i \in \{1, ..., n\}$ ,  $j \in \{1, ..., m\}$ , we have  $a_{ij} = 0$  if and only if  $(i, j) \in C$ . If A is a Cauchon matrix associated with an unspecified Cauchon diagram, we just say that A is a Cauchon matrix.

We conclude this subsection with two results on the application of the Cauchon Algorithm, see Algorithm 1, to TN matrices.

99 THEOREM 9. [12, Theorem B4],[14, Theorem 2.6] Let  $A \in \mathbb{R}^{n,m}$ . Then A is TN 100 if and only if  $\tilde{A}$  is an (entry-wise) nonnegative Cauchon matrix.

**3.2.** TN cells. For  $\mathbb{R}^{n,m}$ , fix a set  $\mathcal{F}$  of minors. The TN cell corresponding to the set  $\mathcal{F}$  is the set of the *n*-by-*m* TN matrices for which all their zero minors are just the ones from  $\mathcal{F}$ . In [14], it is proved that the Cauchon Algorithm provides a bijection between the nonempty TN cells of  $\mathbb{R}^{n,m}$  and  $\mathcal{C}_{n,m}$ . The following theorem gives more details about this mapping.

106 THEOREM 10. [14, Theorem 2.7]

107 (i) Let  $A, B \in \mathbb{R}^{n,m}$  be TN. Then A, B belong to the same TN cell if and only 108 if  $\tilde{A}, \tilde{B}$  are associated with the same Cauchon diagram.

109 (ii) Let  $A \in \mathbb{R}^{n,m}$ . Then A is contained in the TN cell associated with  $C \in \mathcal{C}_{n,m}$ 110 if and only if  $\tilde{a}_{ij} = 0$  if  $(i, j) \in C$  and  $\tilde{a}_{ij} > 0$  if  $(i, j) \notin C$ . **Algorithm 1** (Condensed form of the Cauchon Algorithm) [1, Algorithm 3.3], [4, Algorithm 3.2]

Let  $A = (a_{ij}) \in \mathbb{R}^{n,m}$ . Set  $A^{(n)} := A$ . For  $k = n - 1, \dots, 1$  define  $A^{(k)} = (a_{ij}^{(k)}) \in \mathbb{R}^{n,m}$  as follows: For  $j = 1, \dots, m - 1$ , set  $s_j := \min\left\{h \in \{j + 1, \dots, m\} \mid a_{k+1,h}^{(k+1)} \neq 0\right\}$  (set  $s_j := \infty$  if this set is empty), for  $i = 1, \dots, k$ ,

$$a_{ij}^{(k)} := \begin{cases} a_{ij}^{(k+1)} - \frac{a_{k+1,j}^{(k+1)}a_{i,sj}^{(k+1)}}{a_{k+1,s_j}^{(k+1)}}, & \text{if } s_j < \infty, \\ a_{ij}^{(k+1)}, & \text{if } s_j = \infty, \end{cases}$$

and for i = k + 1, ..., n, j = 1, ..., m, and i = 1, ..., k, j = m

$$a_{ij}^{(k)} := a_{ij}^{(k+1)}.$$

Put  $\tilde{A} := A^{(1)}$ .

**3.3. Lacunary sequences.** We recall from [14] the definition of a lacunary sequence associated with a Cauchon diagram.

113 DEFINITION 11. Let  $C \in \mathcal{C}_{n,m}$ . We say that a sequence

114 (2) 
$$\gamma := ((i_k, j_k), \ k = 0, 1, \dots, t),$$

115 which is strictly increasing in both arguments is a lacunary sequence with respect to

116 C if the following conditions hold:

117 1.  $(i_k, j_k) \notin C, \ k = 1, \dots, t;$ 

118 2.  $(i, j) \in C$  for  $i_t < i \le n$  and  $j_t < j \le m$ .

119 3. Let  $s \in \{1, \dots, t-1\}$ . Then  $(i, j) \in C$  if

- (a) either for all (i, j),  $i_s < i < i_{s+1}$  and  $j_s < j$ , or for all (i, j),  $i_s < i < i_{s+1}$  and  $j_0 \le j < j_{s+1}$ and
  - (b) either for all (i, j),  $i_s < i$  and  $j_s < j < j_{s+1}$ 
    - or for all (i, j),  $i < i_{s+1}$ , and  $j_s < j < j_{s+1}$ .
- 125 We call t the *length* of  $\gamma$ .

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We recall now from [4] and [8] the construction of two special lacunary sequences. In the first case, let  $\delta_{ij} := \det A[i_0, i_1, \dots, i_p \mid j_0, j_1, \dots, j_p]$  be the minor of A associated to the sequence  $\gamma$  given by (2) starting at position  $(i, j) = (i_0, j_0)$  which is formed by the following procedure. We explain the construction only from the starting pair to the next index pair. The process is then continued analogously.

132PROCEDURE 12. [4, Procedure 5.2] Construction of the sequence  $\gamma$  given by (2)133starting at  $(i_0, j_0)$  to the next index pair  $(i_1, j_1)$  for the n-by-m TN matrix A.134If  $i_0 = n$  or  $j_0 = m$  or  $\mathcal{U} := \{(i, j) \mid i_0 < i \le n, j_0 < j \le m, and 0 < \delta_{ij}\}$  is135void then terminate with p := 0;136else

137	if $\delta_{ij_0} = 0$ for all $i = i_0 + 1, \dots, n$ then put $(i_1, j_1) := \min \mathcal{U}$ with
138	respect to the colexicographic order
139	else
140	<b>put</b> $i' := \min \{k \mid i_0 < k \le n \text{ such that } 0 < \delta_{kj_0}\},\$
141	$J := \{k \mid j_0 < k \le mmbox such that \ 0 < \delta_{i',k}\};$
142	if J is not void then put $(i_1, j_1) := (i', \min J)$
143	else put $(i_1, j_1) := \min \mathcal{U}$ with respect to the lexicographic order;
144	end if
145	end if
146	end if.

147 The following proposition provides a representation of the determinant of the 148 submatrix associated to a lacunary sequence with respect to  $C_{\tilde{A}}$ .

149 PROPOSITION 13. [8, Corollary 3.3] Let  $A \in \mathbb{R}^{n,m}$  be such that  $\tilde{A}$  is a Cauchon 150 matrix and let  $\gamma = ((i_0, j_0), (i_1, j_1), \dots, (i_t, j_t))$  be a lacunary sequence with respect to 151  $C_{\tilde{A}}$ . Then the representation

152 (3) 
$$\det A[i_0, i_1, \dots, i_t | j_0, j_1, \dots, j_t] = \tilde{a}_{i_0, j_0} \cdot \tilde{a}_{i_1, j_1} \cdots \tilde{a}_{i_t, j_t}$$

153 *holds*.

The following proposition shows that a certain sequence of zeros in a column or a row of  $\tilde{A}$  is the result of a zero column or row or submatrix in the bottom left or top right part of A.

157 PROPOSITION 14. Let  $A \in \mathbb{R}^{n,m}$  be such that  $\tilde{A} \in \mathbb{R}^{n,m}$  is a Cauchon matrix. 158 Then

(*i*) If  $\tilde{A}[i,...,n \mid j] = 0$  for some  $i \in \{1,...,n\}$  and  $j \in \{1,...,m\}$ , then all entries of  $A[i,...,n \mid 1,...,j]$  are zero or the *j*th column of A is zero.

161 (ii) If  $A[i \mid j, ..., m] = 0$  for some  $i \in \{1, ..., n\}$  and  $j \in \{1, ..., m\}$ , then all 162 entries of  $A[1, ..., i \mid j, ..., m]$  are zero or the *i*th row of A is zero.

Proof. We only give the proof for (i) since the proof of (ii) is parallel. Since  $\tilde{A}$  is a Cauchon matrix and  $\tilde{A}[i, \ldots, n \mid j] = 0$ , we have  $\tilde{A}[i, \ldots, n \mid 1, \ldots, j] = 0$  or  $\tilde{A}[1, \ldots, n \mid j] = 0$ . In the following we assume that  $\tilde{A}[i, \ldots, n \mid 1, \ldots, j] = 0$ . We proceed by decreasing induction on the row index to show that  $a_{st} = 0, s = i, \ldots, n, t = 1, \ldots, j$ . For s = n, by Algorithm 1,  $a_{nt} = \tilde{a}_{nt} = 0, t = 1, \ldots, j$ . Assume that  $a_{ht} = 0, h = s + 1, \ldots, n, t = 1, \ldots, j$ . We show that  $a_{st} = 0, t = 1, \ldots, j$ . From each position  $(s, t), t = 1, \ldots, j$ , we construct by Procedure 12 a lacunary sequence  $\gamma_{st} = ((s, t), (s_1, t_1), \ldots, (s_p, t_p))$  with respect to  $C_{\tilde{A}}$ . If  $\gamma_{st} = ((s, t))$ , then by Proposition 13

$$a_{st} = \det A[s \mid t] = \tilde{a}_{st} = 0.$$

Therefore, we assume in the following that  $\gamma_{st}$  has positive length. By the induction hypothesis and Laplace expansion along the first column of  $A[s, s_1, \ldots, s_p \mid t, t_1, \ldots, t_p]$ , we obtain

$$\det A[s, s_1, \dots, s_p \mid t, t_1, \dots, t_p] = a_{st} \det A[s_1, \dots, s_p \mid t_1, \dots, t_p].$$

163 Since  $\gamma_{st}$  and  $((s_1, t_1), \dots, (s_p, t_p))$  are lacunary sequences, it follows from Proposition 164 13 that

165 (4) 
$$\det A[s, s_1, \dots, s_p \mid t, t_1, \dots, t_p] = \tilde{a}_{st} \cdot \tilde{a}_{s_1, t_1} \cdots \tilde{a}_{s_p, t_p}$$

$$= 0 \cdot \det A[s_1, \dots, s_p \mid t_1, \dots, t_p].$$

Let  $A \in \mathbb{R}^{n,m}$  be such that  $\tilde{A}$  is a Cauchon matrix. Then by the following 172173procedure a uniquely determined lacunary sequence is constructed which is related to the rank of A. 174

#### **PROCEDURE 15.** Let $A \in \mathbb{R}^{n,m}$ be a Cauchon matrix. Construct the sequence 175

176 (5) 
$$\gamma = ((i_p, j_p), \dots, (i_0, j_0))$$

as follows: 177

• Put 
$$(i_{-1}, j_{-1}) := (n+1, m+1)$$
.

• For k = 0, 1, ..., define

$$M_k := \{(i,j) \mid 1 \le i < i_{k-1}, \ 1 \le j < j_{k-1}, \ \tilde{a}_{ij} \ne 0\}$$

If  $M_k = \phi$ , put p := k - 1. Otherwise, put  $(i_k, j_k) := \max M_k$ , where the maximum is taken with respect to the lexicographic order. 180

**PROPOSITION 16.** Let  $A \in \mathbb{R}^{n,m}$  be such that  $\tilde{A}$  is a Cauchon matrix. Then for all  $(i, j) \in \mathcal{S}$ 

$$\operatorname{rank}(A[i, i+1, \dots, n \mid 1, 2, \dots, j]) = \eta + 1$$

where  $\eta$  is the length of the sequence that is obtained by application of Procedure 15 181 to  $A[i, i+1, ..., n \mid 1, 2, ..., j]$ , provided that  $A[i, i+1, ..., n \mid 1, 2, ..., j] \neq 0$ . 182

*Proof.* The matrix that is obtained by application of Algorithm 1 to B := A[i, i + i]183

 $1, \ldots, n \mid 1, 2, \ldots, m$  coincides with  $A[i, i+1, \ldots, n \mid 1, 2, \ldots, m]$ . Hence if we apply 184Procedure 15 to  $\hat{B}[1, ..., n-i+1 \mid 1, ..., j] = \hat{A}[i, i+1, ..., n \mid 1, 2, ..., j]$  and proceed 185

parallel to the proof of [8, Theorem 3.4], we are done. 186

COROLLARY 17. Let  $A \in \mathbb{R}^{n,m}$  be such that  $\tilde{A}$  is a Cauchon matrix. Then for all  $(i,j) \in \mathcal{S}$ 

$$\operatorname{rank}(A[1, 2, \dots, i \mid j, j+1, \dots, m]) = \eta + 1,$$

where  $\eta$  is the length of the sequence that is obtained by application of Procedure 15 187to  $\tilde{A}[1, 2, \dots, i \mid j, j+1, \dots, m]$ , provided that  $A[1, 2, \dots, i \mid j, j+1, \dots, m] \neq 0$ . 188

THEOREM 18. [8, Theorem 3.2] Let  $A \in \mathbb{R}^{n,m}$  be such that  $\tilde{A}$  is a Cauchon 189 matrix. Then for i = 1, ..., n and  $0 \le l \le n - i$ , the rows i, i + 1, ..., i + l of A are 190linearly independent if and only if application of Procedure 15 to  $A[i, \ldots, i+l|1, \ldots, m]$ 191results in a sequence of length l. 192

COROLLARY 19. [8, Corollary 3.2] Let  $A \in \mathbb{R}^{n,m}$  be such that  $\tilde{A}$  is a Cauchon ma-193 trix. Then for j = 1, ..., m and  $0 \le l \le m - j$ , the columns j, j + 1, ..., j + l of A are 194 linearly independent if and only if application of Procedure 15 to  $A[1, \ldots, n|j, \ldots, j+l]$ 195results in a sequence of length l. 196

**3.4.** Descending rank conditions. In this subsection, we link the descending 197 rank conditions, see Definition 6, to Algorithm 1. 198

THEOREM 20. [8, Theorem 4.4] Let  $A \in \mathbb{R}^{n,n}$  and  $B := A^{\#}$ . If A satisfies the 199200 descending rank conditions, then the following statements hold:

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- (i) If  $b_{ij} = 0$  for some  $i \ge j$ , then  $b_{it} = 0$  for all t < j; 201
- (ii) if  $\tilde{b}_{ij} = 0$  for some  $i \leq j$ , then  $\tilde{b}_{tj} = 0$  for all t < i; 202
- (iii) B is a Cauchon matrix. 203

THEOREM 21. [8, Theorem 4.8] Let  $A \in \mathbb{R}^{n,n}$  and  $B := A^{\#}$ . Then the following 204205statements are equivalent:

- (a) A satisfies the descending rank conditions. 206
- (b) B satisfies (i) and (ii) in Theorem 20. 207

4. Relaxing nonsingularity to linear independence of certain rows and 208 columns. For the rest of the paper, we assume for the ease of presentation that the 209given TN matrices do not contain a zero row or column. This is not a restriction 210211 because after deletion of the respective rows and columns the resulting matrix is again TN.212

DEFINITION 22. Let  $A \in \mathbb{R}^{n,m}$  be such that  $\tilde{A}$  is a Cauchon matrix. For a given 213 lacunary sequence  $\gamma = ((i_0, j_0), (i_1, j_1), \dots, (i_p, j_p))$ , the order of the sequence is given 214by215

216 (6) 
$$l := \min\left\{k \mid \tilde{A}[i_k + 1, \dots, n|j_k] = 0 \text{ or } \tilde{A}[i_k|j_k + 1, \dots, m] = 0\right\};$$

we set l := p if the set in (6) is empty. 217

**Condition I.** Let  $A \in \mathbb{R}^{n,m}$  be such that  $\tilde{A}$  is a Cauchon matrix. For all  $(i, j) \in S$ , 218 the rows  $i + 1, \ldots, i + \ell$  and columns  $j + 1, \ldots, j + \ell$  of A are linearly independent 219provided that  $\ell > 0$ , where  $\ell$  is the smallest among the orders of all the lacunary 220 221sequences with respect to  $C_{\tilde{A}}$  that start from (i, j).

In the sequel, it will always be clear from the context to which pairs  $(i, j) \in S$ 223the quantity  $\ell$  refers. Therefore, it will not be necessary to indicate this dependency. 224

LEMMA 23. Let  $A \in \mathbb{R}^{n,m}$  be such that  $\tilde{A}$  is a Cauchon matrix and assume that Condition I holds. Then for any  $(i, j) \in S$  with  $\ell > 0$ , there exists a lacunary sequence 226  $\gamma = ((i, j), (i_1, j_1), \dots, (i_p, j_p))$  with respect to  $C_{\tilde{A}}$  of order  $\ell$  starting from (i, j) such 227that 228

229

2.2.2

(7) 
$$d(i, i_1, \dots, i_\ell) = 0 \quad or \quad d(j, j_1, \dots, j_\ell) = 0$$

where  $\ell$  is given as in Condition I. 230

> *Proof.* Suppose on the contrary that there exists  $(i_0, j_0) \in S$  with  $\ell > 0$  such that for any lacunary sequence  $\gamma = ((i_0, j_0), (i_1, j_1), \dots, (i_p, j_p))$  with respect to  $C_{\tilde{A}}$  of order  $\ell$  we have  $d(i_0, i_1, \ldots, i_\ell) > 0$  and  $d(j_0, j_1, \ldots, j_\ell) > 0$ . Moreover, assume that  $\gamma$  is chosen in such a way that  $(i_0, j_0)$  is the maximum such pair with respect to the lexicographic order. Therefore, we may conclude that

$$d(i_1, \ldots, i_\ell) = 0$$
 or  $d(j_1, \ldots, j_\ell) = 0.$ 

Without loss of generality we may assume that  $d(j_1, \ldots, j_\ell) = 0$  and  $j_1 = j_0 + 2$ . **Case 1.**  $i_{\ell} = n$  or  $\tilde{a}_{s,j_{\ell}} = 0, s = i_{\ell} + 1, \dots, n$ .

If  $\tilde{a}_{s,j_{\ell}} = 0, \ s = i_{\ell} + 1, \dots, n$ , then it follows that  $\tilde{A}[i_{\ell} + 1, \dots, n \mid 1, \dots, j_{\ell}] = 0$ because A is a Cauchon matrix. Hence, in either case it is easy to see that  $(i_{\ell}, j_{\ell})$  is the maximum pair with respect to the lexicographic order of the set

$$\{(u,v) \mid 1 \le u \le n, \ 1 \le v \le j_{\ell}, \ \tilde{a}_{uv} \ne 0\}$$

Moreover, since  $d(j_1, \ldots, j_\ell) = 0$  and  $\gamma = ((i, j), (i_1, j_1), \ldots, (i_p, j_p))$  is a lacunary sequence with respect to  $C_{\tilde{A}}$ , for  $s = 1, \ldots, \ell - 1$ , we have  $(i_s, j_s)$  is the maximum pair with respect to the lexicographic order of the set

$$\{(u, v) \mid 1 \le u < i_{s+1}, \ 1 \le v < j_{s+1}, \ \tilde{a}_{uv} \neq 0\}.$$

Therefore, the sequence which is obtained by the application of Procedure 15 to the columns  $j_1, j_2, \ldots, j_\ell$  coincides with the sequence  $((i_1, j_1), (i_2, j_2), \ldots, (i_\ell, j_\ell))$ . Now we apply Procedure 15 to the columns  $j_0 + 1, j_0 + 2, \ldots, j_0 + \ell$  which coincide with the columns  $j_0 + 1 = j_1 - 1, j_2 - 1, \ldots, j_\ell - 1$ . This results in the lacunary sequence  $((i'_1, j'_1), \ldots, (i'_{\tau}, j'_{\tau}))$ , where  $\tau \leq \ell$ . If  $\tau \leq \ell - 1$ , then by Corollary 19, the columns  $j_0+1, j_0+2, \ldots, j_0+\ell$  are linearly dependent which contradicts Condition I. Therefore, we have  $\tau = \ell$  and hence  $j'_k = j_k - 1, k = 1, 2, \ldots, \ell = \tau$ . Since  $\gamma$  is a lacunary sequence,  $\ell \geq 1$ , A does not have a zero row or column, and  $j_1 = j_0 + 2$ , we have

$$\tilde{a}_{t,i_0+1} = 0, \quad t = 1, 2, \dots, i_1 - 1,$$

which implies that  $i'_1 > i_0$ . Since application of Procedure 15 to the columns  $j_1, j_2, \ldots, j_\ell$ results in the sequence  $((i_1, j_1), (i_2, j_2), \ldots, (i_\ell, j_\ell))$  and  $d(j_1, \ldots, j_\ell) = 0$ , we conclude that for  $g = 0, 1, \ldots, \ell - 1$ , if  $d(i_g, i_{g+1}) > 0$ , then it follows that  $\tilde{a}_{uv} = 0$ ,  $u = i_g + 1, \ldots, i_{g+1} - 1, v = 1, \ldots, i_{g+1} - 1$ . Therefore, we may conclude that

$$i'_k = i_k, \quad k = 1, 2, \dots, \ \ell = \tau.$$

Hence the sequence which is obtained by appending  $((i_0, j_0), (i'_1, j'_1), \dots, (i'_{\ell}, j'_{\ell}))$  to a

lacunary sequence which starts from  $(i'_{\ell}, j'_{\ell})$  is a lacunary sequence with respect to  $C_{\tilde{A}}$ , has order  $\ell$ , and  $d(j_0, j'_1, \ldots, j'_{\ell}) = 0$  which contradicts our assumption.

234 **Case 2.**  $j_{\ell} = m$  or  $\tilde{a}_{i_{\ell},s} = 0, s = j_{\ell} + 1, \dots, m$ .

235 The proof is parallel to the one of Case 1.

LEMMA 24. Let  $A \in \mathbb{R}^{n,m}$  be TN and suppose Condition I holds. Then for any  $(i, j) \in S$  with  $\ell > 0$  we have

$$\det A[i+1, i+2, \dots, i+\ell \mid j+1, j+2, \dots, j+\ell] > 0,$$

236 where  $\ell$  is given as in Condition I.

*Proof.* By Theorem 9,  $\hat{A}$  is a Cauchon matrix. Suppose on the contrary that there exists  $(i_0, j_0) \in S$  such that the determinant of the matrix

$$B := A[i_0 + 1, i_0 + 2, \dots, i_0 + \ell \mid j_0 + 1, j_0 + 2, \dots, j_0 + \ell]$$

vanishes. Moreover, assume that  $(i_0, j_0)$  is the maximum such pair with respect to the lexicographic order and let  $\gamma = ((i_0, j_0), (i_1, j_1), \dots, (i_p, j_p))$  be an associated lacunary sequence with respect to  $C_{\tilde{A}}$  of order  $\ell > 0$  with  $d(i_0, i_1, \dots, i_\ell) = 0$  or  $d(j_0, j_1, \dots, j_\ell) = 0$  which exists by Lemma 23. Without loss of generality, we may assume that  $d(j_0, j_1, \dots, j_\ell) = 0$ . By Lemma 2 and Condition I, the left or the right shadow of B has rank at most  $\ell - 1$ . Since  $((i_1, j_1), \dots, (i_p, j_p))$  is a lacunary sequence with  $\tilde{a}_{i_1, j_1} \neq 0$ , we have by Proposition 13

$$\det A[i_1,\ldots,i_p \mid j_1,\ldots,j_p] \neq 0,$$

and we conclude by Lemma 3 that

$$\det A[i_1,\ldots,i_\ell \mid j_1,\ldots,j_\ell] \neq 0.$$

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Because  $A[i_1, \ldots, i_{\ell} \mid j_1, \ldots, j_{\ell}]$  lies completely in the left shadow of B, the left shadow of B has rank at least  $\ell$ . By Theorem 18, application of Procedure 15 to the rows  $i_0+1, \ldots, i_0+\ell$  results in the lacunary sequence  $((i_0+1, \beta_1), (i_0+2, \beta_2), \ldots, (i_0+\ell, \beta_{\ell}))$ . If  $\beta_1 > j_0$ , then by Corollary 17 the right shadow of  $A[i_0+1, i_0+2, \ldots, i_0+\ell|j_0+1, j_0+2, \ldots, j_0+\ell]$  has rank at least  $\ell$ . Now we assume that  $\beta_1 \leq j_0$ . Let  $s \in \{1, 2, \ldots, \ell\}$  be the smallest integer such that  $\beta_s > j_0$ . Note that  $s \geq 2$ . Define  $(i'_0, j'_0) = (i_0, j_0)$  and for  $k = 1, 2, \ldots, \tau$ , let

$$(i'_k, j'_k) := \min\left\{ (i, j) \mid i = i'_{k-1} + 1, \ j > j_{k-1}, \ \tilde{a}_{ij} > 0 \right\},\$$

where the minimum is taken with respect to the lexicographic order. Consider the 237sequence  $((i'_0, j'_0), (i'_1, j'_1), \ldots, (i'_{\tau}, j'_{\tau}))$ . If  $j'_{\tau} = m$ , then this sequence is a lacunary 238 sequence with respect to  $C_{\tilde{A}}$  since for each  $t = 0, 1, \ldots, \tau - 1, i'_{t+1} = i'_t + 1$  and there 239exists  $\xi_{t+1} < j'_{t+1}$  such that  $\tilde{a}_{i'_{t+1},\xi_{t+1}} > 0$ . Otherwise, we append it to a lacunary 240 sequence starting from  $(i'_{\tau}, j'_{\tau})$  such that the resulting sequence is a lacunary sequence 241with respect to  $C_{\tilde{A}}$ . Hence the order of this sequence is  $\tau$  which is less than  $\ell$  and 242 $d(i'_0, i'_1, \ldots, i'_{\tau}) = 0$  which contradicts our assumption. Therefore,  $\beta_1 > j_0$  and the 243244 right shadow of B has rank at least  $\ell$  which implies by Lemma 2 that det B > 0, a contradiction. Since we have obtained a contradiction both in the event of a left and 245right shadow, the proof is completed. 246Π

Now we turn to the construction of a lacunary sequence with the properties stated in Lemma 23. The procedure is based on the following lemma.

LEMMA 25. Let  $A \in \mathbb{R}^{n,m}$  be such that  $\tilde{A}$  is a Cauchon matrix and suppose Condition I holds. Then for all  $(i, j) \in S$  such that  $\tilde{A}[i+1, \ldots, n|j+1, \ldots, m] \neq 0$ , let

252 
$$s_j := \min \{k \in \{i+1, \dots, n\} \mid \tilde{a}_{kj} \neq 0\},\$$

253 
$$t_i := \min \left\{ k \in \{j+1, \dots, m\} \mid \tilde{a}_{ik} \neq 0 \right\},\$$

provided that both sets are not empty. Then it follows that

$$\tilde{a}_{s_i,j+1} \neq 0$$
 or  $\tilde{a}_{i+1,t_i} \neq 0$ .

261 PROCEDURE 26. Construction of a lacunary sequence  $\gamma = ((i_0, j_0), (i_1, j_1), \dots, (i_p, j_p))$ 262 starting at  $(i_0, j_0) \in S$  to the next index pair  $(i_1, j_1)$  in the n-by-m matrix A such that 263  $\tilde{A}$  is a Cauchon matrix and A satisfies Condition I.

264 If  $\mathcal{U} := \{(i, j) \mid i_0 < i \le n, j_0 < j \le m, and 0 < \tilde{a}_{i,j}\}$  is void then termi-265 nate with p := 0; 266 else 267 if  $\tilde{a}_{i,j_0} = 0$  for all  $i = i_0 + 1, ..., n$  or  $\tilde{a}_{i_0,j} = 0$  for all  $j = j_0 + 1, ..., m$ 268 then put  $(i_1, j_1) := \min \mathcal{U}$  with respect to the colexicographic order and 269 lexicographic order, respectively;

270	else put
271	$i' := \min \{k \mid i_0 < k \le n \text{ such that } \tilde{a}_{k,j_0} \neq 0\},\$
272	$j' := \min \left\{ k \mid j_0 < k \le m \text{ such that } \tilde{a}_{i_0, j} \neq 0 \right\};$
273	if $\tilde{a}_{i',j_0+1} \neq 0$ then $put(i_1,j_1) := (i',j_0+1);$
274	<b>else</b> put $(i_1, j_1) := (i_0 + 1, j');$
275	end if
276	end if
277	end if.

2785. Application to intervals of totally nonnegative matrices. In this section, we consider matrices that satisfy Condition I. In [2], the proof of the interval 279property of the NsTN matrices relies on the fact that the entries of A obtained from 280A by application of Algorithm 1 can be represented as a ratio of contiguous minors 281of A. If we relax the nonsingularity assumption and would like to employ such a 282representation, we have to avoid division by a zero minor. We accomplish this by 283 284 using Lemma 2, where the linear independence of the respective rows and columns is assured by Condition I. Then only the vanishing of the left or the right shadow of a 285zero contiguous minor has to be considered. 286

287

Let  $A \in \mathbb{R}^{n,m}$  be TN. For any  $(i_0, j_0) \in S$ , we can construct a lacunary sequence  $((i_0, j_0), (i_1, j_1), \dots, (i_p, j_p))$  with respect to the Cauchon diagram  $C_{\tilde{A}}$ , and by Proposition 13 we may conclude that

$$\det A[i_0, i_1, \dots, i_p | j_0, j_1, \dots, j_p] = \tilde{a}_{i_0, j_0} \cdot \tilde{a}_{i_1, j_1} \cdots \tilde{a}_{i_p, j_p}.$$

Hence by application of this representation to the lacunary sequence  $((i_1, j_1), \ldots, (i_p, j_p))$  we obtain that

290 (8) 
$$\tilde{a}_{i_0,j_0} = \frac{\det A[i_0, i_1, \dots, i_p | j_0, j_1, \dots, j_p]}{\det A[i_1, \dots, i_p | j_1, \dots, j_p]}$$

1 4 6 . . . . .

 $=\frac{\det A[i_0,i_1,\ldots,i_\ell|j_0,j_1,\ldots,j_\ell]}{\det A[i_1,\ldots,i_\ell|j_1,\ldots,j_\ell]}.$ 

Therefore, each entry of  $\tilde{A}$  can be represented as a ratio of two minors. We want to strengthen this representation in that each entry of  $\tilde{A}$  can even be represented as a ratio of two *contiguous* minors. We call p the *order* of the representation (8).

Now let A in addition satisfy Condition I with  $\ell > 0$ . Then by Procedure 26, for any  $(i_0, j_0) \in S$  we can construct a lacunary sequence  $((i_0, j_0), (i_1, j_1), \dots, (i_p, j_p))$  of order  $\ell$  with (7). Without loss of generality, we may assume that  $d(j_0, j_1, \dots, j_\ell) = 0$ holds. By Proposition 14,  $A[i_\ell+1, \dots, n \mid 1, \dots, j_\ell] = 0$  or  $A[1, \dots, i_\ell \mid j_\ell+1, \dots, m] =$ 0 holds. By (8) and the zero-nonzero pattern of A, we have

$$\begin{split} \tilde{a}_{i_0,j_0} &= \frac{\det A[i_0,i_1,\dots,i_p|j_0,j_1,\dots,j_p]}{\det A[i_1,\dots,i_p|j_1,\dots,j_p]} \\ &= \frac{\det A[i_0,i_1,\dots,i_\ell|j_0,j_1,\dots,j_\ell] \det A[i_{\ell+1},\dots,i_p|j_{\ell+1},\dots,j_p]}{\det A[i_1,\dots,i_\ell|j_1,\dots,j_\ell] \det A[i_{\ell+1},\dots,i_p|j_{\ell+1},\dots,j_p]} \end{split}$$

302 PROPOSITION 27. Let  $A = (a_{ij}) \in \mathbb{R}^{n,m}$  be TN and suppose Condition I holds. Then the entries  $\tilde{a}_{ij}$  of the matrix  $\tilde{A}$  can be represented as 303

304 (10) 
$$\tilde{a}_{i,j} = \frac{\det A[i, i+1, \dots, i+\ell | j, j+1, \dots, j+\ell]}{\det A[i+1, \dots, i+\ell | j+1, \dots, j+\ell]},$$

where  $\ell$  is given in Condition I and is assumed to be positive. 305

*Proof.* By Theorem 9,  $\tilde{A}$  is a nonnegative Cauchon matrix. By the preced-306 ing consideration, for each position  $(i_0, j_0) \in S$ , there exists a lacunary sequence 307  $((i_0, j_0), (i_1, j_1), \dots, (i_p, j_p))$  with respect to the Cauchon diagram  $C_{\tilde{A}}$  of order  $\ell$  such 308 309 that

310 (11) 
$$\tilde{a}_{i_0,j_0} = \frac{\det A[i_0, i_1, \dots, i_\ell | j_0, j_1, \dots, j_\ell]}{\det A[i_1, \dots, i_\ell | j_1, \dots, j_\ell]}.$$

Using Lemma 23, we can assume without loss of generality that  $d(j_0, j_1, \ldots, j_\ell) = 0$ . By Proposition 13 and Lemma 3, det  $A[i_1, i_2, \ldots, i_\ell | j_1, j_2, \ldots, j_\ell] \neq 0$  holds, since  $((i_1, j_1), \dots, (i_p, j_p))$  is a lacunary sequence and det  $A[i_1, i_2, \dots, i_p | j_1, j_2, \dots, j_p] \neq 0.$ By Proposition 16, the rank of the matrix  $B := A[i_0 + 1, i_0 + 2, \dots, n|1, 2, \dots, j_\ell]$  is  $\ell$ . Let  $R_{i_0+1}, R_{i_0+2}, \ldots, R_n$  be the rows of the matrix B. Hence we may represent  $R_h = \sum_{s=1}^{\ell} \alpha_{h,s} R_{i_s}, \ h = i_0 + 1, i_0 + 2, \dots, i_0 + \ell.$  Therefore, we may conclude

$$A[i_0 + 1, i_0 + 2, \dots, i_0 + \ell \mid 1, 2, \dots, j_\ell] = CA[i_1, \dots, i_\ell \mid 1, 2, \dots, j_\ell],$$

311 where 
$$C = (c_{t_1,t_2}) \in \mathbb{R}^{\ell,\ell}$$
 with  $c_{t_1,t_2} = \alpha_{i_0+t_1,t_2}, t_1, t_2 = 1, 2, \dots, \ell$ .

In particular, we obtain for a special choice of the column vectors 312

313 
$$A[i_0+1, i_0+2, \dots, i_0+\ell | j_0+1, j_0+2, \dots, j_0+\ell] = CA[i_1, i_2, \dots, i_\ell | j_0+1, j_0+2, \dots, j_0+\ell]$$
  
314  $= CA[i_1, i_2, \dots, i_\ell | j_1, j_2, \dots, j_\ell],$ 

whence 315

316 (12) 
$$\det A[i_0 + 1, i_0 + 2, \dots, i_0 + \ell | j_0 + 1, j_0 + 2, \dots, j_0 + \ell] = 317 \quad \det C \det A[i_1, i_2, \dots, i_\ell | j_1, j_2, \dots, j_\ell].$$

Since by Lemma 24

$$\det A[i_0+1, i_0+2, \dots, i_0+\ell | j_0+1, j_0+2, \dots, j_0+\ell] \neq 0$$

and

$$\det A[i_1, i_2, \dots, i_{\ell} | j_1, j_2, \dots, j_{\ell}] \neq 0,$$

we conclude that  $\det C \neq 0$ . 318 Moreover, we obtain 319

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$$A[i_0, i_0 + 1, \dots, i_0 + \ell | j_0, j_0 + 1, \dots, j_0 + \ell] = C' A[i_0, i_1, \dots, i_\ell | j_0, j_1, \dots, j_\ell]$$

where  $C' \in \mathbb{R}^{\ell+1,\ell+1}$  is given by

$$C' = \left[ \begin{array}{cc} 1 & 0 \\ 0 & C \end{array} \right]$$

which yields 321

322 (13) 
$$\det A[i_0, i_0 + 1, \dots, i_0 + \ell | j_0, j_0 + 1, \dots, j_0 + \ell]$$
  
323 
$$= \det C' \det A[i_0, i_1, \dots, i_\ell | j_0, j_1, \dots, j_\ell].$$

Since det  $C' = \det C$ , the representation follows now from (11)-(13). 324

THEOREM 28. Let  $A = (a_{kj}), B = (b_{kj}) \in \mathbb{R}^{n,m}$  be TN such that Condition I holds and  $A \leq^* B$ . Then  $\tilde{A} \leq^* \tilde{B}$  and the entries  $\tilde{a}_{kj}$  and  $\tilde{b}_{kj}$  of  $\tilde{A}$  and  $\tilde{B}$ , respectively, can be represented as ratios of contiguous minors of the same order,  $k = 1, \ldots, n$ ,  $j = 1, \ldots, m$ .

*Proof.* Let A and B be TN. Then by Theorem 9,  $\hat{A}$  and  $\hat{B}$  are nonnegative 329 Cauchon matrices. We show by decreasing induction with respect to the lexicographic 330 order on (k, j) that if  $\tilde{a}_{kj}$  and  $\tilde{b}_{kj}$  have representations as in (10) of order l and l', 331 respectively, then both of them can be represented as ratios of contiguous minors 332 of the same order and  $(-1)^{k+j}\tilde{a}_{kj} \leq (-1)^{k+j}\tilde{b}_{kj}$ . For k=n or j=m, the result is trivial and follows by the application of Algorithm 1 and the assumption that 334  $A \leq B$ . Suppose the claim holds for all  $(k^{\circ}, j^{\circ})$  such that  $(k^{\circ}, j^{\circ}) > (k, j)$  with 335 respect to the lexicographic order. We show that the claim holds for the entries in the 336 position (k, j). Let  $((k, j), (k_1, j_1), \dots, (k_p, j_p))$  and  $((k, j), (k'_1, j'_1), \dots, (k'_{p'}, j'_{p'}))$  be 337 the lacunary sequences that start from the position (k, j) with respect to the Cauchon 338 diagrams  $C_{\tilde{A}}$  and  $C_{\tilde{B}}$ , respectively. Then by Proposition 27,  $\tilde{a}_{kj}$  and  $b_{kj}$  allow the 339following representations<sup>1</sup> 340

341 (14) 
$$\tilde{a}_{kj} = \frac{\det A[k, \dots, k+l|j, \dots, j+l]}{\det A[k+1, \dots, k+l|j+1, \dots, j+l]},$$

343 (15) 
$$\tilde{b}_{kj} = \frac{\det B[k, \dots, k+l'|j, \dots, j+l']}{\det B[k+1, \dots, k+l'|j+1, \dots, j+l']},$$

where l and l' are defined as in Condition I.

Let k+j be even; the proof of the case that k+j is odd is parallel. Then the following three cases are possible:

**Case 1**: Suppose that l = l'. Then by (14), (15), and Lemma 5, we have

$$\tilde{a}_{kj} \leq b_{kj}$$

**Case 2**: Suppose that l < l'. By Lemma 23 and without loss of generality, we may assume that  $d(j_0, j_1, \ldots, j_l) = 0$ . If k = n - 1, then l' = 1, l = 0. Hence  $\tilde{A}[n \mid 1, \ldots, j] = 0$  or  $\tilde{A}[1, \ldots, n - 1 \mid j + 1, \ldots, m] = 0$  which implies by Proposition 14 that  $A[n \mid 1, \ldots, j] = 0$  or  $A[1, \ldots, n - 1 \mid j + 1, \ldots, m] = 0$ . In particular,  $a_{nj} = 0$  or  $a_{n-1,j+1} = 0$ . Thus  $b_{nj} = 0$  or  $b_{n-1,j+1} = 0$  since n + j is odd and  $A \leq^* B$  which implies that  $B[n \mid 1, \ldots, j] = 0$  or  $B[1, \ldots, n - 1 \mid j + 1, \ldots, m] = 0$ . Therefore,  $\tilde{B}[n \mid 1, \ldots, j] = 0$  or  $\tilde{B}[1, \ldots, n - 1 \mid j + 1, \ldots, m] = 0$ . Whence l' = 0 which is a contradiction. Let  $h := \min\{s : \tilde{a}_{k_s+1,j_s} = 0\}$ . The sequence  $((k_h+1, j_h), (k_{h+1}, j_{h+1}), \ldots, (k_p, j_p))$  is a lacunary sequence since  $d(j_0, j_1, \ldots, j_\ell) = 0$ . Because  $\tilde{a}_{k_h+1,j_h} = 0$  and  $d(j_0, j_1, \ldots, j_\ell) = 0$ , we conclude by the induction hypothesis and Proposition 13 that

$$\det A[k_h + 1, k_h + 2, \dots, k_h + 1 + l - h|j_h, j_h + 1, \dots, j_h + l - h] = 0.$$

Since  $k_h = k + h$  and  $j_h = j + h$ , we obtain

$$\det A[k+h+1, k+h+2, \dots, k+1+l|j+h, j+h+1, \dots, j+l] = 0$$

By Lemma 3, it follows that

$$\det A[k+1,...,k+l+1|j,...,j+l] = 0,$$

<sup>1</sup>If l = 0 or l' = 0, we employ the convention that the respective denominator is 1.

and consequently by Lemma 4,

$$\det B[k+1, \dots, k+l+1 | j, \dots, j+l] = 0$$

since otherwise we would have det  $A[k+1, \ldots, k+l+1|j, \ldots, j+l] > 0$ . By using Sylvester's Identity and again Lemma 3, we obtain

$$\begin{aligned}
346 \quad \tilde{b}_{kj} &= \frac{\det B[k, \dots, k+l'|j, \dots, j+l']}{\det B[k+1, \dots, k+l'|j+1, \dots, j+l']} \\
347 \quad &= \frac{\det B[k, \dots, k+l'-1|j, \dots, j+l'-1] \det B[k+1, \dots, k+l'|j+1, \dots, j+l']}{\det B[k+1, \dots, k+l'-1|j+1, \dots, j+l'] \det B[k+1, \dots, k+l'-1|j+1, \dots, j+l'-1]} \\
348 \quad &- \frac{\det B[k, \dots, k+l'-1|j+1, \dots, j+l'] \det B[k+1, \dots, k+l'|j, \dots, j+l'-1]}{\det B[k+1, \dots, k+l'|j+1, \dots, j+l'] \det B[k+1, \dots, k+l'-1|j+1, \dots, j+l'-1]} \\
\end{aligned}$$

349 
$$= \frac{\det B[k, \dots, k+l-1]j, \dots, j+l-1]}{\det B[k+1, \dots, k+l'-1]j+1, \dots, j+l'-1]}.$$

If l' = l + 1, then  $\tilde{b}_{kj}$  has order l. Otherwise, apply Sylvester's Identity repeatedly to obtain the required order.

**Case 3**: Suppose that l' < l. Without loss of generality assume that  $d(j'_0, j'_1, \ldots, j'_{l'}) = 0$ . Let  $A_1 := A[k+1, \ldots, k+l|j+1, \ldots, j+l]$  and  $B_1 := B[k+1, \ldots, k+l|j+1, \ldots, j+l]$ , then  $A_1$  is NsTN and  $A_1 \leq^* B_1$ . By Lemma 4, we obtain

$$0 < \det A_1 \le \det B_1$$

We conclude that  $B_1$  is nonsingular.

Let  $h := \max \{s : d(k'_0, k'_1, \dots, k'_s) = 0\}$ . Then define the sequence

$$((k'_{h}+1,j'_{h}),(k'_{h+1},j'_{h+1}),\ldots,(k'_{p'},j'_{p'}))$$

which is a lacunary sequence. By the induction hypothesis, det 
$$B[k'_h + 1, \ldots, k'_h +$$

351  $l'|j'_h, \dots, j'_h + l' - 1] = 0$ . By Lemma 3, det  $B[k'_h + 1, \dots, k'_h + l' + s|j'_h, \dots, j'_h + l' - 1 + s] =$ 352  $0, s = 1, 2, \dots$ 

353 By using Sylvester's Identity if l = l' + 1, we obtain

354

355  

$$\tilde{b}_{kj} = \frac{\det B[k, k+1 \dots, k+l'+1|j, j+1, \dots, j+l'+1]}{\det B[k+1, \dots, k+l'+1|j+1, \dots, j+l'+1]}$$

$$= \frac{\det B[k, k+1 \dots, k+l|j, j+1, \dots, j+l]}{\det B[k+1, \dots, k+l|j+1, \dots, j+l]}.$$

If 
$$l > l' + 1$$
, we apply Sylvester's Identity repeatedly to arrive at the required order.

THEOREM 29. Let  $A, B, Z \in \mathbb{R}^{n,n}$  be such that  $A \leq^* Z \leq^* B$ . Let A, B be TN and satisfy the descending rank conditions, and let  $A^{\#}, B^{\#}$  satisfy Condition I. Then Z is TN and satisfies the descending rank conditions.

Proof. Put  $A_1 := A^{\#}, B_1 := B^{\#}, Z_1 := Z^{\#}$ . Then  $A_1 \leq^* Z_1 \leq^* B_1, A_1, B_1$  are TN, and by assumption, Condition I holds for both  $A_1$  and  $B_1$ . Then by Theorem 9,  $\tilde{A}_1 = (\tilde{a}_{ij})$  and  $\tilde{B}_1 = (\tilde{b}_{ij})$  are nonnegative Cauchon matrices and satisfy conditions (i)-(ii) in Theorem 20. By Theorems 9 and 21 it suffices to show that  $\tilde{Z}_1$  is a nonnegative Cauchon matrix and satisfies conditions (i)-(ii) in Theorem 20.

By Theorem 28,  $\tilde{a}_{ij}$  and  $\tilde{b}_{ij}$  can be represented as ratios of contiguous minors of the same order, i.e.,

$$\tilde{a}_{ij} = \frac{\det A_1[i, i+1, \dots, i+\ell | j, j+1, \dots, j+\ell]}{\det A_1[i+1, \dots, i+\ell | j+1, \dots, j+\ell]},$$
$$\tilde{b}_{ij} = \frac{\det B_1[i, i+1, \dots, i+\ell | j, j+1, \dots, j+\ell]}{\det B_1[i+1, \dots, i+\ell | j+1, \dots, j+\ell]},$$

for some  $\ell$ . By Lemma 5, 366

367 (16) 
$$A_1 \leq^* Z' \leq^* B_1,$$

where  $Z' = (z'_{ij})$  with

$$z'_{ij} := \frac{\det Z_1[i, i+1, \dots, i+\ell | j, j+1, \dots, j+\ell]}{\det Z_1[i+1, \dots, i+\ell | j+1, \dots, j+\ell]}$$

From (16) it follows that  $Z' \ge 0$ . If  $z'_{ii} = 0$ , then by (16),  $\tilde{a}_{ii} = 0$ . Since A satisfies the 368 descending rank conditions we can apply Theorem 20 to conclude that  $\tilde{a}_{si} = \tilde{a}_{it} = 0$ , 369  $s, t = 1, \ldots, i$ . Again by (16), we conclude that  $b_{i-1,i} = b_{i,i-1} = 0$  and since B satisfies 370the descending rank conditions, we obtain that  $\tilde{b}_{it} = \tilde{b}_{si} = 0, s, t = 1, \dots, i-1$ . Hence 371  $z'_{it} = z'_{si} = 0, s, t = 1, \dots, i$ . We proceed in the same way if  $z'_{ij} = 0, i < j$  or i > j, to 372 obtain: 373

(i) If  $z'_{ij} = 0$  for some  $i \ge j$ , then  $z'_{it} = 0$  for all t < j; (ii) if  $z'_{ij} = 0$  for some  $i \le j$ , then  $z'_{tj} = 0$  for all t < i. 374

375

Therefore, Z' is a Cauchon matrix. If we are able to show that  $Z' = \tilde{Z}_1$ , then by 376 Theorems 9 and 21 we are done. 377

Claim:  $Z' = Z_1$ . 378

We proceed by decreasing induction with respect to the lexicographic order on the 379 pairs (i, j),  $i, j = 1, \ldots, n$ . By definition,  $z'_{nj} = z_{nj} = \tilde{z}_{nj}$  for all  $j = 1, \ldots, n$ . 380 Suppose that we have shown the claim for each pair  $(i^{\circ}, j^{\circ})$  such that  $i^{\circ} = i+1, \ldots, n$ , 381  $j^{\circ} = 1, \ldots, n$  and  $i^{\circ} = i, j^{\circ} = j + 1, \ldots, n$ . Without loss of generality we may 382 assume that i + j is even. Let  $((i, j), (i''_1, j''_1), \ldots, (i''_{p_1}, j''_{p_1}))$  be a lacunary sequence 383 with respect to  $C_{Z'}$  such that  $\ell''$  is the minimum order and  $d(i, i''_1, \ldots, i''_{\ell''}) = 0$  or 384  $d(j, j''_1, \dots, j''_{\ell''}) = 0$ . Without loss of generality, assume that  $d(j, j''_1, \dots, j''_{\ell''}) = 0$  and 385  $i \geq j$ . By (9) we have the following representation 386

387 (17) 
$$\tilde{z}_{ij} = \frac{\det Z_1[i, i''_1, \dots, i''_{\ell''}|j, j''_1, \dots, j''_{\ell''}]}{\det Z_1[i''_1, \dots, i''_{\ell''}|j''_1, \dots, j''_{\ell''}]}.$$

By Proposition 16,  $\operatorname{rank}(Z_1[i''_1, i''_1 + 1, \dots, n|j''_1, \dots, j''_{\ell''}]) = \ell''$  since the lacunary sequence  $((i''_1, j''_1), \ldots, (i''_{\ell''}, j''_{\ell''}))$  coincides with the one that is constructed by Procedure 15 applied to the columns  $j''_1, \ldots, j''_{\ell''}$  of Z'. Hence

$$Z_1[i+1,i+2,\ldots,i+\ell''|j+1,j+2\ldots,j+\ell''] = CZ_1[i_1'',i_2'',\ldots,i_{\ell''}'|j_1'',j_2'',\ldots,j_{\ell''}'],$$

for some  $C \in \mathbb{R}^{\ell'',\ell''}$ . We distinguish the following three cases: 388

- Case 1:  $\ell = \ell''$ 389
- We get from Lemma 4 390

391 
$$0 < \det A_1[i+1, i+2, \dots, i+\ell | j+1, j+2\dots, j+\ell]$$
  
392  $\leq \det Z_1[i+1, i+2, \dots, i+\ell'' | j+1, j+2\dots, j+\ell'']$ 

and conclude that det  $C \neq 0$ . Proceeding as in the proof of Proposition 27, we arrive at

395

$$\tilde{z}_{ij} = \frac{\det Z_1[i, i+1, \dots, i+\ell | j, j+1, \dots, j+\ell]}{\det Z_1[i+1, \dots, i+\ell | j+1, \dots, j+\ell]} = z'_{ij}$$

Case 2:  $\ell'' < \ell$ By Lemma 3,

$$\det A_1[i+1,\ldots,i+\ell''+s \mid j+1,\ldots,j+\ell''+s] > 0$$

because  $A_1[i+1,\ldots,i+\ell''+s \mid j+1,\ldots,j+\ell''+s]$  are leading principal submatrices in  $A_1[i+1,\ldots,i+\ell \mid j+1,\ldots,j+\ell]$  for all  $s=0,1,\ldots,\ell-\ell''$ . By Lemma 4,

det 
$$Z_1[i+1,\ldots,i+\ell''+s \mid j+1,\ldots,j+\ell''+s] > 0, \ s=0,1,\ldots,\ell-\ell''.$$

396 We proceed parallel to Case 1 to arrive at

39

$$z_{ij} = \frac{1}{\det Z_1[i''_1, \dots, i''_{\ell''}|j''_1, \dots, j''_{\ell''}]} \\ = \frac{\det Z_1[i, i+1, \dots, i+\ell''|j, j+1, \dots, j+\ell'']}{\det Z_1[i+1, \dots, i+\ell''|j+1, \dots, j+\ell'']}.$$

By the induction hypothesis,  $Z_1[i+1,\ldots,n \mid j, j+1,\ldots,n]$  is TN. By arguing as in Case 3 in the proof of Theorem 28 we may conclude that det  $Z_1[i+1,\ldots,i+\ell''+1]$ 

 $\det Z_1[i, i''_1, \dots, i''_{\ell''} | j, j''_1, \dots, j''_{\ell''}]$ 

401  $j, j + 1, \dots, j + \ell'' = 0$ . By Lemma 3, we have

:

402 det 
$$Z_1[i+1,\ldots,i+\ell''+1+s \mid j,j+1,\ldots,j+\ell''+s] = 0, \ s=1,\ldots,\ell-\ell''-1.$$

403 Application of Sylvester's Identity step by step to the representation of  $\tilde{z}_{ij}$  that is 404 given in (18), we obtain

406

$$\tilde{z}_{ij} = \frac{\det Z_1[i, i+1, \dots, i+\ell''|j, j+1, \dots, j+\ell'']}{\det Z_1[i+1, \dots, i+\ell''|j+1, \dots, j+\ell'']} \\ = \frac{\det Z_1[i, i+1, \dots, i+\ell''+1|j, j+1, \dots, j+\ell''+1]}{\det Z_1[i+1, \dots, i+\ell''+1|j+1, \dots, j+\ell''+1]}$$

$$= \frac{\det Z_1[i, i+1, \dots, i+\ell | j, j+1, \dots, j+\ell]}{\det Z_1[i+1, \dots, i+\ell | j+1, \dots, j+\ell]}$$
  
=  $z'_{ij}$ .

Case 3:  $\ell < \ell''$ 

Define  $W := Z_1[i+1, i+2, \ldots, i+\ell''|j+1, j+2, \ldots, j+\ell'']$ . If det  $W \neq 0$ , then  $\tilde{z}_{ij}$  can be written as in (18). Otherwise, by [15, Proposition 1.15] the rows  $i+1, \ldots, i+\ell''$  of  $Z_1$  are linearly dependent or the right shadow of W in  $Z_1[i+1, i+2, \ldots, n|1, 2, \ldots, m]$ has rank at most  $\ell'' - 1$  since by the induction hypothesis the later submatrix is TNand  $d(j, j''_1, \ldots, j''_{\ell''}) = 0$ . If i = j, then define  $(\alpha_0, \beta_0) := (i, j)$  and for  $k = 1, \ldots, \tau$ , let

$$(\alpha_k, \beta_k) := \min\left\{ (\alpha, \beta) \mid \alpha = \alpha_{k-1} + 1, \ \beta > \beta_{k-1}, \ z'_{\alpha, \beta} \neq 0 \right\},\$$

where the minimum is taken with respect to the lexicographic order. This sequence is a lacunary sequence or a part of a lacunary sequence of order  $\tau$  since the entries of

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Z' satisfy the conditions (i) and (ii) above with possible gaps between columns and  $\tau < \ell''$  which is a contradiction. Hence if i = j, det  $W \neq 0$ . If i > j, then j < i - 1 since i + j is even. It is easy to see that the order of the sequence at the position (i, j) is less than or equal to that of (i, j + 1). Hence by the induction hypothesis, the rows  $i + 1, \ldots, i + \ell''$  cannot be linearly dependent and the right shadow of W in  $Z_1[i + 1, i + 2, \ldots, n|1, 2, \ldots, m]$  has not rank less than  $\ell''$ . Thus det  $W \neq 0$  and we conclude that det  $C \neq 0$ . Therefore,  $\tilde{z}_{ij}$  can be written as in (18). Proceeding as in the proof of Theorem 28, Case 2 and by Lemma 3, we arrive at

$$\det Z_1[i+1,\ldots,i+\ell+1+s \mid j,\ldots,j+\ell+s] = 0, \ s = 0, 1,\ldots,\ell''-\ell-1.$$

410 Now use Sylvester's Identity to decrease step by step the order of the representation

similarly as in (19) to obtain  $\tilde{z}_{ij} = z'_{ij}$ . This completes the proof.

412 THEOREM 30. Let  $A, B, Z \in \mathbb{R}^{n,m}$  be such that  $A \leq^* Z \leq^* B$ . If A, B are TN, 413 belong to the same TN cell, and both satisfy Condition I, then Z is TN, satisfies 414 Condition I, and belongs to the same TN cell that includes A and B.

The proof of this theorem is parallel to the proof of the Theorem 29 and therefore omitted.

417

The follwing example illustrates the difference between Theorem 29 and Theorem 419 30.

420 EXAMPLE 31. Let

421

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 2 & 3 & 7 \end{bmatrix}, \quad and \quad Z = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 2 & 3 & 4 \end{bmatrix}.$$

Then we have

$$A \leq^* Z \leq^* B$$

422 and obtain

423 
$$\tilde{A} = \begin{bmatrix} \frac{1}{3} & 0 & 1\\ 0 & 0 & 3\\ 2 & 3 & 3 \end{bmatrix} \quad and \quad \tilde{B} = \begin{bmatrix} \frac{1}{3} & 0 & 1\\ 0 & \frac{12}{17} & 3\\ 2 & 3 & 7 \end{bmatrix}$$

424 A, B are TN but belong to two different TN cells and satisfy the descending rank 425 conditions.  $A^{\#}$ ,  $B^{\#}$  fulfill Condition I. Z is TN.

In [2], two relaxations of the nonsingularity assumption are presented. The following example shows that Theorem 29 covers a different situation.

428 EXAMPLE 32. Let

429

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 5 & 10 & 5 \\ 1 & 2 & 1 \end{bmatrix} \quad and \quad B = \begin{bmatrix} 2 & 2 & 1 \\ 5 & 10 & 5 \\ 1 & 2 & 13 \end{bmatrix}$$

Then we have

 $A \leq^* B$ 

Π

430 and obtain

431

$$\tilde{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 5 \\ 1 & 2 & 1 \end{bmatrix} \quad and \quad \tilde{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & \frac{120}{13} & 5 \\ 1 & 2 & 13 \end{bmatrix}$$

A and B are TN, both  $A(=A^{\#})$  and  $B(=B^{\#})$  satisfy Condition I as well as the descending rank conditions. Hence all matrices in [A, B] are TN. Neither [2, Theorem 3.6] nor [2, Corollary 3.7] can be used to draw this conclusion since A is singular and

$$\det A[1,2] = \det A[2,3] = 0.$$

Unfortunately, Condition I alone is not strong enough to guarantee the intervalproperty as the following example documents.

434 EXAMPLE 33. Let

435

	[3	2	2	2			4	2	2	1				5	2	2	1	
A =	6	5	5	5	,	Z =	6	5	5	5	,	and	B =	5	5	5	5	
	3	3	3	3			3	3	3	3				3	3	3	3	

436 A and B are TN, satisfy Condition I, and  $A \leq^* Z \leq^* B$ . But Z is not TN since 437 det Z[1,2,3 | 1,2,4] = -3 < 0.

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