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ORIGINAL PAPER

Transformation of global spherical harmonic models of the gravity field to a local adjusted spherical cap harmonic model

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Abstract The new global gravity models represented by global spherical harmonics like EGM2008 require a high degree and order in their coefficients to resolve the gravity field in local areas; therefore, there are interests to represent the regional or local field by less parameters and to develop a parameter transformation from the global model to a local kind of spherical harmonic model. The authors use local spherical cap harmonics for the regional gravity potential representation related to a local pole and a local spherical coordinate system. This allows to model regional gravity potential with less parameters and less memory requirements in computation and storage. From different kinds of representations of spherical cap harmonics, we have selected the so-called adjusted spherical cap harmonics (ASCH). This is the most appropriate for the presented mathematical model of deriving its coefficients from global gravity models. In that way, the global gravity models can fully be exploited and mapped to regional ASCH, in particular with respect to the computation of regional geoid models with improved solution.

Keywords Global spherical harmonics (SH) · Spherical cap harmonics (SCH) · Adjusted spherical cap harmonics (ASCH) · Regional quasigeoid/geoid computations

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Global gravity models

Presently, there are several global geopotential field models available from various sources and with different spatial resolution. The International Centre for Global Earth Models (ICGEM) provides the access to the various satellite only or combined models on behalf of the International Association of Geodesy (http://icgem.gfz-potsdam.de/ICGEM/ICGEM.html). For example, EGM2008 has a maximum degree and order of 2190 and Eigen05C and EGM96 have maximum degree and order of 360. The common way for representing the gravitational potential V in a global model is to use spherical harmonics (SH), which satisfy Laplace's equation (Eq. 1; Hofmann-Wellenhof and Moritz 2005):

$$\Delta V = \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} + \frac{\partial^2 V}{\partial Z^2} = 0 \quad (1)$$

In Eq. (1), V is the gravitational potential, (X, Y and Z) are the Cartesian coordinates of a position and ΔV is the Laplace operator. The differential Eq. (1) is satisfied by the spherical harmonic representation of the gravitational

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potential at point P with parameters $\overline{C}_{nm}$ and $\overline{S}_{nm}$ in Eq. (2), (Torge 2001), reading:

$$V=\frac{GM}{r}+\frac{GM}{r}\sum_{n=2}^{\max n}\left(\frac{a}{r}\right)^n\sum_{m=0}^{n}\left(\overline{C}_{n,m}\cos(m\lambda)+\overline{S}_{n,m}\sin(m\lambda)\right)\overline{P}_{n,m}(\sin\overline{\phi}) \tag{2}$$

$\overline{C}_{nm}$ and $\overline{S}_{nm}$ are the fully normalized spherical harmonic parameters. $(\lambda,\overline{\phi},r)$ are the geocentric spherical coordinates of point P. $\overline{P}_{nm}(\sin\overline{\phi})$ is the fully normalized associated Legendre function. $\overline{P}_{nm}(\sin\overline{\phi})$ can be calculated by the following recursive formulas (Eq. 3), with the abbreviations $t=\sin\overline{\phi}$ and $u=\cos\overline{\phi}$ (Holmes and Featherstone 2002).

$$\begin{aligned}
&\overline{P}_{n,m}=a_{n,m}t\overline{P}_{n-1,m}-b_{n,m}\overline{P}_{n-2,m}\\
&a_{n,m}=\sqrt{\frac{(2n-1)(2n+1)}{(n-m)(n+m)}}\\
&b_{n,m}=\sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(n-m)(n+m)(2n-3)}}\\
&\overline{P}_{0,0}=1\quad,\quad\overline{P}_{1,0}=\sqrt{3}t\quad,\overline{P}_{1,1}=\sqrt{3}u\\
&\text{If}\, n=m\\
&\overline{P}_{m,m}=u\sqrt{\frac{2m+1}{2m}}\overline{P}_{m-1,m-1}
\end{aligned} \tag{3}$$

Equation (2) represents the gravitational potential V caused by the mass distribution in the earth. The other component of the potential is centrifugal potential Z in Eq. (4). It is caused by the rotation of the earth with an angular velocity $\omega=7292115\times10^{-11}\ rad/s$ as given by the definition parameters of GRS80 reference system (Torge 2001).

$$Z=\frac{1}{2}\omega^2\left(X^2+Y^2\right) \tag{4}$$

The total potential W (gravity potential) is the summation of both ($W=V+Z$; Heiskanen and Moritz 1967). Using Eqs. (2) and (4), one can easily derive other components related to the potential W. These quantities are the gravity g, the quasigeoid height N_{QG}, the geoid height N_{G}, and the deflections of vertical in the east and north directions (η, ζ) etc. (Fan 2004). They can be computed according to Eq. (5).

$$\begin{aligned}
&g_{\mathrm{LGV}}=\begin{bmatrix}\frac{\partial W}{\partial r} & \frac{\partial W}{r\cos\overline{\phi}\partial\lambda} & \frac{\partial W}{r\partial\overline{\phi}}\end{bmatrix}\\
&N_{\mathrm{QG}}=\frac{T}{\gamma_{\mathrm{Q}}}\\
&N_{\mathrm{G}}=N_{\mathrm{QG}}+\frac{\overline{g}-\overline{\gamma}}{\overline{\gamma}}H_{\mathrm{p}}\\
&\zeta=-\frac{\partial N}{\partial S_{\mathrm{North}}}=-\frac{1}{\gamma_{\mathrm{Q}}(M+h)}\frac{\partial T}{\partial\phi}\\
&\eta=-\frac{\partial N}{\partial S_{\mathrm{East}}}=-\frac{1}{\gamma_{\mathrm{Q}}(N+h)\cos\phi}\frac{\partial T}{\partial\lambda}
\end{aligned} \tag{5}$$

In Eq. (5), T is the disturbing potential, defined as the difference between the gravity potential W and the ellipsoidal normal potential U. γ_{Q} is the ellipsoidal normal gravity for point Q on the so-called telluroid with same latitude and longitude as the calculation point and height of $h_{\mathrm{Q}}=H_{\mathrm{P}}=h_{\mathrm{P}}-N_{\mathrm{QG}}$. $\overline{g}$ is the mean gravity along the plumb line of point P. $\overline{\gamma}$ is the mean ellipsoidal normal gravity from the earth surface down to the geoid/quasigeoid along the plumb line of point Q. S_{North} and S_{East} are the differential distance elements towards North and East, respectively. Finally, M and N are the ellipsoidal radii of curvature in the meridian and in the prime vertical, respectively.

Spherical cap harmonics

A modification of the spherical harmonics model was introduced by G. Haines in 1985 to be used in a local area for modelling the potential V using the so-called spherical cap harmonic (SCH) represented by (S'_{nm}, C'_{nm}). These spherical cap harmonics are suited for the area of a local cap covering the region of interest on the sphere instead of the whole sphere, see Fig. 1. The cap position is described by a local spherical coordinate system (α, θ, r). Here, α is the azimuth of the line from the cap pole to the point, θ is the spherical distance from the cap pole to the point P, and r is the radial distance from the sphere centre to the point P. The relation between global coordinates and local coordinates is reading:

$$\begin{aligned}
&\tan\alpha=\frac{\cos\overline{\phi}.\sin(\lambda-\lambda_0)}{\sin\overline{\phi}\cos\overline{\phi}_0-\cos\overline{\phi}\sin\overline{\phi}_0\cos(\lambda-\lambda_0)}\\
&\cos\theta=\sin\overline{\phi}\sin\overline{\phi}_0-\cos\overline{\phi}\cos\overline{\phi}_0\cos(\lambda-\lambda_0)
\end{aligned} \tag{6}$$

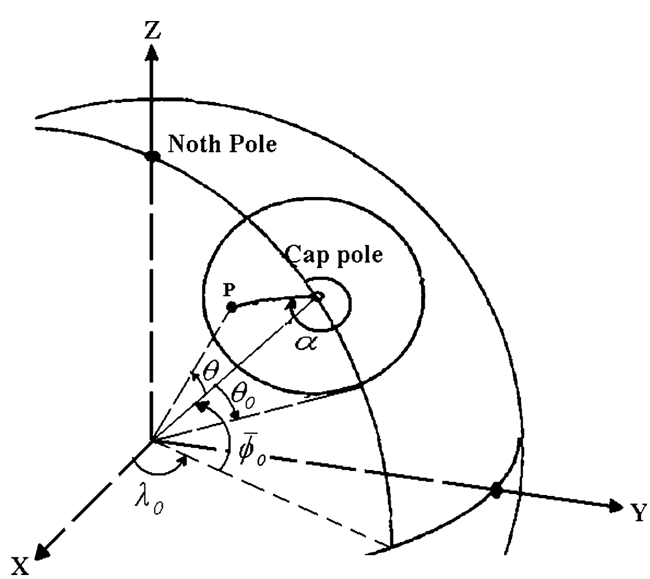

Fig 1 Spherical cap area, with its own pole located at the origin of the interest area

In Eq. (6), $(\lambda, \overline{\phi})$ are the geocentric longitude and latitude of the point. $(\lambda_0, \overline{\phi}_0)$ are the geocentric longitude and latitude of the cap pole.

For the potential V in spherical cap harmonics, we obtain:

$$V(r,\alpha,\theta) = \frac{GM}{r} \sum_{k=0}^{k\max} \left(\frac{R}{r}\right)^{n(k)} \sum_{m=0}^{k} \left(C'_{nm} \cos m\alpha + S'_{nm} \sin m\alpha\right) P_{n(k),m}(\cos\theta) \tag{7}$$

The advantage of spherical cap harmonics in Eq. (7) is that the number of parameters necessary for a particular resolution for the local cap area is much smaller than the number of parameters needed in an ordinary global SH development for the same resolution (Haines 1988).

The SCH have an integer order m and a real degree $n(k)$, where the real degree $n(k)$ is the root of the Legendre function. Legendre function and its derivative have to satisfy the orthogonality conditions of the function in the cap area according to Eq. (8) (Haines 1985). In Eq. (8), k is the degree, m is the order, and θ_0 is the angular spherical distance from the pole of the cap area to the border of the area of interest.

$$\begin{aligned} \left.\frac{dP_{n(k),m}(\cos\theta_0)}{d\theta}\right|_{\theta_0} &= 0, \qquad k-m = \text{even} \\ P_{n(k),m}(\cos\theta_0)\big|_{\theta_0} &= 0, \qquad k-m = \text{odd} \end{aligned} \tag{8}$$

The calculation of the Legendre function with the real degree $n(k)$ and the integer m cannot be accomplished by direct and recursive formulas, as it is in the case of integer degree and order as in Eq. (3). Instead, it is defined by an infinite power series (Haines 1988), which has to be elaborated iteratively introducing some approximations that will introduce additional errors (Oliver and Smith 1983).

When applying spherical cap harmonics, some difficulties and problems are to be considered. First problem: We need to search for the real degrees $n(k)$ according to the conditions in Eq. (8), whereby different non-direct and iterative algorithms have to be used. This introduces additional errors and consumes time (De Santis et al. 1999). Second problem: The calculation of Legendre functions and their derivatives with non-integer degrees is again a time-consuming iterative process (Schneid 2006), alternative algorithms may suffer from some different approximations (Haines 1985).

To avoid the above problems, a modified approach of SCH was introduced by De Santis (1992). It is called adjusted spherical cap harmonics (ASCH). This approach aims at using the well-known integer order and degree Legendre functions. Its principle enlarges the cap area to a hemisphere using Eq. (9), where the pole of the hemisphere is the pole of the cap itself, with new coordinates (r', α', ϑ)defined in (9).

$$\begin{aligned} s &= \tfrac{0.5\pi}{\theta_0} \\ \vartheta &= s \cdot \theta \\ \alpha' &= \alpha \quad \text{and} \quad r' = r \end{aligned} \tag{9}$$

According to the ASCH definition, Eq. (7) is modified to a new formula in Eq. (10). The new formula looks like the conventional spherical harmonics series. It is advantageous that there is no more the need to calculate the Legendre functions with real degree and integer order.

$$V(r,\alpha,\vartheta) = \frac{GM}{r} \sum_{k=0}^{k\max} \left(\frac{R}{r}\right)^{n(k)} \sum_{m=0}^{k} \left(C'_{nm} \cos m\alpha + S'_{nm} \sin m\alpha\right) P_{n,m}(\cos\vartheta) \tag{10}$$

ASCH in Eq. (10) have the following advantages compared to the normal SCH in Eq. (7): First, the well-known Legendre function with its recursive formulas is used. Second, there is no need to calculate the roots $n(k)$ of Legendre function and its derivative according to the conditions in Eq. (8), which is time-consuming (De Santis 1992). This means, as mentioned before, the conditions in Eq. (8) do not exist anymore. De Santis has proven and introduced the direct formula of Eq. (11) to be used to compute the numbers $n(k)$ (De Santis 1992).

$$n(k) = \sqrt{s^2 k(k+1) + 0.25} - 0.5 \tag{11}$$

In Eq. (11), s is the scale factor computed from Eq. (9). The parameter k is the degree in an ASCH model. Following Haines (1988), an approximate formula of Eq. (11) may be used for low degree and order ASCH models.

$$n(k) = s(k + 0.5) \tag{12}$$

Transformation of SH to ASCH

Existing methods

Different methods have been developed and proposed in the past to transform the SH coefficients of global geopotential models of type (2) to the local SCH of type (7). When the SH and SCH have the same pole, only a transformation of the Legendre function with integer degree and order to the Legendre function with real degree and integer order is required as shown in Eq. (13) (De Santis et al. 1999). This leads to:

$$P_{nm}(\theta) = \sum_{k=m}^{\infty} A_k^{n,m} P_{n(k),m}(\theta) \tag{13}$$

The parameters $A_k^{n,m}$ are then the transformation parameters for the transformation from the global to the cap system. These parameters can be calculated using a grid of points over the cap area. In that case, the local spherical cap harmonic coefficients (S'_{nm}, C'_{nm}) can directly be calculated from the global spherical harmonic coefficients $(\overline{S}_{nm}, \overline{C}_{nm})$ using the transformation parameters $A_k^{n,m}$ in Eq. (14).

$$\begin{bmatrix} C'_{nm} \\ S'_{nm} \end{bmatrix} = \sum_{n=m}^{\infty} A_k^{m,n} \begin{bmatrix} \overline{C}_{nm} \\ \overline{S}_{nm} \end{bmatrix} \tag{14}$$

In the general case, the transformation of spherical harmonic parameters to spherical cap harmonic parameters requires to consider different poles (De Santis et al. 1996). In the case of ASCH, no algorithm has been developed up to now, and the authors of this paper close the gap in the following section.

Proposed method to transform SH to ASCH

The above discussed methods of transformation do not work for transforming SH to ASCH. The reason is that the coordinates are not only related to different poles, but are also scaled according to Eq. (9). A straightforward method to transform SH to ASCH is to set up a system of linear equations for a number of positions $P(r, \vartheta, \lambda)$, reading:

$$V_{\mathrm{ASCH}}(r, \vartheta, \lambda) = \frac{GM}{r} \sum_{k=0}^{k\max} \left(\frac{R}{r}\right)^{n(k)+1} \sum_{m=0}^{k} \left(C'_{nm} \cos m\alpha + S'_{nm} \sin m\alpha\right) P_{n,m}(\cos\vartheta) = V_{\mathrm{Global}}(r, \vartheta, \lambda) \tag{15}$$

The solution of the system of equations based on Eq. (15) is linear with respect to m coefficients (S'_{nm}, C'_{nm}) by using at least m number positions $(r, \vartheta, \lambda)_i$ $\left(m = (k_{\max} + 1)^2\right)$. This method is derived in Jäger (2010). The extension of that approach, presented here, takes into account the fact that both SH (Eq. 2) and ASCH (Eq. 15) are truncated series. This means that V for (SH) computed by Eq. (2) and V_{ASCH} computed by Eq. (15) are inconsistent. So the computation of the coefficients (S'_{nm}, C'_{nm}) has to be controlled and optimized

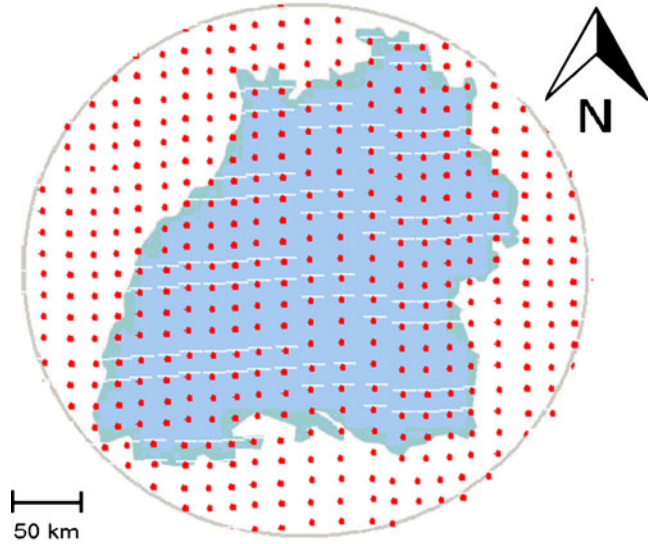

Fig 2 Distribution of a sample grid points over the cap area for the example of Baden–Württemberg state in Germany

at the same time. This is done by a least squares estimation of (S'_{nm}, C'_{nm}) related to Eq. (15) set up in the following way: A 3D grid of points is generated over the cap area, where the minimum number of required grid points is the same or more than the number of unknown parameters $(k_{\max}+1)^2$. Figure 2 shows an example of grid points distributed all over a cap covering the state of Baden–Württemberg in Germany. This area is the test area used in this paper.

The potential values $V(r, \vartheta, \lambda)_i$ at the grid points P_i are taken from a global model V_{Global} using Eq. (2), and they are used as observations in Eq. (15).

The ASCH coefficients (S'_{nm}, C'_{nm}) are the unknown parameters to be estimated. The number of unknowns in a spherical cap harmonic model is $(k_{\max}+1)^2$ (Schneid 2006). The least squares solution of the over determined problem related to Eq. (15) reads:

$$\hat{\mathrm{x}} = \left(\mathrm{A}^T \mathrm{C}_l^{-1} \mathrm{A}\right)^{-1} \mathrm{A}^T \mathrm{C}_l^{-1} l \tag{16a}$$

Table 1 ASCH defining parameters for the state of Baden–Württemberg

Parameter	Value
Gravitation constant of the earth (GM)[a]	3986005×10^8 $m^3 s^{-2}$
Selected reference radius (R)[b]	6366166.378729511 m
Latitude of origin	48°.361439
Longitude of origin	8°.9805804
Scale (s)	90°/1°.136=79.22535211

[a] GRS80

[b] Here, the radius selected as distance to the geocenter of the origin of the cap area is chosen as the ellipsoidal height of the earth surface at the origin

Table 2 RMS of residuals as a function of maximum degree of the ASCH expansion

Degree	5	10	20	30	45	60
RMSE (m)	2.145	0.953	0.042	0.031	0.018	0.009

The design-matrix A is reading:

$$A = \begin{bmatrix} \frac{GM}{r_1} & \frac{GM}{r_1}\frac{R}{r_1}P_{10} & \frac{GM}{r_1}\frac{R}{r_1}\cos\alpha_1 P_{11} & \frac{GM}{r_1}\frac{R}{r_1}\sin\alpha_1 P_{10} & & \\ \frac{GM}{r_2} & \frac{GM}{r_2}\frac{R}{r_2}P_{10} & \frac{GM}{r_2}\frac{R}{r_2}\cos\alpha_2 P_{11} & \frac{GM}{r_2}\frac{R}{r_2}\sin\alpha_2 P_{10} & \cdots & \cdots \\ \frac{GM}{r_3} & \frac{GM}{r_3}\frac{R}{r_3}P_{10} & \frac{GM}{r_3}\frac{R}{r_3}\cos\alpha_3 P_{11} & \frac{GM}{r_3}\frac{R}{r_3}\sin\alpha_3 P_{10} & & \\ & \vdots & & & \ddots & \\ & \vdots & & & & \\ & \vdots & & & & \frac{GM}{r_m}\left(\frac{R}{r_m}\right)^n \sin m\alpha_m P_{nn} \end{bmatrix} \quad (16b)$$

The vector of unknowns $\hat{\mathbf{x}}$ and the vector of observations l are reading:

$$\hat{\mathbf{x}} = \left[C'_{00} \quad C'_{10} \quad C'_{11} \quad S'_{11} \quad C'_{20} \cdots\cdots S'_{nn}\right]^T \quad (16c)$$

$$l = [V_1 \quad V_2 \quad V_3 \quad V_4 \quad \cdots\cdots \quad V_m]^T \quad (16d)$$

Each observation leads to a row in the so-called design-matrix A (Eq. 16a), and the elements of each row are the coefficients of the unknown parameters $\hat{\mathbf{x}}$ (Eq. 16c). The column vector l is observations computed from the input V_{Global}. C_1 is the fully correlated covariance matrix of the observations $V(r,\vartheta,\lambda)_i$ and has to be computed on applying the law of error propagation to Eq. (2), using the covariance matrix $C_{\overline{C}_{nm},\overline{S}_{nm}}$ of the coefficients of the global spherical harmonic model (Migliaccio et al. 2010).

Numerical investigations and results of the proposed method

Tests for the method (Eq. 16a–d) were applied to transform the EGM2008 model with degree and order 2190 to a local ASCH model for the state of Baden–Württemberg in Germany (Fig. 2). The maximum spherical angle is equal to 1.136°, meaning an area extension with a radius of nearly 126 km. This leads to the characteristics of the cap coordinate system according to the "Spherical Cap Harmonics" section and Eq. (9) as given in Table 1.

The calculations were done using different values for the ASCH degree to test the convergence of the solution and to find the optimal degree needed to represent the EGM2008 spherical harmonic coefficients in the local adjusted spherical cap harmonic model. As a measure to assess the quality of the production of the quasigeoid/geoid of the area of interest, the RMS error of the chosen grid points was used. It was found that, for an area like Baden–Württemberg, it is sufficient to

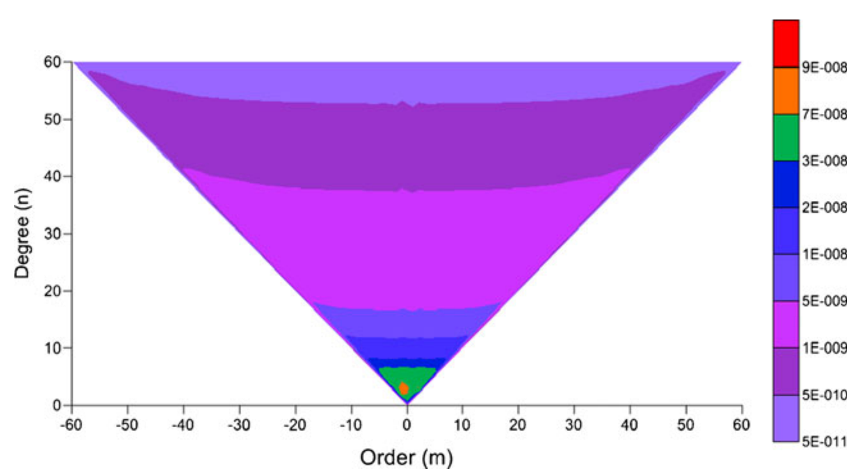

Fig 3 Standard deviations of the ASCH parameters C'_{nm} for $m \geq 0$ and S'_{nm} for $m<0$

estimate the coefficients to a maximal degree of 60 to obtain a RMS of about 9 mm for the resulting quasigeoid/geoid, as a chosen reference function for the transition of $(\overline{S}_{nm}, \overline{C}_{nm})$ global to (S'_{nm}, C'_{nm}) local, see Table 2.

As the accuracy of quasigeoid heights from EGM2008 is 5 cm at maximum (Pavlis et al. 2008), the degree of 60 is sufficient to get a RMSE of less than 1 cm. The solution was computed with equal weights due to the unavailability of the full variance–covariance matrix of the EGM2008 SH parameters. The standard deviations of the calculated ASCH parameters are shown in Fig. 3. These values can be used for a direct error propagation to calculate the standard deviations of the derived quantities by a direct error propagation problem.

Conclusions

Using Eqs. (15) and (16a–d) the global EGM2008 gravity field model can be transformed to a local model. But instead of a maximum degree of 2190, which corresponds to 4800481 parameters $(\overline{S}_{nm}, \overline{C}_{nm})$ in the original EGM2008 development, an ASCH model has been used with maximum degree of 60, with only 3721 (S'_{nm}, C'_{nm}) parameters to map the SH to ASCH in such a way that the resolution of the model in the area of interest, the quasigeoid in our case, is fully exploited. The reduction of the number of parameters can significantly reduce the amount of memory required and reduce dramatically the calculation time for the gravity field parameters at new points or grids.

In the approach presented here gravity field-related computations, such as geoid or quasigeoid calculations in local, regional and even in continental dimension, the a priori information from a global gravity model is based on the ASCH coefficients (S'_{nm}, C'_{nm}). In contrast to the use of "selected observables" (gravity values, vertical deflections etc.) from a global model, such as, e.g. EGM 2008, the use of coefficients transformed to an ASCH with coefficients (S'_{nm}, C'_{nm}) preserves the full physical information content of the original SH model. The covariance matrix of the ASCH coefficients can be derived by the introduction of the inverse covariance matrix of the SH coefficients as weight matrix in Eq. (16a). A further advantage of the transition from a SH model to a regional ASCH is that memory and computation time are saved for all kinds of operations related to the translated ASCH instead of the SH. The translated ASCH coefficients (S'_{nm}, C'_{nm}) (Eq. 16a–d) can now be introduced as direct observations of the respective unknowns in any integrated approach related to gravity field and geoid/quasigeoid determination, like what is shown at the example of the DFHBF approach (www.dfhbf.de) of Karlsruhe University of Applied Sciences by Jäger (2010).

Using the coefficients (S'_{nm}, C'_{nm}) to calculate new points close to the boundary of the spherical may cause somewhat larger deviations and oscillations, probably due to the simplified boundary conditions implied by the method. Tests have shown that these effects can be mitigated by choosing a cap area larger than the area of interest by a factor of 1.2. That extension provides a sufficiently large distance of the area of interest from the border of the calculation cap.

For the calculation of parameters, the Cholesky decomposition was used. As the number of unknowns becomes higher, it is difficult to use traditional inversion algorithms due to time and memory limits (Ghilani and Wolf 2006). Also for much higher degree and order, for example if the maximum degree is 120 or higher, block matrix Cholesky decomposition was used to avoid numerical and memory limits (Schaefer 2003).

Within the representation of gravity potential related observations, such as the terrestrial gravity observations, the ASCH coefficients transformed from the global model contribute as a first part of direct observations. Further coefficients related to higher degrees have to be introduced in addition to parameterize the observations in a higher and adequate resolution. For example, in Baden–Württemberg (Fig. 2), the terrestrial gravity observations have to be parameterized to order and degree of 220 to enable a resolution of 0.01 mgal that corresponds to the measurement accuracy.

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