Palestine Journal of Mathematics

Vol. 6(2)(2017) , 528–532

GENERALIZATIONS OF B.BERGGREN AND PRICE MATRICES

Muneer Jebreel Karama

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 20M99, 13F10; Secondary 13A15, 13M05.

Keywords and phrases: B.Berggren Matrices , Price Matrices, Pythagorean triple, Linear Transformation, Matrix Power
.

Abstract. The aim of this paper is to generalize B.Berggren Matrices, and Price Matrices through a general formula for matrices power; so B.Berggren Matrices and Price Matrices will become a special case when the power of a matrix is reduced to the power of one.

1 Introduction

It is well known that if a right triangle has legs of length a and b its hypotenuse has length c, then $a^2 + b^2 = c^2$, when a is odd and b is even, then $(a, b) = 1$ and in this case we call (a, b, c) a primitive Pythagorean triple.

Overmars [1] stated that B.Berggren discovered a structure of a rooted tree, i.e. a ternary tree that generates all primitive Pythagorean triples, Where F.J.M Bariny [2] showed that when any of the tree matrices $A, B, andC$ are:

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}, C = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix}$$

multiplied on the right by column a vector which is a Pythagorean triple, then we get a different Pythagorean triple. For example if

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}, v = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

so

$$Av = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \\ 13 \end{bmatrix}$$

also if

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}, v = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

then

$$Bv = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 21 \\ 20 \\ 29 \end{bmatrix}$$

also if

$$C = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix}, v = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

then

$$Cv = \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \\ 17 \end{bmatrix}$$

and so on. Price [3] used the following different matrices A', B', C' as shown below:

$$A' = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix}, B' = \begin{bmatrix} 2 & 1 & 1 \\ 2 & -2 & 2 \\ 2 & -1 & 3 \end{bmatrix}, C' = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

In this paper, we will generalize B.Berggren and Price matrices, and show they are just a special case.

To do so, first we find the k^th power of all above matrices. It is suffices to take the matrix A' and findA'^k, because all the matrices above can be treated in the same manner . Assume that

$$A' = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix}$$

. We need to find eigenvalues and eigenvectors of it. Let $\lambda \in R$ be an eigenvalue of the matrixA'. So there exists a non-zero column vector v such that $A'v = \lambda v$, i.e. determent$(A' - \lambda)v = 0$. Now we have

$$A' = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix}, v = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

So, $|A' - \lambda|v = \begin{vmatrix} 2-\lambda & 1 & -1 \\ -2 & 2-\lambda & 2 \\ -2 & 1 & 3-\lambda \end{vmatrix} = 0$. We know $(A' - \lambda)v = 0$ has non-zero solution, namely $(\lambda^2 - 5\lambda + 4)(2 - \lambda) = 0$, so our eigenvalues are **1,2,4**. For each eigenvalue1,2,4, we have an eigenvector so our diagonal matrix say D,

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

. Now we continue to find our three eigenvectors as follows.

For $\lambda = 1$, we have $A' - \lambda I$, where I is the identity matrix, so we have $A' - \lambda I =$

$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & 1 & 2 \\ -2 & 1 & 2 \end{bmatrix}$$

since $A' - \lambda I = 0$, so we have a homogeneous system of linear equation , we solve it by Gaussian Elimination , i.e. $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -2 & 1 & 2 & 0 \\ -2 & 1 & 2 & 0 \end{array}\right)$ finally we got $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

So we have $x_1 - x_3 = 0$, and $x_2 = 0$ so our first eigenvector is

$$\begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix}$$

For $\lambda = 2$, we have $A' - \lambda I =$

$$\begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 2 \\ -2 & 1 & 1 \end{bmatrix}$$

which implies $\left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ -2 & 0 & 2 & 0 \\ -2 & 1 & 1 & 0 \end{array}\right)$ solving by Gaussian Elimination, $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

So we have $x_1 - x_3 = 0, x_2 - x_3 = 0$. Hence our second eigenvector is

$$\begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix}$$

For $\lambda = 4$, we have $A' - \lambda I =$

$$\begin{bmatrix} -2 & 1 & -1 \\ -2 & -2 & 2 \\ -2 & 1 & -1 \end{bmatrix}$$

which implies $\left(\begin{array}{ccc|c} -2 & 1 & -1 & 0 \\ -2 & -2 & 2 & 0 \\ -2 & 1 & -1 & 0 \end{array}\right)$ solving by Gaussian Elimination, $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

So we have $x_1 = 0, x_2 - x_3 = 0$. Thus our third eigenvector is

$$\begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix}$$

Hence our eigenvector matrix is

$$\begin{bmatrix} x_3 & x_3 & 0 \\ 0 & x_3 & x_3 \\ x_3 & x_3 & x_3 \end{bmatrix}$$

. Hence we may take

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

. Hence

$$P^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

Thus

$$A' = PDP^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

Hence we have $A'^k = (PDP^{-1})^k$, where $k = 1, 2, 3, ..., n$ applying the power to both sides we have

$$A'^k = \begin{bmatrix} 2^k & 2^k - 1 & 1 - 2^k \\ 2^k - 4^k & 2^k & 4^k - 2^k \\ 2^k - 4^k & 2^k - 1 & 1 + 4^k - 2^k \end{bmatrix}$$

. By the a similar method, we can find $A^k, B^k, C^k, B'^k, andC'^k$.

Theorem 1.1. *When any of the tree matrices $A^k, B^k, C^k, A'^k, B'^k$, and C'^k are multiplied on the right by column vector which is a Pythagorean triple, then we get a different Pythagorean triple, where :*

$$A^k = \begin{bmatrix} 1 & -2k & 2k \\ 2k & 1-2k^2 & 2k^2 \\ 2k & -2k^2 & 1+2k^2 \end{bmatrix}$$

$$B^k = \begin{bmatrix} \frac{1}{4}(3-2\sqrt{2})^{k} + \frac{1}{4}(3+2\sqrt{2})^{k} + \frac{1}{2}(-1)^{k} & \frac{1}{4}(3+2\sqrt{2})^{k} + \frac{1}{4}(3-2\sqrt{2})^{k} + \frac{1}{2}(-1)^{k+1} & \frac{1}{4}\sqrt{2}((3+2\sqrt{2})^{k} - (3-2\sqrt{2})^{k}) \\ \frac{1}{4}(3+2\sqrt{2})^{k} + \frac{1}{4}(3-2\sqrt{2})^{k} + \frac{1}{2}(-1)^{k+1} & \frac{1}{4}(3-2\sqrt{2})^{k} + \frac{1}{4}(3+2\sqrt{2})^{k} + \frac{1}{2}(-1)^{k} & \frac{1}{4}\sqrt{2}((3+2\sqrt{2})^{k} - (3-2\sqrt{2})^{k}) \\ \frac{1}{4}\sqrt{2}((3+2\sqrt{2})^{k} - (3-2\sqrt{2})^{k}) & \frac{1}{4}\sqrt{2}((3+2\sqrt{2})^{k} - (3-2\sqrt{2})^{k}) & \frac{1}{2}(3+2\sqrt{2})^{k} + \frac{1}{2}(3-2\sqrt{2})^{k} \end{bmatrix}$$

$$C^k = \begin{bmatrix} 1-2k^2 & 2k & 2k^2 \\ -2k & 1 & 2k \\ -2k^2 & 2k & 1+2k^2 \end{bmatrix}$$

$$A'^k = \begin{bmatrix} 2^k & 2^k - 1 & 1 - 2^k \\ 2^k - 4^k & 2^k & 4^k - 2^k \\ 2^k - 4^k & 2^k - 1 & 1 + 4^k - 2^k \end{bmatrix}$$

$$B'^k = \begin{bmatrix} \frac{1}{9}2^{2+2k} + \frac{4}{9} + \frac{1}{9}(-1)^k 2^k & \frac{1}{3} + \frac{1}{3}(-1)^{1+k} 2^k & \frac{1}{9}(-1)^k 2^k + \frac{1}{9}2^{2+2k} - \frac{5}{9} \\ \frac{1}{3}(-1)^{1+k} 2^k + \frac{1}{3}4^k & (-1)^k 2^k & \frac{1}{3}(-1)^{1+k} 2^k + \frac{1}{3}4^k \\ \frac{5}{9}4^k + \frac{1}{9}(-1)^{k+1} 2^k - \frac{4}{9} & -\frac{1}{3} + \frac{1}{3}(-1)^k 2^k & \frac{1}{9}(-1)^{k+1} 2^k + \frac{5}{9} + \frac{5}{9}4^k \end{bmatrix}$$

$$C'^k = \begin{bmatrix} 2^k & 1 - 2^k & -1 + 2^k \\ -2^k + 4^k & 2^k & 4^k - 2^k \\ -2^k + 4^k & 2^k - 1 & 1 + 4^k - 2^k \end{bmatrix}$$

. We will use induction to prove the case A^k (other can be proved similarly). So we need to prove if

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}$$

,then it is true for

$$A^k = \begin{bmatrix} 1 & -2k & 2k \\ 2k & 1-2k^2 & 2k^2 \\ 2k & -2k^2 & 1+2k^2 \end{bmatrix}$$

Assume $k = 1$.

$$A^1 = \begin{bmatrix} 1 & -2(1) & 2(1) \\ 2(1) & 1-2(1)^2 & 2(1)^2 \\ 2(1) & -2(1)^2 & 1+2(1)^2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}$$

, this is true for k =1 .

Assume it is true for k=n, i.e.$A^n = \begin{bmatrix} 1 & -2n & 2n \\ 2n & 1-2n^2 & 2n^2 \\ 2n & -2n^2 & 1+2n^2 \end{bmatrix}$

Consider k=n+1,$A^{n+1} = (A^n)(A^1) = \begin{bmatrix} 1 & -2n & 2n \\ 2n & 1-2n^2 & 2n^2 \\ 2n & -2n^2 & 1+2n^2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} =$

$\begin{bmatrix} 1 & -2(n+1) & 2(n+1) \\ 2(n+1) & 1-2(n+1)^2 & 2(n+1)^2 \\ 2(n+1) & -2(n+1)^2 & 1+2(n+1)^2 \end{bmatrix} = A^{n+1}$, which is true for k =n+1, if it is true for k=n.

It is important to note that all matrices ($A^k, B^k, C^k, A'^k, B'^k$, and C'^k) are unimodular because they have only integer entries and their determinants are 1 or -1, thus all their inverses are unimodular , i.e.$(A^{-k}, B^{-k}, C^{-k}, A'^{-k}, B'^{-k}$, and $C'^{-k})$, which implies if $u, v \in Z^3$,

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

,

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

, such that $u_1^2 + u_2^2 = u_3^2$, $v_1^2 + {v_2}^2 = {v_3}^2$, since we have linear transformation say $T : Z^3 \to Z^3$, such that : $T(u^{\to}) = A^k v^{\to}$, which yields to $v^{\to} = A^{-k} T(u^{\to})$, this is for all$(A^k, B^k, C^k, A'^k$, $B'^k, C'^k, A^{-k}, B^{-k}, C^{-k}, A'^{-k}, B'^{-k}$, and C'^{-k}.

Finally, if we multiply any of the above matrices together an arbitrary number of times, then we get a matrix, say F. If v is a Pythagorean triple vector, then Fv gives a different Pythagorean triples.

References

[1] Overmars, Anthony, and Lorenzo Ntogramatzidis. [*A new parameterization of Pythagorean triples in terms of odd and even series*]. arXiv preprint, arXiv:1504.03163 ,2015.

[2] Barning, F. J. M. [*Over pythagorese en bijna Â■pythagorese driehoeken en een generatieproces met behulp van unimodulaire matrices (in Dutch)*]. Math. Centrum Amsterdam Afd. Zuivere Wisk. ZW, Â■011: 37 ,1963.

[3] Price, H.Lee. [*The Pythagorean Tree: A New Species*]. arXiv preprint, arXiv:0809.4324 ,2008.

Author information

Muneer Jebreel Karama, Department of Applied Mathematics, Palestine Polytechnic University , PO Box 198, Palestine, West Bank, Hebron, Palestine.
E-mail: muneerk@ppu.edu

Received: April 22, 2016.
Accepted: January 7, 2017.