ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-858 Proposed by Muneer Jebreel Karama, Hebron, Palestine

Show that

$$\frac{1}{2}\left((2F_nF_{n+1})^8 + (F_{n-1}F_{n+2})^8 + F_{2n+1}^8\right)$$

is a perfect square for all $n \geq 0$.

H-859 Proposed by Robert Frontczak, Stuttgart, Germany

Prove that

$$\sum_{n\geq 1} \zeta(2n+1)\frac{F_{2n}}{5^n} = \frac{1}{2},$$

where $\zeta(k) = \sum_{n\geq 1} 1/n^k$ for $k \geq 2$ is the Riemann zeta function.

H-860 Proposed by Robert Frontczak, Stuttgart, Germany

Let $(B_n)_{n\geq 0}$ be the Bernoulli numbers defined by

$$\frac{z}{e^z - 1} = \sum_{n\geq 0} B_n \frac{z^n}{n!} \quad (|z| < 2\pi).$$

Show that for all $n \geq 0$, we have

$$\sum_{\substack{k=0 \\ k\equiv n \pmod 2}}^{n} \binom{n}{k}(2^k L_k - 2)5^{(n-k)/2}\frac{B_{n-k+2}}{n-k+2} = \frac{2^{n+2}L_{n+2} - 2}{5(n+1)(n+2)} - 1.$$

H-861 Proposed by David Terr, Oceanside, CA

For arbitrary constants a, b, c, define the sequence $(G_n)_{n\geq 0}$ by $G_0 = a$, $G_1 = b$, $G_2 = c$, and the recurrence $G_n = G_{n-1} + G_{n-2} + G_{n-3}$ for $n \geq 3$. Find a closed form expression for

$$\sum_{j=0}^{n} G_{2j}G_{2j+1} \quad \text{valid for all} \quad n \geq 0.$$

Advance Problem

Problem proposed by : Muneer Jebreel Karama, Palestine Polytechnic University,Hebron,Palestine

Show that : $(2F_nF_{n+1})^8 + (F_{n-1}F_{n+2})^8 + (F_{2n+1})^8 = 2\ Squares$

Proof

let $F(n) = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$, **then** *we have the following defintions*

$$> F(n) := \frac{(1+\mathrm{sqrt}(5))^n - (1-\mathrm{sqrt}(5))^n}{2^n \cdot \mathrm{sqrt}(5)}$$

$$F := n \rightarrow \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} \quad (1)$$

$$> g := simpilfy(seq((F(n), n = 0..10)))$$

$$g := simpilfy\Big(0, 1, \frac{1}{20}\left((\sqrt{5}+1)^2 - (1-\sqrt{5})^2\right)\sqrt{5}, \frac{1}{40}\left((\sqrt{5}+1)^3 - (1-\sqrt{5})^3\right)\sqrt{5}, \frac{1}{80}\left((\sqrt{5}+1)^4 - (1-\sqrt{5})^4\right)\sqrt{5}, \frac{1}{160}\left((\sqrt{5}+1)^5 - (1-\sqrt{5})^5\right)\sqrt{5}, \frac{1}{320}\left((\sqrt{5}+1)^6 - (1-\sqrt{5})^6\right)\sqrt{5}, \frac{1}{640}\left((\sqrt{5}+1)^7 - (1-\sqrt{5})^7\right)\sqrt{5}, \frac{1}{1280}\left((\sqrt{5}+1)^8 - (1-\sqrt{5})^8\right)\sqrt{5}, \frac{1}{2560}\left((\sqrt{5}+1)^9 - (1-\sqrt{5})^9\right)\sqrt{5}, \frac{1}{5120}\left((\sqrt{5}+1)^{10} - (1-\sqrt{5})^{10}\right)\sqrt{5}\Big) \quad (2)$$

$$> simplify(g)$$

$$simpilfy(0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55) \quad (3)$$

(4)

$$> F(n+1) := \frac{(1+\mathrm{sqrt}(5))^{n+1} - (1-\mathrm{sqrt}(5))^{n+1}}{2^{n+1} \cdot \mathrm{sqrt}(5)}$$

$$F(n+1) := \frac{1}{5}\,\frac{\left((\sqrt{5}+1)^{n+1} - (1-\sqrt{5})^{n+1}\right)\sqrt{5}}{2^{n+1}} \quad (5)$$

$$> F(n+2) := \frac{(1+\mathrm{sqrt}(5))^{n-1} - (1-\mathrm{sqrt}(5))^{n-1}}{2^{n-1} \cdot \mathrm{sqrt}(5)}$$

$$F(n+2) := \frac{1}{5}\,\frac{\left((\sqrt{5}+1)^{n-1} - (1-\sqrt{5})^{n-1}\right)\sqrt{5}}{2^{n-1}} \quad (6)$$

$$> F(n\text{-}1) := \frac{(1+\mathrm{sqrt}(5))^{n+1} - (1-\mathrm{sqrt}(5))^{n+1}}{2^{n+1} \cdot \mathrm{sqrt}(5)}$$

$$F(n-1) := \frac{1}{5}\,\frac{\left((\sqrt{5}+1)^{n+1} - (1-\sqrt{5})^{n+1}\right)\sqrt{5}}{2^{n+1}} \quad (7)$$

> $F(2n+1) := \frac{(1+\text{sqrt}(5))^{2n+1}-(1-\text{sqrt}(5))^{2n+1}}{2^{2n+1}\cdot\text{sqrt}(5)}$

$$F(2n+1) := \frac{1}{5}\,\frac{\left(\left(\sqrt{5}+1\right)^{2n+1}-\left(1-\sqrt{5}\right)^{2n+1}\right)\sqrt{5}}{2^{2n+1}} \qquad \textbf{(8)}$$

> $a := 2\cdot F(n)\cdot F(n+1)$

$$a := \frac{2}{5}\,\frac{\left(\left(\sqrt{5}+1\right)^{n}-\left(1-\sqrt{5}\right)^{n}\right)\left(\left(\sqrt{5}+1\right)^{n+1}-\left(1-\sqrt{5}\right)^{n+1}\right)}{2^{n}\,2^{n+1}} \qquad \textbf{(9)}$$

> $b := F(2n+1)$

$$b := \frac{1}{5}\,\frac{\left(\left(\sqrt{5}+1\right)^{2n+1}-\left(1-\sqrt{5}\right)^{2n+1}\right)\sqrt{5}}{2^{2n+1}} \qquad \textbf{(10)}$$

> $c := F(n-1)\cdot F(n+2)$

$$c := \frac{1}{5}\,\frac{\left(\left(\sqrt{5}+1\right)^{n+1}-\left(1-\sqrt{5}\right)^{n+1}\right)\left(\left(\sqrt{5}+1\right)^{n-1}-\left(1-\sqrt{5}\right)^{n-1}\right)}{2^{n+1}\,2^{n-1}} \qquad \textbf{(11)}$$

> $k := a^8+b^8+c^8$

$$k := \frac{256}{390625}\,\frac{\left(\left(\sqrt{5}+1\right)^{n}-\left(1-\sqrt{5}\right)^{n}\right)^{8}\left(\left(\sqrt{5}+1\right)^{n+1}-\left(1-\sqrt{5}\right)^{n+1}\right)^{8}}{\left(2^{n}\right)^{8}\left(2^{n+1}\right)^{8}} + \frac{1}{625}\,\frac{\left(\left(\sqrt{5}+1\right)^{2n+1}-\left(1-\sqrt{5}\right)^{2n+1}\right)^{8}}{\left(2^{2n+1}\right)^{8}} + \frac{1}{390625}\,\frac{\left(\left(\sqrt{5}+1\right)^{n+1}-\left(1-\sqrt{5}\right)^{n+1}\right)^{8}\left(\left(\sqrt{5}+1\right)^{n-1}-\left(1-\sqrt{5}\right)^{n-1}\right)^{8}}{\left(2^{n+1}\right)^{8}\left(2^{n-1}\right)^{8}} \qquad \textbf{(12)}$$

> $k1 := 4^8+5^8+3^8$

$$k1 := 462722 \qquad \textbf{(13)}$$

> $ifactor(k1)$

$$(2)\,(13)^2\,(37)^2 \qquad \textbf{(14)}$$

> $d := |combine(a^8+b^8+c^8)|$

$$d := \left|\frac{1}{390625}\,2^{-16n}\left(\left(\sqrt{5}+1\right)^{n+1}-\left(1-\sqrt{5}\right)^{n+1}\right)^{8}\left(\left(\sqrt{5}+1\right)^{n}-\left(1-\sqrt{5}\right)^{n}\right)^{8} + \frac{1}{160000}\left(\left(\sqrt{5}+1\right)^{2n+1}-\left(1-\sqrt{5}\right)^{2n+1}\right)^{8}2^{-16n} + \frac{1}{390625}\,2^{-16n}\left(\left(\sqrt{5}+1\right)^{n+1}-\left(1-\sqrt{5}\right)^{n+1}\right)^{8}\left(\left(\sqrt{5}+1\right)^{n-1}-\left(1-\sqrt{5}\right)^{n-1}\right)^{8}\right| \qquad \textbf{(15)}$$

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