Palestine Polytechnic University

College of Engineering & Technology

Civil and Architectural Engineering Department

Surveying and Geomatics Engineering

Graduation Project

GPS Height Integration

By

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Murad Talahmeh

Supervisor:

M.Sc. Ghadi Zakarneh

Hebron-Palestine

May-2007

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CERTIFICATION

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Project Team

Fadi Najajrah Fahid Tarayrah
Murad Talahmeh

By the guidance of our supervisor, and the approval of members of testing committee, this project is delivered to the department of civil and architectural engineering, in the college of engineering and technology to be as partial fulfillment of the requirements of the department for the degree of B.Sc in surveying and geomatics engineering.

Supervisor signature
Name: …….……...

Head of dep. Signature:
Name: ………………

Committee member's signature:

Name: ………………….. Name: ………………… Name: …………..

Hebron- Palestine
May -2007

الإهداء

الى روح الاب والمعلم والقائد الشهيد ابو عمار رحمه الله...
إلى روح الأستاذ كمال غطاشة رحمه الله ...
إلى الذين ساروا مع الفجر ليخطوا لنا طريق الحرية ...
إلى رواد العلم المبدع، طلاب اليوم، بناة المستقبل، قادة الغد ...
إلى نبع الحنان و المحبة .... أمي الحبيبة
إلى نبع البذل و العطاء ... أبي الحبيب
إلى من رووا بدمائهم الزكية تراب هذا الوطن الغالي ...
إلى أسرى الحرية داخل زنازين القمع الصهيوني
إلى الذين حملوا أرواحهم على اكفهم من اجل وطنهم ...
إلى حبيبتي "فلسطين" ...
إلى كل الأوفياء والمخلصين...
إلى طلاب جامعة بوليتكنيك فلسطين من إخوتنا وأخواتنا ... من خالفنا منهم الرأي أو من اتفق معنا وانطلق لآفاق التجدد و الإبداع.

نهديهم جمعيا هذا العمل المتواضع .

"قل إن صلاتي ونسكي ومحياي ومماتي لله رب العالمين"

Acknowledgment

بداية نحمد الله العزيز القادر الذي بحمده تدوم النعم

نتقدم بأرفع آيات الشكر والامتنان إلى جامعة بوليتكنك فلسطين

والى كلية الهندسة
دائرة الهندسة المدنية والمعمارية

إلى هيئة التدريس في الجامعة

إلى مدرسي هندسة المساحة

ونخص بالذكر الاستاذ ماهر العويوي

والمهندس اشرف زبن والمهندس معتز قفيشة

وشكر خاص إلى رئيس الدائرة الدكتور نبيل ألجولاني

والى نبع العطاء المستمر ...الأستاذ المهندس
غادي زكا رنة

والى كل الذين قدموا لنا العون والمساعدة .

Abstract

GPS Height Integration

Project team

Fadi Najajrah Fahid Tarayrah
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Project supervisor

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This project aims to find a geoid model for 10*10km area in Hebron city, using at least twenty reference point, with known (E, N, H) coordinate from traditional survey to a polynomial estimation for the geoid.

The model using reference points will be built using polynomials of different degrees (1^{st} , 2d , 3d , 4th) this model will be compared to the global geoid model (EGM 96) with spherical harmonics representing the equipotential surface (Gravimetric Geoid model).

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INTRODUCTION

CHAPTER ONE

Introduction

1.1 Background

This project aims to find a geoid model for 10*10km area in Hebron city, using at least twenty reference point, with known (E, N, H) coordinate from traditional survey to a polynomial estimation for the geoid.

Using GPS , (λ , f , h) will be measured for the reference points with respect to two different datum's , the first is the Palestine coordinate system with Clark ellipsoid , and the second is world geodetic system (WGS 84) or (GRS 80) .a height reference system nearly geoid is applied Using the ground surveying height H (orthometric height) and the GPS height that represent by heights above the ellipsoid.

The model using reference points will be built using polynomials of different degrees (1^{st}, 2d, 3d, 4th) this model will be compared to the global geoid model (EGM 96) with spherical harmonics representing the equipotential surface (Gravimetric Geoid model).

1.2 Objectives

This project deals with visual basic program including data analysis to built a geoid model for 10*10km area in Hebron city, this model depends on the difference between the oerthometric height and ellipsoidal height for the reference points.

The difference between the ellipsoid and the geoid at a point is called the geoidal separation or geoid undulation (N). Using Global Positioning System (GPS) measurements, the height of a point on the earth above the ellipsoid can be accurately measured. This height is called the ellipsoidal height or geodetic height (a geometric height) and it is normal to the ellipsoid. The elevation of a point on the earth is the height above the geoid; this height is the orthometric height. The relationship described in figure 1.1 and shown in equation 1.1:

$$H = h - N \tag{1.1}$$

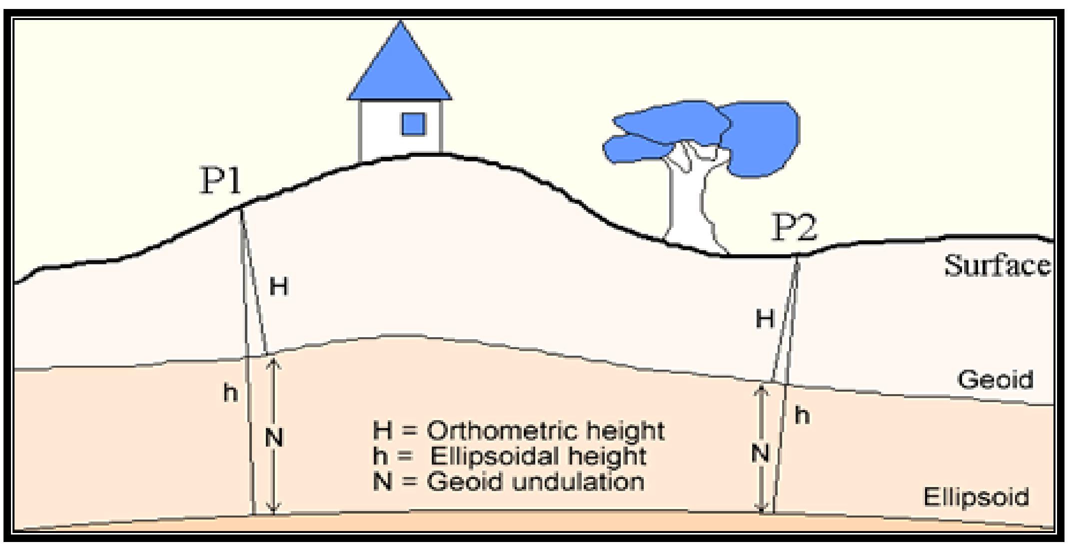

Figure 1.1: The relationship between H, h and N

The project work will be achieved by the following steps:

1- Measurement of reference points coordinates for both systems, the Palestinian system (E, N, H) and the GPS system (λ, f , h).
2- Modeling the geoid model as a polynomial estimation of the 1st, 2nd, 3rd or 4th degree as required for the user of the software programmed in this project.
3- Applying the global geoid model (EGM) for the given area and it is comparison to the computed height reference system.
4- The end-user of the software will have the choice to select to use the computed height reference system or the global geoid model.
5- Two separate tools will be programmed; the first is the height reference system modeling tool, with ability to store results in different format like grid or polynomial parameters. The second tool is to calculate the height H for the newly GPS measured GPS point.

1.3 Methodology

This project has the following scope:

Chapter 1; Introduction to the project, objectives, and methodology.

Chapter 2; Introduction to GPS, Galileo, Glonass.

Chapter3; Introduction to local and gravity geoid model, global Geoid model and Principle of gravity measurements, and height systems.

Chapter 4; modeling height reference surface.

Chapter 5; Results and testing.

Chapter 6; Conclusion and Recommendation.

Appendix; The EGM96 parameters up todegree 36.

Prove that the gravity potential satisfies Laplace equation.

Software code

1.4 Study area

The study area is definition by square that is one side is 10KM in that contain Hebron city and a rounding area as shown in figure 1.2.

Figure 1.2: Study area

1.5 previous studies

In our university there is no previous study dealing with this project aims, there are many institutions in the world governmental or educational, ones that concern in producing Geoid models locally or globally for example EGM96. one of the previous study is the digital finite element height reference surface (DFHRS) research project of the hochschule karlsruhe university of applied sciences.

1.6 Problems and limitations

There is a problem present in this project due to:

- Lack of accurate coordinate for trig point, the accurate coordinate that covers this project doesn't available in the whole country, and the elevation of this trigs is not correct

1.7 Equipment necessary to carryout the survey

§ **GPS Receiver with their accessories**

The following list may be taken as a guide:

- GPS receiver, antenna and associated cabling.
- External batteries (including spares). Battery charger.
- Antenna tripod, tribrachs or adaptors for mounting antenna on pillars.
- Field books, access details, observation schedule instructions, etc.
- Useful ancillary equipment: camera, communications equipment.

§ Total Station

1.8 Schedule and time planning

Table 1.1: Timetable

Activities	Time(week)	Starting date (approximate)	Ending date (approximate)	Done (or not)
1- Problem definition	4	20-9-2006	18-10-2006	done
2- literature Review	2	18-10-2006	2-11-2006	done
3- data Collecting	6	2-11-2006	17-12-2006	done
4- data Editing	5	17-12-2006	25-1-2007	done
5-Preparing research plan	2	25-1-2007	10-2-2007	Not yet
6-Field work	3	10-2-2007	1-3-2007	Not yet
7-Formatting and editing	4	1-3-2007	1-4-2007	Not yet
8-Covering and printing	2	1-4-2007	16-4-2007	Not yet
9- Final report	29	20-4-2007	Final date	Not yet

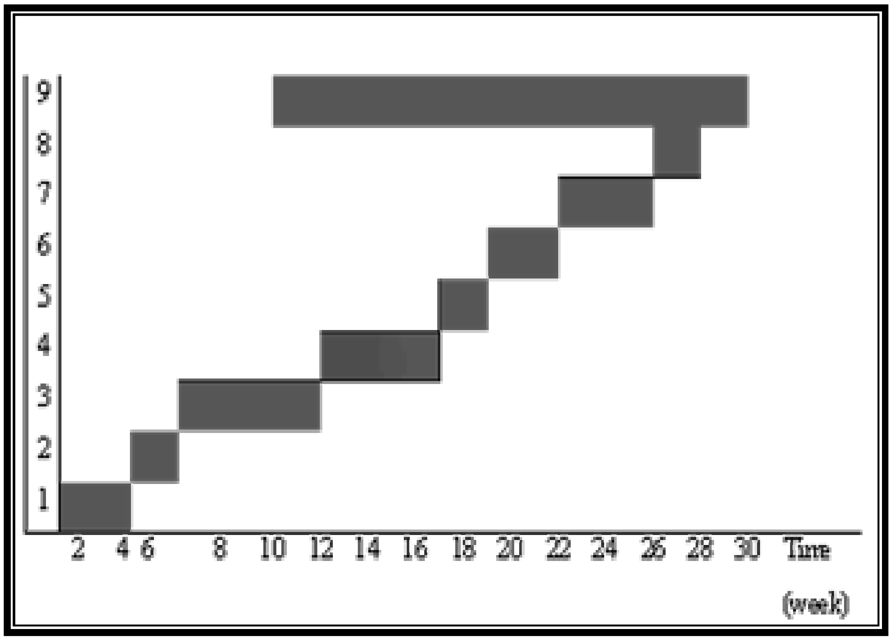

Figure 1.3: Time planning

CHAPTER

1

INTRODUCTION

CHAPTER

2

GLOBAL POSITIONING SYSTEM (GPS)

CHAPTER TWO

Global Positioning Systems
GPS

2.1 Introduction

Since earliest times, man has concerned itself with where is he sat and where he is going. One of the earliest techniques that travelers used were simple rock cairns marking the trail, either for finding their way back, repeating their path, or for others to follow. A better method was the use of a clay tablet or piece of parchment which could be moved from one-person to another. These were the earliest maps. Today, we live in a world of precision. We expend great amounts of intellectual and monetary currency on ever-smaller units of measurement. Knowledge of where we are and where we are going has, for the past several thousand years, relied on highly trained and skilled surveyors. The science of surveying has achieved phenomenal levels of precision but, unfortunately, only for those very few whose needs have outweighed the ever-increasing cost necessary to achieve that precision.

2.2 Definition of the GPS

The Global Positioning System is a space-based navigation and positioning system that was designed by the U.S. Military to allow the soldiers to autonomously determine their position within 10 to 20 meters accuracy without any other radio (or otherwise) communications.

The system provides worldwide coverage, and that the coverage be available 24 hours a day. At the same time, it had to be militarily safe, so that the U.S. Military had to have the ability to deny any hostiles' use of the system without degrading their own use.

The plane was that each soldier and each military vehicle will be equipped with a GPS receiver. Therefore, it was necessary that the receivers be sufficiently low in cost to meet this end.

2.3 GPS civil applications

Although GPS was originally envisioned for military use, it soon became obvious that there would be numerous civilian applications. Some of the civilian applications are:

1. Surveying,
2. Marine navigation,
3. Scientific,
4. Commercial,
5. Recreational uses,
6. Truck fleet management,
7. Control of construction, and
8. Monitoring of deformation.

2.4 GPS segments

The Global Positioning System consists of three major segments as follows:

1. Space Segment, the satellite or space vehicles.
2. Control Segment, ground station run by the DOD (Department Of Defense).
3. User segment, all users and there GPS receivers.

These three segments are illustrated in figure 2.1.

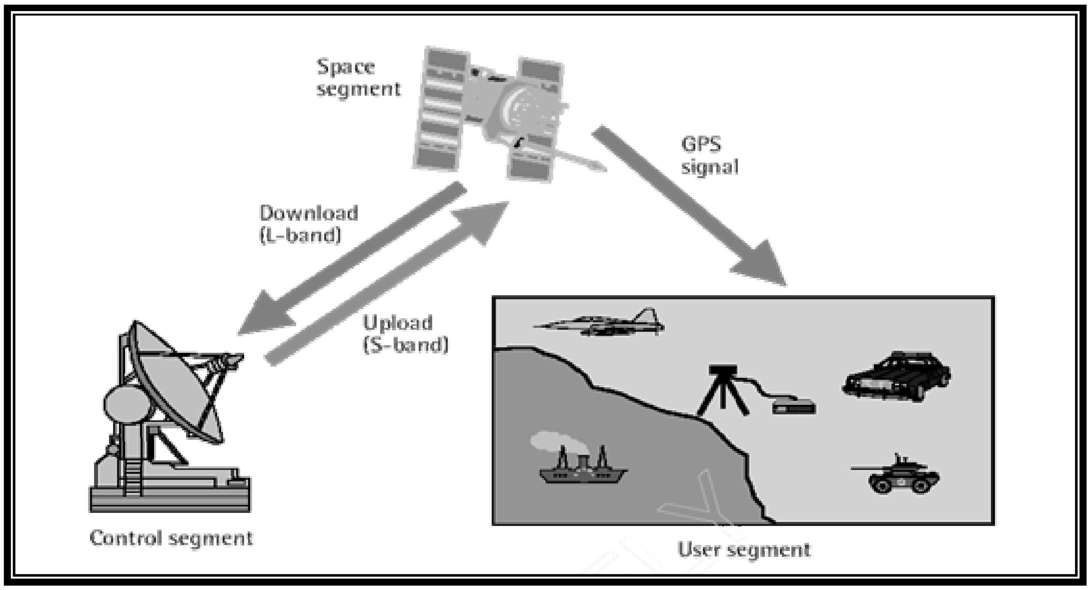

Figure 2.1: GPS segments. [2]

2.4.1 The control segment

The control segment of the Global Positioning System consists of one Master Control Station (MCS) in Colorado, and four unmanned monitor stations located strategically around the world, located more or less equidistant around the equator, as shown in figure 2.2.

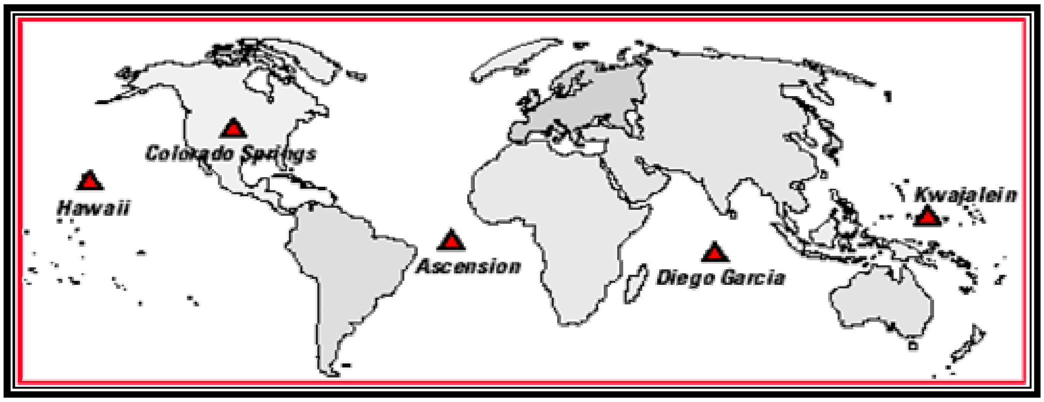

Figure 2.2: Control Segment Station Locations [6]

The monitor stations passively track all GPS satellites visible to them at any given moment, collecting signal (ranging) data from each. This information is then passed on to the Master Control Station. The Master Control Station then periodically sends the corrected position and clock-timing data to the appropriate ground antennas which then upload those data to each of the satellites. Finally, the satellites use that corrected information in their data transmissions down to the end user. This sequence of events occurs every few hours for each of the satellites to help insure that any possibility of error creeping into the satellite positions or their clocks is minimized.

2.4.2 The Space Segment.

The space segment consists of the complete constellation of orbiting Navistar GPS satellites.

Facts about GPS:

- Each satellite weighs approximately 900 kilograms and is about five meters wide with the solar panels fully extended.
- The base size of the constellation includes 21 operational satellites with three orbiting backups, for a total of 24.
- They are located in six orbit satellites approximately 20,200 kilometers altitude. Each of the six orbits is inclined 55 degrees up

from the equator, and is spaced 60 degrees apart, with four satellites located in each orbit.

- The orbital period is 12 hours, meaning that each satellite completes two full orbits each 24-hour day. The space segments are shown in the following figures.

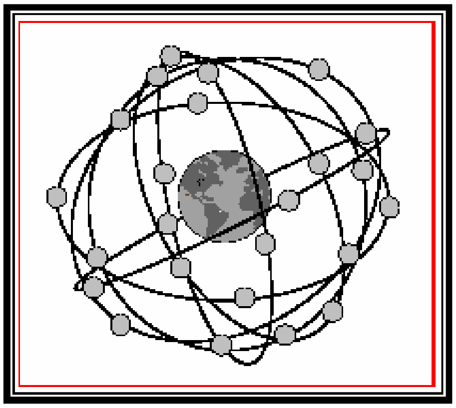

Figure 2.3: GPS satellite constellation [6]

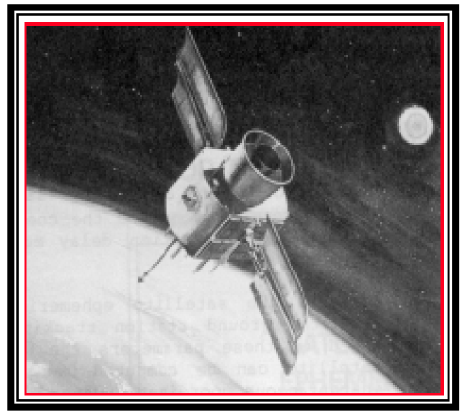

Figure 2.4: GPS satellite [6]

2.4.3 User Segment

The user segment includes all military and civilian users. With a GPS receiver connected to a GPS antenna, a user can receive the GPS signals, which can be used to determine his or her position anywhere in the world. GPS is currently available to all users worldwide for free.

2.5 The principle of GPS positioning

The concept of the GPS is very simple. The GPS is a distance (ranging) system. This means that the only thing that the user is trying to do is determine the distance to any given satellite. The GPS satellite sends signals in all directions, although there is an excellent orientation toward the Earth. The GPS operates on the principle of three dimension trilateration. In trilateration, the position of an unknown point is determined by measuring the lengths of the sides of a triangle between the unknown point and three or more known points (i.e., the satellites).

The satellites transmit radio signal code that is unique to each satellite. Receivers on the ground passively receive each visible satellite's radio signal and measures the time that it takes for the signal to travel to the receiver.

Distance is then simply computed according to equation:

$$R = V * T \tag{2.1}$$

Where;

R: Distance.

T: Time in transit.

V: Velocity of transit (speed of light) fixed at 300,000 kilometers per second.

Therefore, the only thing needed by the user to calculate distance from any given satellite is a measurement of the time it took for a radio signal to travel from the satellite to the receiver. See Figure 2.5.

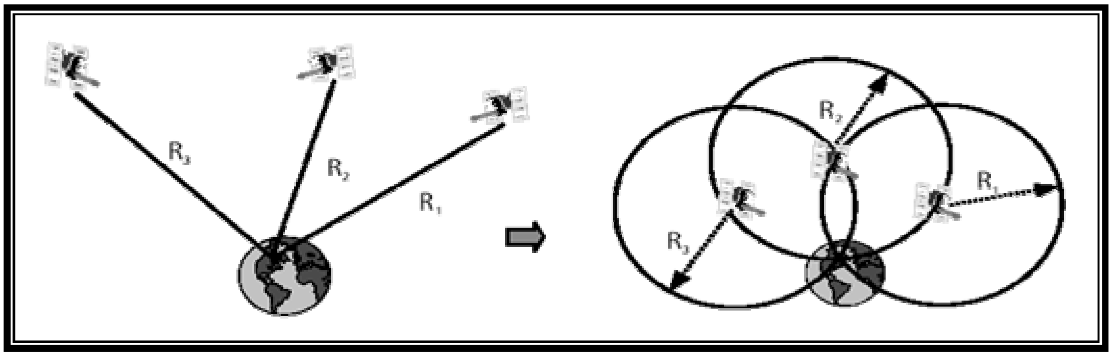

Figure 2.5: Basic idea of GPS positioning [2]

Because we measure the time it takes a radio signal to travel from a satellite transmitter down to our receiver. To acquire an accurate position, we have to make very precise time measurements. The signal needs about l/15 of a second from a satellite to reach our receiver on the ground. With radio waves traveling at some 300,000 Km/s, only 1/1,000,000 (one one-millionth) of a second of error in measuring the travel time translates into approximately 300 meters of error in our position.

To keep very accurate time, each satellite has four atomic clocks on, two rubidium and two cesium. These clocks are accurate to within billionths of a second per month. But they are not really practical for our ground-based receivers.

It is critical that the satellite and receiver both start counting time at exactly the same moment and continue to count time at the same rate since it's the time it takes for a signal to reach us that we're trying to measure. It turns out that we can insure this by adding a fourth satellite that acts as a time referee.

2.6 GPS signals

Each GPS satellite has several very accurate atomic clocks on broad. The clocks operate at a fundamental frequency of 10.23MHz. This is use to generate the signal s that are broadcast from the satellite. The satellites broadcast two carrier waves constantly. These carrier waves are in the L-Band (used for radio), and travel to earth at the speed of light. These carrier waves are derived from the fundamental frequency, generated by a very precise atomic clock:

· The L1 carrier is broadcast at 1575.42 MHz (10.23 x 154)

· The L2 carrier is broadcast at 1227.60 MHz (10.23 x 120).

The L1 carrier then has two codes modulated upon it. The C/A Code or Coarse/Acquisition Code is modulated at 1.023MHz (10.23/10) and the P-code or Precision Code is modulated at 10.23MHz. The L2 carrier has just one code modulated upon it. The L2 P-code is modulated at 10.23 MHz.as shown in figure 2.6.

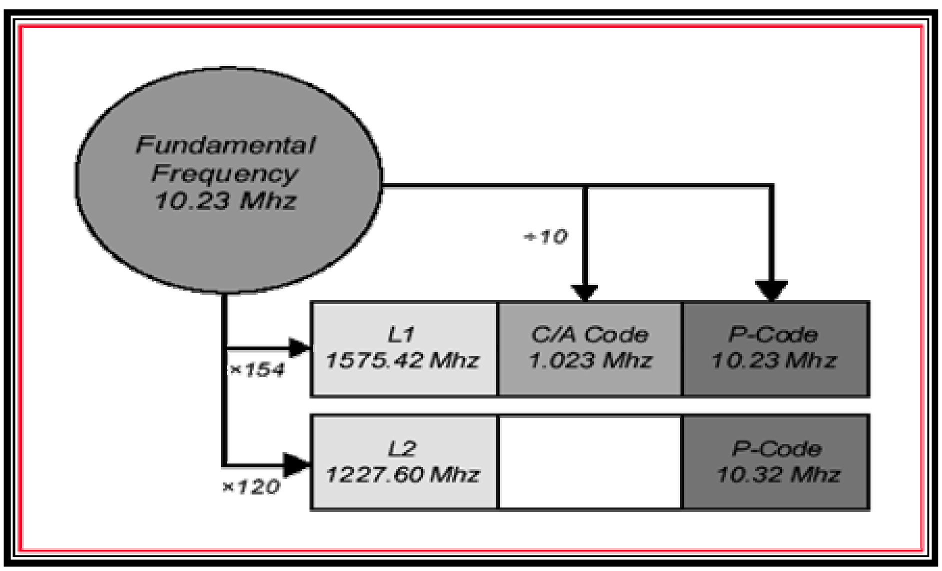

Figure 2.6: GPS Signal Structure [6]

2.7 GPS error budget

There are several sources of error that degrade the accuracy of all forms of GPS positioning. They include:

1. Satellite clock timing error,
2. Satellite position error (ephemeris error),
3. Ionosphere and troposphere refraction,
4. Receiver noise,
5. Multi path,
6. Selective Availability or SA,
7. Geometric arrangement of the satellites.

2.7.1 Satellite position and clock timing errors

These errors have typically low sense. The Air Force constantly monitors each satellite and sends up correction data every few hours. The accuracy of satellite position is about 1 meter. This uncertainty in position can introduce errors of similar magnitude in the final position calculation.

2.7.2 Ionospheric and tropospheric refraction

The ionosphere is a layer of electrically charged particles between around 50 and 200 kilometers altitude. The troposphere is simply what we usually think of as the atmosphere, extending from the surface up to between eight and 16 kilometers altitude. Both ionosphere and troposphere cause bending of the signals. This bending of radio waves is called refraction. The problem with the Ionosphere is the electrically charged particles that drag on the incoming signal. In the troposphere, the problem is with the water vapor content which does the same thing. These problems are even further exacerbated when a satellite is low on the horizon. This is because a line tangent to the surface of the Earth (or nearly so) passes through a much thicker layer of atmosphere than if that line were pointing straight up.

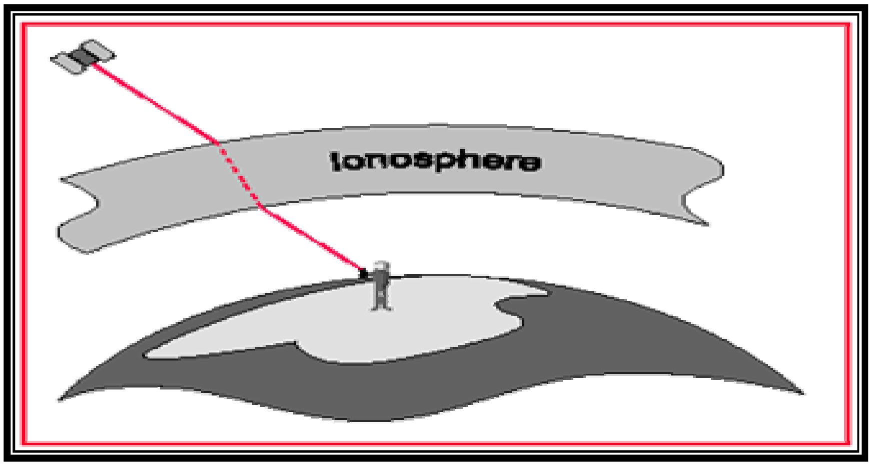

Figure 2.7: Ionosphere and troposphere refraction [6]

To deal with refractions the satellite's NAV-massage includes an atmospheric refraction model that compensates for as much as 50-70% of the error and to use a dual-frequency receiver which simultaneously collects the signals on both the Ll and L2 carriers. Because the amount of refraction that a radio wave experiences is inversely proportional to its frequency, using two different frequencies transmitted through the same atmosphere at the same time makes it relatively easy to compute the amount of refraction taking place and compensate it.

$$d_{TORP} = \frac{K_d}{\sin\sqrt{E^2 + 1.5^2}} + \frac{K_W}{\sin\sqrt{E^2 + 1.5^2}} \tag{2.2}$$

$$K_d = 77.6\frac{P}{T}\left[40136 + 148.72((T - 273.16) - h)2.10^{-7}\right] \tag{2.3}$$

$$K_d = 77.6\frac{4810}{T^2}\left[(1100 - h)2 * 10^{-7}\right] \tag{2.4}$$

HOPFIELD MODELL

E=Satellite Elevation

h=Satellite height

P=Pressure (mbar)

e=Partial vapour pressure (mbar)

T=Temperature (Kelvin)

The ionosphere is modeled for both code and phase observations as shown in equation 2.5 and 2.6

$$d_{IONO[m]} = -\frac{1}{\cos(Z)} * \frac{40.3 * VTEC}{f^2} - PhaseOpservation \quad (2.5)$$

$$d_{IONO}[m] = +\frac{1}{\cos(Z)} * \frac{40.3 * VTEC}{f^2} - CodeOpservation \quad (2.6)$$

Where f : frequency

VTEC: Vertical total electron content in ionosphere given by maps from IGS (**International GNSS Service)**

2.7.3 Mask angle

This means that the receiver can be set to ignore any satellite signals that come from below a user-definable angle above the horizon, or mask them out. The most typical mask angle is usually somewhere between 10-15 degrees. See figure 2.8

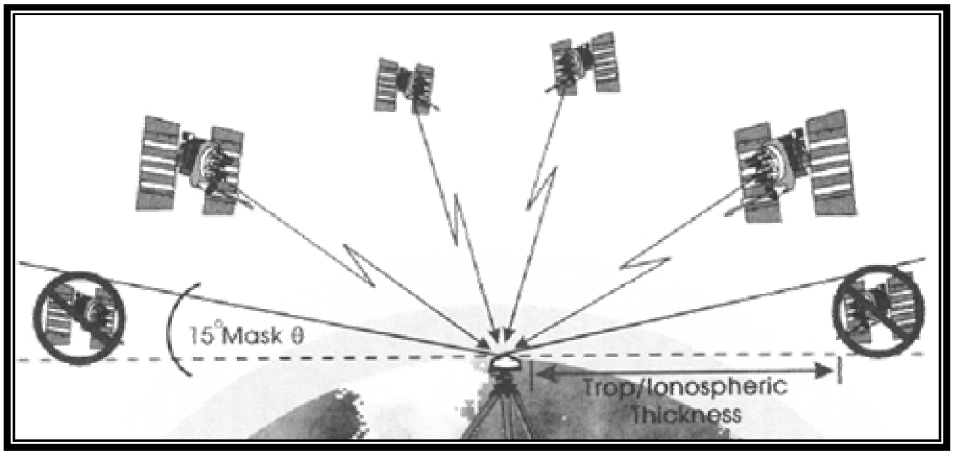

Figure 2.8: Mask Angle [2]

2.7.4 Receiver noise or electronic noise

Produced by the receiver itself that interferes with the very weak incoming signal, While this error is highly variable among receiver brands, most have some kind of internal filtering designed to minimize the problem some better than others.

2.7.5 Multi-path errors

Multi-Path is simply the reception of a reflected satellite signal. With multi-path reception, the receiver collects both the direct signal from the satellite and a fractionally delayed signal of some nearby reflective surface then reached the receiver. The problem is that the path of the signal that has reflected off some surface is longer than the direct line to the satellite. As shown in the figure 2.9.

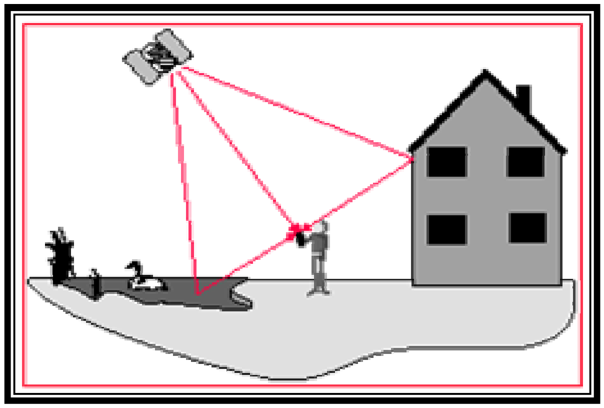

Figure 2.9: Multi-Path Errors [6]

There are several ways to deal with this problem. Most receivers have some way of seeing and comparing the correct and incorrect incoming signal. Since the reflected multi-path signal has traveled a longer path, it will arrive a fraction of a second later, and a fraction weaker than the direct signal, or by using semi-directional, ground-plane antennas to reduce the amount of multi-path that the receiver will have to deal with. Semi-directional antennas are designed to reject any signal below a tangent to the surface of the Earth

2.7.6 Selective availability

Selective Availability is a process applied by the U.S. Department of Defense to the GPS signal. This reduces civilian accuracy levels to a maximum of about 100 meters. Additionally, the ephemeris (or path that the satellite will follow) is broadcast as being slightly different from what it is in reality. Users of differential systems are not significantly affected by S/a.

2.7.7 Dilution of precision (DOP)

The effect of satellite geometry is quantified in the measure called dilution of precision, or DOP. When satellites are widely spaced the overlap area of the two zones of possible satellites range error is relatively small, this area called area of positional ambiguity. Figure 2.10 illustrates the low DOP, while figure 2.11shows high DOP.

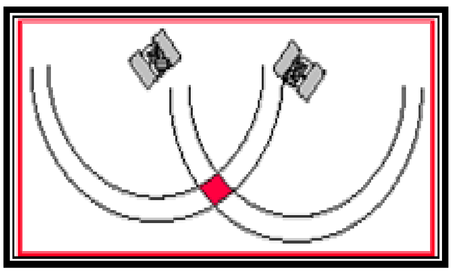

Figure 2.10: Well spaced satellites
Low uncertainty of position

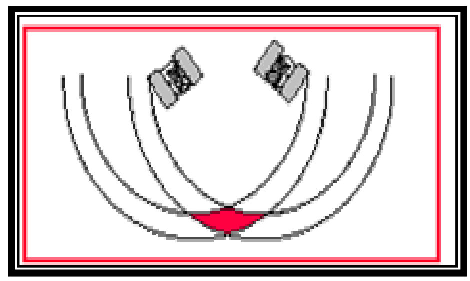

Figure 2.11: Poorly spaced
High uncertainty of position [6]

The best way to minimize the effect of DOP is to observe as many satellites as possible. And these are the values of dilution of precision:

1. A DOP value less than 2 is considered excellent.
2. A DOP value between 2 and 3 is considered very good.
3. A DOP value between 3 and 5 is considered good.
4. A DOP value greater than 5 and less than 6 is considered fair.

Different types of Dilution of Precision or DOP can be calculated depending on the dimension; these values are calculated by the covariance matrix of the position generated from least squares adjustment:

1. VDOP. Vertical Dilution of Precision. Gives accuracy degradation in vertical direction.

$$\text{VDOP} = \sqrt{q_{zz}} \quad (2.7)$$

2. HDOP. Horizontal Dilution of Precision. Gives accuracy degradation in horizontal direction.

$$\text{HDOP} = \sqrt{q_{xx} + q_{yy}} \tag{2.8}$$

3. PDOP. Positional Dilution of Precision. Gives accuracy degradation in 3D position.

$$\text{PDOP} = \sqrt{q_{xx} + q_{yy} + q_{zz}} \tag{2.9}$$

4. GDOP. Geometric Dilution of Precision. Gives accuracy degradation in 3D position and time.

$$\text{GDOP} = \sqrt{q_{xx} + q_{yy} + q_{zz} + q_{tt}} \tag{2.10}$$

5. TDOP. Time dilution of precision. Gives accuracy in time.

$$\text{TDOP} = \sqrt{q_{tt}} \tag{2.11}$$

2.8 GPS positioning modes

2.8.1 Absolute positioning

Absolute positioning use only one receiver to collect data from satellites to determine user's position using a technique called trilateration. The user's measures the distance between satellites and the earth, so the position is determined by the intersection of the observed ranges to the satellites. At least three satellite ranges are required to compute a 3-D position. Using this method will not provide the accuracies needed by users.

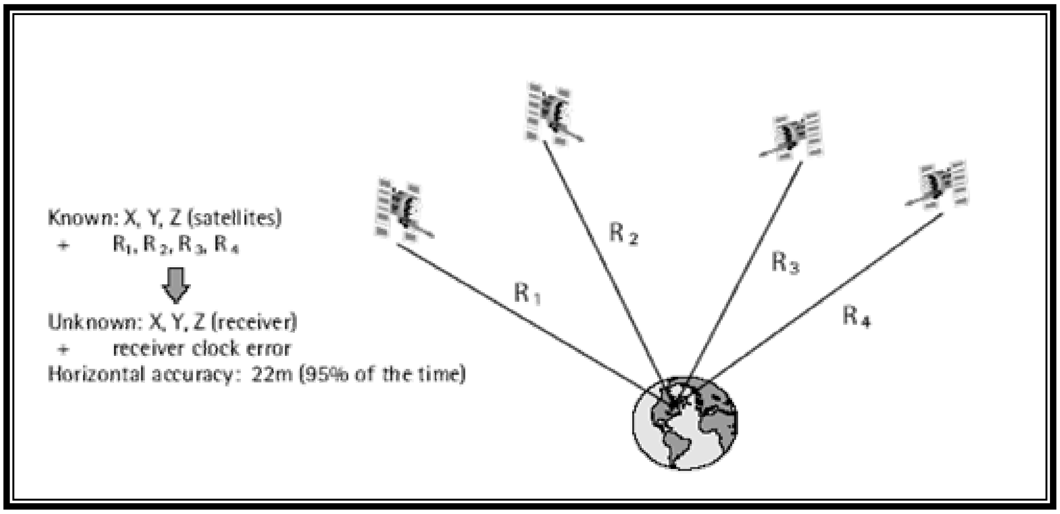

Figure 2.12: Principle of GPS point positioning. [2]

Two type of measuring:

1. Code observation using time measure:

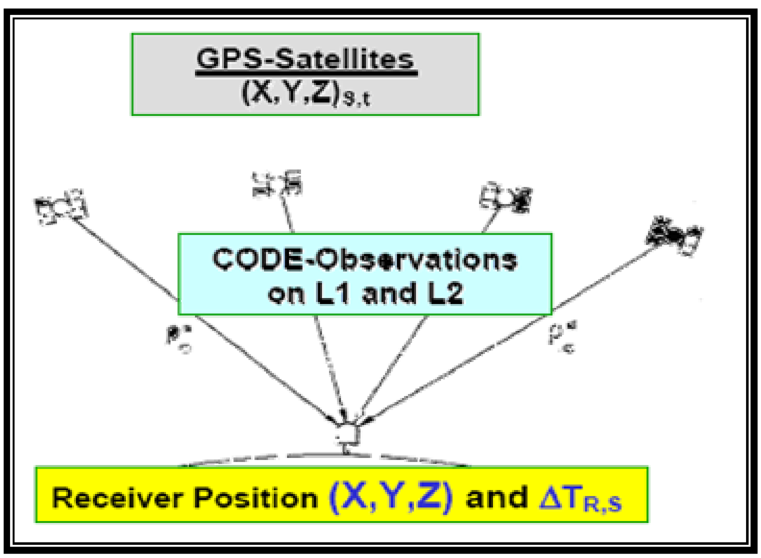

Figure 2.13: Code observation using time measure [12]

$$rR = \sqrt{(X - X_S)^2 + (Y - Y_S)^2 + (Z - Z_S)^2} + \Delta T_{S,R}.C + d_{ion} + d_{trop} \qquad (2.12)$$

2. Phase observations (Relative Positioning Observation Equation):

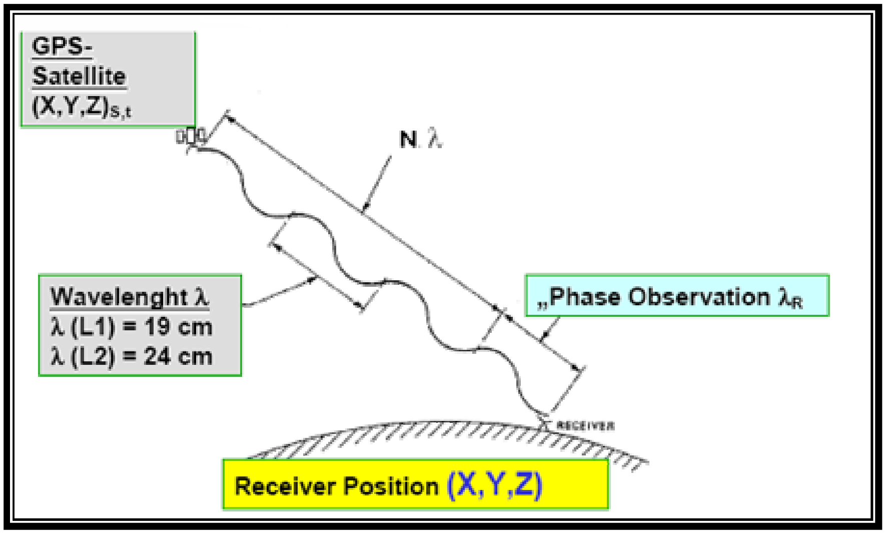

Figure 2.14: Phase Observation [12]

$$l\,R = \sqrt{(X - X_S)^2 + (Y - Y_S)^2 + (Z - Z_S)^2} \text{ -N. } l + d_{ion} + d_{trop} + d_{tS-tR} \qquad (2.13)$$

Where:

X, Y, Z: receiver position

X_s, Y_s, Z_s: satellites position
ΔT: time difference between and receiver
C: light speed
N: number of ambiguity
λ: wavelength

2.8.2 Differential GPS (Differential or relative GPS)

DGPS (Differential GPS) is achieved by using two receiver to have better accuracy, on of them called base receiver which places over a known point (National Geodetic Survey NGS). The second receiver is called a rover which collects data from the unknown points in the field. Both the base and a rover collect the same data at the same time from the same satellites. The base receiver knows its exact location, it is also receives a position from satellites (with the same component of errors).

From the difference between the known position and the GPS derived position a vector displacement of errors can be applied to the rover's position data. The base line length which is the distance between the base and the rover are best kept below 20 Km to get the required accuracy because if the base line is greater than 20Km, its possible that the two receivers might be observing one or more different satellite, also Ionospheric and tropospheric correction may be different.

The observation equations of the DGPS are:

$$\Delta X = X\ (\text{Rover}) - X\ (\text{Base}) \tag{2.14}$$

$$\Delta Y = Y\ (\text{Rover}) - Y\ (\text{Base}) \tag{2.15}$$

$$\Delta Z = Z\ (\text{Rover}) - Z\ (\text{Base}) \tag{2.16}$$

GPS elevation has less accuracy than the accuracy of the horizontal position. So height in GPS is the weakest component, this is due to the orbital geometry of the X, Y, Z positional determinations of satellite orbit. And the accuracy is:

$$\sigma\ \text{plan} : \sigma\ \text{height} = 1 : 1.5$$

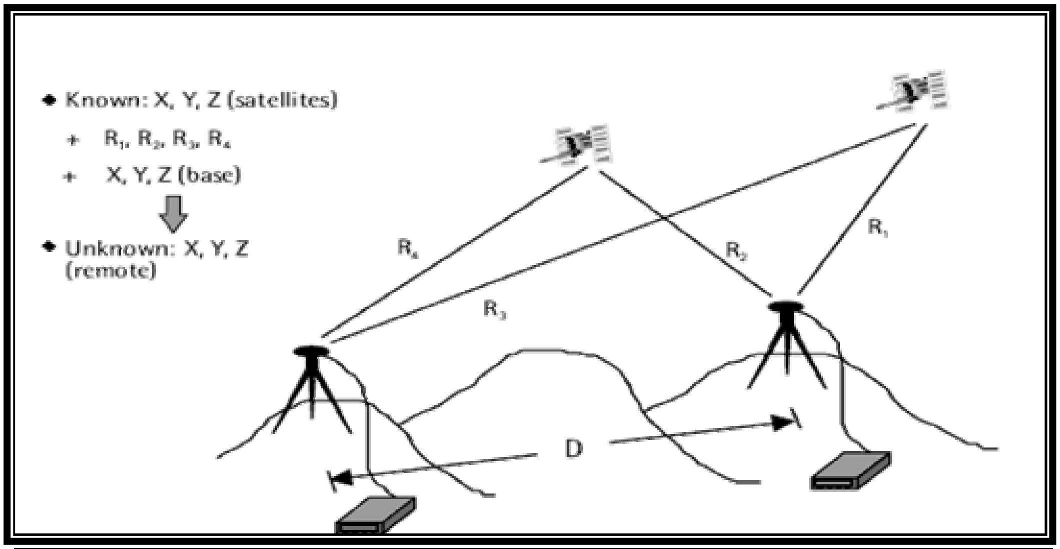

Figure 2.15: Principle of GPS relative positioning. [2]

Differential GPS carrier phase surveying is used to obtain the highest precision from GPS and has direct application to most topographic and engineering survey activities. DGPS uses three Different GPS differential surveying techniques:

1. Static.
2. Fast Static.
3. Real Time Kinematics.

2.8.2.1 Static GPS survey techniques

In Static GPS surveying two GPS receivers are used to measure a GPS baseline distance. GPS receiver pairs are set up over stations of either known or unknown location.

Typically one of the receivers is positioned over a point whose coordinates are known, and the second is positioned over another point whose coordinates are unknown, but are desired. Both GPS receivers must receive signals from the same four (or more) satellites for a period of time that can range from a few minutes to several hours, depending on the conditions of observation and precision required.

Accuracy of GPS static surveys is the most accurate and can be used for any order survey.

2.8.2.2 Fast Static Surveying Techniques

Fast static surveying is similar to static surveying methods. The rover receiver spends only a short time on each station, loss of lock is allowed between stations, and accuracy is similar to static.

Fast static surveying requires that one receiver be placed over a known control point. A rover receiver occupies each unknown station for 5-20 min, depending on the number of satellites and their geometry.

The accuracy of fast static surveys is similar to static surveys of 0.03 feet (1 centimeter) or less. This method can be used for medium-to high accuracy survey.

2.8.2.3 RTK surveying techniques

RTK surveying requires two receivers, recording observations simultaneously, and allows the rover receiver to be moving. RTK surveying techniques also use dual-frequency LI/L2 GPS observations and can handle loss of lock.

RTK surveying requires dual frequency LI/L2 GPS receivers. One of the GPS receivers is set over a known point, while the other receiver may be free to travel from point to point. If the survey is performed in real time, a radio link and a processor or data collector are needed. The radio link is used to transfer the raw data from the reference station to the rover.

RTK surveys can be accurate to within 0.05 to 0.10 feet (2– 3 centimeters), providing a good static network and calibration were performed prior to performing the RTK survey.

2.9 GPS reference system

The GPS satellite are referenced to the world geodetic system of 1984 (WGS 84) ellipsoid. For surveying purposes, this earth-centered WGS 84 coordinate system must be transformed to a user-defined ellipsoid and datum, such as Clarck ellipsoid that is used in Palestine as reference datum.

Table2.1: WGS84 and Clarck 1880 geometric parameter.

Ellipsoid name	Semi major axis (a in meters)	Reciprocal of flattening (1/f)
WGS84	6378137	298.257223563
Clarck 1880	6378300.74	293.4669074

The other geometric parameters are computed using the following equations:

$$r = a(1+n^2/4)/(1+n) \tag{2.17}$$

$$n = f/(2-f) \tag{2.18}$$

$$e^2 = f(2-f) \tag{2.19}$$

$$e'^2 = e^2/(1-f)^2 \tag{2.20}$$

$$b = a\,(1-f)$$

The absolute positions obtained from GPS are based on the 3-D WGS84 ellipsoid. Coordinate outputs are on a Cartesian system(X-Y-Z) relative to WGS84 rectangular coordinate. These coordinate can be transformed to λ, f , and h by an iterative solution where:

$$l = \tan^{-1} \frac{Y}{X} \tag{2.21}$$

$$f = \tan^{-1}\left(\frac{Z}{\sqrt{X^2 + Y^2}} \left(1 - e^2 \frac{N}{N+h} \right)^{-1} \right) \tag{2.22}$$

$$h = \frac{\sqrt{X^2 + Y^2}}{\cos f} - N \tag{2.23}$$

$$N = \frac{a^2}{\sqrt{a^2 \cos^2 f + b^2 \sin^2 f}} \tag{2.24}$$

As initial value to start the iterative solution;

$$f = \tan^{-1} \frac{Z}{\sqrt{X^2 + Y^2}} \left(1 - e^2\right)^{-1} \tag{2.25}$$

h = 0

The inverse problem to find the X, Y, and z, from λ, f , and h;

$$X = (N + h) \cos f \cos l \tag{2.26}$$

$$Y = (N + h) \cos f \sin l \tag{2.27}$$

$$Z = ((1 - e^2) N + h) \sin f \tag{2.28}$$

These coordinates can be transformed to local datum system using 3D similarity transformation as shown in Fig 2.16:

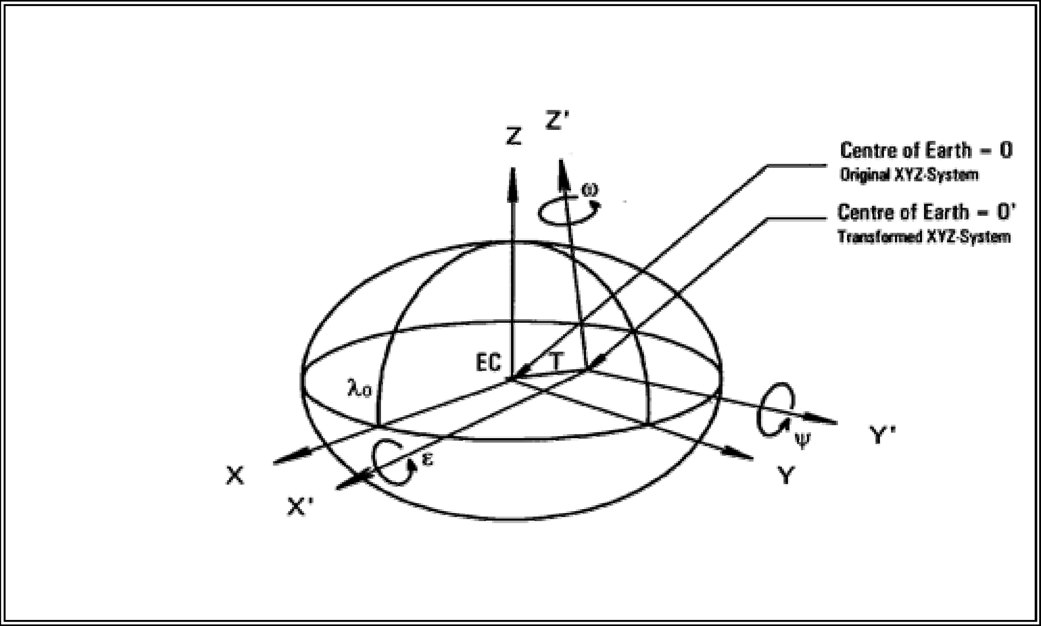

Fig 2.16: Translated and rotated 3D co-ordinate system [7]

The 3D similarity has 7 parameters that are; scale factor, three rotation and three translations, the equations for the 3D similarity transformation are:

$$X = S(m_{11}x + m_{21}y + m_{31}z) + T_x \tag{2.29}$$

$$Y = S(m_{12}x + m_{22}y + m_{32}z) + T_y \tag{2.30}$$

$$Z = S(m_{13}x + m_{23}y + m_{33}z) + T_z \tag{2.31}$$

Where:

$$m_{11} = \cos(f)\cos(k) \tag{2.32}$$

$$m_{12} = \sin(w)\sin(f)\cos(k) + \cos(w)\sin(k) \tag{2.33}$$

$$m_{13} = -\cos(w)\sin(f)\cos(k) + \sin(w)\sin(k) \tag{2.34}$$

$$m_{21} = -\cos(f)\sin(k) \tag{2.35}$$

$$m_{22} = -\sin(w)\sin(f)\sin(k) + \sin(w)\cos(k) \tag{2.36}$$

$$m_{23} = \cos(w)\sin(f)\sin(k) + \sin(w)\cos(k) \tag{2.37}$$

$$m_{31} = \sin(f) \tag{2.38}$$

$$m_{32} = -\sin(w)\cos(f) \tag{3.39}$$

$$m_{33} = \cos(w)\cos(f) \tag{3.40}$$

In Palestine coordinate system, the transformation is defined by three translations. The values of the translation are taken from Trimble geomatics office software.

According to the following equations:

$$X_{Palestine} = X_{WGS84} + \Delta X \tag{2.41}$$

$$Y_{Palestine} = Y_{WGS84} + \Delta Y \tag{2.42}$$

$$Z_{Palestine} = Z_{WGS84} + \Delta Z \tag{2.43}$$

Where;

$\Delta X = 230.00$ m, $\Delta Y = 71.00$ m, $\Delta Z = -273$ m

Here it is assumed that the scale factor =1, and the three rotations are equal to zero.

2.10 Other satellite navigation systems

2.10.1 GLONASS satellite system

GLONASS is a global navigation satellite system developed by Russia. The GLONASS satellite system has much in common with the GPS system.

GLONASS can be described as follows:

- The constellation of the GLONASS system consists of 21operational satellites and three spares at a nominal altitude of 19,100 km.
- Eight GLONASS satellites are arranged in each of three orbital planes.
- GLONASS orbits are approximately circular, with an orbital period of 11 hours and 15 minutes and an inclination of 64.8 Similar to GPS.
- Each GLONASS satellite transmits a signal that has a number of components: two L-band carriers, C/A-code on L1, P-code on both L1 and L2, and a navigation message. GLONASS codes are the same for all the satellites. As such, GLONASS receivers use the frequency channel rather than the code to distinguish the satellites.
- The signal of GLONASS system is not affected by either SA or ant spoofing.
- There are two problems with GPS/GLONASS integration. The first one is that both GPS and GLONASS systems use different coordinate frames to express the position of their satellites. GPS uses the WGS 84 system, while GLONASS uses the Earth Parameter System1990 (PZ-90) system. The second problem with the GPS/GLONASS integration is that both systems use different reference times.

Table 2.2 shows a comparison between GPS and GLONASS [3]

Parameter	GLONASS	NAVSTAR GPS
Satellites		
Number of satellites in the baseline constellation	21 + 3 spares	21 + 3 spares
Number of orbital planes	3	6
Inclination	64.8°	55°
Orbital altitude	19 100 km	20 180 km
Orbital radius	25 510 km	26 560 km
Orbital period (sidereal time)	11 hours 15 min.	12 hours
Repeat ground tracks	every sidereal day	every 8 sidereal days
Navigation message		
Ephemeris representation	9 parameters (position, velocity, acceleration) in the ECEF Cartesian system	Keplerian elements and interpolation coefficients
Geodetic datum	PZ-90	WGS 84
Time base	GLONASS system time	GPS system time
Related system time	$UTC_{[SU]}$	$UTC_{[USNO]}$
Almanac transmission	2.5 minutes	12.5 minutes
Signals		
Satellite signal division	Frequency division	Code division
Frequency band L1	1.602–1.615 MHz	1.575 MHz
Frequency band L2	1.246–1.256 MHz	1.228 MHz
Codes	same for all satellites C/A-code on L1 P-code on L1, L2	different for all satellites C/A-code on L1 P-code on L1, L2
Code type	PRN sequence	Gold code
Code frequency C/A-code	0.511 MHz	1.023 MHz
Code frequency P-code	5.11 MHz	10.23 MHz
Clock data	clock offset frequency offset	clock offset frequency offset and rate

2.10.2 Chinese regional satellite navigation system (Beidou system)

China has recently launched two domestically built navigation satellites, which form the first generation of a satellite-based navigation system. It is an all-weather regional navigation system, which is known as the Beidou Navigation System. The satellites are placed in geostationary orbits at an altitude of approximately 36,000 km above the Earths surface.

The primary use of the system is in land and marine transportation. China is also planning to build its second-generation satellite positioning and navigation system, which will have more satellites and more coverage area.

2.10.3 Galileo system

Galileo is a satellite-based global-navigation system proposed by Europe. Galileo is a civil-controlled satellite system to be delivered through a public-private partnership. Three different constellation types were investigated to ensure the optimum selection of the Galileo architecture, namely low Earth orbits (LEO), medium Earth orbits (MEO), and inclined geosynchronous orbits (IGSO). The Galileo decision makers adopted a constellation of 30 MEO satellites. The satellites will be evenly distributed over three orbital planes at an altitude of about 23,000km.

The Galileo system is now in the testing phase, and it is expected to start in 2008. Figure 2.16 shows the planned schedule for Galileo positioning system.

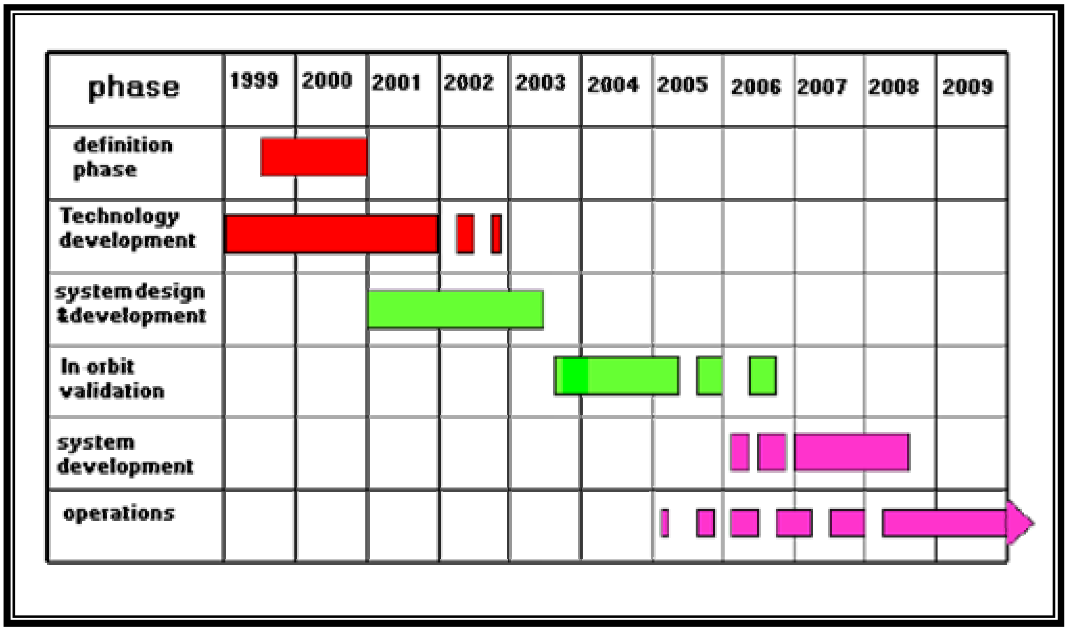

Figure 2.17: planed schedule for Galileo positioning system [12]

Galileo has three frequencies (Wavelength λ):

- λ (L1) = 19 cm
- λ (L2) = 24 cm
- λ (L3) = 25 cm

CHAPTER

3

PHYSICAL GEODESY

CHAPTER THREE

Physical Geodesy

3.1 Introduction

Physical geodesy concerns studies of the figure and the gravity field of the earth, using measurements of physical quantities (e.g. gravity). Most of geodetic measurements (except slope distances) are influenced by the gravity field of the earth.

3.2 Gravitational fields of simple mass distributions

3.2.1 Newton's law

Let m1 and m2 be two point masses at distance l. According to Newton's gravitational law, the two masses will attract each other by the gravitational force F:

$$F=G\frac{m_1 * m_2}{l^2} \tag{3.1}$$

Where:

G: gravitational constant ($G\approx 0.667*10^{-7}cm^3g^{-1}s^{-2}$)

m_1, m_2: are the mass points.

l: the distance between the two points.

$$L = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \tag{3.2}$$

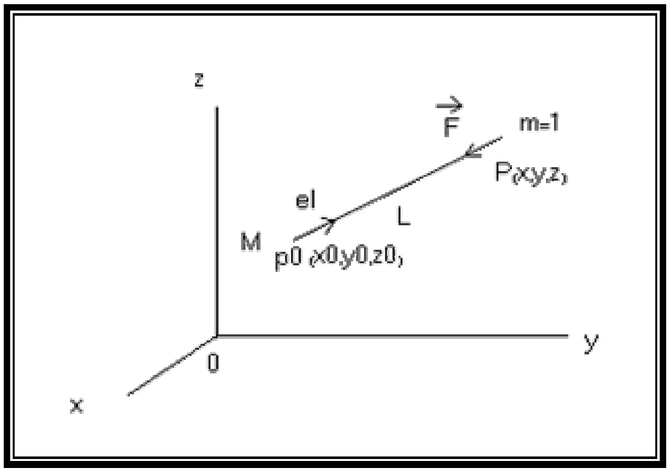

Figure 3.1: Gravitational force of a point mass [4]

To study the gravitational field generated by a point mass M, the gravitational force of point mass M attracting a unit mass m (m = 1unit mass), as shown in figure 3.1, will be:

$$F = \frac{GM}{l^2} \tag{3.3}$$

For the earth GM ≈ 3.986005 x 10^{14} m^3/s^2 and radius R ≈ 6371 km, the approximate gravitational force (**gravitational acceleration**) on the earth surface:

$$F = \frac{GM}{R^2} = \frac{3.986005 * 10^{14} m^3/s^2}{6.371^2 * 10^{12} m^2} = 9.820251921 \text{ m/s}^2 \tag{3.4}$$

Other unit for the gravitational acceleration can be used that is called Gal, where:

Gal = 1 cm/s^2

= 1000 mGal

= 1000 000 m Gal

Then: F calculated in equation 3.4 will be:

F = 982. 025 192 1 Gal

= 982 025. 192 1 mGal

= 982 025 192. 1 m Gal

The most accurate instruments (gravimeters) can measure the gravitational acceleration on the earth surface at accuracy of several m Gal.

The vector of the gravitational force (acceleration) $\vec{F}$ can be defined by it's magnitude F and 3 D component as follows:

$$\vec{F} = \begin{bmatrix} Fx \\ Fy \\ Fz \end{bmatrix} = \text{F} \, (- \overrightarrow{el}\;) = \text{-F*} \begin{bmatrix} (x - x_0)/l \\ (y - y_0)/l \\ (z - z_0)/l \end{bmatrix} = -\frac{GM}{l^2} \begin{bmatrix} (x - x_0)/l \\ (y - y_0)/l \\ (z - z_0)/l \end{bmatrix} \tag{3.5}$$

Where:

$\overrightarrow{el}$ Denotes the unit vector from P_0 to P.

To avoid dealing with acceleration because it is a vector, we deal the **gravitation potential** V (x, y, z) that is a scalar value, where V is define as follows:

$$\vec{F} = \begin{bmatrix} Fx \\ Fy \\ Fz \end{bmatrix} = \begin{bmatrix} \frac{\partial n}{\partial x} \\ \frac{\partial n}{\partial y} \\ \frac{\partial n}{\partial z} \end{bmatrix} \tag{3.6}$$

And the gradient vector of V (x, y, z) is the gravitation force.

$$\vec{F} = \text{grad (V)} \tag{3.7}$$

V can be calculated as follows:

$$V(x, y, z) = \frac{GM}{l} \tag{3.8}$$

If the point P is influenced by n mass points then:

$$V(x, y, z) = \sum_{i=1}^{n} Vi = \sum_{i=1}^{n} \frac{Gm_i}{l_i} \tag{3.9}$$

If the point P is influenced by a body with volume v, and density ρ. Using the mass element dm and the volume dv:

$$dV = \frac{Gdm}{l} = \frac{Grdv}{l} \tag{3.10}$$

And the total gravitational potential of the body is then:

$$V = \int dV = G\iiint_V \frac{pdv}{l} \tag{3.11}$$

3.3 Laplace's equation

For a function V(x, y, z), the Laplace equation for this function is:

$$\Delta(V) \equiv \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{3.12}$$

Where:

Δ(.)= Laplace operator

Using spherical coordinates $(r, \overline{f}, l)$, Laplace's equation can be transformed to:

$$r^2\frac{\partial^2 V}{\partial r^2}+2r\frac{\partial V}{\partial r}+\frac{\partial^2 V}{\partial \bar{f}^2}-\tan\bar{f}\frac{\partial V}{\partial \bar{f}}+\frac{1*\partial^2 V}{\cos^2\bar{f}\partial l^2}=0 \quad (3.13)$$

For V(x, y, z), the density ρ is constant, dv is the same for all elements, then only l is changing for each element. The Laplace operator for the gravitation:

$$\Delta(V)=\Delta\left(G\iiint_v \frac{r\,dn}{l}\right)=G\iiint_v \Delta(\frac{1}{l})\,r\,dn\ =0 \quad (3.14)$$

Then V(x, y, z) satisfies Laplace equation if:

$$\Delta(\frac{1}{l})=0$$

(3.15)

Appendix-B shows the proof that V satisfies Laplace equation.

To fined the general solution to Laplace's equation, the solution can be decomposed of three functions; f_1, f_2 , and f_3:

$$V(r,f,l)=f_1(r).\ f_2(f)\ .\ f_3(l) \quad (3.16)$$

$$f_1(r)=\frac{1}{r^{n+1}}\ \text{or}\ r^n \qquad (n = 0,1,2,\ldots) \quad (3.17)$$

$$f_2(f)=p_{nm}(\sin f) \qquad (n=0,1,2,\ldots,m=0,1,2\ldots,n-1,n) \quad (3.18)$$

$$f_3(l)=\cos m\lambda\ \text{or}\ \sin m\lambda \qquad (m=0,1,2,\ldots.n-1,n) \quad (3.19)$$

Where:

$p_{nm}(\sin f)$: Legendry functions of degree n and order m.

$f_1(r) = r^n$ can be neglected , and taking $f_1(r)=\frac{1}{r^{n+1}}$ the general solution for v:

$$V(r,f,l)=\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\sum_{m=-n}^{n}A_{nm}Y_{nm}(f,l) \quad (3.20)$$

Where:

$$\mathrm{Y}_{nm}(f,l) = \begin{cases} \cos m\,l\,P_{n|m|}(\sin f) & for.....m \le 0 \\ \sin m\,l\,P_{nm}(\sin f) & for.....m > 0 \end{cases} \tag{3.21}$$

$$\mathrm{A}_{nm} = \begin{cases} a_{n|m|} & for.....m \le 0 \\ b_{nm} & for.....m > 0 \end{cases} \tag{3.22}$$

Then:

$$\mathrm{V}(\mathrm{r},\ f, l) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum (a_{nm} \cos \mathrm{m}\ l + b_{nm} \sin \mathrm{m}\ l)\, p_{nm}(\sin f) \tag{3.23}$$

For the Legendre function $\mathrm{P}_{nm}(\mathrm{t})$:

$$\mathrm{P}_{nm}(\mathrm{t}) = (1-\mathrm{t}^2)^{m/2} \frac{d^m P_n(t)}{dt^m} \tag{3.24}$$

This can be solved in a recursive formulas as shown in table 3.1.

Table 3.1: Recursive formulas for Legendre functions

$\mathrm{P}_n(t) = -\frac{n-1}{n} P_{n-2}(t) + \frac{2n-1}{n} t\ \ P_{n-1}(t)$	(n ≥ 2 ; m =0)
$\mathrm{P}_{nm}(t) = -\frac{n+m-1}{n-m} P_{n-2,m}(t) + \frac{2n-1}{n-m} t\ \ P_{n-1,m}(t)$	(n ≥ 3 ; 1 ≤ m ≤ $n-2$)
$\mathrm{P}_{n,n-1}(t) = (2n-1) t P_{n-1,n-1}(t)$	(n ≥ 1; $m = n-1$)
$\mathrm{P}_{nn}(t) = (2n-1)\sqrt{1-t^2}\, P_{n-1,n-1}(t)$	(n ≥ 2 ; m =n)

In a normalized form:

$$\overline{Y}_{nm}(f,l) = f_{nm} Y_{nm}(f,l) \tag{3.25}$$

$$\overline{P}_{nm}(t) = f_{nm} P_{nm}(t) \tag{3.26}$$

$$\mathrm{V}(\mathrm{r}, f, l) = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=-n}^{n} \overline{A}_{nm} \overline{Y}_{nm}(f,l) \tag{3.27}$$

$$\bar{A}_{nm} = \frac{A_{nm}}{f_{nm}} \tag{3.28}$$

Where:

$$f_{nm} = \begin{cases} \sqrt{2n+1} & for......m = 0 \\ \sqrt{2(2n+1)\dfrac{(n-m)!}{(n+m)!}} & for......m \neq 0 \end{cases} \tag{3.29}$$

Table 3.2: Recursive formulas for normalized Legendre functions

$\bar{P}_n(t) = -\frac{\sqrt{2n+1}}{n}\frac{n-1}{\sqrt{2n-3}}\bar{P}_{n-2}(t) + \frac{\sqrt{2n+1}}{n}\sqrt{2n-1}\ \ t\,\bar{P}_{n-1}(t)$ (n≥ 2 ; m=0)
$\bar{P}_{nm}(t) = \sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(2n-3)(n+m)(n-m)}}\bar{P}_{n-2,m}(t) + \sqrt{\frac{(2n+1)(2n-1)}{(n+m)(n-m)}}t\,\bar{P}_{n-1,m}(t)$
(n $\geq 3;\ 1 \leq m \leq n-2$)
$\bar{P}_{n,n-1}(t) = \sqrt{2n+1}\ \ t\,\bar{P}_{n-1,n-1}(t)$ (n≥ 1; m=n-1)
$\bar{P}_{nn}(t) = \sqrt{\frac{2n+1}{2n}}\sqrt{1-t^2}\,\bar{P}_{n-1,n-1}(t)$ (n≥ 2; m=n)

Alternative method [1]:

$$V(r, f, l) = \sum_{n=0}^{\infty}(\frac{R}{r})^{n+1}\sum_{m=-n}^{n}\bar{A}_{nm}Y_{nm}(f, l) \tag{3.30}$$

$$\bar{A}_{nm} = \frac{\bar{A}_{nm}}{R^{n+1}} \tag{3.31}$$

Explicit expressions of Legendre functions to degree and order 3 can be calculated as shown in table 3.3.

Table 3.3: Explicit expressions of Legendre functions to degree and order 3

n	m	$P_{nm}(t)$	f_{nm}	$\bar{P}_{nm}(t)$
0	0	1	1	1
1	0	t	$\sqrt{3}$	$\sqrt{3}\ t$
1	1	$\sqrt{1-t^2}$	$\sqrt{3}$	$\sqrt{3}\ \sqrt{1-t^2}$
2	0	$(3/2)t^2 - 1/2$	$\sqrt{5}$	$\sqrt{5}\,((3/2)t^2 - 1/2)$
2	1	$3t\sqrt{1-t^2}$	$1/3\sqrt{15}$	$\sqrt{15}\ t\sqrt{1-t^2}$
2	2	$3(1-t^2)$	$1/6\sqrt{15}$	$1/2\sqrt{15}\,(1-t^2)$
3	0	$5/2t^3 - 3/2t$	$\sqrt{7}$	$\sqrt{7}\,(5/2t^3 - 3/2t)$
3	1	$\sqrt{1-t^2}\,(15/2t^2 - 3/2)$	$\sqrt{\frac{7}{6}}$	$\sqrt{\frac{7}{6}}\ \sqrt{1-t^2}\,(15/2t^2 - 3/2)$
3	2	$15t(1-t^2)$	$1/30\sqrt{105}$	$1/2\sqrt{105}\,(1-t^2)$
3	3	$15t(1-t^2)^{3/2}$	$(¼)(1/15)\sqrt{70}$	$(1-t^2)^{3/2}$

3.4 Gravity field of the earth

3.4.1 Introduction

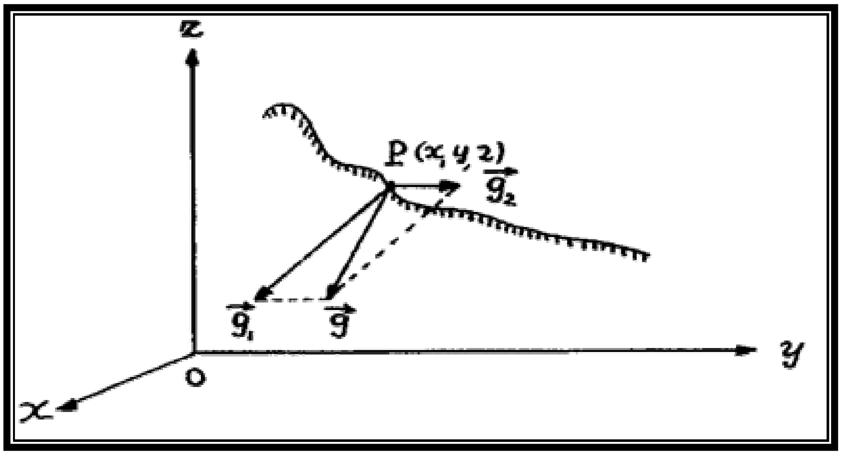

Figure 3.2: Gravitational force, centrifugal force and gravity force [1]

For the point P in the earth surface, two forces affect the point:

- Gravitational force $\vec{g}_1$
- And centrifugal force $\vec{g}_2$

The gravity vector is:

$$\vec{g} = \vec{g}_1 + \vec{g}_2 \tag{3.32}$$

This result gravitation potential W (geopotentential) that equals to sum of the gravitational potential V and the centrifugal potential Ω :

$$W=V+\Omega \tag{3.33}$$

Where:

$$V = G\iiint_{v} \frac{p\,dv}{l} \tag{3.34}$$

Centrifugal potential:

$$\Omega = \frac{1}{2}w^2(x^2 + y^2) = \frac{1}{2}w^2 r^2 \cos^2 \bar{f} \tag{3.35}$$

Where, w is angular velocity of earth rotation nearly $0.7292115 * 10^{-4} s^{-1}$.

And:

$$\vec{g}_2 = grad(\Omega) = \begin{bmatrix} w^2 x \\ w^2 y \\ 0 \end{bmatrix} = \begin{bmatrix} w^2 r \cos \bar{f} \cos l \\ w^2 r \cos \bar{f} \sin l \\ 0 \end{bmatrix} \tag{3.36}$$

$$g_2 = |\vec{g}_2| = \sqrt{\left(\frac{\partial \Omega}{\partial x}\right)^2 + \left(\frac{\partial \Omega}{\partial y}\right)^2 + \left(\frac{\partial \Omega}{\partial z}\right)^2} = w\sqrt{x^2 + y^2} = w^2.r.\cos\bar{f} \tag{3.37}$$

The gravity vector $\vec{g}$ is defined as the gradient of the potential W:

$$\vec{g} = \text{grad (W)} = \begin{bmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial z} \end{bmatrix} \tag{3.38}$$

The magnitude of g is:

$$g = |\vec{g}| = \sqrt{(\frac{\partial W}{\partial x})^2 + (\frac{\partial W}{\partial y})^2 + (\frac{\partial W}{\partial z})^2} \tag{3.39}$$

Taking W(x, y, z) = constant C, this defines equipotential surface (**Level surface**), if the level surface coincides approximately with the global mean of the earth sea level, the surface is called **Geoid**. Figure 3.3 shows the equipotential surfaces that are always perpendicular to the plump line, the equipotential surface not necessary to be parallel.

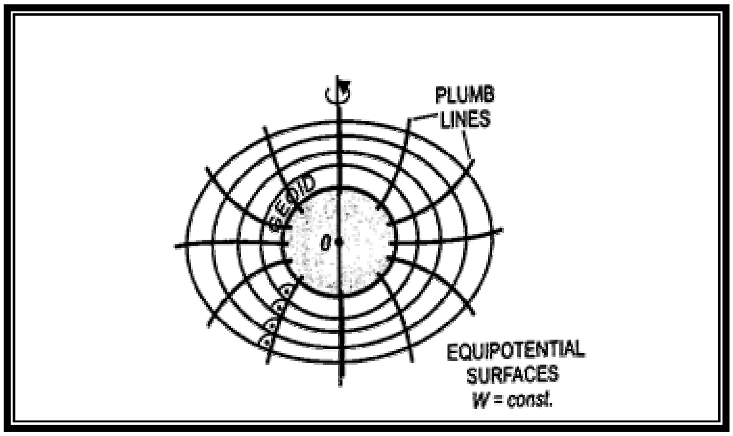

Figure 3.3: equipotential surface and plump line close to the earth [10]

3.4.2 Gravity measurements

It is necessary to measure the gravity field of the earth to define the heights. The gravity measurement has basically two types; first is the absolute gravity where we measure the gravity acceleration of a station directly by the gravimeter. And second is relative gravity, where we measure the difference in the gravity between two stations. Here we have two examples of the principal of the two types:

1. Absolute gravity by free-falling body.

Defined by the measurement in the change in the vertical distance h in a given time, depend on Newton motion equation:

$$h = h_0 + v_0 t + \frac{1}{2} g t^2 \tag{3.40}$$

If we observe different heights h_1, h_2, h_3 and their times t_1, t_2, t_3 , then g is calculated:

$$g = 2\frac{s_2/\Delta t_2 - s_1/\Delta t_1}{\Delta t_2 - \Delta t_1} \tag{3.41}$$

$$s_1 = h_2 - h_1, \quad s_2 = h_3 - h_1 \quad \Delta t_2 = t_3 - t_1 \tag{3.42}$$

2. Relative gravity measurement by spring:

If a spring with constant k, and initial length s_0, a mass m is attached. The spring will have anew length s where:

$$k(s - s_0) = m.g \tag{3.43}$$

If the spring has two lengths at two different points then:

$$g_2 - g_1 = \frac{k}{m}(s_2 - s_1) \tag{3.44}$$

This is called the relative gravity, and the instruments are called relative gravimeters.

3.4.3 Normal gravity of the earth

The normal gravity field of the earth is generated by an ellipsoid of revolution with semi-major axis (a) and semi-minor axis (b, such that:

- the total mass of the reference ellipsoid is equal to that of the Earth;
- the reference ellipsoid is rotating around its minor axis at the same angular velocity as the earth rotation;
- the surface of the reference ellipsoid is an equipotential surface. The normal potential U_O on the reference ellipsoid is equal to the geopotential W_O on the geoid.

Many ellipsoids were defined by physical definition depending on the principle of normal gravity. Examples of physically defined ellipsoids are GRS67, GRS80, and WGS84. The defining parameters for GRS80 ellipsoid are shown in table 3.4.

Table 3.4: The defining parameters of the GRS80 ellipsoid

Notation	Constant	Unit	Numerical value
a	Semi-major axis	m	6378137.000
GM	Product of G and total mass M	m^3s^{-2}	$0.3986005.10^{15}$
J2	Dynamic from factor $\frac{C-A}{Ma^2}$		0.00108263
w	Angular velocity	s^{-1}	$0.72921151.\ 10^{-4}$
b	Semi-minor axis	meter	6356752.3141
f	Geometrical flattening		0.003352810681 1/298.257222101
e^2	First eccentricity squared		0.006694380023
e'^2	second eccentricity squared	$\sec^{-1}$	0.006739496775
U_0	Normal potential on the ellipsoid	$m^2.\sec^2$	62636860.850
g_p	Normal gravity on the poles	Gal	983.21863685
g_e	Normal gravity on the equator	Gal	978.03267715
f^*	Gravity flattening		0.005302440112 1/188.592417552
k	$(bg_p - ag_e)/(ag_e)$		0.001931851353
m	$w^2a^2b/(GM)$		0.003449786003 1/289.873052743
g_{45}	Normal gravity at latitude 45^0	Gal	980.6199203
$\bar{g}$		Gal	979.7644656

As the ellipsoid is defined as an equipotential surface, it has a potential U, that is a result of the ellipsoidal gravitational potential V', and the centrifugal potential Ω.

$$U = V' + \Omega \tag{3.45}$$

Where:

V': Gravitational potential [1]

$$V'(r,\bar{f},l) = \frac{GM}{r}\left(1+\sum_{n=1}^{\infty}(\frac{a}{r})^{2n}\overline{J_{2n}}\overline{P}_{2n}(\sin\bar{f})\right) \tag{3.46}$$

$$\bar{J}_{2n} = (-1)^n \frac{3.e^{2n}\sqrt{4n+1}}{(2n+1)(2n+3)}\left(1-n+\frac{5n}{e^2}.J_2\right) \tag{3.47}$$

Ω: Centrifugal potential

$$\Omega = \frac{1}{2}w^2(x^2+y^2) \tag{3.48}$$

Approximate Calculation for U:

$$\text{U(x, y, z) =U (u, } b\text{)} = \frac{GM}{E}\tan^{-1}\frac{E}{u} + \frac{1}{2}w^2a^2\frac{q}{q_0}(\sin^2 b - \frac{1}{3}) + \frac{1}{2}w^2(u^2+E^2)\cos^2 b \tag{3.49}$$

$$\text{E}^2 = a^2 - b^2 \tag{3.50}$$

$$\text{q}=\frac{1}{2}\left[\left(1+3\frac{u^2}{E^2}\right)\tan^{-1}\frac{E}{u} - \frac{1}{3}\frac{u}{E}\right]$$

(3.51)

$$q_0 = \frac{1}{2}\left[\left(1+3\frac{b^2}{E^2}\right)\tan^{-1}\frac{E}{b} - \frac{1}{3}\frac{b}{E}\right] \tag{3.52}$$

The Reference gravity potential U_0, can be calculated by assuming u=b:

$$U_0 = U(b,b) = \frac{GM}{E}\tan^{-1}\frac{E}{b} + \frac{1}{3}w^2a^2 \tag{3.53}$$

The normal gravity at the surface of the ellipsoid can be calculated as follows:

$$g = g_e \frac{1 + K\sin^2 f}{\sqrt{1 - e^2\sin^2 f}} \tag{3.54}$$

$$K = \frac{bg_p}{ag_e} - 1 \tag{3.55}$$

The gravity at a given height h above the ellipsoid:

$$g_h = g\left[1 - \frac{2}{a}(1 + f + m - 2f\sin^2 f)h + \frac{3}{a^2}h^2\right] \tag{3.56}$$

$$m = \frac{w^2a^2b}{GM} \tag{3.57}$$

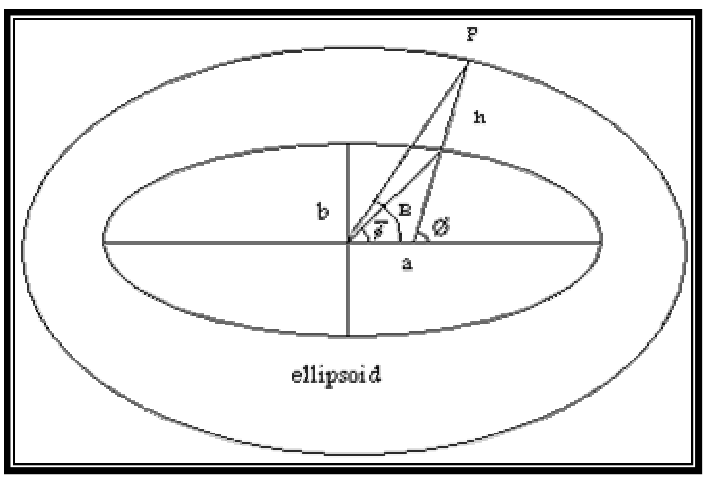

Figure 3.4: $\overline{f}$ the reduced latitude of f

$$\tan \bar{f} = (1 - e^2) \tan f \tag{3.58}$$

$$tab = \frac{b}{a} \tan f \tag{3.59}$$

3.4.4 Gravity field anomalies

To fined the difference between the two equipotential surfaces; the Geoid and the ellipsoid, that is called the Geoid undulation we define the following:

1. Disturbing potential T_P:

$$T_P = W_P - U_P \tag{3.60}$$

$$T_P = (V_P + \Omega_P) - (V'_P + \Omega_P) = V_P - V'_P \tag{3.61}$$

2. Gravity disturbance:

$$dg_p = g_p - g_p \tag{3.62}$$

3. Gravity anomaly:

$$\Delta g_p = g_p - g_q \tag{3.63}$$

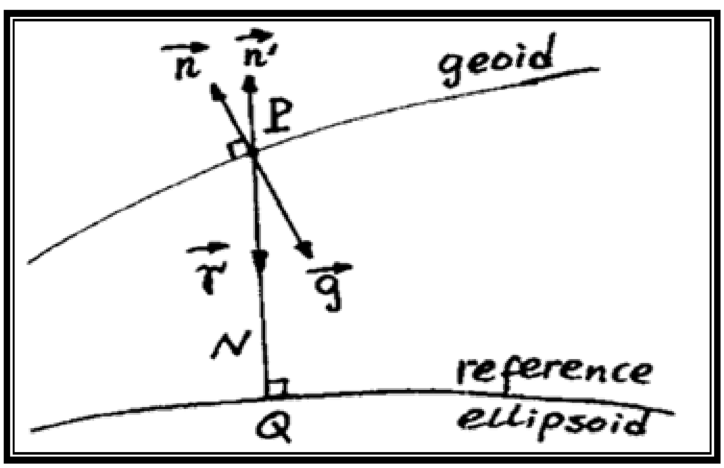

Figure 3.5: Geoid versus the reference ellipsoid [4]

If P is the point in the Geoid and Q is the same point in the ellipsoid in the ellipsoid, where $W_P = U_q$, then we can fined the normal potential of point P using the normal potential of point Q (can be calculated using equation 3.53).

$$U_P = U_q + \frac{\partial U}{\partial n'} N + \frac{1}{2!}\frac{\partial^2 U}{\partial n'^2} N^2 + ... \approx U_q + \frac{\partial U}{\partial n'} N = U_q - g_q.N \qquad (3.64)$$

$$U_p - U_q = -gN$$

$$N = \frac{U_q - U_p}{g}$$

$$N = \frac{W_p - U_p}{g}$$

$$\mathrm{N} = \frac{T_p}{g_q} \qquad (3.65)$$

U_q is equal to U_0 because Q is on the ellipsoid surface (h=0), this can be done by the use of Tylor series:

Gravity disturbance:

$$dg_p = g_p - g_p = \left(-\frac{\partial W_P}{\partial n}\right) - \left(-\frac{\partial U_P}{\partial n'}\right) \approx \left(-\frac{\partial W_p}{\partial r}\right) - \left(-\frac{\partial U_p}{\partial r}\right) = -\frac{\partial (W_p - U_{P)}}{\partial r}$$

$$dg_p = -\frac{\partial T_P}{\partial r} \qquad (3.66)$$

Gravity anomaly:

$$\Delta g_p = g_p - g_q = \left(-\frac{\partial W_P}{\partial n}\right) - \left(-\frac{\partial U_q}{\partial n'}\right) \approx \left(-\frac{\partial (T_p + U_P)}{\partial r}\right) - \left(-\frac{\partial U_q}{\partial r}\right) = -\frac{\partial T_p}{\partial r} - \frac{\partial}{\partial r}\{U_q - g_q.N\}$$

$$+\frac{\partial U_q}{\partial r} = -\frac{\partial T_p}{\partial r} + \frac{\partial g_q}{\partial r}.N \qquad (3.67)$$

$$\frac{\partial g_q}{\partial r} \approx \frac{\partial}{\partial r}\left(\frac{GM}{r^2}\right) = -\frac{2}{r}\frac{GM}{r^2} \approx -\frac{2}{r} g_q \tag{3.68}$$

$$\Delta g_p = -\frac{\partial T_p}{\partial r} - \frac{2}{r} T_p \tag{3.69}$$

3.5 Harmonic expansion of the earth gravity field

3.5.1 Introduction

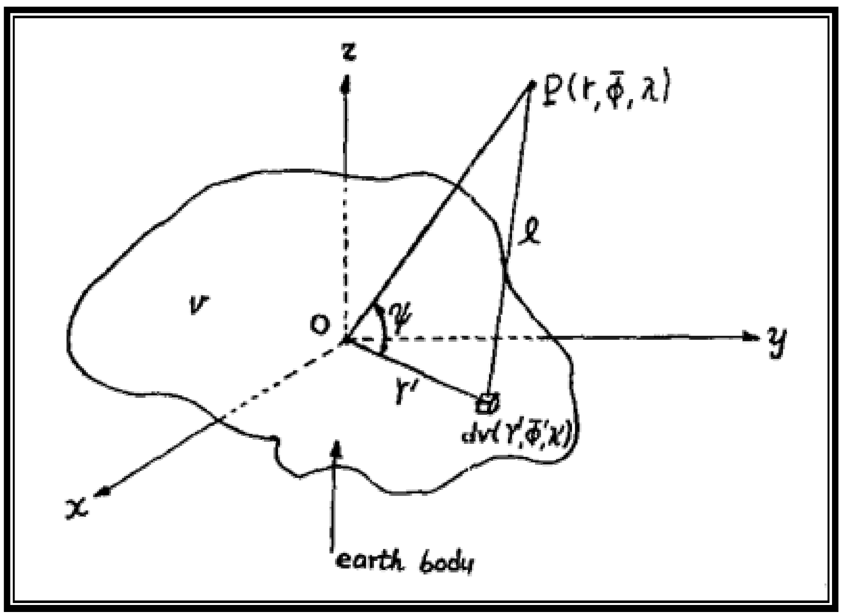

Figure 3.6: The external gravitational potential of the earth [4]

According to the graph above, the gravitational potential of the earth, V, at a point P with spherical coordinates $(r, \bar{f}, l)$, is given by the gravitational law:

$$V(r,\bar{f},l) = G\iiint_v \frac{p(r',\bar{f}'l').dv(r',\bar{f}',l')}{\mathbf{l}} \tag{3.70}$$

V can be written as in equation 3.71:

$$V(r,\bar{f},l)=\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\sum_{m=0}^{n}(\bar{a}_{nm}\cos ml+\bar{b}_{nm}\sin ml)\bar{P}_{nm}(\sin\bar{f}) \qquad (3.71)$$

The value of $\bar{a}_{nm}$ and $\bar{b}_{nm}$ can written as:

$$\bar{a}_{nm}=\frac{G}{2n+1}\iiint_v (r')^n.\cos ml'\bar{P}_{nm}(\sin\bar{f}').Pdv \qquad (3.72)$$

$$\bar{b}_{nm}=\frac{G}{2n+1}\iiint_v (r')^n.\sin ml'\bar{P}_{nm}(\sin\bar{f}').pdv \qquad (3.73)$$

To find $\bar{a}_{00}$:

$$\bar{a}_{00}=G\iiint_v pdv=GM \qquad (3.74)$$

According to equation 3.71 we need it while m=0 then cos(mλ)=0.

Then:

$$V_{00}=\frac{GM}{r} \qquad (3.75)$$

For $\bar{a}_{10},\bar{a}_{11},and\bar{b}_{11}$:

$$\bar{a}_{10}=\frac{G}{3}\iiint_v r'.\sqrt{3}\sin\bar{f}'.dm \qquad (3.76)$$

$$\bar{a}_{11}\ \frac{G}{3}\iiint_v r'.\cos l'.\sqrt{3}\cos\bar{f}'.dm \qquad (3.77)$$

$$\bar{b}_{11}=\frac{G}{3}\iiint_v r'.\sin l'.\sqrt{3}\cos\bar{f}'.dm \qquad (3.78)$$

We have:

$$r'\sin f'=z' \qquad (3.79)$$

$$r'\cos f'\cos l'=x' \qquad (3.80)$$

$$r'\cos f'\sin l'=y' \qquad (3.81)$$

Then:

$$\bar{a}_{10} = \frac{G}{\sqrt{3}} \iiint z'.dm \tag{3.82}$$

$$\bar{a}_{11} = \frac{G}{\sqrt{3}} \iiint x'.dm \tag{3.83}$$

$$\bar{b}_{11} = \frac{G}{\sqrt{3}} \iiint y'.dm \tag{3.84}$$

In mechanics:

$$x_0 = \frac{1}{M} \iiint_V x'.dm \tag{3.85}$$

$$y_0 = \frac{1}{M} \iiint_V y'.dm \tag{3.86}$$

$$z_0 = \frac{1}{M} \iiint_V z'.dm \tag{3.87}$$

By referring to equations 3.85 to 3.87, and multiplying the three equations M/M we get:

$$\bar{a}_{10} = \frac{GM}{\sqrt{3}} z_0 \tag{3.88}$$

$$\bar{a}_{11} = \frac{GM}{\sqrt{3}} x_0 \tag{3.89}$$

$$\bar{b}_{11} = \frac{GM}{\sqrt{3}} y_0 \tag{3.90}$$

For the earth $x_0 = y_0 = z_0 = 0$; the values of $\bar{a}_{10}, \bar{a}_{11}, b_{10}, b_{11}$. This leads to:

$$v_{10} = v_{11} = 0$$

For V, the values $\bar{a}_{nm}$ and $\bar{b}_{nm}$ have and they can be very large number, then instead of equation:

$$V(r, \bar{f}, l) = \sum_{ى=2}^{\infty} \frac{1}{r^{n+1}} \sum_{m=0}^{n} (\bar{a}_{nm} \cos ml + \bar{b}_{nm} \sin ml) \bar{P}_{nm}(\sin \bar{f}) \tag{3.91}$$

We can use:

$$V((r,\bar{f},l)=\frac{GM}{r}+\frac{GM}{a}\sum_{n=2}^{\infty}(\frac{a}{r})^{n+1}\sum_{m=0}^{n}(\bar{A}_{nm}\cos ml+\bar{B}_{nm}\sin ml)\bar{P}_{nm}(\sin\bar{f}) \quad (3.92)$$

$$\bar{A}_{nm}=\frac{1}{a^{n.GM}}.\bar{a}_{nm} \quad (3.93)$$

$$\bar{B}_{nm}=\frac{1}{a^{n.GM}}.\bar{b}_{nm}$$

3.5.2 Normal potential

As described previously the normal gravitational potential can be written as:

$$V'(r,\bar{f},l)=\frac{GM}{r}(1+\sum_{n=1}^{\infty}(\frac{a}{r})^{2n}\bar{J}_{2n}\bar{P}_{nm}(\sin\bar{f})) \quad (3.94)$$

$$\bar{J}_{2n}=(-1)^{n}\frac{3.e^{2n}\sqrt{4n+1}}{(2n+1)(2n+3)}(1-n+\frac{5n}{e^{2}}.J_{2}) \quad (3.95)$$

This can be written in other way:

$$V(r,\bar{f},l)=\frac{GM}{r}(1+\sum_{n=2}^{\infty}(\frac{a}{r})^{n}\sum_{m=0}^{n}(\bar{J}_{nm}\cos ml+\bar{K}_{nm}\sin ml)\bar{P}_{nm}(\sin\bar{f}) \quad (3.96)$$

Where:

$\bar{J}_{nm}=0$ If n is odd or m$\neq o$

$\bar{J}_{nm}=\bar{J}_{n}$ If n is even and m=0

$\bar{K}_{nm}=0$ For all n, m

3.5.3 Anomalies

According to equations 3.92 and 3.96, the anomalies in section 3.4.4 can be written in the following forms:

1. Disturbing potential T_p :

$$\mathrm{T}(r,\bar{f},l)=W-U=V-V'=\frac{GM}{a}\sum_{n=2}^{\infty}(\frac{a}{r})^{n+1}\sum_{m=0}^{n}\left\{\bar{C}_{nm}\cos ml+\bar{S}_{nm}\sin ml\right\}\bar{P}_{nm}(\sin\bar{f})$$

(3.97)

Where:

$$\bar{C}_{nm}=\bar{A}_{nm}-\bar{J}_{nm}\qquad (\mathrm{n}\geq 2,\ 0\leq m\leq n) \qquad (3.98)$$

$$\bar{S}_{nm}=\bar{B}_{nm}\qquad (\mathrm{n}\geq 2,\ 0\leq m\leq n)$$

(3.99)

2. The Geoid Undulation N:

$$\mathrm{N}(r,\bar{f},l)=\frac{T}{g}=\frac{GM}{ag}\sum_{n=2}^{\infty}(\frac{a}{r})^{n+1}\sum_{m=0}^{n}\left\{\bar{C}_{nm}\cos ml+\bar{S}_{nm}\sin ml\right\}\bar{P}_{nm}(\sin\bar{f}) \qquad (3.100)$$

3. Gravity distance $dg_p = g_p - g_p$:

$$4.\ gg\ (r,\bar{f},l)=-\frac{\partial T}{\partial r}=\frac{GM}{a^2}\sum_{n=2}^{\infty}(n+1)(\frac{a}{r})^{n+2}\sum_{m=0}^{n}\left\{\bar{C}_{nm}\cos ml+\bar{S}_{nm}\sin ml\right\}\bar{P}_{nm}(\sin\bar{f}) \qquad (3.101)$$

5. The Gravity Anomaly $\Delta g_p = g_p - g_q$:

$$\Delta g\ (r,\bar{f},l)=-\frac{\partial T}{\partial r}-\frac{2}{r}T=\frac{GM}{a^2}\sum_{n=2}^{\infty}(n+1)(\frac{a}{r})^{n+2}\sum_{m=0}^{n}\left\{\bar{C}_{nm}\cos ml+\bar{S}_{nm}\sin ml\right\}\bar{P}_{nm}(\sin\bar{f}) \quad (3.102)$$

3.5.4 Methods for determining harmonic coefficients

1. **satellite altimeter:**

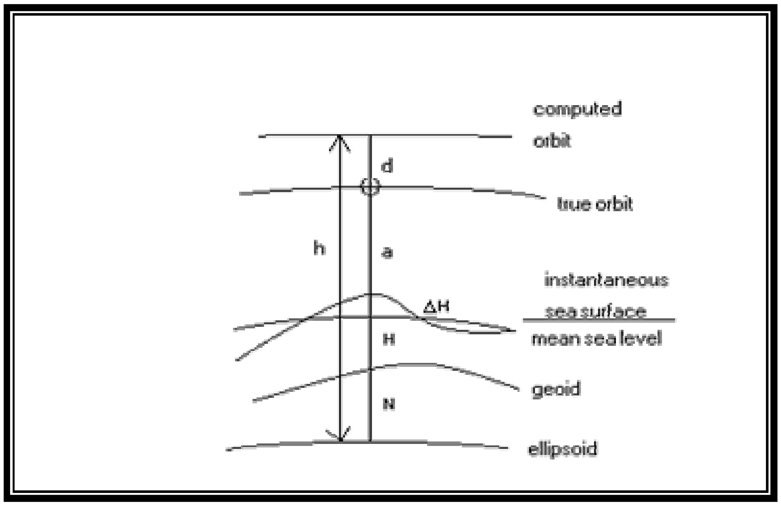

Figure 3.7: determining Harmonic coefficients using satellite altimeter [4]

The satellite send the waves to the sea surface and receive them back, the model described in the above graph can be written as:

$$h=N+H+\Delta H+a+d \qquad (3.103)$$

h : ellipsoidal height of the altimeter satellite, based on orbit computation,

N : geoidal height,

H : sea surface topography,

ΔH : instantaneoue tidal effects,

a : altimeter measurement,

b : discrepancy between the computed orbit and the actual orbit.

For the mean sea surface it has difference 1-2m from the geoid, but as initial approximations it can be used as geoid undulations.

The coefficients can be calculated as:

$$\bar{C}_{nm} = \frac{ag}{4\pi GM} \iint_{\sigma} N(a, \bar{\varphi}', \lambda') \bar{Y}_{nm}(\bar{\varphi}', \lambda') d\sigma(\bar{\varphi}', \lambda') \qquad (3.104)$$

2. Gravity data:

Using measured gravity data g and theoretical γ we can fined the gravity anomaly $\Delta g = g_p - g_q$, we can fined the harmonic coefficients:

$$\bar{C}_{nm} = \frac{a^2}{4\pi GM(n-1)} \iint_{\sigma} \Delta g(a, \bar{\varphi}', \lambda') \bar{Y}_{nm}(\bar{\varphi}', \lambda') d\sigma(\bar{\varphi}', \lambda') \qquad (3.105)$$

3. Combination methods:

Both data from satellite and terrestrial gravity can be used in combination.

3.6 Global gravitational model (GGM)

By GGM we mean a set of normalized spherical harmonic coefficients (C_{nm}, S_{nm}) representing the earth's gravitational potential.

Today EGM96 is regarded as the most common global gravitational model and it is available for free, it also was used for defining WGS84. Table 3.5 lists sum of GGM.

Table 3.5: examples of existing Geoid model

Name	N_{max}	Authors	year	Remarks
GEM 9	20	GSFC	1977	Satellite data only
GEM 10C	180	GSFC	1978	Combination solution
GEM T1	36	GSFC	1988	Satellite data only
GEM T2	36	GSFC	1990	Satellite data only
GEM T3	50	GSFC	1993	Satellite data only
JGM 3	70	GSFC,UTA,OSU,CNES	1994	Combination solution
Rapp 81	180	Rapp, OSU	1981	Combination solution
OSU86C	250	Rapp, OSU	1986	Combination solution
OSU89A	360	Rapp, OSU	1989	Combination solution
OSU91A	360	Rapp, OSU	1991	Combination solution
WGS84	42	DOD,USA	1984	For GPS ephemeris
GFZ93A	360	Gruber,GFZ,Po	1993	
WDM94	360	Ning,WTUSM	1994	Combination, with Chinese data
EGM96	360	NASA,NIMA, OSU,etc	1996	Combination solution
GPM98C	1800	Wenzel,Hanno	1998	Combination solution
EIGEN-3P	65	GFZ	2003	Using 3-year CHAMP data
EIGEN-GRACE01S	120	GFZ	2003	Using 39-days GRACE data
GGM01S	120	CSR	2003	Using 111-days GRACE data

The appreciations of the authors names are explained in table 3.5

Table 3.6: explained symbols in table 3.5

CNES	Center national d`Etudes spatiales,france
DOD	Department of defence, USA
GFZ	GeoForschungsZentrum,Potsdam,Germany
GRIM	Geodetic Research Institute munich, Germany
GSFC	Goddard Space Flight Center ,NASA, Greenbe,Maryland,USA
OSU	Ohio State University, Department of Geodetic science and Surveying,USA
CSR	Center for Space Research, Univercity of Texas
UTA	Univercity of Texas in Austin, USA
WTUSM	Wuhan Technical University of Surveying and Mapping, China

CHAPTER

4

GEOID MODELING

CHAPTER FOUR

Geoid Modeling

4.1 Introduction

The geoid is a surface of constant potential energy that coincides with mean sea level over the oceans. This definition is not very rigorous. First, mean sea level is not quite a surface of constant potential due to dynamic processes within the ocean. Second, the actual equipotential surface under continents is affected by the gravitational attraction of the overlying mass. In geodesy we define the geoid as though that mass were below the Geoid instead of above it. The main function of the geoid in geodesy is to serve as a reference surface for leveling. The elevation measured by surveying is relative to the Geoid.

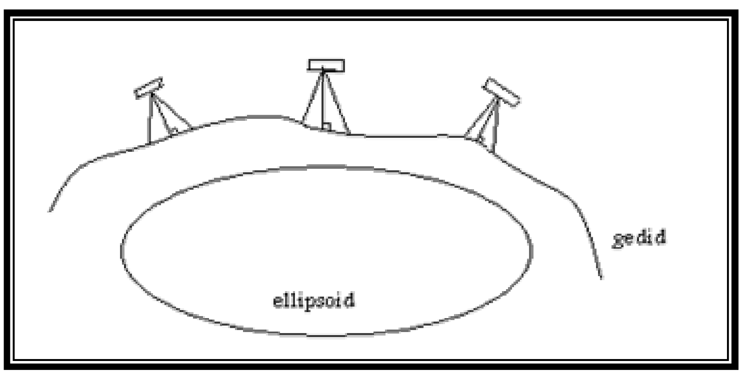

Figure4.1: plump line perpendicular to the Geoid

Geodetic applications require three different surfaces to be clearly defined as shown in Figure 4.2 they are:

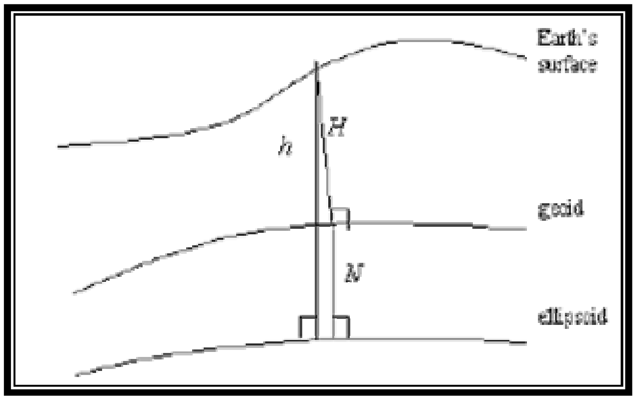

Figure 4.2: the ellipsoid, Geoid, earth surface relationship [4]

- the highly irregular topographic surface (the landmass topography as well as the ocean bathymetry),
- a geometric or mathematical reference surface called the ellipsoid,
- And the geoid, the equipotential surface that mean sea level follows.

The difference between the surfaces at a point defines the different heights:

- The difference between the topographic surface and the Geoid is called the orthometric height, normally measured by leveling.
- The difference between the topographic surface and the ellipsoid is called the ellipsoid. The height in absolute GPS is ellipsoidal height.
- The difference between the Geoid and the ellipsoid is called the Geoid undulation.

4.2 Geoid model

4.2.1 Introduction

Equation (4.1) connects *h* (the ellipsoid height relative to the ellipsoid), N (the geoid undulation relative to the ellipsoid), and *H* (the elevation relative to the geoid). See Figure 4.2.

$$h=H+N \tag{4.1}$$

The Geoid undulations range worldwide from -107 m to 85 m relative to the WGS 84 ellipsoid. The primary goal of physical geodesy is to develop a geoid model, which is then used to connect the three values. Given N, we can compute H or h from the other.

In general, the global or large-scale features of the geoid are expressed by a spherical harmonic expansion of the gravitational potential. Its higher terms are well defined by the ground gravity data, and the lower terms by the satellite tracking data. The Earth Gravitational Model 1996 (EGM96) is one of the most common global models. It is complete through degree and order 360. And have an error range of ± 0.5 to ± 1.0 m worldwide.

4.2.2 Global geoid model

The Global Geoid Models are represented by using the combination of satellite altimeter and the gravity data that described in chapter 3.

Gravity field quantities (e.g. N, Δg, γg) are related to the disturbing potential T and thus their harmonic expansions can be derived from that for T, where:

The Geoid Undulation N:

$$\mathrm{N}(r,\bar{f},l)=\frac{T}{g}=\frac{GM}{ag}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^{n+1}\sum_{m=0}^{n}\left\{\bar{C}_{nm}\cos ml+\bar{S}_{nm}\sin ml\right\}\bar{P}_{nm}(\sin\bar{f}) \qquad (4.2)$$

And the Gravity distance $dg_p = g_p - g_p$:

$$gg\ (r,\bar{f},l)=-\frac{\partial T}{\partial r}=\frac{GM}{a^2}\sum_{n=2}^{\infty}(n+1)\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left\{\bar{C}_{nm}\cos ml+\bar{S}_{nm}\sin ml\right\}\bar{P}_{nm}(\sin\bar{f}) \quad (4.3)$$

And The Gravity Anomaly $\Delta g_p = g_p - g_q$:

$$\Delta g\ (r,\bar{f},l)=-\frac{\partial T}{\partial r}-\frac{2}{r}T=\frac{GM}{a^2}\sum_{n=2}^{\infty}(n+1)\left(\frac{a}{r}\right)^{n+2}\sum_{m=0}^{n}\left\{\bar{C}_{nm}\cos ml+\bar{S}_{nm}\sin ml\right\}\bar{P}_{nm}(\sin\bar{f}) \quad (4.4)$$

The degree of the geoid is defined by the number n. The higher n is more details the global gravitational model contains about the geoid and the gravity field of the earth.

In our project we use EGM96 up to degree36 (n=36). So the total number of coefficients for a global gravitational model of degree n will be:

$n^2 + 2n - 3$

This mean we are dealing with $(36^2 + 2*36 - 3 = 1365)$ parameters. These parameters are stored in an ASCII file. We have made visual basic program to read these parameters $(\overline{C}_{nm}, \overline{S}_{nm})$, and calculating the geoid undulation with respect to the GRS80 datum and converting these undulation geometrically to the Palestinian datum, where Clarck 1880 ellipsoid is used. the parameters of EGM96 are shown in the appendix-A.

4.2.3 Gravimetric geoid (local) (stocks formula)

Stocks formula is one of the most fundamental formulas in physical geodesy. it gives us the possibility to determine the geoid height N from terrestrial gravity measurements.

Stocks assumptions:

1- No mass outside the geoid.

2- Gravity measurements all over the world.

The disturbing potential $T(r, f, l)$ and the gravity anomaly $\Delta g(r, f, l)$ have the following harmonic expansion over the unit sphere:

$$T(r, f, l) = \sum_{n=2}^{\infty} (\frac{R}{r})^{n+1} \sum_{m=-n}^{n} \overline{C}_{nm} \overline{Y}_{nm}(f, l) \tag{4.5}$$

$$\Delta g(r, f, l) = -\frac{\partial T}{\partial r} - \frac{2}{r} T = \sum_{n=2}^{\infty} \frac{n-1}{R} (\frac{R}{r})^{n+2} \sum_{m=-n}^{n} \overline{C}_{nm} \overline{y}_{nm}(f, l) \tag{4.6}$$

The geopotential coefficients $\overline{C}_{nm}$ can be estimated from the boundary data $\Delta g(R, f, l)$:

$$\overline{C}_{nm} = \frac{R}{4p(n-1)} \iint_{s} (\Delta g(R, f', l') \overline{y}_{nm}(f', l') ds(f', l') \tag{4.7}$$

Inserting $\overline{C}_{nm}$ into the harmonic expansions of $T(r, f, l)$ leads to:

$$T(r, f, l) = \frac{R}{4p} \iint_{s} S(r, y).\Delta g(R, f', l').ds(f', l') \tag{4.8}$$

Where $S(r, y)$ is called the extended Stokes function.

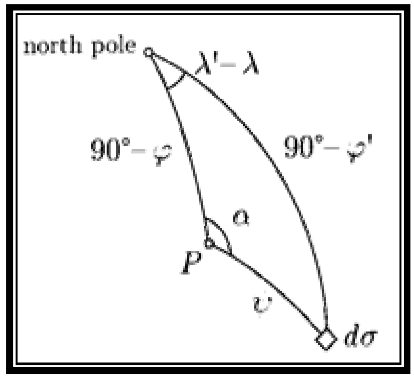

Figure4.3: spherical distance ψ [10]

The stock's function is defined as a weighting function, and it depends on the spherical distance ψ between the point P and th.

When $r \rightarrow R$, we get the disturbing potential on the geoid:

$$T(R,f,l) = \frac{R}{4p}\iint_s S(y).\Delta g \qquad (4.9)$$

The geoid height N is then obtained by the famous stocks formula:

$$N = \frac{T(R,f,l)}{g} = \frac{R}{4pg}\iint_s S(y).\Delta g \qquad (4.10)$$

Where g denotes the normal gravity and Sy is called stokes function:

$$Sy = \left[S(r,y)\right]_{r\rightarrow R} = \sum_{n=2}^{\infty}\frac{2n+1}{n-1}P_n(\cos y) \qquad (4.11)$$

Also:

$$S(y) = \frac{1}{\sin\frac{y}{2}} - 6\sin\frac{y}{2} + 1 - 5\cos y - 3\cos y\, Ln(\sin\frac{y}{2} + \sin^2\frac{y}{2}) \qquad (4.12)$$

If we introduce a new variable $t = \cos y$, stokes function can also be written as:

$$S(t) = \sqrt{\frac{2}{1-t}} - 6\sqrt{\frac{1-t}{2}} + 1 - 5t - 3tLn\left(\sqrt{\frac{1-t}{2}} + \frac{1-t}{2}\right) \qquad (4.13)$$

Stokes’ formula may be numerically evaluated by the grid method in the following way, see figure4.5:

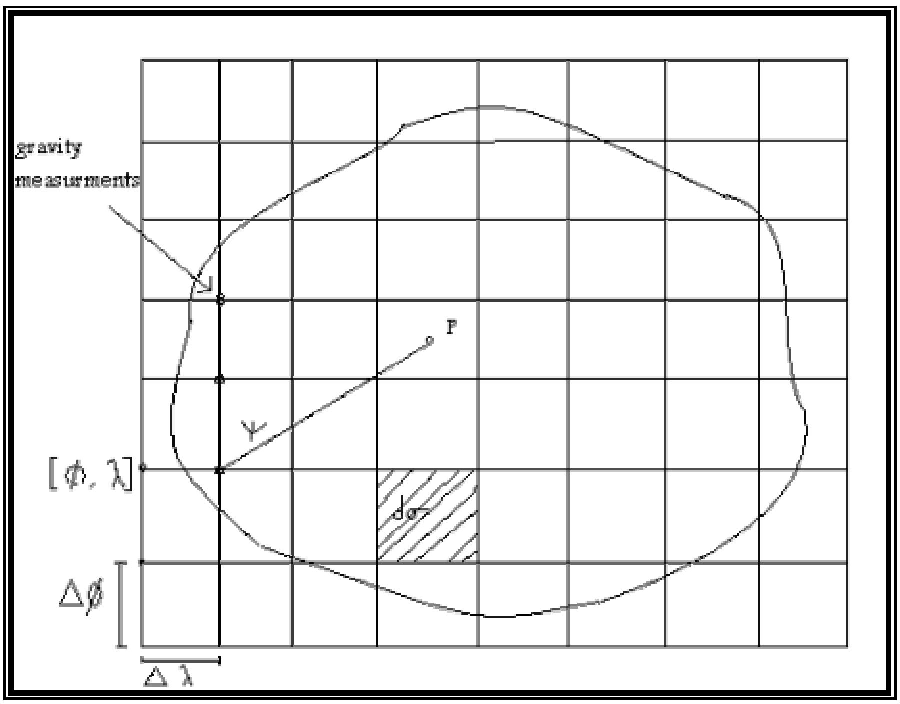

Figure4.4: stocks formula (grid method)

$$N = \frac{R}{4pg}\iint_{s0} \Delta g S(y)ds = \frac{R}{4pg}\sum_I \sum_J \iint_{sIJ} \Delta g S(y).ds_{ij}$$

$$\approx \frac{R}{4pg}\sum_I \sum_I \iint_{sIJ} \Delta\bar{g}_{ij} S(y_{IJ}).ds_{ij} = \frac{R}{4pg}\sum_I \sum_I \Delta\bar{g}_{ij} S(y_{IJ})\iint_{sIJ} ds$$

$$\frac{R}{4pg}\sum_I \sum_I \Delta\bar{g}_{ij} S(y_{IJ})\iint_{sIJ} ds_{ij} = \frac{R}{4pg}\sum_I \sum_I (\Delta\bar{g}_{ij} S(y_{ij})A_{ij}) \qquad (4.14)$$

Where:

R = mean radius of the earth

g = normal gravity of the reference ellipsoid

$\Delta\overline{g}_{ij}$ = mean gravity anomaly for block s_{ij}

y_{ij} = spherical distance from the computation point (f, l) to the block center of s_{ij}

$f_{\min}, l_{\min}$ = the minimum latitude and minimum longitude of the integration area

$\Delta f, \Delta l$ = block sizes (latitude/longitude difference of a block)

A_{ij} = area of block s_{ij}

Some of the above quantities can be computed as follows:

$$\cos y_{ij} = \sin f \sin f_i + \cos f \cos f_i \cos(l - l_j)$$

$$f_i = f_{\min} + (i - \frac{1}{2})\Delta f \qquad (4.15)$$

$$l_j = l_{\min} + (j - \frac{1}{2})\Delta l \qquad (4.16)$$

$$A_{ij} = \iint_{sij} ds = 2.\Delta l .\sin \frac{\Delta f}{2} \cos q_i \qquad (4.17)$$

4.2.4 Polynomial representation

Using total station, we will measure (E, N, H) for really the twenty references point that chosen in the study area. After that using GPS, (λ, f , h) will be measured for the reference points.

From the difference between the ellipsoidal height or geodetic height, and the orthometric height or the height above the geoid, geoid undulation (N) will be calculated from equation (4.23):

$$N=h-H \tag{4.18}$$

The model using reference points will be built using polynomials of different degrees (1st, 2d, 3d, and 4th),using least square solution.

$$X = (A^T.A)^{-1}.A^T.W.L$$

1. Constant, where the geoid is assumed to be flat surface parallel to the ellipsoid.

$$N = a_0 \tag{4.19}$$

The matrices for the least square solution are:

$$A=\begin{bmatrix}1\\1\\.\\.\\.\\1\end{bmatrix} \quad X=\begin{bmatrix}a_0\end{bmatrix} \quad L=\begin{bmatrix}N_1\\N_2\\.\\.\\.\\N_{20}\end{bmatrix}$$

2. Linear, where the geoid is in first order surface

$$N = a_0 + a_1 x + b_1 y \tag{4.20}$$

The matrices for the least square solution are:

$$A=\begin{bmatrix}1 & x_1 & y_1\\1 & x_2 & y_2\\. & . & .\\. & . & .\\. & . & .\\1 & x_{20} & y_{20}\end{bmatrix} \quad X=\begin{bmatrix}a_0\\a_1\\b_1\end{bmatrix} \quad L=\begin{bmatrix}N_1\\N_2\\.\\.\\.\\N_{20}\end{bmatrix}$$

3. Quadratic, where the geoid is in second order surface.

$$N = a_0 + a_1 \ x + b_1 y + a_2 x^2 + b_2 y^2 + c_1 xy \tag{4.21}$$

The matrices for the least square solution are:

$$A = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & y_1^2 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2^2 & y_2^2 & x_2 y_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & x_{20} & y_{20} & x_{20}^2 & y_{20}^2 & x_{20} y_{20} \end{bmatrix} \quad X = \begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \\ c_1 \end{bmatrix}$$

$$L = \begin{bmatrix} N_1 \\ N_2 \\ . \\ . \\ . \\ N_{20} \end{bmatrix}$$

4. Cubic

$$N = a_0 + a_1 x + b_1 y + a_2 x^2 + b_2 y^2 + c_1 xy + a_3 x^3 + b_3 y^3 + c_2 x^2 y + c_3 xy^2 \tag{4.22}$$

The matrices for the least square solution are:

A=

$$\begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & y_1^2 & x_1 y_1 & x_1^3 & y_1^3 & x_1^2 y_1 & x_1 y_1^2 \\ 1 & x_2 & y_2 & x_2^2 & y_2^2 & x_2 y_2 & x_2^3 & y_2^3 & x_2^2 y_2 & x_2 y_2^2 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 1 & x_{20} & y_{20} & x_{20}^2 & y_{20}^2 & x_{20} y_{20} & x_{20}^3 & y_{20}^3 & x_{20}^2 y_{20} & x_{20} y_{20}^2 \end{bmatrix}$$

$$X=\begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \\ c_1 \\ a_3 \\ b_3 \\ c_2 \\ c_3 \end{bmatrix} \qquad L=\begin{bmatrix} N_1 \\ N_2 \\ . \\ . \\ . \\ N_{20} \end{bmatrix}$$

4.3 Area definition

We chose an area equal to (10*10)(15*) 15Km and it is relatively small area, because in this area we assume the ellipsoid is flat, if the area is more than 15*15Km this assumption is wrong. And the area is divided into meshes.

The meshes can be connected to each others by common control points to satisfy the continuity of the surface.

If the area is larger than we deal with patches, where each patch has a group of meshes with special datum transformation parameters for high accuracy applications.

CHAPTER

5

DATA ANALYSIS

CHAPTER FIVE

Data Analysis

5.1 Introduction

This chapter includes the analysis of the results of geoid representation using Polynomials, by finding the parameters for the geoid model using least square solution depending on a visual basic program. The results using different polynomial degrees were compared.

The model was built using polynomial of different degrees using visual basic program, the difference between the ellipsoidal height and the height above geoid is used, and then geoid undulation (N) obtained from the equation 5.1. This undulation is used as the observation equation for the least squares solution.

$$N=h-H \tag{5.1}$$

The parameters of the HRS are calculated by EXCEL using least square solution and by VB6 program, these parameters are then used to calculate the geoid undulation and height above geoid for the degrees representing the polynomial using a separate program.

5.2 Artificial data

Before using real data, the program was tested using artificial data; these data were created in the study area. These data were used to compute the HRS parameters for different degrees in EXCEL, and then these points were computed by using visual basic program.

The results using these points were used to determine whether the program runs properly or not. The artificial data are listed in table 5.1.

Table 5.1: Artificial data coordinates and ellipsoidal and geoidal heights

Point	X	Y	H (Clark ellips.)	H (Geoid)
1	157864.100	104755.110	997.320	995.680
2	157721.120	104648.210	975.650	974.130
3	157630.310	104102.650	943.560	941.460
4	157666.580	103859.710	901.250	899.260
5	157213.640	103422.530	894.620	892.630
6	157011.960	102215.120	880.520	878.530
7	156889.560	101999.820	871.560	869.570
8	156765.430	101522.230	854.230	852.240
9	156421.210	100799.540	830.650	828.660
10	156157.840	100445.650	812.450	810.460
11	155955.960	100122.565	799.360	797.370
12	155487.320	99985.556	783.120	781.130
13	154921.000	99241.560	765.520	763.530
14	154001.031	99003.540	685.210	683.220
15	153654.580	98456.120	642.520	640.530
16	153213.021	98994.510	623.150	621.160
17	152898.720	98621.350	595.320	593.330
18	152212.540	97256.910	560.120	558.130
19	151684.790	97100.310	526.350	524.360
20	149654.315	96585.610	489.320	487.330

5.3 Results of artificial data

The HRS parameters were computed for artificial data using excel and VB6 to find the parameters of the least square solution. Then the results are compared. The same results were got by both methods; the visual basic program was used for the real data.

5.3.1 Artificial result for zero order

In this method, the geoid is assumed to be flat surface parallel to the ellipsoid, and the parameters are obtained using equation (5.2). This solution is no more than the average separation between the ellipsoid and the geoid for the control point used.

$$N = a_0 \tag{5.2}$$

The matrices for the least square solution built by EXCEL are:

$$A=\begin{bmatrix}1\\1\\.\\.\\.\\1\end{bmatrix} \quad X=\begin{bmatrix}a_0\end{bmatrix} \quad L=\begin{bmatrix}N_1\\N_2\\.\\.\\.\\N_{20}\end{bmatrix}$$

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1.640 \\ 1.520 \\ 2.100 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \end{bmatrix}$$

$A^T*A = \boxed{20}$

$(A^T*A)INV = \boxed{0.05}$

$A^T*L = \boxed{39.09}$

Parameters:

$X = \boxed{1.9545}$

The solution using the VB program is shown in figure 5.1.

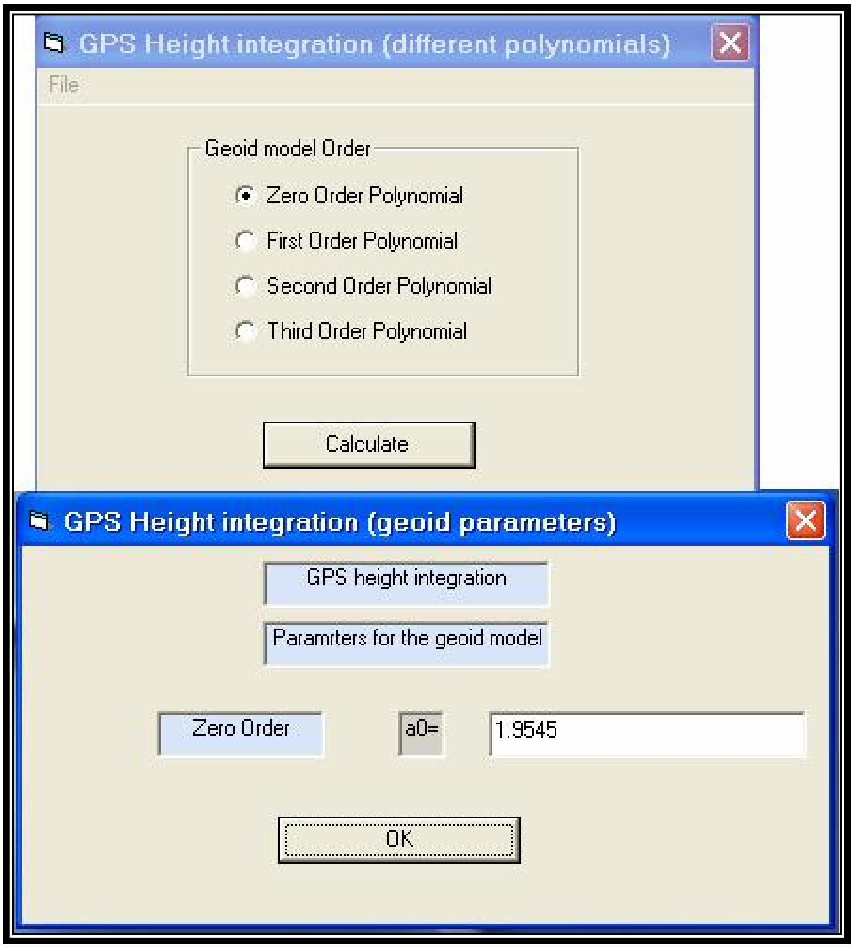

Figure 5.1: The solution using the VB program for zero order

We can clearly see that the parameter is the same as in EXCEL solution.

5.3.2 Artificial result for first order

The geoid is in first order surface (linear), in this method the geoid is assumed to be flat, but inclined little from horizontal. And the parameters are obtained using equation (5.3).

$$N = a_0 + a_1 x + b_1 y \tag{5.3}$$

The matrices for the least square solution are:

$$A=\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ . & . & . \\ . & . & . \\ . & . & . \\ 1 & x_{20} & y_{20} \end{bmatrix} \quad X=\begin{bmatrix} a_0 \\ a_1 \\ b_1 \end{bmatrix} \quad L=\begin{bmatrix} N_1 \\ N_2 \\ . \\ . \\ . \\ N_{20} \end{bmatrix}$$

A =

1	157864.100	104755.110
1	157721.120	104648.210
1	157630.310	104102.650
1	157666.580	103859.710
1	157213.640	103422.530
1	157011.960	102215.120
1	156889.560	101999.820
1	156765.430	101522.230
1	156421.210	100799.540
1	156157.840	100445.650
1	155955.960	100122.565
1	155487.320	99985.556
1	154921.000	99241.560
1	154001.031	99003.540
1	153654.580	98456.120
1	153213.021	98994.510
1	152898.720	98621.350
1	152212.540	97256.910
1	151684.790	97100.310
1	149654.315	96585.610

L =

1.640
1.520
2.100
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990

AT*A =

20	3105025.03	2013138.601
3105025.03	4.8217E+11	3.1265E+11
2013138.6	3.1265E+11	2.02762E+11

(A^T*A)INV =

420.6537981	-0.004929654	0.003424813
-0.004929654	7.14938E-08	-6.12959E-08
0.003424813	-6.12959E-08	6.05169E-08

$$A^T * L = \boxed{\begin{array}{c} 39.09 \\ 6066957.776 \\ 3931748.16 \end{array}}$$

Parameters:

$$X = \boxed{\begin{array}{c} 0.856179189 \\ 5.00086E\text{-}05 \\ -6.62208E\text{-}05 \end{array}}$$

The solution using the VB program is shown in figure 5.2:

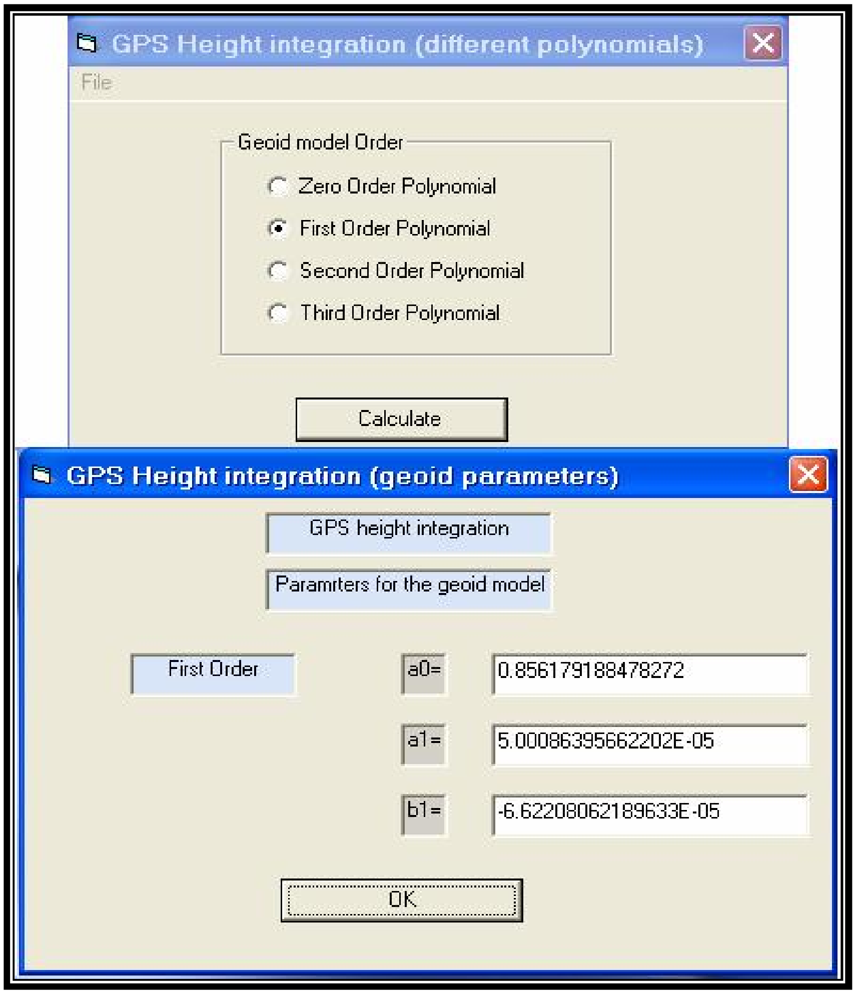

Figure 5.2: The solution using the VB program for first order.

We can clearly see that the parameters are the same as in EXCEL solution.

5.3.3 Artificial result for second order

In this method, the geoid is assumed to be a second order surface. And the parameters are obtained using equation (5.4).

$$N = a_0 + a_1 \; x + b_1 y + a_2 x^2 + b_2 y^2 + c_1 xy \tag{5.4}$$

The matrices for the least square solution are:

$$A=\begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & y_1^2 & x_1 y_1 \\ 1 & x_2 & y_2 & x_2^2 & y_2^2 & x_2 y_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & x_{20} & y_{20} & x_{20}^2 & y_{20}^2 & x_{20} y_{20} \end{bmatrix} \quad X=\begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \\ c_1 \end{bmatrix}$$

$$L=\begin{bmatrix} N_1 \\ N_2 \\ . \\ . \\ . \\ N_{20} \end{bmatrix}$$

Note: both X and Y is divided by 1000.

A =

1	157.864	104.755	24921.07	10973.6331	16537.07116
1	157.721	104.648	24875.95	10951.2479	16505.23289
1	157.630	104.103	24847.31	10837.3617	16409.73299
1	157.667	103.860	24858.75	10786.8394	16375.20528
1	157.214	103.423	24716.13	10696.2197	16259.4324
1	157.012	102.215	24652.76	10447.9308	16048.99633
1	156.890	102.000	24614.33	10403.9633	16002.70688
1	156.765	101.522	24575.40	10306.7632	15915.17604
1	156.421	100.800	24467.59	10160.5473	15767.18601
1	156.158	100.446	24385.27	10089.3286	15685.37574
1	155.956	100.123	24322.26	10024.5280	15614.71074
1	155.487	99.986	24176.31	9997.1114	15546.48614
1	154.921	99.242	24000.52	9848.8872	15374.60172
1	154.001	99.004	23716.32	9801.7009	15246.64723
1	153.655	98.456	23609.73	9693.6076	15128.23377
1	153.213	98.995	23474.23	9799.9130	15167.24794
1	152.899	98.621	23378.02	9726.1707	15079.07818
1	152.213	97.257	23168.66	9458.9065	14803.7213
1	151.685	97.100	23008.28	9428.4702	14728.64013
1	149.654	96.586	22396.41	9328.7801	14454.4533

L =

1.640
1.520
2.100
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990
1.990

A^T*A =

20	3105.025027	2013.138601	482165.3022	202761.9105	312649.9362
3105.025027	482165.3022	312649.9362	74889574.86	31500671.4	48566585.06
2013.138601	312649.9362	202761.9105	48566585.06	20434719.13	31500671.4
482165.3022	74889574.86	48566585.06	11634297221	4894930911	7545884370
202761.9105	31500671.4	20434719.13	4894930911	2060729369	3175776897
312649.9362	48566585.06	31500671.4	7545884370	3175776897	4894930911

(A^T*A)INV =

2636336.107	-37391.1260	4701.767498	173.0341211	93.52885666	-155.0833122
-37391.1238	1507.382422	-1582.366749	-12.76640138	-10.9665243	24.41067228
4701.764142	-1582.36670	2359.647218	16.30257129	15.11418729	-34.72305095
173.0340955	-12.7664009	16.30257112	0.127303808	0.124604723	-0.266389287
93.52882886	-10.9665235	15.11418656	0.124604719	0.137776599	-0.275906396
-155.083254	24.41067092	-34.72305	-0.266389282	-0.27590640	0.580825998

A^T*L =

39.09
6066.957776
3931.74816
941828.0827
395700.4536
610433.0093

Parameters:

X =

-6.052808657
5.049531119
-7.672683571
-0.075676634
-0.103881515
0.183845433

The solution using the VB program is shown in figure 5.3.

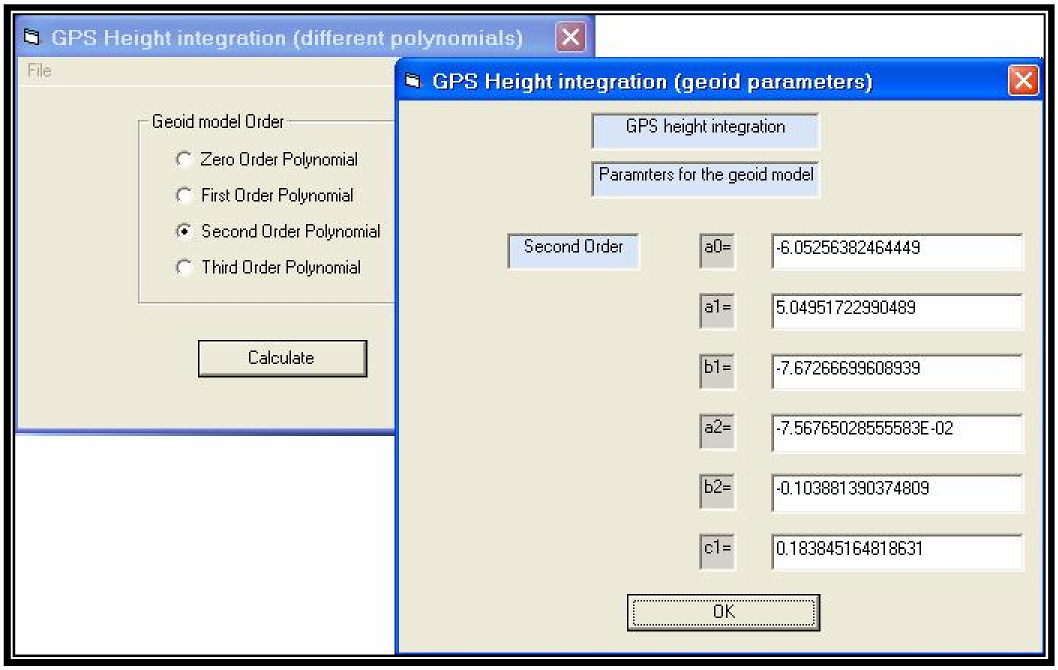

Figure 5.3: The solution using the VB program for second order.

We can clearly see that the parameters are the same as in EXCEL solution.

5.3.4 Artificial result for third order

In this method the geoid is assumed to be a third order surface, and the parameters are obtained using equation (5.5).

$$N = a_0 + a_1 x + b_1 y + a_2 x^2 + b_2 y^2 + c_1 xy + a_3 x^3 + b_3 y^3 + c_2 x^2 y + c_3 xy^2 \qquad (5.5)$$

The matrices for the least square solution are:

A=

$$\begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & y_1^2 & x_1 y_1 & x_1^3 & y_1^3 & x_1^2 y_1 & x_1 y_1^2 \\ 1 & x_2 & y_2 & x_2^2 & y_2^2 & x_2 y_2 & x_2^3 & y_2^3 & x_2^2 y_2 & x_2 y_2^2 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 1 & x_{20} & y_{20} & x_{20}^2 & y_{20}^2 & x_{20} y_{20} & x_{20}^3 & y_{20}^3 & x_{20}^2 y_{20} & x_{20} y_{20}^2 \end{bmatrix}$$

$$X=\begin{bmatrix} a_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \\ c_1 \\ a_3 \\ b_3 \\ c_2 \\ c_3 \end{bmatrix} \qquad L=\begin{bmatrix} N_1 \\ N_2 \\ . \\ . \\ . \\ N_{20} \end{bmatrix}$$

Note: both X and Y is divided by 1000.

$$L = \begin{bmatrix} 1.640 \\ 1.520 \\ 2.100 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \\ 1.990 \end{bmatrix} \qquad A^T*L = \begin{bmatrix} 39.09 \\ 6066.957776 \\ 3931.74816 \\ 941828.0827 \\ 395700.4536 \\ 610433.0093 \\ 146240112.3 \\ 39848219.02 \\ 94794809.46 \\ 61456124.52 \end{bmatrix}$$

Parameters:

$$X = \begin{bmatrix} 3813.585938 \\ 502.6385498 \\ -895.1818848 \\ -17.29336166 \\ -24.81369019 \\ 43.57383728 \\ 0.129324853 \\ -0.147514388 \\ -0.428333521 \\ 0.445812106 \end{bmatrix}$$

The solution using the VB program is shown in figure 5.4.

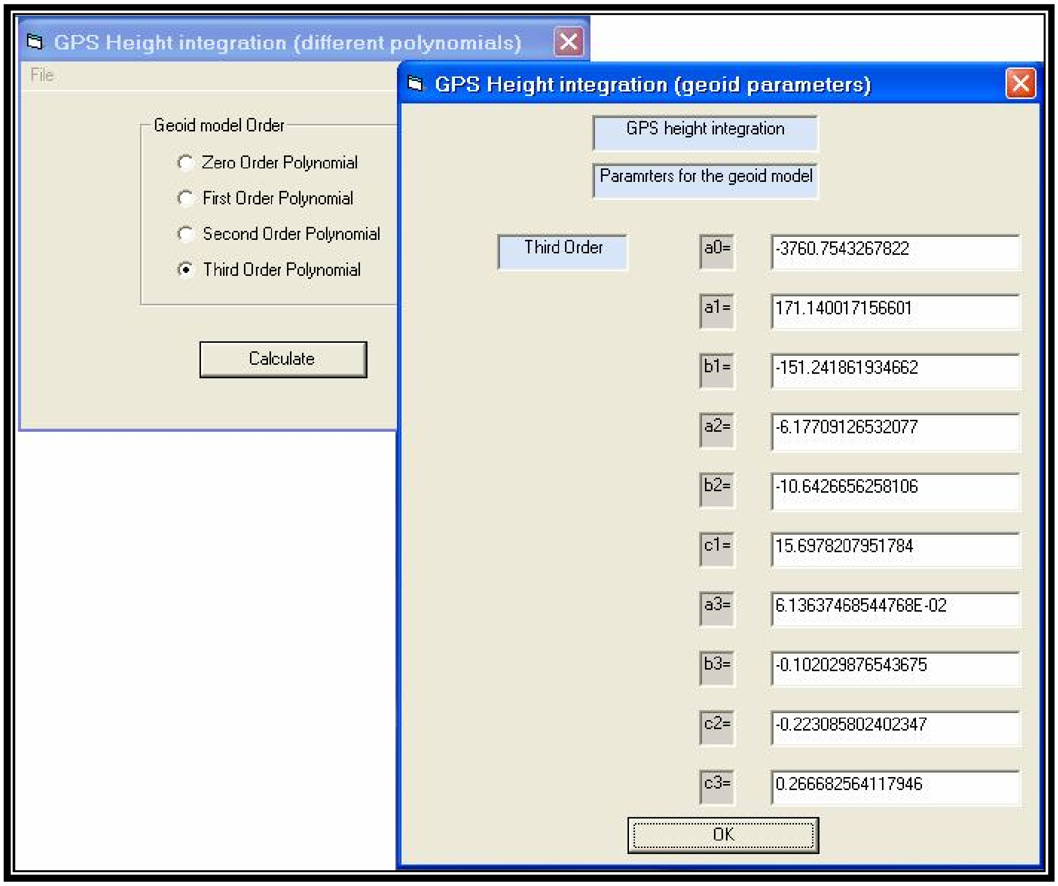

Figure 5.4: The solution using the VB program for third order.

We can clearly see that there is a difference between the parameters related to the technical differences between VB and excel.

5.4 Real data

After the program was tested using artificial data in the study area, in order to examine whether the program runs properly. It was found that it runs properly. According to this result the program was applied to real data, were the GPS height and the geoid height were measured for the control points.

The real data were used to compute the HRS parameters for different degrees, and then these parameters were saved and used to compute the geoid undulation and height above geoid in another VB program for new points. The real data point's observations and coordinates are listed in table 5.2.

Table 5.2: Real data coordinates and ellipsoidal and geoidal heights

Point	X	Y	H(ellipsoid)	H(geoid)
1	155960.356	104235.580	897.910	895.799
2	158266.343	106672.048	960.010	957.715
3	155917.432	107507.029	845.300	843.145
4	158239.302	104210.633	961.221	958.980
5	160833.893	107388.752	972.520	970.210
6	151356.277	104384.964	783.490	781.310
7	162095.515	101693.583	883.330	881.130
8	161812.411	103627.078	935.525	933.180
9	153446.000	101673.541	856.050	853.880
10	157174.690	101676.372	879.176	876.900
11	161818.411	106095.212	947.437	945.180
12	159105.846	100355.835	954.030	951.870
13	155872.637	108642.889	821.863	819.600
14	159602.147	108749.722	975.399	973.130
15	158546.092	101692.212	892.409	890.155
16	158465.322	101512.331	900.189	897.973
17	158817.726	101265.672	901.002	898.805
18	159114.346	101345.795	866.491	864.245
19	158993.973	101694.213	889.669	887.430
20	158555.128	106885.354	985.234	982.990

5.5 Test of real data

The real data were tested for different polynomials, and we get different values of geoid undulation according to the degree of the polynomial we use. The results for geoid undulation are shown below using surfer program.

5.5.1 Testing for zero order

Figure 5.5 shows the shape for the geoid model in the study area, using the polynomial of zero order. The geoid is flat and parallel to the horizontal plane (in small area the ellipsoid is assumed to be flat).

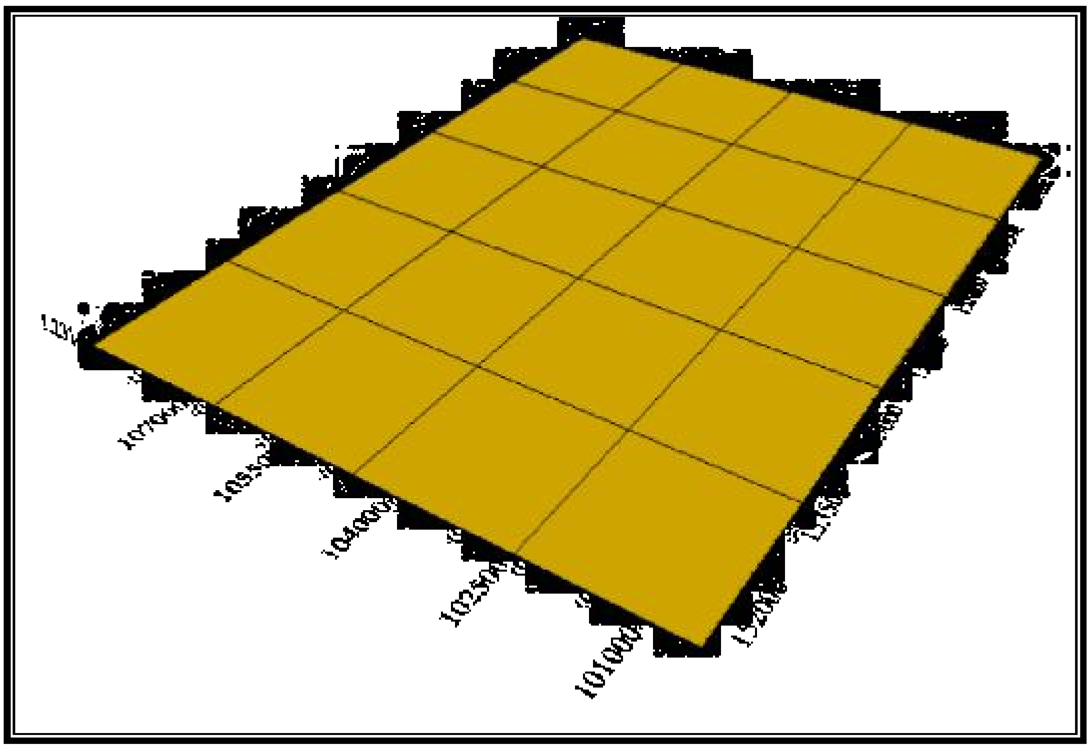

Figure 5.5: goid model for zero order

The parameters are shown in figure 5.6.

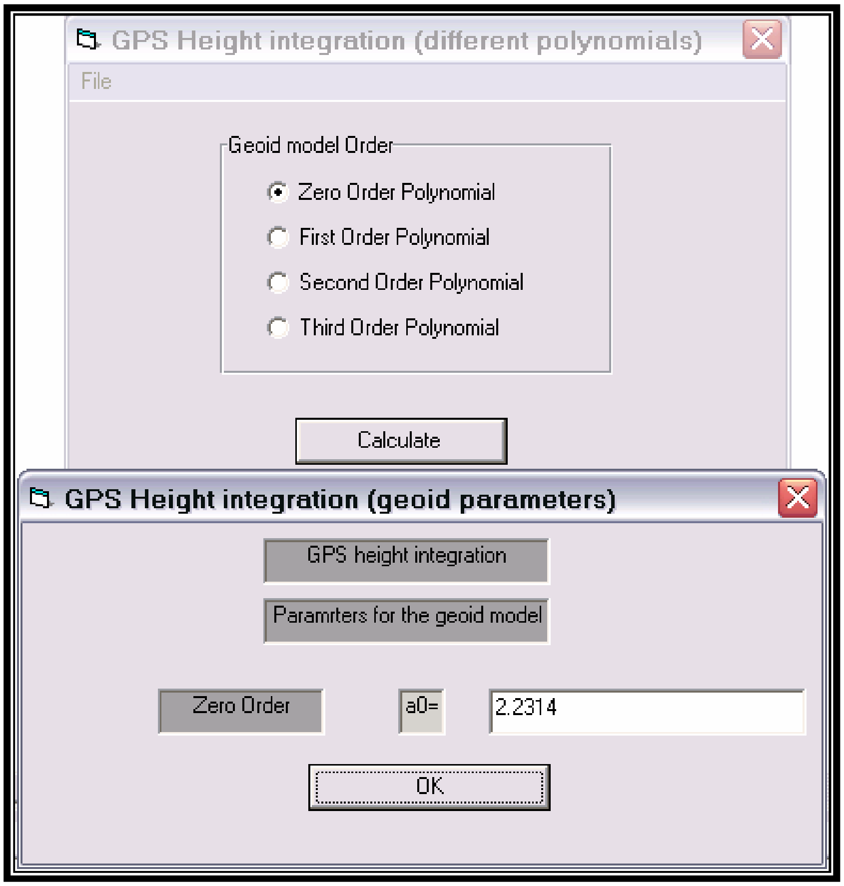

Figure 5.6: parameters for zero order

The results were tested using points with known geoidal height (H), the ellipsoidal height (h) for these points were measured using GPS. Using equation (5.6), H was calculated and the 95% RMSE was found in table 5.3.

$$H=h-N \tag{5.6}$$

Table 5.3: RMSE for zero order

point	H(fixed)	H(cal.)	dH
1	895.799	895.6786	0.120
2	957.715	957.7786	-0.064
3	843.145	843.0686	0.076
4	958.980	958.9896	-0.010
5	970.210	970.2886	-0.079
6	781.310	781.2586	0.051
7	881.130	881.0986	0.031
8	933.180	933.2936	-0.114
9	853.880	853.8186	0.061
10	876.900	876.9446	-0.045
11	945.180	945.2056	-0.026
12	951.870	951.7986	0.071
13	819.600	819.6316	-0.032
14	973.130	973.1676	-0.038
15	890.155	890.1776	-0.023
16	897.973	897.9576	0.015
17	898.805	898.7706	0.034
18	864.245	864.2596	-0.015
19	887.430	887.4376	-0.008
20	982.990	983.0026	-0.013

$$RMSE = \sqrt{\frac{\sum (H_{fixed} - H_{cal.})^2}{n}} = \pm 0.0563324m$$

So we can say that the geoid undulation (N) has accuracy of **6cm**.

5.5.2 Testing for first order

Figure 5.7 shows the scape of the geoid model in the study area. Where the geoid is assumed to a flat surface inclined little from the horizontal plane.

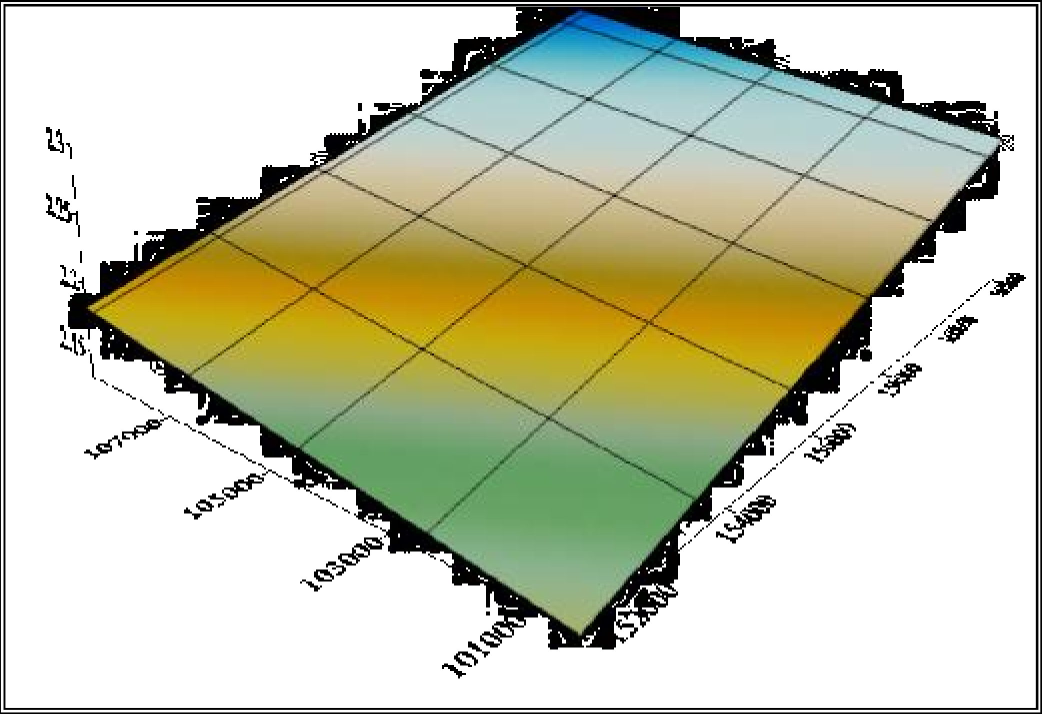

Figure 5.7: geoid model for first order

The parameters are shown in figure 5.8.

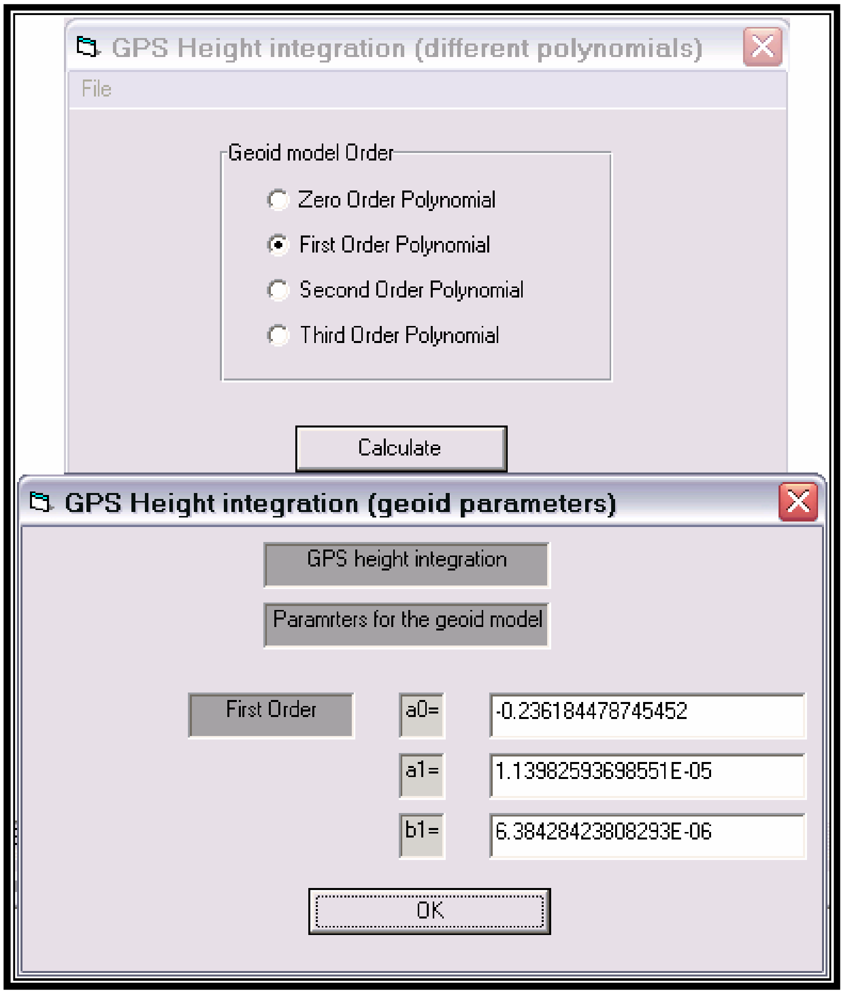

Figure 5.8: parameters for first order

RMSE was found in table 5.4.

Table 5.4: RMSE for first order

point	H(fixed)	H(cal.)	dH
1	895.799	895.703	0.096
2	957.715	957.7612	-0.046
3	843.145	843.0726	0.072
4	958.980	958.9882	-0.008
5	970.210	970.2374	-0.027
6	781.310	781.3346	-0.025
7	881.130	881.0693	0.061
8	933.180	933.2552	-0.075
9	853.880	853.8881	-0.008
10	876.900	876.9715	-0.072
11	945.180	945.1514	0.029
12	951.870	951.812	0.058
13	819.600	819.6289	-0.029
14	973.130	973.1217	0.008
15	890.155	890.1888	-0.034
16	897.973	897.9709	0.002
17	898.805	898.7814	0.024
18	864.245	864.2665	-0.021
19	887.430	887.4437	-0.014
20	982.990	982.9805	0.010

$$RMSE = \sqrt{\frac{\sum (H_{fixed} - H_{cal.})^2}{n}} = \pm 0.0447565m$$

So we can say that the geoid undulation (N) has accuracy of **4cm**.

5.5.3 Testing for second order

Figure 5.9 shows the shape for the geoid model in the study area. Where the geoid is assumed to be in second order surface.

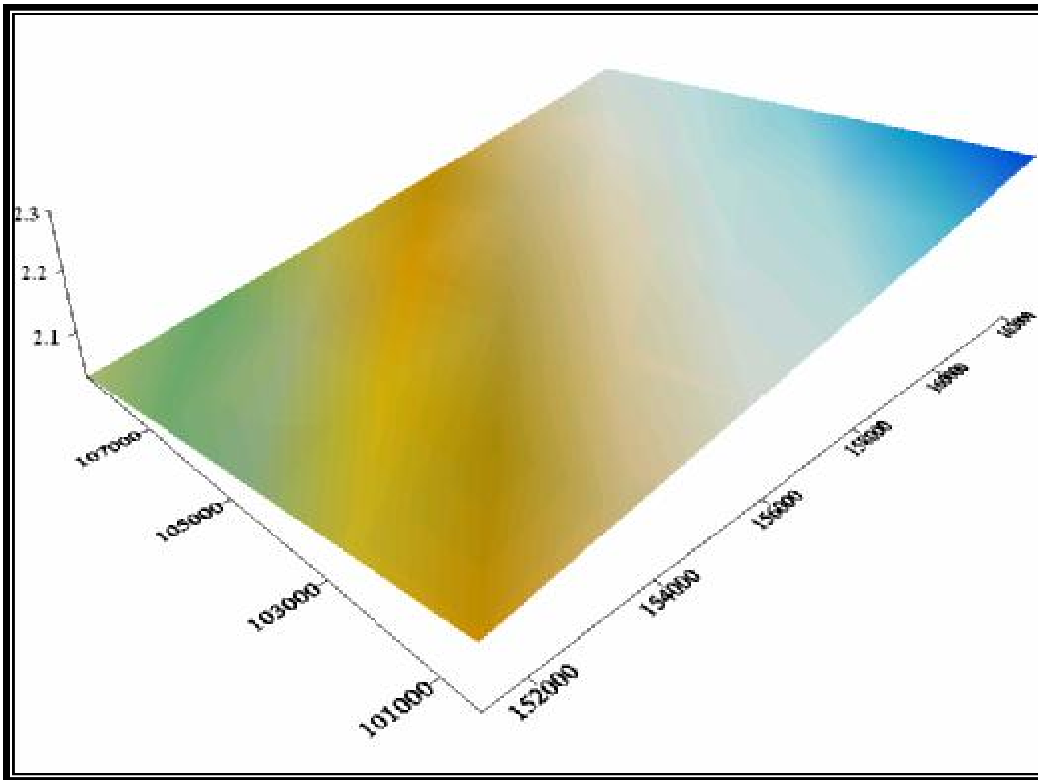

Figure 5.9: geoid model for second order

The parameters are shown in figure 5.10.

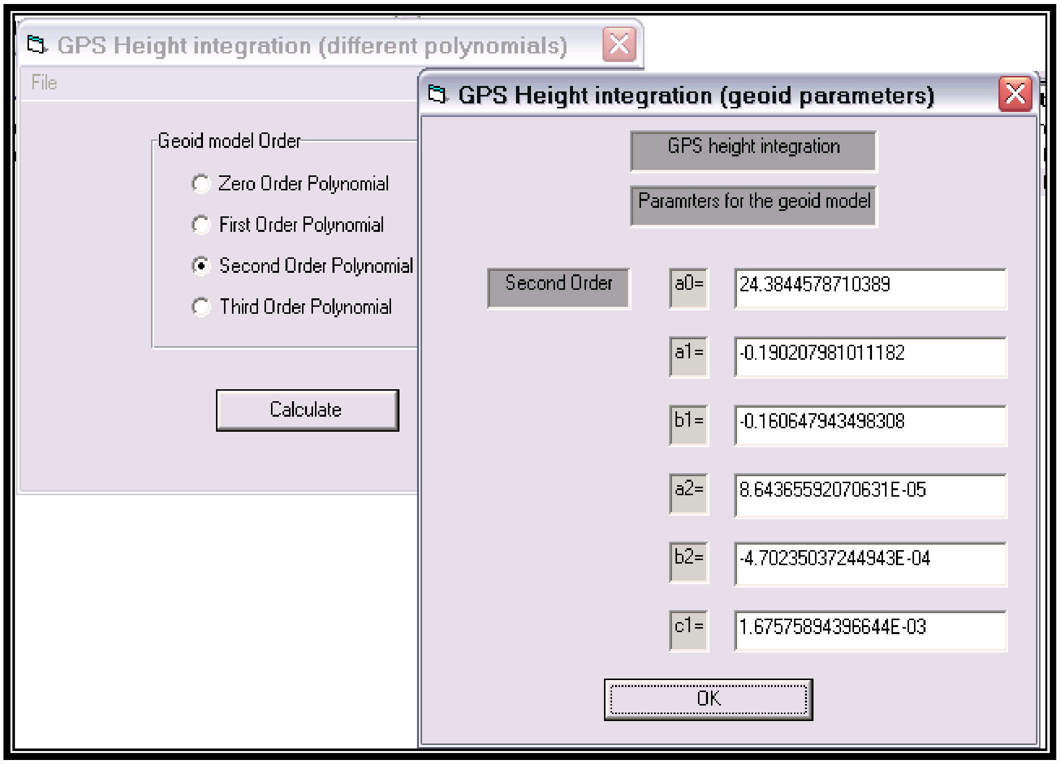

Figure 5.10: parameters for first order

RMSE was found in table 5.6.

Table 5.5: RMSE for second order

point	H(fixed)	H(cal.)	dH
1	895.799	895.7002	0.099
2	957.715	957.7602	-0.045
3	843.145	843.0872	0.058
4	958.980	958.9849	-0.005
5	970.210	970.2229	-0.013
6	781.310	781.3317	-0.022
7	881.130	881.0828	0.047
8	933.180	933.2531	-0.073
9	853.880	853.8675	0.013
10	876.900	876.9673	-0.067
11	945.180	945.1357	0.044
12	951.870	951.8213	0.049
13	819.600	819.6522	-0.052
14	973.130	973.1164	0.014
15	890.155	890.1899	-0.035
16	897.973	897.9722	0.001
17	898.805	898.7851	0.020
18	864.245	864.2712	-0.026
19	887.430	887.4465	-0.017
20	982.990	982.9786	0.011

$$RMSE = \sqrt{\frac{\sum (H_{fixed} - H_{cal.})^2}{n}} = \pm 0.0435952m$$

So we can say that the geoid undulation (N) has accuracy of **4cm**.

5.5.4 Testing for third order

Figure 5.8 shows the shape for the geoid model in the study area.

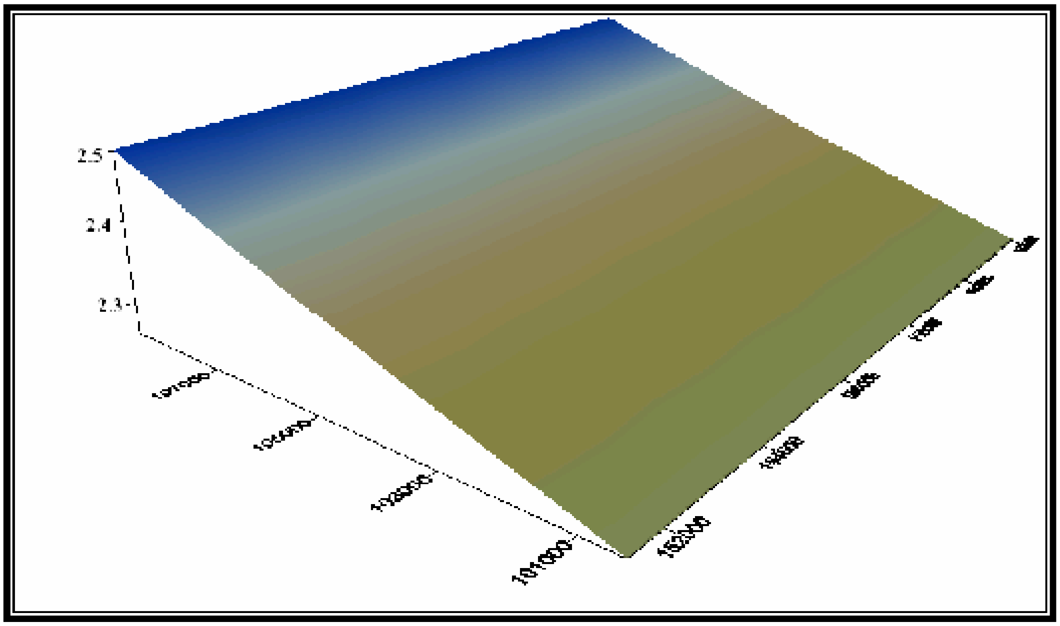

Figure 5.11: geoid model for third order

RMSE was found in table 5.5.

Table 5.6: RMSE for third order

point	H(fixed)	H(cal.)	dH
1	895.799	895.7864	0.013
2	957.715	957.7597	-0.045
3	843.145	843.131	0.014
4	958.980	958.9577	0.022
5	970.210	970.2434	-0.033
6	781.310	781.316	-0.006
7	881.130	881.1363	-0.006
8	933.180	933.1986	-0.019
9	853.880	853.8659	0.014
10	876.900	876.9739	-0.074
11	945.180	945.1463	0.034
12	951.870	951.8706	-0.001
13	819.600	819.6056	-0.006
14	973.130	973.1162	0.014
15	890.155	890.1629	-0.008
16	897.973	897.9526	0.020
17	898.805	898.7713	0.034
18	864.245	864.2514	-0.006
19	887.430	887.412	0.018
20	982.990	982.9711	0.019

$$RMSE = \sqrt{\frac{\sum (H_{fixed} - H_{cal.})^2}{n}} = \pm 0.0262123m$$

Therefore, we can say that the geoid undulation (N) has accuracy of **3cm**.

5.6 Accuracy testing

According to the results shown in tables (5.3, 5.4, 5.5, 5.6), the curve was drawn to the accuracy for the different polynomial degrees for calculating height reference surface.

Table 5.7: Tested Accuracy and the polynomial order

Order	Order0	Order1	Order2	Order3
Accuracy (cm)	5.6	4.4	4.3	2.5

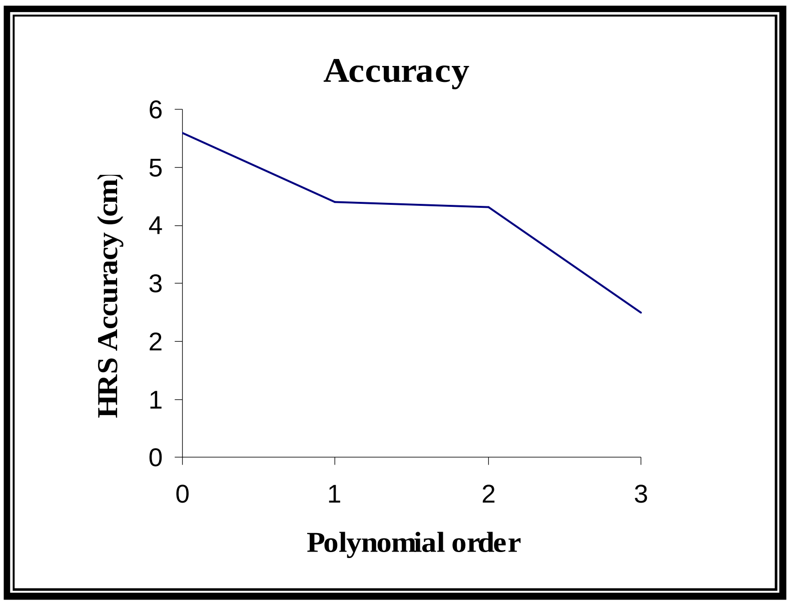

Figure 5.12: Accuracy testing

CHAPTER

6

CONCLUSIONS & RECOMMENDATIONS

CHAPTER SIX

Conclusions and Recommendations

6.1 Conclusions

According to the different tests and results of this project shown in chapter five, we get the following conclusions:

1. The project calculates the height reference system (HRS) using polynomials of the degrees: zero order, first order, second order and third order.

2. The software of this project could calculate the HRS parameters properly. As tested using EXCEL solutions, as well it uses the calculated parameters to find H using measured h.

3. In this project, it was assumed that the ellipsoid for the area to be flat, and we could get good results that support this assumption.

4. The project runs using projected coordinates system, but not the longitude and latitude. And it does not matter if the system real or auxiliary.

5. EGM96 geoid model has shown very bad results, because of in our area little number of gravity data was used for that model.

6. The results have shown that the accuracy is always getting higher as the degree of the polynomial gets higher. as shown in chapter 5.6.

6.2 Recommendations

According to the work done in this project, we recommend the following to be considered in HRS representation:

1. The test of results using higher polynomial degrees (fourth order, fifth order, ...).

2. this project can only be applied for small areas 10*10km or maximum15*15km, for larger areas (national level or regional levels), the model can be divided into small areas (10*10km), but the continuity condition has to bf applied, and the region can be divided to patches (e.g. 50*50km), so each patch has separate datum transformation parameters.

3. In the adjustment, it is recommended gravity and astronomical data (deflection of vertical), so that the HRS model can be more realistic, because these data originally depend on the geoid.

4. For the people of surveying in Palestine, it is recommended to inform these people about the difference between different types of heights and coordinate systems.

APPENDIX A

EGM 96 Parameters Up To 36 Degree

APPENDIX A

EGM 96 Parameters Up To 36 Degree

Table A-1: EGM96 parameters

n	m	C_{nm}	S_{nm}
2	0	0.15548961E-08	0.00000000E+00
2	1	0.00000000E+00	0.00000000E+00
2	2	0.24390067E-05	-0.14000870E-05
3	0	0.95703311E-06	0.00000000E+00
3	1	0.20307524E-05	0.24960266E-06
3	2	0.90353915E-06	-0.61898576E-06
3	3	0.72150732E-06	0.14137252E-05
4	0	-0.25039630E-06	0.00000000E+00
4	1	-0.53525575E-06	-0.47413322E-06
4	2	0.34825956E-06	0.66402365E-06
4	3	0.99131081E-06	-0.20142885E-06
4	4	-0.18936775E-06	0.30896802E-06
5	0	0.68688331E-07	0.00000000E+00
5	1	-0.60759526E-07	-0.95025830E-07
5	2	0.65608000E-06	-0.32412978E-06
5	3	-0.45185048E-06	-0.21707107E-06
5	4	-0.29504971E-06	0.51356076E-07
5	5	0.17190752E-06	-0.66905926E-06
6	0	-0.14792196E-06	0.00000000E+00
6	1	-0.77140451E-07	0.25319721E-07
6	2	0.52446631E-07	-0.37524337E-06
6	3	0.58415398E-07	0.68782457E-08
6	4	-0.88826681E-07	-0.47112551E-06

6	5	-0.26608318E-06	-0.53688302E-06
6	6	0.96977647E-08	-0.23696573E-06
7	0	0.90084749E-07	0.00000000E+00
7	1	0.28185033E-06	0.96226629E-07
7	2	0.32097572E-06	0.95694116E-07
7	3	0.25239511E-06	-0.20959979E-06
7	4	-0.27424825E-06	-0.12419677E-06
7	5	-0.10203298E-09	0.19694196E-07
7	6	-0.35855234E-06	0.15158761E-06
7	7	-0.14861559E-08	0.25282302E-07
8	0	0.48380019E-07	0.00000000E+00
8	1	0.24572133E-07	0.58448181E-07
8	2	0.69506807E-07	0.67298789E-07
8	3	-0.16565131E-07	-0.87073521E-07
8	4	-0.24310439E-06	0.67017055E-07
8	5	-0.23643946E-07	0.87145479E-07
8	6	-0.64860212E-07	0.31015485E-06
8	7	0.68906444E-07	0.74722875E-07
8	8	-0.12143992E-06	0.12066191E-06
9	0	0.28440334E-07	0.00000000E+00
9	1	0.14211255E-06	0.25365458E-07
9	2	0.28492603E-07	-0.34982892E-07
9	3	-0.16130770E-06	-0.85673561E-07
9	4	-0.12181511E-07	0.25861246E-07
9	5	-0.24086651E-07	-0.57528523E-07
9	6	0.66747223E-07	0.22336337E-06
9	7	-0.12292539E-06	-0.95140873E-07
9	8	0.18819537E-06	-0.37058420E-08
9	9	-0.61348629E-07	0.97042530E-07
10	0	0.54967314E-07	0.00000000E+00

10	1	0.83155949E-07	-0.13561995E-06
10	2	-0.81935873E-07	-0.50106549E-07
10	3	-0.40960213E-08	-0.16041175E-06
10	4	-0.93942739E-07	-0.68807042E-07
10	5	-0.48920380E-07	-0.45803528E-07
10	6	-0.34483699E-07	-0.78375345E-07
10	7	0.90179229E-08	-0.22191507E-08
10	8	0.42160554E-07	-0.92853576E-07
10	9	0.12438558E-06	-0.38941169E-07
10	10	0.96639411E-07	-0.18940944E-07
11	0	-0.51937400E-07	0.00000000E+00
11	1	0.18789966E-07	-0.30238895E-07
11	2	0.12580638E-07	-0.91994434E-07
11	3	-0.31030480E-07	-0.13177790E-06
11	4	-0.36318793E-07	-0.70227447E-07
11	5	0.40492533E-07	0.58378070E-07
11	6	-0.22096118E-08	0.28007585E-07
11	7	0.32534717E-08	-0.87447075E-07
11	8	-0.59488520E-08	0.23754140E-07
11	9	-0.40130795E-07	0.43269605E-07
11	10	-0.52991922E-07	-0.21362210E-07
11	11	0.45505963E-07	-0.64589547E-07
12	0	0.34091827E-07	0.00000000E+00
12	1	-0.54252561E-07	-0.44203534E-07
12	2	0.67317039E-08	0.31802951E-07
12	3	0.40396775E-07	0.17586390E-07
12	4	-0.63292441E-07	-0.45521470E-08
12	5	0.37269665E-07	0.44109379E-08
12	6	-0.21614079E-08	0.42793605E-07
12	7	-0.15995328E-07	0.34851984E-07

12	8	-0.23372086E-07	0.14836903E-07
12	9	0.42181160E-07	0.23658502E-07
12	10	-0.84005909E-08	0.31990145E-07
12	11	0.97893862E-08	-0.84029464E-08
12	12	-0.51824553E-08	-0.11107629E-07
13	0	0.42987279E-07	0.00000000E+00
13	1	-0.57834903E-07	0.45047532E-07
13	2	0.53154463E-07	-0.63139706E-07
13	3	-0.16896080E-07	0.83374341E-07
13	4	-0.89250536E-08	-0.17502520E-08
13	5	0.48617903E-07	0.54172922E-07
13	6	-0.23195715E-07	0.10615923E-08
13	7	-0.31865755E-08	-0.56731814E-08
13	8	-0.10158126E-07	-0.99146240E-08
13	9	0.18299923E-07	0.46225214E-07
13	10	0.40034144E-07	-0.41474931E-07
13	11	-0.43036318E-07	-0.29798284E-10
13	12	-0.31032309E-07	0.84030873E-07
13	13	-0.62529956E-07	0.68046019E-07
14	0	-0.20874614E-07	0.00000000E+00
14	1	-0.16546828E-07	0.27125379E-07
14	2	-0.38492401E-07	-0.43780286E-10
14	3	0.39164418E-07	0.20310608E-07
14	4	-0.79895326E-08	-0.59321157E-09
14	5	0.24567479E-07	-0.14832612E-07
14	6	-0.95222536E-08	0.72856795E-08
14	7	0.36671119E-07	-0.26792336E-08
14	8	-0.34592625E-07	-0.19095498E-07
14	9	0.34550116E-07	0.29056740E-07
14	10	0.36478662E-07	-0.31315939E-08

14	11	0.13804735E-07	-0.40430631E-07
14	12	0.78663794E-08	-0.32021215E-07
14	13	0.32021299E-07	0.45438329E-07
14	14	-0.51267444E-07	-0.50594096E-08
15	0	0.80777815E-09	0.00000000E+00
15	1	0.12217677E-07	0.11128027E-07
15	2	-0.25156244E-07	-0.28812382E-07
15	3	0.40168575E-07	0.26259721E-07
15	4	-0.43516908E-07	0.75970620E-08
15	5	0.10001336E-07	0.17570483E-07
15	6	0.25691563E-07	-0.51539931E-07
15	7	0.60775338E-07	0.15287728E-07
15	8	-0.33107615E-07	0.23560736E-07
15	9	0.11337024E-07	0.44530058E-07
15	10	0.90272628E-08	0.11380171E-07
15	11	0.21578428E-08	0.23386649E-07
15	12	-0.32173283E-07	0.87246056E-08
15	13	-0.29069917E-07	-0.41929657E-08
15	14	0.67593124E-08	-0.24725886E-07
15	15	-0.18982400E-07	-0.57180203E-08
16	0	-0.69674419E-08	0.00000000E+00
16	1	0.29762444E-07	0.24056477E-07
16	2	-0.14090720E-07	0.24868930E-07
16	3	-0.32614384E-07	-0.43991617E-07
16	4	0.39603946E-07	0.47384625E-07
16	5	-0.67955599E-08	0.18427955E-08
16	6	0.63436101E-08	-0.28497461E-07
16	7	-0.95300107E-09	-0.12637220E-07
16	8	-0.18206427E-07	0.30948957E-08
16	9	-0.18818589E-07	-0.38217071E-07

16	10	-0.12755829E-07	0.83993615E-08
16	11	0.18356918E-07	-0.43301942E-08
16	12	0.20003068E-07	0.53417125E-08
16	13	0.13469324E-07	0.11027380E-08
16	14	-0.19497010E-07	-0.38630934E-07
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30	14	0.23110867E-08	-0.14278791E-08
30	15	0.15724093E-08	-0.94383819E-08
30	16	-0.10942744E-08	0.17714132E-08
30	17	0.10170832E-08	0.53292523E-09
30	18	-0.71912599E-09	-0.76170867E-09
30	19	-0.56279331E-08	-0.21905613E-08
30	20	-0.41996988E-08	0.55677192E-08
30	21	-0.11771006E-07	-0.59538518E-08
30	22	-0.33342773E-08	-0.28411420E-08
30	23	-0.32034038E-08	-0.12062012E-08
30	24	-0.39236997E-08	-0.49783238E-09
30	25	0.54745310E-08	-0.48252642E-08
30	26	-0.35853392E-08	0.91820446E-08
30	27	-0.60250856E-08	0.11270688E-07
30	28	-0.12152854E-08	-0.15663863E-08
30	29	0.10563570E-08	0.55391012E-08
30	30	0.13675204E-08	0.46966008E-09
31	0	0.76192595E-08	0.00000000E+00
31	1	0.16403745E-08	-0.18644011E-08
31	2	0.48821085E-08	0.17028426E-08
31	3	0.26006006E-08	-0.30160639E-08
31	4	-0.27794109E-08	-0.89986332E-09
31	5	-0.29226564E-08	-0.20584988E-08
31	6	-0.22216409E-09	-0.41377404E-09
31	7	-0.13590974E-08	-0.91873345E-09
31	8	-0.18442681E-08	-0.23340156E-08
31	9	0.14198109E-08	-0.18379699E-08

31	10	0.36988335E-08	-0.48314042E-08
31	11	0.28309241E-09	0.84056410E-08
31	12	0.21424623E-09	0.23222394E-08
31	13	0.88911339E-08	0.28272096E-08
31	14	-0.11537941E-07	0.25878382E-08
31	15	-0.49724582E-09	-0.36711081E-08
31	16	-0.38187234E-08	0.15217096E-09
31	17	-0.82264316E-08	0.61006394E-08
31	18	0.58082460E-09	-0.10289827E-08
31	19	0.25790851E-08	0.21631482E-08
31	20	-0.26252502E-09	0.26803430E-08
31	21	-0.27361157E-08	0.55944573E-08
31	22	-0.72539614E-08	-0.79895124E-08
31	23	0.96422250E-08	0.94387292E-08
31	24	-0.38311754E-08	0.14092951E-08
31	25	-0.12019741E-07	-0.31623143E-08
31	26	-0.13746385E-07	-0.39036675E-08
31	27	-0.16460855E-08	0.66679623E-08
31	28	0.56474009E-08	0.34776765E-08
31	29	-0.27889768E-08	-0.44736234E-08
31	30	0.22103028E-08	0.35575482E-08
31	31	-0.33367360E-08	-0.27472820E-08
32	0	-0.27771389E-08	0.00000000E+00
32	1	-0.90313940E-08	-0.12788546E-08
32	2	-0.29572566E-08	0.40290672E-08
32	3	0.47848309E-09	0.23345913E-08
32	4	0.24109664E-08	-0.40055677E-08
32	5	-0.31226603E-09	-0.31440438E-08
32	6	-0.45792557E-08	0.13562585E-08
32	7	-0.25447419E-08	0.43308093E-08

32	8	0.13931823E-08	0.28456157E-08
32	9	-0.99747191E-09	0.64412499E-10
32	10	0.10293338E-08	-0.31557064E-08
32	11	-0.46493344E-08	-0.24289795E-08
32	12	-0.43837438E-08	0.61777853E-08
32	13	0.94404856E-08	0.23561402E-08
32	14	0.43713497E-08	0.94319682E-08
32	15	0.50396041E-08	-0.40358629E-08
32	16	0.46017953E-08	0.36466599E-08
32	17	-0.10429059E-07	0.39701515E-08
32	18	0.48077086E-08	-0.18534929E-08
32	19	0.51994923E-08	-0.33663340E-08
32	20	0.73649603E-09	-0.32606590E-09
32	21	-0.17671937E-08	0.87412552E-08
32	22	-0.32033107E-08	-0.21634488E-08
32	23	0.17026247E-08	-0.29883533E-09
32	24	-0.38089947E-08	0.14151100E-08
32	25	-0.15137520E-07	0.49203186E-08
32	26	-0.11721419E-08	-0.24841620E-08
32	27	-0.34561919E-08	-0.45753379E-08
32	28	0.61368102E-08	0.13247260E-08
32	29	-0.55609110E-09	0.48120177E-08
32	30	0.44685775E-08	-0.95183284E-09
32	31	-0.74715979E-09	0.64653561E-09
32	32	0.84059624E-09	-0.93473786E-09
33	0	0.51093253E-08	0.00000000E+00
33	1	-0.12149713E-08	0.43943928E-09
33	2	-0.47376262E-08	0.25501496E-08
33	3	-0.33878721E-08	-0.80388230E-09
33	4	0.25399097E-08	0.34818242E-09

33	5	-0.26538144E-08	0.28220750E-08
33	6	-0.26597001E-09	-0.40241622E-08
33	7	-0.10350151E-08	0.13572287E-09
33	8	-0.23213274E-08	0.12361935E-08
33	9	-0.79210705E-10	0.19664622E-08
33	10	0.66245034E-10	0.64304550E-10
33	11	0.38344470E-08	-0.14330538E-08
33	12	0.34476303E-08	0.37404590E-08
33	13	0.31546646E-08	0.65764483E-08
33	14	0.33425248E-08	0.46783306E-09
33	15	-0.35965693E-08	0.20240954E-08
33	16	0.14492704E-08	0.12667778E-08
33	17	-0.40567620E-09	0.55681226E-08
33	18	-0.47096180E-08	-0.22672311E-08
33	19	0.61747179E-08	-0.56761134E-09
33	20	0.21108563E-08	-0.11466619E-08
33	21	0.14492665E-08	0.70413395E-09
33	22	-0.11824474E-07	-0.33280060E-08
33	23	0.19180993E-08	-0.33511996E-08
33	24	0.61512096E-08	-0.84360721E-09
33	25	0.11943460E-08	-0.10979681E-07
33	26	0.92989975E-08	-0.14859889E-08
33	27	-0.27449912E-08	0.21797197E-08
33	28	-0.27311200E-08	0.12972504E-08
33	29	-0.15760081E-07	0.30203015E-08
33	30	0.49395445E-08	-0.20281755E-07
33	31	0.25623402E-08	-0.47244740E-09
33	32	0.42550616E-08	0.41997068E-08
33	33	-0.12968597E-08	-0.10797466E-08
34	0	-0.57588303E-08	0.00000000E+00

34	1	-0.33414485E-08	0.56753685E-09
34	2	0.37065126E-08	0.38544022E-08
34	3	-0.13019276E-08	0.35500052E-08
34	4	0.42592905E-08	-0.33682532E-08
34	5	-0.64546460E-09	0.17484852E-08
34	6	0.21435602E-09	-0.12583338E-08
34	7	0.34242762E-08	0.11937261E-08
34	8	0.44469619E-09	-0.71688027E-09
34	9	-0.10896647E-09	0.12859885E-09
34	10	-0.40002376E-08	-0.18077981E-08
34	11	0.19072761E-08	-0.29263704E-08
34	12	0.67673277E-09	0.46956808E-08
34	13	-0.61985707E-08	0.42769731E-08
34	14	-0.34999592E-08	0.25978189E-08
34	15	0.36656651E-08	0.20311531E-08
34	16	0.53231482E-08	-0.11762108E-08
34	17	-0.36912981E-08	0.53904652E-08
34	18	-0.23261398E-08	-0.21200182E-08
34	19	0.21380683E-08	-0.15243883E-08
34	20	0.44075446E-08	-0.23144000E-08
34	21	0.26418229E-08	-0.13922496E-08
34	22	0.31804205E-08	0.29674538E-08
34	23	-0.14496844E-08	-0.41273873E-08
34	24	0.73031171E-08	0.38279677E-08
34	25	0.87043354E-08	-0.46267863E-08
34	26	0.10342814E-08	-0.11663136E-07
34	27	0.11370770E-07	-0.23310682E-08
34	28	0.26300672E-08	-0.15192068E-07
34	29	0.15414895E-08	-0.50391923E-08
34	30	-0.99882783E-08	-0.22579711E-08

34	31	0.24132580E-08	0.11637732E-08
34	32	-0.16884916E-08	-0.15230092E-08
34	33	0.31091915E-08	0.31186708E-08
34	34	-0.42763338E-10	-0.37042016E-09
35	0	0.78140709E-08	0.00000000E+00
35	1	-0.36385091E-08	0.22889240E-08
35	2	-0.37194650E-08	0.18562674E-08
35	3	-0.68583570E-10	0.21049570E-08
35	4	0.32155058E-08	0.37217157E-09
35	5	0.11993433E-08	-0.40829291E-09
35	6	-0.49340501E-09	-0.18727810E-08
35	7	0.38916829E-09	-0.10328130E-08
35	8	-0.39349558E-09	0.21223159E-09
35	9	-0.14617120E-08	-0.22985099E-08
35	10	-0.12013829E-08	0.68667386E-09
35	11	-0.13492819E-08	-0.58743271E-08
35	12	0.87059592E-09	-0.36949735E-08
35	13	-0.14376756E-08	0.50513549E-08
35	14	-0.67126612E-08	-0.71066687E-09
35	15	-0.24510852E-08	0.53901362E-08
35	16	0.21675012E-08	0.22862779E-08
35	17	0.64255214E-08	-0.48058960E-08
35	18	0.36589693E-09	-0.20174921E-08
35	19	-0.31480773E-08	0.14427499E-09
35	20	-0.52647288E-09	-0.66312429E-09
35	21	0.57826433E-08	0.22280867E-08
35	22	0.13101644E-08	0.42641191E-08
35	23	-0.52468531E-08	-0.24676198E-08
35	24	0.36354954E-08	0.30476082E-08
35	25	0.28269199E-08	0.24094585E-08

35	26	-0.59897058E-08	-0.10138420E-08
35	27	0.12339753E-07	-0.11902514E-07
35	28	0.26951940E-08	-0.14395718E-07
35	29	0.87215303E-08	0.14403586E-10
35	30	-0.10179106E-08	0.77066029E-09
35	31	0.31634884E-08	0.58545933E-08
35	32	-0.85880053E-08	-0.66387592E-08
35	33	0.35673813E-08	0.74216103E-08
35	34	-0.23872702E-08	-0.20956762E-08
35	35	0.42637983E-09	0.29389224E-09
36	0	-0.47918308E-08	0.00000000E+00
36	1	-0.99445727E-10	0.30538768E-08
36	2	0.19970045E-08	-0.58078790E-09
36	3	-0.33624318E-08	0.44238665E-09
36	4	0.21730027E-08	-0.58078917E-09
36	5	-0.81620587E-09	0.17293874E-08
36	6	-0.52218190E-09	-0.25178137E-08
36	7	0.31298424E-09	-0.73549557E-09
36	8	-0.17376981E-08	0.60802581E-09
36	9	-0.29007396E-09	-0.39043954E-09
36	10	-0.16218821E-08	-0.19381700E-08
36	11	0.59042928E-09	0.13283648E-08
36	12	-0.12515391E-08	0.27424158E-09
36	13	-0.14462633E-08	0.37767969E-08
36	14	-0.34693609E-08	-0.34282737E-08
36	15	0.21794520E-08	-0.13811280E-08
36	16	0.17441111E-08	-0.23831942E-09
36	17	0.29224034E-08	-0.13638775E-08
36	18	0.30582160E-10	0.16655299E-09
36	19	-0.69952820E-09	0.77208428E-09

36	20	-0.19394269E-08	-0.20748737E-09
36	21	0.23105800E-08	-0.43550635E-08
36	22	-0.45541772E-09	0.16395183E-08
36	23	-0.20414901E-08	-0.15927978E-08
36	24	0.34765079E-09	-0.29341497E-08
36	25	0.91178048E-09	0.90987939E-08
36	26	0.30418097E-08	0.64559344E-08
36	27	-0.84166488E-08	0.53536590E-08
36	28	0.14082796E-08	-0.17119633E-08
36	29	0.13412224E-08	-0.91713750E-09
36	30	-0.64072900E-08	0.12875954E-08
36	31	-0.21613715E-08	-0.87760559E-09
36	32	0.19452455E-09	0.28940299E-08
36	33	-0.40936686E-08	-0.58626914E-08
36	34	-0.21684159E-08	0.18188953E-08
36	35	0.46852355E-09	-0.22694319E-08
36	36	0.38624235E-09	0.36811811E-09

APPENDIX B

Proof That V Satisfies Laplace Equation

APPENDIX B

Proof that V satisfies Laplace equation

This is the proof that V satisfies Laplace equation (see section 3.3)

$$F = \frac{1}{l} = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)}}$$

$$\frac{\partial F}{\partial x} = \frac{-(x-x_0)}{((x-x_0)^2 + (y-y_0)^2 + (z-z_0))^{3/2}}$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{3(x-x_0)^2 - ((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2)}{((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2)^{5/2}}$$

$$= \frac{3(x-x_0)^2 - l^2}{l^5}$$

$$\frac{\partial^2 F}{\partial y^2} = \frac{3(y-y_0)^2 - l^2}{l^5}$$

$$\frac{\partial^2 F}{\partial z^2} = \frac{3(z-z_0)^2 - l^2}{l^5}$$

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = 0$$

$$\Rightarrow \frac{3(x-x_0)^2 - l^2 + 3(y-y_0)^2 - l^2 + 3(z-z_0)^2 - l}{l^5} = \frac{3l^2 - 3l^2}{l^5} = 0$$

APPENDIX C

Software Code

APPENDIX C

Software Code

C.1 Software Cods For HRS Parameters:

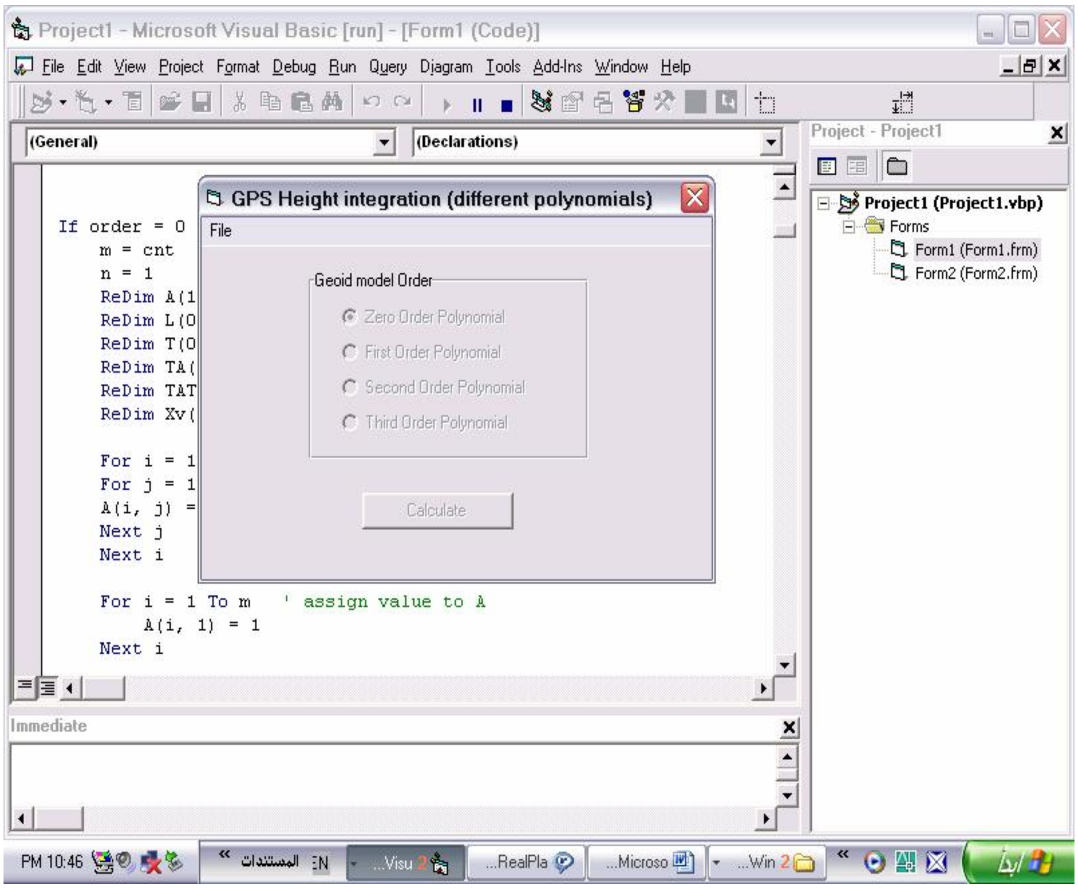

Figure C-1: Software Cods For HRS Parameters

```
Dim A(), Xv() As Double
Dim order As Integer
Dim x(0 To 1000), y(0 To 1000), he(0 To 1000), H(0 To
1000) As Double
Dim L(), T(), TA(), TAT() As Double
Dim ptnr As Integer
Dim cnt, m, n, j, k As Integer

Private Sub Command1_Click()

Form2.Text1.Visible = False
Form2.Text2.Visible = False
Form2.Text3.Visible = False
Form2.Text4.Visible = False
Form2.Text5.Visible = False
Form2.Text6.Visible = False
Form2.Text7.Visible = False
Form2.Text8.Visible = False
Form2.Text9.Visible = False
Form2.Text10.Visible = False
Form2.Label2.Visible = False
Form2.Label3.Visible = False
Form2.Label4.Visible = False
Form2.Label5.Visible = False
Form2.Label6.Visible = False
Form2.Label7.Visible = False
Form2.Label8.Visible = False
Form2.Label9.Visible = False
Form2.Label10.Visible = False
Form2.Label11.Visible = False
```

```
If order = 0 Then     'zero order start
    m = cnt
    n = 1
    ReDim A(1 To m, 0 To 1)
    ReDim L(0 To 2, 0 To m)
    ReDim T(0 To 1, 0 To m)
    ReDim TA(0 To 1, 0 To 1)
    ReDim TAT(0 To 1, 0 To m)
    ReDim Xv(0 To 1, 0 To 1)

    For i = 1 To m          ' make A= zeros
    For j = 1 To 1
    A(i, j) = 0
    Next j
    Next i

    For i = 1 To m   ' assign value to A
        A(i, 1) = 1
    Next i

'    L matrix
     For i = 1 To m
            L(1, i) = he(i) - H(i)
     Next i

 '   A transpose
    For i = 1 To 1
        For j = 1 To m
        T(i, j) = A(j, i)
        Next j
```

```
    Next i

 '   A transpose * A
    For i = 1 To 1
        For j = 1 To 1
        TA(i, j) = #0
        For k = 1 To m
        TA(i, j) = TA(i, j) + T(i, k) * A(k, j)
        Next k
        Next j
    Next i

 '    At*A inverse
    For k = 1 To 1
    For j = 1 To 1
        If j <> k Then TA(k, j) = TA(k, j) / TA(k, k)
    Next j
    TA(k, k) = 1 / TA(k, k)
    For i = 1 To 1
    If i <> k Then
      For j = 1 To 1
        If j <> k Then TA(i, j) = TA(i, j) - TA(i, k) *
TA(k, j)
        Next j
        TA(i, k) = -TA(i, k) * TA(k, )
        End If
    Next i
    Next k

    For i = 1 To 1
```

```
        For j = 1 To m
        TAT(i, j) = 0
        For k = 1 To 1
        TAT(i, j) = TAT(i, j) + TA(i, k) * T(k, j)
        Next k
        Next j
    Next i

 ''    inv(A transpose * A) *L
    For i = 1 To 1
       Xv(1, i) = 0
        For k = 1 To m
       Xv(1, i) = Xv(1, i) + TAT(i, k) * L(1, k)
       Next k
    Next i

End If    ' zero order end


If order = 1 Then     'First order start
    m = cnt
    n = 3
    ReDim A(1 To m, 0 To 3)
    ReDim L(0 To 2, 0 To m)
    ReDim T(0 To 3, 0 To m)
    ReDim TA(0 To 3, 0 To 3)
    ReDim TAT(0 To 3, 0 To m)
    ReDim Xv(0 To 1, 0 To 3)

    For i = 1 To m          ' make A= zeros
```

```
    For j = 1 To n
    A(i, j) = 0
    Next j
    Next i

    For i = 1 To m   ' assign value to A
        A(i, 1) = 1
        A(i, 2) = x(i)
        A(i, 3) = y(i)
    Next i

'    L matrix
     For i = 1 To m
            L(1, i) = he(i) - H(i)
     Next i

 '   A transpose
    For i = 1 To n
        For j = 1 To m
        T(i, j) = A(j, i)
        Next j
    Next i

 '   A transpose * A
    For i = 1 To n
        For j = 1 To n
        TA(i, j) = #0
        For k = 1 To m
        TA(i, j) = TA(i, j) + T(i, k) * A(k, j)
        Next k
```

```
        Next j
    Next i

 '    At*A inverse
    For k = 1 To n
    For j = 1 To n
        If j <> k Then TA(k, j) = TA(k, j) / TA(k, k)
    Next j
    TA(k, k) = 1 / TA(k, k)
    For i = 1 To n
    If i <> k Then
      For j = 1 To n
        If j <> k Then TA(i, j) = TA(i, j) - TA(i, k) *
TA(k, j)
        Next j
        TA(i, k) = -TA(i, k) * TA(k, k)
        End If
    Next i
    Next k

    For i = 1 To n
        For j = 1 To m
        TAT(i, j) = 0
        For k = 1 To n
        TAT(i, j) = TAT(i, j) + TA(i, k) * T(k, j)
        Next k
        Next j
    Next i

 ''    inv(A transpose * A) *L
```

```
    For i = 1 To n
       Xv(1, i) = 0
        For k = 1 To m
       Xv(1, i) = Xv(1, i) + TAT(i, k) * L(1, k)
       Next k
    Next i

End If   ' First order end

If order = 2 Then     '2nd order start
    m = cnt
    n = 6
    ReDim A(1 To m, 0 To n)
    ReDim L(0 To 2, 0 To m)
    ReDim T(0 To n, 0 To m)
    ReDim TA(0 To n, 0 To n)
    ReDim TAT(0 To n, 0 To m)
    ReDim Xv(0 To 1, 0 To n)

    For i = 1 To m          ' make A= zeros
    For j = 1 To n
    A(i, j) = 0
    Next j
    Next i

    For i = 1 To m   ' assign value to A
        A(i, 1) = 1
        A(i, 2) = x(i) / 1000
        A(i, 3) = y(i) / 1000
        A(i, 4) = (x(i) ^ 2) / 1000000
```

```
        A(i, 5) = (y(i) ^ 2) / 1000000
        A(i, 6) = (x(i) * y(i)) / 1000000
    Next i

'    L matrix
     For i = 1 To m
           L(1, i) = he(i) - H(i)
     Next i
 '        A transpose
    For i = 1 To n
        For j = 1 To m
        T(i, j) = A(j, i)
        Next j
    Next i

 '   A transpose * A
    For i = 1 To n
        For j = 1 To n
        TA(i, j) = #0
        For k = 1 To m
        TA(i, j) = TA(i, j) + T(i, k) * A(k, j)
        Next k
        Next j
    Next i

 '    At*A inverse
    For k = 1 To n
    For j = 1 To n
        If j <> k Then TA(k, j) = TA(k, j) / TA(k, k)
    Next j
```

```
    TA(k, k) = 1 / TA(k, k)
    For i = 1 To n
    If i <> k Then
      For j = 1 To n
        If j <> k Then TA(i, j) = TA(i, j) - TA(i, k) *
TA(k, j)
        Next j
        TA(i, k) = -TA(i, k) * TA(k, k)
        End If
    Next i
    Next k

    For i = 1 To n
        For j = 1 To m
        TAT(i, j) = 0
        For k = 1 To n
        TAT(i, j) = TAT(i, j) + TA(i, k) * T(k, j)
        Next k
        Next j
    Next i

 ''    inv(A transpose * A) *L
    For i = 1 To n
       Xv(1, i) = 0
        For k = 1 To m
       Xv(1, i) = Xv(1, i) + TAT(i, k) * L(1, k)
       Next k
    Next i

End If    ' 2nd  order end
```

```
If order = 3 Then     '3d order start
    m = cnt
    n = 10
    ReDim A(1 To m, 0 To n)
    ReDim L(0 To 2, 0 To m)
    ReDim T(0 To n, 0 To m)
    ReDim TA(0 To n, 0 To n)
    ReDim TAT(0 To n, 0 To m)
    ReDim Xv(0 To 1, 0 To n)

    For i = 1 To m          ' make A= zeros
    For j = 1 To n
    A(i, j) = 0
    Next j
    Next i

    For i = 1 To m   ' assign value to A
        A(i, 1) = 1
        A(i, 2) = x(i) / 1000
        A(i, 3) = y(i) / 1000
        A(i, 4) = (x(i) ^ 2) / 1000000
        A(i, 5) = (y(i) ^ 2) / 1000000
        A(i, 6) = (x(i) * y(i)) / 1000000
        A(i, 7) = (x(i) ^ 3) / 1000000000
        A(i, 8) = (y(i) ^ 3) / 1000000000
        A(i, 9) = ((x(i) ^ 2) * y(i)) / 1000000000
        A(i, 10) = ((x(i) * (y(i) ^ 2))) / 1000000000

    Next i
```

```
'    L matrix
     For i = 1 To m
            L(1, i) = he(i) - H(i)
     Next i

 '   A transpose
    For i = 1 To n
        For j = 1 To m
        T(i, j) = A(j, i)
        Next j
    Next i

 '   A transpose * A
    For i = 1 To n
        For j = 1 To n
        TA(i, j) = #0
        For k = 1 To m
        TA(i, j) = TA(i, j) + T(i, k) * A(k, j)
        Next k
        Next j
    Next i

 '    At*A inverse
    For k = 1 To n
    For j = 1 To n
        If j <> k Then TA(k, j) = TA(k, j) / TA(k, k)
    Next j
    TA(k, k) = 1 / TA(k, k)
    For i = 1 To n
    If i <> k Then
```

```
      For j = 1 To n
        If j <> k Then TA(i, j) = TA(i, j) - TA(i, k) *
TA(k, j)
        Next j
        TA(i, k) = -TA(i, k) * TA(k, k)
        End If
    Next i
    Next k

    For i = 1 To n
        For j = 1 To m
        TAT(i, j) = 0
        For k = 1 To n
        TAT(i, j) = TAT(i, j) + TA(i, k) * T(k, j)
        Next k
        Next j
    Next i

 ''    inv(A transpose * A) *L
    For i = 1 To n
       Xv(1, i) = 0
        For k = 1 To m
       Xv(1, i) = Xv(1, i) + TAT(i, k) * L(1, k)
       Next k
    Next i

End If   ' 3d  order end

Form2.Show
```

```
If order = 0 Then
    Form2.Text1.Text = Xv(1, 1)
    Form2.Text1.Visible = True
    Form2.Label2.Visible = True
    Form2.Label12.Visible = True
    Form2.Label13.Visible = False
    Form2.Label14.Visible = False
    Form2.Label15.Visible = False

    End If

If order = 1 Then
        Form2.Text1.Text = Xv(1, 1)
        Form2.Text1.Visible = True
        Form2.Text2.Text = Xv(1, 2)
        Form2.Text2.Visible = True
        Form2.Text3.Text = Xv(1, 3)
        Form2.Text3.Visible = True
        Form2.Label2.Visible = True
        Form2.Label3.Visible = True
        Form2.Label4.Visible = True
        Form2.Label12.Visible = False
        Form2.Label13.Visible = True
        Form2.Label14.Visible = False
        Form2.Label15.Visible = False
End If

If order = 2 Then
        Form2.Text1.Text = Xv(1, 1)
```

```
        Form2.Text1.Visible = True
        Form2.Text2.Text = Xv(1, 2)
        Form2.Text2.Visible = True
        Form2.Text3.Text = Xv(1, 3)
        Form2.Text3.Visible = True
        Form2.Text4.Text = Xv(1, 4)
        Form2.Text4.Visible = True
        Form2.Text5.Text = Xv(1, 5)
        Form2.Text5.Visible = True
        Form2.Text6.Text = Xv(1, 6)
        Form2.Text6.Visible = True
        Form2.Label2.Visible = True
        Form2.Label3.Visible = True
        Form2.Label4.Visible = True
        Form2.Label5.Visible = True
        Form2.Label6.Visible = True
        Form2.Label7.Visible = True
        Form2.Label12.Visible = False
        Form2.Label13.Visible = False
        Form2.Label14.Visible = True
        Form2.Label15.Visible = False
End If

If order = 3 Then
    Form2.Text1.Text = Xv(1, 1)
    Form2.Text1.Visible = True
    Form2.Text2.Text = Xv(1, 2)
    Form2.Text2.Visible = True
    Form2.Text3.Text = Xv(1, 3)
    Form2.Text3.Visible = True
```

```
    Form2.Text4.Text = Xv(1, 4)
    Form2.Text4.Visible = True
    Form2.Text5.Text = Xv(1, 5)
    Form2.Text5.Visible = True
    Form2.Text6.Text = Xv(1, 6)
    Form2.Text6.Visible = True
    Form2.Text7.Text = Xv(1, 7)
    Form2.Text7.Visible = True
    Form2.Text8.Text = Xv(1, 8)
    Form2.Text8.Visible = True
    Form2.Text9.Text = Xv(1, 9)
    Form2.Text9.Visible = True
    Form2.Text10.Text = Xv(1, 10)
    Form2.Text10.Visible = True
    Form2.Label2.Visible = True
    Form2.Label3.Visible = True
    Form2.Label4.Visible = True
    Form2.Label5.Visible = True
    Form2.Label6.Visible = True
    Form2.Label7.Visible = True
    Form2.Label8.Visible = True
    Form2.Label9.Visible = True
    Form2.Label10.Visible = True
    Form2.Label11.Visible = True
    Form2.Label12.Visible = False
    Form2.Label13.Visible = False
    Form2.Label14.Visible = False
    Form2.Label15.Visible = True
End If
```

```
parsave.Enabled = True

End Sub

Private Sub ex_Click()
End
End Sub

Private Sub Form_Load()
order = 0
Option1.Value = True
Command1.Enabled = False

Form2.Text1.Visible = False
Form2.Text2.Visible = False
Form2.Text3.Visible = False
Form2.Text4.Visible = False
Form2.Text5.Visible = False
Form2.Text6.Visible = False
Form2.Text7.Visible = False
Form2.Text8.Visible = False
Form2.Text9.Visible = False
Form2.Text10.Visible = False

            Option1.Enabled = False
            Option2.Enabled = False
            Option3.Enabled = False
            Option4.Enabled = False

End Sub
```

```
Private Sub Option1_Click()
If Option1.Value = True Then
order = 0
   If cnt < 3 Then
            Option1.Enabled = True
            Option2.Enabled = False
            Option3.Enabled = False
            Option4.Enabled = False
            End If

            If cnt < 1 Then
            Option1.Enabled = False
            Option2.Enabled = False
            Option3.Enabled = False
            Option4.Enabled = False
            End If
End If

End Sub
Private Sub Option2_Click()
If Option2.Value = True Then
order = 1
 If cnt < 6 Then
            Option1.Enabled = True
            Option2.Enabled = True
            Option3.Enabled = False
            Option4.Enabled = False
            End If

            If cnt < 3 Then
```

```
            Option1.Enabled = True
            Option2.Enabled = False
            Option3.Enabled = False
            Option4.Enabled = False
            End If

            If cnt < 1 Then
            Option1.Enabled = False
            Option2.Enabled = False
            Option3.Enabled = False
            Option4.Enabled = False
            End If

End If

End Sub
Private Sub Option3_Click()
If Option3.Value = True Then
order = 2
            If cnt < 10 Then
            Option1.Enabled = True
            Option2.Enabled = True
            Option3.Enabled = True
            Option4.Enabled = False
            End If

            If cnt < 6 Then
            Option1.Enabled = True
            Option2.Enabled = True
            Option3.Enabled = False
```

```
            Option4.Enabled = False
            End If

            If cnt < 3 Then
            Option1.Enabled = True
            Option2.Enabled = False
            Option3.Enabled = False
            Option4.Enabled = False
            End If

            If cnt < 1 Then
            Option1.Enabled = False
            Option2.Enabled = False
            Option3.Enabled = False
            Option4.Enabled = False
            End If

End If

End Sub
Private Sub Option4_Click()
If Option4.Value = True Then
order = 3
       If cnt >= 10 Then
            Option1.Enabled = True
            Option2.Enabled = True
            Option3.Enabled = True
            Option4.Enabled = True
            End If
```

```
            If cnt < 10 Then
            Option1.Enabled = True
            Option2.Enabled = True
            Option3.Enabled = True
            Option4.Enabled = False
            End If

            If cnt < 6 Then
            Option1.Enabled = True
            Option2.Enabled = True
            Option3.Enabled = False
            Option4.Enabled = False
            End If

            If cnt < 3 Then
            Option1.Enabled = True
            Option2.Enabled = False
            Option3.Enabled = False
            Option4.Enabled = False
            End If

            If cnt < 1 Then
            Option1.Enabled = False
            Option2.Enabled = False
            Option3.Enabled = False
            Option4.Enabled = False
            End If

End If
```

```
End Sub

Private Sub osf_Click()
CommonDialog1.Filter = "Geoid Data Source File
"|*.*|(*.*)
CommonDialog1.ShowOpen

If CommonDialog1.FileName <> "" Then
            Open CommonDialog1.FileName For Input As #1
            Command1.Enabled = True

            cnt = 0
            Do Until EOF(1)
                      cnt = cnt + 1
 Input #1, ptnr, x(cnt), y(cnt), he(cnt), H(cnt)

                  Loop
            Close #1

            If cnt >= 10 Then
            Option1.Enabled = True
            Option2.Enabled = True
            Option3.Enabled = True
            Option4.Enabled = True
            End If

            If cnt < 10 Then
            Option1.Enabled = True
            Option2.Enabled = True
            Option3.Enabled = True
```

```
            Option4.Enabled = False
            End If

            If cnt < 6 Then
            Option1.Enabled = True
            Option2.Enabled = True
            Option3.Enabled = False
            Option4.Enabled = False
            End If

            If cnt < 3 Then
            Option1.Enabled = True
            Option2.Enabled = False
            Option3.Enabled = False
            Option4.Enabled = False
            End If

            If cnt < 1 Then
            Option1.Enabled = False
            Option2.Enabled = False
            Option3.Enabled = False
            Option4.Enabled = False
            End If
End If

End Sub

Private Sub parsave_Click()
CommonDialog2.Filter = "Geoid Parameters (*.HRS)|*.HRS"|
CommonDialog2.ShowSave
```

```
If CommonDialog2.FileName <> "" Then
            Open CommonDialog2.FileName For Output As #2

                If order = 0 Then
Print #2, order, Xv(1, 1)
                End If

                If order = 1 Then
 Print #2, order, Xv(1, 1), Xv(1, 2), Xv(1, 3)
                End If

                If order = 2 Then
 Print #2, order, Xv(1, 1), Xv(1, 2), Xv(1, 3), Xv(1, 4),
Xv(1, 5), Xv(1, 6)
                End If

                If order = 3 Then
Print #2, order, Xv(1, 1), Xv(1, 2), Xv(1, 3), Xv(1, 4),
Xv(1, 5), Xv(1, 6), Xv(1, 7), Xv(1, 8), Xv(1, 9), Xv(1,
10)
                End If


            Close #2

End If

End Sub
```

C.2 Software Cods For N Calculations:

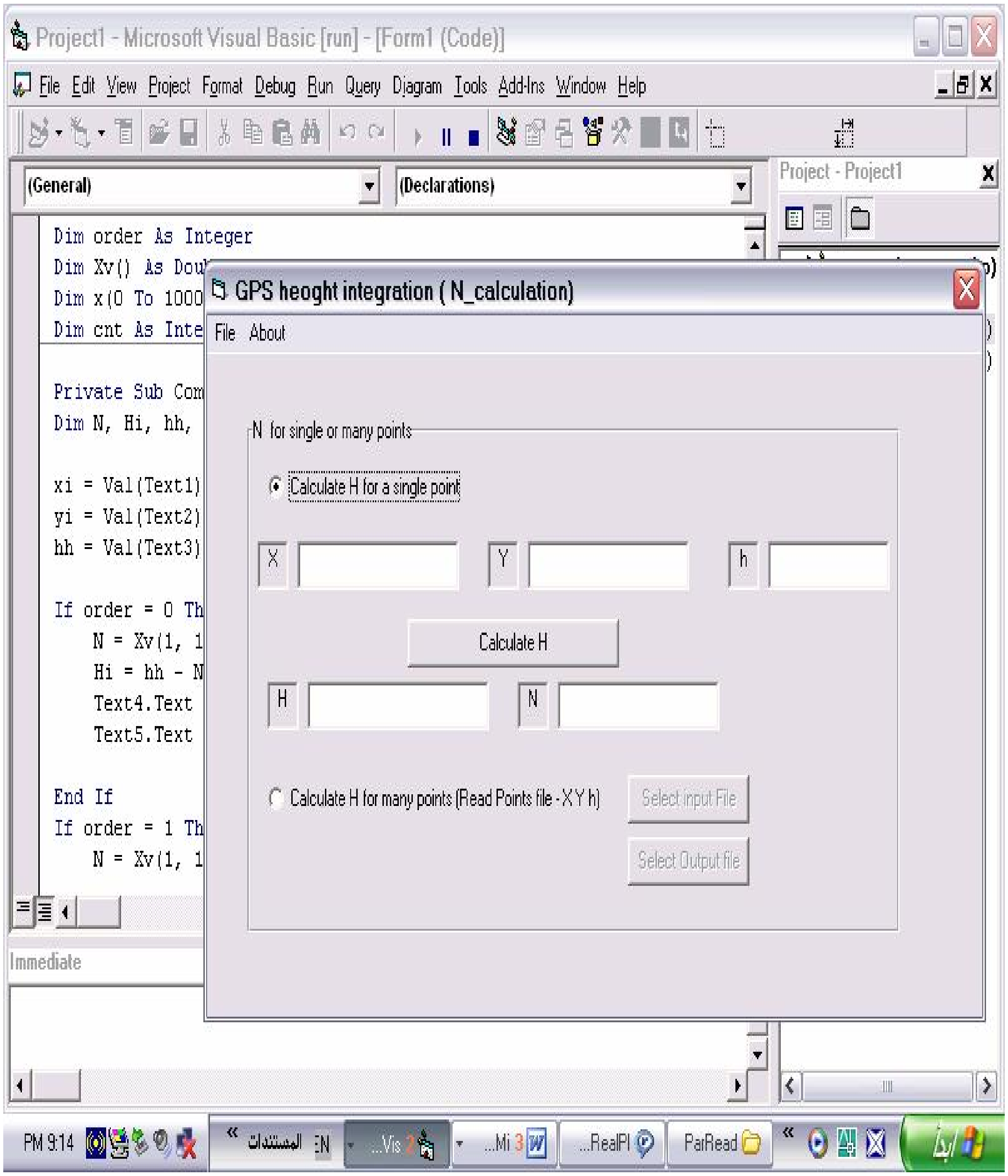

Figure C-2: Software Cods For N Calculations

```
Dim order As Integer
Dim Xv() As Double
Dim x(0 To 10000), y(0 To 10000), he(0 To 10000), ptnr(1
To 10000) As Double
Dim cnt As Integer

Private Sub Command1_Click()
Dim N, Hi, hh, xi, yi As Double

xi = Val(Text1)
yi = Val(Text2)
hh = Val(Text3)

If order = 0 Then
    N = Xv(1, 1)
    Hi = hh - N
    Text4.Text = Hi
    Text5.Text = N

End If
If order = 1 Then
    N = Xv(1, 1) + Xv(1, 2) * xi + Xv(1, 3) * yi
    Hi = hh - N
    Text4.Text = Hi
    Text5.Text = N
End If

If order = 2 Then
```

```
    N = Xv(1, 1) + Xv(1, 2) * xi / 1000 + Xv(1, 3) * yi /
1000 + (Xv(1, 4) * xi ^ 2) / 1000000 + (Xv(1, 5) * yi ^
2) / 1000000 + (Xv(1, 6) * xi * yi) / 1000000
    Hi = hh - N
    Text4.Text = Hi
    Text5.Text = N
End If

If order = 3 Then
    N = Xv(1, 1) + Xv(1, 2) * xi / 1000 + Xv(1, 3) * yi /
1000 + (Xv(1, 4) * xi ^ 2) / 1000000 + (Xv(1, 5) * yi ^
2) / 1000000 + (Xv(1, 6) * xi * yi) / 1000000 + (Xv(1, 7)
* xi ^ 3) / 1000000000 + (Xv(1, 8) * yi ^ 3) / 1000000000
+ (Xv(1, 9) * xi ^ 2 * yi) / 1000000000 + (Xv(1, 10) * xi
* yi ^ 2) / 1000000000
    Hi = hh - N
    Text4.Text = Hi
    Text5.Text = N
End If
End Sub

Private Sub Command2_Click()
CommonDialog2.Filter = "Pt# X Y h"|*.*|(*.*)
CommonDialog2.ShowOpen

If CommonDialog2.FileName <> "" Then
            Open CommonDialog2.FileName For Input As #2
            Command3.Enabled = True
            cnt = 0
           Do Until EOF(2)
```

```
                      cnt = cnt + 1
                     Input #2, ptnr(cnt), x(cnt), y(cnt),
he(cnt)

                  Loop
            Close #2

End If

End Sub

Private Sub Command3_Click()
Dim N, Hi, hh, xi, yi As Double
Dim i As Integer

CommonDialog3.Filter = "Pt# X Y h (*.txt)|*.txt"|
CommonDialog3.ShowSave

If CommonDialog3.FileName <> "" Then
        Open CommonDialog3.FileName For Output As #3

        For i = 1 To cnt
                xi = x(i)
                yi = y(i)
                hh = he(i)

                If order = 0 Then
                    N = Xv(1, 1)
                    Hi = hh - N
```

```
                End If
                If order = 1 Then
                    N = Xv(1, 1) + Xv(1, 2) * xi + Xv(1, 
3) * yi
                    Hi = hh - N

                End If

                If order = 2 Then
                    N = Xv(1, 1) + Xv(1, 2) * xi / 1000 + 
Xv(1, 3) * yi / 1000 + (Xv(1, 4) * xi ^ 2) / 1000000 + 
(Xv(1, 5) * yi ^ 2) / 1000000 + (Xv(1, 6) * xi * yi) / 
1000000
                    Hi = hh - N

                End If

                If order = 3 Then
                    N = Xv(1, 1) + Xv(1, 2) * xi / 1000 + 
Xv(1, 3) * yi / 1000 + (Xv(1, 4) * xi ^ 2) / 1000000 + 
(Xv(1, 5) * yi ^ 2) / 1000000 + (Xv(1, 6) * xi * yi) / 
1000000 + (Xv(1, 7) * xi ^ 3) / 1000000000 + (Xv(1, 8) * 
yi ^ 3) / 1000000000 + (Xv(1, 9) * xi ^ 2 * yi) / 
1000000000 + (Xv(1, 10) * xi * yi ^ 2) / 1000000000
                    Hi = hh - N

                End If
         Print #3, ptnr(i), xi, yi, Hi
        Next i
```

```
End If
Close #3

End Sub

Private Sub Form_Load()
 Command2.Enabled = False
 Command3.Enabled = False
End Sub

Private Sub Option1_Click()
If Option1.Value = True Then
    Text1.Enabled = True
    Text2.Enabled = True
    Text3.Enabled = True
    Text4.Enabled = True
    Text5.Enabled = True
    Command1.Enabled = True
        Command2.Enabled = False
    Command3.Enabled = False
End If
If Option1.Value = False Then
    Text1.Enabled = False
    Text2.Enabled = False
    Text3.Enabled = False
    Text4.Enabled = False
    Text5.Enabled = False
    Command1.Enabled = False
End If
```

```
End Sub

Private Sub Option2_Click()
If Option2.Value = True Then
    Text1.Enabled = False
    Text2.Enabled = False
    Text3.Enabled = False
    Text4.Enabled = False
    Text5.Enabled = False
    Command1.Enabled = False
    Command2.Enabled = True

End If
End Sub

Private Sub paropen_Click()
CommonDialog1.Filter = "Geoid Parameters (*.HRS)|*.HRS"|
CommonDialog1.ShowOpen


If CommonDialog1.FileName <> "" Then
            Open CommonDialog1.FileName For Input As #1

                Input #1, order

                If order = 0 Then
                    ReDim Xv(0 To 1, 0 To 1)
                    Input #1, Xv(1, 1)

                End If
```

```
                If order = 1 Then
                    ReDim Xv(0 To 1, 0 To 3)
                    Input #1, Xv(1, 1), Xv(1, 2), Xv(1,
3)

                End If

                If order = 2 Then
                    ReDim Xv(0 To 1, 0 To 6)
                    Input #1, Xv(1, 1), Xv(1, 2), Xv(1,
3), Xv(1, 4), Xv(1, 5), Xv(1, 6)

                End If

                If order = 3 Then
                    ReDim Xv(0 To 1, 0 To 10)
                    Input #1, Xv(1, 1), Xv(1, 2), Xv(1,
3), Xv(1, 4), Xv(1, 5), Xv(1, 6), Xv(1, 7), Xv(1, 8),
Xv(1, 9), Xv(1, 10)

                End If


            Close #1

End If
End Sub
```