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Ideals In Skew Lattices

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Dedications

This work is dedicated to all who helped me and stood beside me in this life.

Walaa

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Abstract

This theses aims to develop a better understanding of skew lattice and its ideals. We present the definitions of lattice, skew lattice, ideals and filters.

Due to the importance of orders and ordered sets in the study of both lattice and skew lattice, the definitions of orders and ordered sets are presented, then we define a lattice in two ways and present the connection between them, furthermore, we discuss the concept of ideal and filter in a lattice. The algebraic definition of a skew lattice and some of its properties are introduced. We discuss the three Green's relations on a skew lattice. Also we study an order structure of a skew lattice, and the related results and the characterizations theorems are studied.

Furthermore we introduce the concept of ideal, filter, skew ideal and principal ideal in a skew lattice. The characterizations theorems for each of these concepts and the results connecting between them are discussed.

Contents

Introduction

In the first half of the nineteenth century, through George Booles research work on formalization of propositional logic, he introduced the concept of Boolean algebra. At the end of that century, Charles S. Pierce and Ernst Schroeder found it useful to introduce the concept of lattice while independently, Richard Dedekinds research led to the same concept. In fact, Dedekind submit two papers about the subject of lattice, but then the subject stayed recumbent and did not attracted the attention of mathematicians until the thirties of the twentieth century.

In the mid-thirties of the 20th century, the lattice theory was further developed by Garrett Birkhoff. In a series of Birkhoffs papers, it is demonstrated that the lattice theory provides a unified framework for unrelated developments in many mathematical disciplines. Birkhoff himself, V. Glivenko, Karl Menger, John von Neumann, Oystein Ore, George Gratzer and many authors have developed enough for this field.

In 1949, noncommutative lattices have studied by Pascual Jordan who was motivated by certain questions in quantum mechanics. It is later approached by Slavk and Cornish who refer to a special variety of noncommutative lattices, namely skew lattices. Skew lattices turned out to be the most fruitful class of noncommutative lattices and thus have received the most attention. A more general version of skew lattices can be found in Jonathan Leech's 1989 paper.

Joao Pita Costa [29] introduced the concept of ideal (filter) in a skew lattice. In his paper the concept of a skew ideal in a skew lattice was introduced. Also he introduced the concept of principal ideal in a skew lattice.

The material of this thesis lies in three chapters, each one contains basic definitions, examples and important theorems.
Chapter one: In this chapter we begin with basic definitions needed in this work. We introduce the concept of partially ordered set, preorder, totally ordered set, duality, lattice as partially ordered set, lattice as algebraic structure and sublattice. Examples and theorems are also introduced to illustrate these concepts. In addition special elements in partially ordered set such as least and greatest, minimal and maximal, upper and lower bounds are given. We learn how to represent any finite partially ordered set graphically. Finally, special subsets of lattices such as ideals, filters, and principal ideals are introduced.

Chapter two: In this chapter we introduce the concept of skew lattice. We present an algebraic structure of a skew lattice and derive its relation with the lattice. We discuss Greens relations on skew lattices and several theorems regarding $\mathcal{D}$ relation. Some properties of skew lattices such as rectangular, left- and right-handed, symmetric, normal, conormal, binormal, distributive, and cancellative will be studied. Also the two famous first and second decomposition theorems will be given. In addition Skew Boolean algebra are defined with some examples. We also present an order structure of a skew lattice. The notions of natural partial order and the natural preorder on a skew lattice will be introduced with characterizations theorems. Furthermore, representing the skew lattice geometrically will be illustrated.

Chapter three: In this chapter we talk about ideals in skew lattices. We give the definitions of an ideal, filter, and skew ideal of a skew lattice with examples and characterizations of each concept. Also the definition of principal ideal and principal skew ideal of a skew lattice will be introduced with theorems connecting between these concepts.

Chapter 1

Preliminaries

This chapter is devoted mainly to presenting some definitions and theorems that will be used in posterior chapters. The basic concepts introduced in this chapter are partially ordered set, lattice and ideal and filter in a lattice. Theorems and examples are discussed to illustrate these concepts.

1.1 Partially Ordered Sets

This section describes the basic theory of partially ordered sets. We give the definitions of a partially ordered set and totally ordered set. We present extreme elements in a partially ordered set. Also we illustrate how to present a partially ordered set graphically. Finally, the very important concept of duality will be introduced.

1.1.1 Basic Definitions

Definition 1.1.1. [3] Suppose that P is a set and that $\leq_P$ a binary relation on P. Then $\leq_P$ is a **partial order** if it is **reflexive**, **antisymmetric**, and **transitive**, i.e., for all a, b and c in P, we have that

P_1: $a \leq_P a$. (reflexivity)

P_2: if $a \leq_p b$ and $b \leq_p a$ then $a = b$. (antisymmetry)

P_3: if $a \leq_p b$ and $b \leq_p c$ then $a \leq_p c$. (transitivity)

A set P with a partial order $\leq_P$ on it is called a **partially ordered set** or simply a **poset**, and is denoted by $(P, \leq_P)$.

Example 1.1.1. *For any set X, consider the power set $P(X)$. The set inclusion relation $\subseteq$ on $P(X)$ is a partial order, and $(P(X), \subseteq)$ is a poset.*

Remark 1.1.1. Any subset of a poset is itself a poset under the same relation.

Remark 1.1.2. The symbol $\leq_P$ is read **related to**. The notation, $a <_P b$ means that $a \leq_P b$, but $a \neq b$. The relation $a \leq_P b$ is also written $b \geq_p a$.

Definition 1.1.2. [17] A relation $\leq_P$ on a set P which is reflexive and transitive but not necessarily antisymmetric is called a **quasiorder** or a **preorder**.

Example 1.1.2. *The proper subset relation $\subset$ on the power set $P(X)$ of a set X is a preorder.*

Remark 1.1.3. A pair of elements a, b are **comparable** if $a \leq_p b$ or $b \leq_p a$. Otherwise, a and b are incomparable, denoted by $a \parallel b$.

Definition 1.1.3. A poset $(P, \leq_P)$ is **totally ordered** if every $a, b \in P$ are comparable, that is $a \leq_P b$ or $b \leq_P a$.

A totally ordered set is also called a **chain**. In contrast, a poset $(P, \leq_P)$ is **antichain** if every distinct pair of elements is incomparable.

Example 1.1.3. *The divisibility relation* $|$ *on the set of natural numbers* $\mathbb{N} = \{1, 2, \ldots\}$ *defined by* $a \mid b$ *if and only if* a *divides* b *, is a partial order and* $(\mathbb{N}, |)$ *is a poset, but it is not a chain, since not every* $a, b \in \mathbb{N}$ *are comparable. For example,* $2 \parallel 3$*.*

Example 1.1.4. *The sets* $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ *of natural, integer, rational, and real numbers with the usual order relation* $\leq$ *(less than or equal) form chains.*

Example 1.1.5. *The set of prime numbers which is partially ordered by division relation is an antichain.*

1.1.2 Extreme Elements in Posets

Various types of extreme elements in a partially ordered set $(P, \leq_P)$ will be defined and examples will be introduced.

Definition 1.1.4. [1] Let $(P, \leq_P)$ be a poset, and $x \in P$. Then:

1. x is called a **minimal element** if $a \in P$ and $a \leq_P x$ implies that $a = x$.

2. x is called a **maximal element** if $a \in P$ and $x \leq_P$ a implies that $x = a$.

Example 1.1.6. *Let* $P = \{2, 4, 5, 10, 12\}$ *be partially ordered by divisibility* $|$*. Then* $|$ *is given by* $\{(2,2), (4,4), (5,5), (10,10), (12,12), (2,4), (2,10), (2,12), (4,12), (5,10)\}$. *It is clear that* 10 *and* 12 *are maximal elements; while,* 2 *and* 5 *are minimal elements.*

Definition 1.1.5. [2] Let $(P, \leq_P)$ be a partially ordered set.

1. An element $m \in P$ is called a **greatest element** of P if $p \leq_P m$ for all $p \in P$.

2. An element $n \in P$ is called a **least element** of P if $n \leq_P p$ for all $p \in P$.

The least and greatest elements of a poset are also called bottom and top and they are represented by the symbols $\perp$ and $\top$, respectivey. A poset that has both bottom and top is called a **bounded** poset.

Remark 1.1.4. A poset has at most one greatest element and at most one least element, i.e., if greatest and least elements exist, they are unique.

Example 1.1.7. *In the poset* $(P(X), \subseteq)$, *we have a top* $\top = X$ *and a bottom* $\perp = \emptyset$. *Thus,* $(P(X), \subseteq)$ *is a bounded poset.*

Example 1.1.8. *A finite chain always has bottom and top elements, but an infinite chain need not have. For example, the chain* $(\mathbb{N}, \leq)$ *has bottom element* 1, *but no top, while the chain* $(\mathbb{Z}, \leq)$ *have neither bottom nor top.*

Note that a least element must be minimal and a greatest element must be maximal, but the converse is not necessarily true. For instance, in Example 1.1.6, 10 is not a greatest element since $4 \nmid 10$ and 5 is not a least element since $5 \nmid 12$.

Definition 1.1.6. [1] Let $(P, \leq_P)$ be a partially ordered set and let $S \subseteq P$.

1 . An **upper bound** for S is an element $u \in P$ for wich

$$s \leq_P u \qquad \forall s \in S$$

The set of all upper bounds for S is denoted by S^u. If S^u has a least element, it is called the **join** or **least upper bound** or **supremum** of S and is denoted by $\bigvee S$. The join of a finite set $S = \{a_1, \ldots, a_n\}$ is denoted by

$$a_1 \vee \ldots \vee a_n$$

2 . A **lower bound** for S is an element l for which

$$l \leq_P s \qquad \forall s \in S$$

The set of all lower bounds for S is denoted by S^l. If S^l has a greatest element, it is called the **meet** or **greatest lower bound** or **infimum** of S and is denoted by $\bigwedge S$. The meet of a finite set $S = \{a_1, \ldots, a_n\}$ is denoted by

$$a_1 \wedge \ldots \wedge a_n.$$

The least upper bound and the greatest lower bound of S can be shortened by $l.u.b.(lub)$ and $g.l.b.(glb)$ respectively. If $l.u.b.$ and $g.l.b.$ exist, they are unique.

Example 1.1.9. *In the poset* $(\mathbb{N}, |)$ *the supremum of any two elements* a *and* b *or* $a \vee b$ *is the least common multiple of* a *and* b*. The infimum of* a *and* b *or* $a \wedge b$ *is their greatest common divisor.*

The upper and lower bounds may fail to exist, this can be seen in the following example.

Example 1.1.10. *Consider the poset* $(\mathbb{R}, \leq)$*. The subset* $\mathbb{Z} \subseteq \mathbb{R}$ *has no upper bound and no lower bound.*

1.1.3 Diagrams

Just as we represent sets, functions, and relations with diagrams to make them more understandable, we can represent (finite) partially ordered sets informatively with a particular diagram. To illustrate a diagrammatic representation of a poset, we need the idea of covering.

Definition 1.1.7. [20] In the poset $(P, \leq_P)$, a **covers** b or b is **covered by** a, in notation, $a \sqsupset b$ or $b \sqsubset a$ iff $a >_P b$ and for no x, $a >_P x >_P b$.

Observe that, if the poset $(P, \leq_P)$ is finite, $b \leq_P a$ if and only if there exists a finite sequence $x_1, x_2, \ldots, x_n$ of elements of P such that

$$b \sqsubset x_1 \sqsubset x_2 \sqsubset \ldots \sqsubset x_n \sqsubset a.$$

Thus, in the finite case, the partial order on P determines, and is determined by, the covering relation.

Now, using covering relation, we can get graphical representation of any finite poset P as follows:

- Elements of P are points in the plane.
- If $b <_P a$, then a is drawn above b.
- If $b \sqsubset a$, then a and b are connected by a line.

The resulting figure is called a **diagram** or a **Hasse diagram** of P.

Example 1.1.11. *Figure 1.1 shows the Hasse diagram of the poset* $\{1, 2, 3, 4, 5, 6, 7, 8\}$ *ordered by divisibility.*

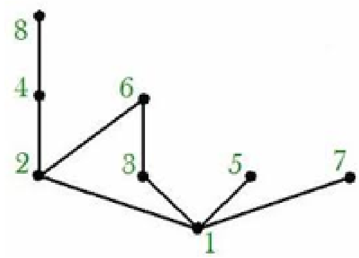

Figure 1.1:

Note that 1 *is a minimal and least element of the above poset and* $5, 6, 7, 8$ *are maximal elements. But there is no greatest element*

Example 1.1.12. *Figure 1.2 shows the Hasse diagram of the poset* $P(\{a, b, c\})$ *under set inclusion.*

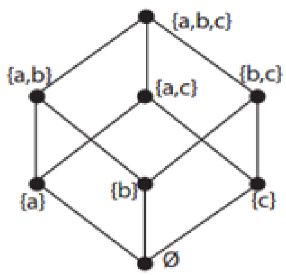

Figure 1.2:

Note that $\emptyset$ *is a minimal and least element of the above poset and* $\{a, b, c\}$ *is a maximal and greatest element.*

1.1.4 Duality

Given any partially ordered set P we can form a new partially ordered set P^∂ (the dual of P) by defining $x \leq_P y$ to hold in P^∂ if and only if $y \leq_P x$ holds in P (such partial order on P^∂ is called the converse partial order.

Definition 1.1.8. [28] The dual of a poset P is that poset P^∂ defined by the converse partial ordering relation on the same elements.

Let Φ be a statement about a poset P. One can define the *dual statement* to Φ to be the statement formed by replacing each occurrence of $\leq_P$ by $\geq_P$ and vice versa. Thus, least upper bound is replaced by greatest lower bound, maximal by minimal and so on.

If a statement holds for a poset $(P, \leq_P)$, its dual holds for the dual poset $(P^\partial, \geq_P)$. For example, x is an upper bound for the set K in the poset $(P, \leq_P)$ if and only if x is a lower bound for K in $(P^\partial, \geq_P)$. Similarly, u is a supremum of K in the poset $(P, \leq_P)$ if and only if u is an infimum of K in $(P^\partial, \geq_P)$. This allows us to formulate the following principle.

The Duality Principle for Posets: Given a statement Φ about posets which is true in all posets, then the dual statement Φ^∂ is true in all posets.
" The validity of this principle follows from the fact that any poset can be regarded as the dual of some other poset. The duality principle allows us to simplify proofs of certain statements that concern posets. For statements involving both a concept and its dual we need to prove only half of the statement; the other half follows by applying the duality principle. For instance, once we prove the statement "any subset of a poset can have at most one least upper bound," the dual statement "any subset of a poset can have at most one greatest lower bound" follows. " [18]

Remark 1.1.5. The two statements Φ and Ψ that are true in exactly the same posets (i.e. Φ is true in a poset P if and only if Ψ is true in the same poset P) are called **logically equivalent**. If a statement is logically equivalent to its dual, it is called a **self-dual**.

Note that the dual of the dual poset is the original poset $\left[\left(P^{\partial}\right)^{\partial} = P \right]$ and that the dual of the dual statement is the original statement also $\left[\left(\Phi^{\partial}\right)^{\partial} = \Phi \right]$.

For a finite poset P, we can get a Hasse diagram for P^{∂} by turning upside down a Hasse diagram for P. Figure 1.3 gives an example.

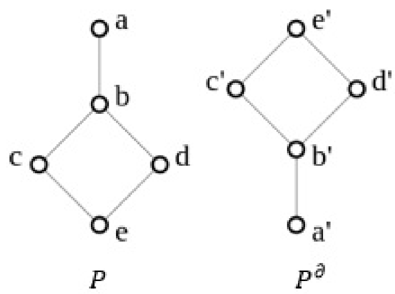

Figure 1.3:

1.2 Lattices

In this section the concepts of lattice and sublattice are introduced where lattices are defined in two ways, one as partially ordered sets and the other as algebraic structures. Theorems and examples are presented to illustrate these concepts. Furthermore, we discuss the concept of ideal and filter in a lattice.

1.2.1 Lattices as Partially Ordered Sets

Many important properties of a poset P are expressed in terms of the existence of certain upper bounds or lower bounds of subsets of P. One of the most important classes of posets defined in this way is lattices.

In partially ordered sets, the least upper bound $x \vee y$ or greatest lower bound $x \wedge y$ of $\{x, y\}$ may fail to exist for two different reasons, one of them is that x and y may have no common upper bound or have no common lower bound. The other reason is that x and y have no least upper bound or have no greatest lower bound.

A special structure arises when every pair of elements in a poset has both a least upper bound and a greatest lower bound.

Definition 1.2.1. [7] A **lattice** is a partially ordered set $(L, \leq_P)$ where every pair x, y in L has a supremum and an infimum (in L).

Note that the join (supremum) and meet (infimum) are called *lattice operations.*

Definition 1.2.2. [21] A lattice is **trivial** if it has only one element; otherwise it is **nontrivial**.

Example 1.2.1. *Every chain is a lattice. For a two elements x and y, without loss of generality, say $x \leq_P y$, then $x \wedge y = x$ and $x \vee y = y$. For example, the set $\mathbb{Z}$ of integers with the usual order relation is a lattice, where for all x and y in $\mathbb{Z}$*

$$x \wedge y = min(x, y) \quad \text{and} \quad x \vee y = max(x, y)$$

Example 1.2.2. *The set of natural numbers $\mathbb{N}$ under divisibility forms a lattice, where for any x and y in $\mathbb{N}$*

$$x \wedge y = gcd(x, y) \quad \text{and} \quad x \vee y = lcm(x, y)$$

Example 1.2.3. *Let L be the set of linear subspaces of a vector space V. Then L is lattice under set inclusion, where for all S and T in L*

$$S \wedge T = S \cap T \quad \text{and} \quad S \vee T = S + T = \{v + w \colon v \in S, w \in T\}.$$

Example 1.2.4. *Let S be the set of all subgroups of a group G. Then S is a lattice under set inclusion, where for any H and K in S*

$$H \wedge K = H \cap K \quad \text{and} \quad H \vee K = \langle H \cup K \rangle$$

where $\langle H \cup K \rangle$ means the intersection of all subgroups of G that contains $H \cup K$. Similar statements can be made for other algebraic objects, such as the submodules of a module, the subrings of a ring or the subfields of a field.

Note that not all posets are lattices. For example, consider the poset $(\{1, 2, 3, 4, 5\}, |)$ whose Hasse diagram is given in Figure 1.4. The supremum of $\{2, 3\}$ and $\{3, 5\}$ do not exist. Thus this poset is not a lattice.

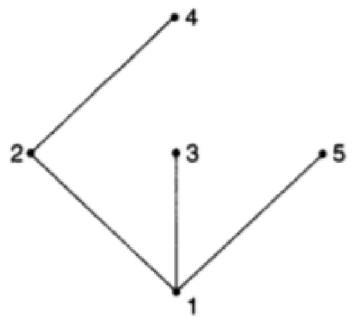

Figure 1.4:

Theorem 1.2.1. *[24] If $(L, \leq_P)$ is a lattice then for any $a, b, c \in L$, the following results hold:*

(L_1)	$a \wedge a = a$	*and*	$a \vee a = a$	*(Idempotent)*
(L_2)	$a \wedge b = b \wedge a$	*and*	$a \vee b = b \vee a$	*(Commutative)*
(L_3)	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$	*and*	$(a \vee b) \vee c = a \vee (b \vee c)$	*(Associative)*
(L_4)	$a \wedge (a \vee b) = a$	*and*	$a \vee (a \wedge b) = a$	*(Absorption)*

Proof. We prove the results for the meet operation. And the results of the join operation can be proven similarly.

(L_1) $a \wedge a = inf\{a, a\} = a$. Thus $a \wedge a = a$.

(L_2) $a \wedge b = inf\{a, b\} = inf\{b, a\} = b \wedge a$.

(L_3) Let $b \wedge c = d$ then $d = inf\{b, c\}$

$\Rightarrow d \leq_P b$ and $d \leq_P c$

Now, let $a \wedge d = e$ then $e = inf\{a, d\}$

$\Rightarrow e \leq_P a$ and $e \leq_P d$

$\Rightarrow e \leq_P a$ and $e \leq_P b$ and $e \leq_P c$ [since $\leq_P$ is transitive and $d \leq_P b, d \leq_P c$]

$\Rightarrow e$ is a lower bound of $\{a, b, c\}$

If l is any lower bound of $\{e, b, c\}$ then

$$l \leq_P a \text{ and } l \leq_P b \text{ and } l \leq_P c$$

Now,

$$l \leq_P b, l \leq_P c \text{ and } d = inf\{b, c\} \text{ give } l \leq_P d$$
$$l \leq_P a \text{ and } e = inf\{a, d\} \text{ give } l \leq_P e.$$

Hence, $e = inf\{a, b, c\}$. Similarly, we can show $(a \wedge b) \wedge c = inf\{a, b, c\}$. Thus

$$(a \wedge b) \wedge c = a \wedge (b \wedge c).$$

(L_4) To prove $a \wedge (a \vee b) = a$, we know that $a \leq_P a$ and $a \leq_P a \vee b$

$$\Rightarrow a \leq_P a \wedge (a \vee b) \tag{1.1}$$

again

$$a \wedge (a \vee b) = inf\{a, a \vee b\} \leq_P a \tag{1.2}$$

From 1.1 and 1.2, by antisymmetry, we have, $a \wedge (a \vee b) = a$.

□

Theorem 1.2.2. *[24] For any a, b, c, d in a lattice $(L, \leq_P)$, the following properties hold:*

(a) $a \leq_P b$ *and* $c \leq_P d \Rightarrow a \vee c \leq_P b \vee d$

(b) $a \leq_P b$ *and* $c \leq_P d \Rightarrow a \wedge c \leq_P b \wedge d$.

Proof. (a) Suppose that $a \leq_P b$ and $c \leq_P d$. By definition of join operation $\vee$, we have

$$b \leq_P b \vee d \text{ and } d \leq_P b \vee d$$

Now,

$$a \leq_P b \text{ and } b \leq_P b \vee d \Rightarrow a \leq_P b \vee d \text{ by transitivity.}$$

Similarly,

$$c \leq_P d \text{ and } d \leq_P b \vee d \Rightarrow c \leq_P b \vee d.$$

Thus $b \vee d$ is an upper bound of a and c. Since $a \vee c$ is the least upper bound of a and c, we have, $a \vee c \leq_P b \vee d$. The proof of (b) is similar.

□

Corollary 1.2.1. *[24] For any a, b, c in a lattice $(L, \leq_P)$, the following properties hold.*

(a) $a \leq_P b \Rightarrow a \vee c \leq_P b \vee c$

(a) $a \leq_P b \Rightarrow a \wedge c \leq_P b \wedge c$.

Proof. The proofs of (a) and (b) follow by taking $d = c$ in the above theorem. □

The following theorem gives a connection between the partial ordering relation $\leq_P$ and the two binary operations $\vee$ and $\wedge$ in a lattice $(L, \leq_P)$.

Theorem 1.2.3. *[24] Let $(L, \leq_P)$ be a lattice in which $\wedge$ and $\vee$ denote the operations of meet and join. Then for any a, b*

(a) $a \leq_P b \Leftrightarrow a \wedge b = a$

(b) $a \leq_P b \Leftrightarrow a \vee b = b.$

Proof. (a) Suppose that $a \wedge b = a$. Since $a \wedge b = inf\{a, b\}$, therefore $a \wedge b \leq_P b$

$$\Rightarrow a \leq_P b. \qquad \qquad [\text{since } a \wedge b = a\ ]$$

Conversely, suppose that $a \leq_P b$. Since $\leq_P$ is reflexive, we have, $a \leq_P a$.

Now, $a \leq_P b$ and $a \leq_P a \Rightarrow a$ is a lower bound of $\{a, b\}$

$$\Rightarrow a \leq_P inf\{a, b\} = a \wedge b.$$

Since $a \wedge b$ is infimum of $\{a, b\}$, $a \wedge b \leq_P a$. Hence, by antisymmetry, $a = a \wedge b$.

Part (b) can be proved in a way similar to the proof (a).

□

Corollary 1.2.2. *[24] Let L be a lattice and $a, b \in L$. Then*

$$a \wedge b = a \text{ if and only if } a \vee b = b.$$

Proof. The proof follows from parts (a) and (b) of the above theorem. □

1.2.2 Lattices as Algebraic Structures

A set together with certain operations defined on it is usually referred to as an algebraic structure. Lattices were introduced as partially ordered sets. Now, an alternative definition of a lattice can be formulated as algebraic structure satisfying certain axiomatic identities.

Definition 1.2.3. [31] An **algebraic lattice** $(L; \vee, \wedge)$ is a set L with two binary operations meet and join, $\wedge$ and $\vee$, such that both operations are commutative and associative, and the absorption law holds. i.e. $\forall a, b, c \in L$,

1 . $a \wedge b = b \wedge a, a \vee b = b \vee a$ (Commutivity)

2 . $(a \wedge b) \wedge c = a \wedge (b \wedge c), (a \vee b) \vee c = a \vee (b \vee c)$ (Associativity)

3 . $a = a \wedge (a \vee b) = a \vee (a \wedge b)$ (Absorption law)

Actually, absorption causes both operations to be idempotent. Given that $\vee$ and $\wedge$ are satisfying the absorption identities,

$$a \wedge a = a \wedge (a \vee (a \wedge b)) = a \quad \text{and} \quad a \vee a = a \vee (a \wedge (a \vee b)) = a.$$

So the idempotently is implicitly contained and it can be neglected in the definition.

Theorem 1.2.4. *[9] The two definitions of a lattice given in Definitions1.2.1 and 1.2.3 are equivalent. Equivalently, a lattice defined as a poset $(L, \leq_P)$ (see Definition1.2.1) is an algebraic system with two binary operations which satisfy commutative, associative and absorption laws; a lattice defined as an algebraic system $(L; \vee, \wedge)$ (see Definition1.2.3) is a lattice defined as a poset where $l.u.b.\{a, b\}$ and $g.l.b.\{a, b\}$ are $a \vee b$ and $a \wedge b$ respectively.*

Proof. Let $(L, \leq_P)$ be a lattice satisfying Definition 1.2.1 Then the $l.u.b.$ and $g.l.b.$ viewed as binary operations on L satisfy commutative, associative and absorption laws (see Theorem 1.2.1). Then $(L; \vee, \wedge)$ is a lattice according to Definition1.2.3, where $a \vee b = l.u.b.\{a, b\}$ and $a \wedge b = g.l.b.\{a, b\}$. This proves the first part of the theorem.

For proving the second part, we start with the algebraic system $(L; \vee, \wedge)$ where $\vee$ and $\wedge$ satisfy commutative, associative and absorption laws. We have to (*i*) Define a partial ordering on L.

(*ii*) Prove that $a \wedge b = g.l.b.\{a, b\}$ realized from $\leq_P$.

(*iii*) Prove that $a \vee b = l.u.b.\{a, b\}$ realized from $\leq_P$.

We define a relation $\leq_P$ on L by

$$a \leq_P b \text{ iff } a \vee b = b.$$

By idempotent law, $a \vee a = a$. Hence $a \leq_P a$, proving that $\leq_P$ is reflexive.
If $a \leq_P b$ and $b \leq_P a$, then $a \vee b = b$ and $b \vee a = a$. As $a \vee b = b \vee a$, $a = b$, proving antisymmetry of $\leq_P$.
To prove transitivity of $\leq_P$, assume $a \leq_P b$ and $b \leq_P c$. This means that $a \vee b = b$ and $b \vee c = c$. Then

$$\begin{aligned} a \vee c &= a \vee (b \vee c) \\ &= (a \vee b) \vee c && \text{(by associativity)} \\ &= b \vee c \\ &= c. \end{aligned}$$

Hence $a \leq_P c$, proving the transitivity of $\leq_P$. Thus we have defined a partial ordering $\leq_P$ on L.
Before proving (*ii*), we prove the following claim:

$$a \leq_P b \Leftrightarrow a \vee b = b \Leftrightarrow a \wedge b = a.$$

The proof of this claim is as follows:
The first equivalence is simply the definition of the relation $\leq_P$. Assume $a \vee b = b$, then

$$a \wedge b = a \wedge (a \vee b) = a.$$

So

$$a \vee b = b \Rightarrow a \wedge b = a.$$

Assume $a \wedge b = a$. Then

$$a \vee b = (a \wedge b) \vee b = b.$$

So

$$a \wedge b = a \Rightarrow a \vee b = b.$$

Hence the claim is established.

For proving (ii), we use absorption laws. By L_4, $a \vee (a \wedge b) = a$ and $b \vee (b \wedge a) = b$, so $a \wedge b \leq_P a$ and $a \wedge b = b \wedge a \leq_P b$, implying that $a \wedge b$ is a lower bound for a and b. If c is any lower bound for a and b, then $c \leq_P a$ and $c \leq_P b$, this means $c \vee a = a$ and $c \vee b = b$. So

$$\begin{aligned} c &= (c \vee a) \wedge c && \text{(by absorption law)} \\ &= a \wedge c. && \text{(since } c \vee a = a\text{)} \end{aligned}$$

Similarly

$$c = (c \vee b) \wedge c = b \wedge c. \qquad \text{(since } c \vee b = b\text{)}$$

So

$$\begin{aligned} c = c \wedge c &= (a \wedge c) \wedge (b \wedge c) \\ &= c \wedge (a \wedge b). && \text{(by commutative and associative laws)} \end{aligned}$$

By the claim, $c \leq_P a \wedge b$. Thus any lower bound c of a and b satisfies $c \leq_P a \wedge b$. Hence $g.l.b.\{a, b\} = a \wedge b$. (iii) can be proved similarly. Thus we have proved the second part of the theorem. □

Let $(L, \leq_P)$ be a lattice. It may has a greatest and least elements. If we think of a lattice as algebraic structure $(L; \vee, \wedge)$, it is suitable to see these elements from an algebraic viewpoint.

A lattice L has a **one** element if there exists $1 \in L$ such that $x \wedge 1 = 1 \wedge x = x$, $\forall x \in L$. Dually, a lattice L has a **zero** element if there exists $0 \in L$ such that $x \vee 0 = 0 \vee x = x$, $\forall x \in L$. A lattice $(L, \leq_P)$ has a greatest element if and only if

$(L; \vee, \wedge)$ has a one element. A dual statement holds for zero and least element.

Definition 1.2.4. [2] A lattice $(L; \vee, \wedge)$ possessing 0 and 1 is called **bounded** lattice.

Note that a finite lattice is always bounded, with $0 = \bigwedge L$ and $1 = \bigvee L$.

Example 1.2.5. *The lattice* $(\mathbb{N}_0; lcm, gcd)$*, where* $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ *is bounded with greatest element* $1 = 0$ *and least element* $0 = 1$*. While the lattice* $(\mathbb{N}_0; max, min)$ *is not bounded since* 1 *does not exist.*

1.2.3 Sublattices

Definition 1.2.5. [22] A **sublattice** of a lattice L is a subset of L that is closed under infimums and supremums.

Equivalently, a sublattice $(M; \vee, \wedge)$ of a lattice $(L; \vee, \wedge)$ is a subset M of L that is closed under both the operations $\vee$ and $\wedge$. i.e. if $a, b \in M$ then $a \vee b \in M$ and $a \wedge b \in M$.

Example 1.2.6. *A one-element subset of a lattice is a sublattice. In general, any nonempty chain in a lattice is a sublattice.*

Example 1.2.7. *The set of positive integers* $\mathbb{Z}^+$ *forms a lattice under divisibility where* $a \vee b = lcm(a, b)$ *and* $a \wedge b = gcd(a, b)$*. Let* S_n *be the set of all positive divisors of any positive integer* n*. Then* $(S_n, |)$ *is a sublattice of* $(\mathbb{Z}^+, |)$*.*

Note that a sublattice of a lattice is also a lattice. However, a subset M of a lattice L can be a lattice under the same order relation and need not be a sublattice of L. We demonstrate this fact by the following example.

Example 1.2.8. *The division relation on the set* $L = \{1, 2, 3, 6, 12\}$ *forms a lattice whose Hasse diagram is given in Figure 1.5. The subset* $T = \{1, 2, 3, 6\}$ *is a sublattice of* L*. The subset* $S = \{1, 2, 3, 12\}$ *is a lattice under division, but not a sublattice of* L *since* $2 \vee 3 = 6 \notin S$*.*

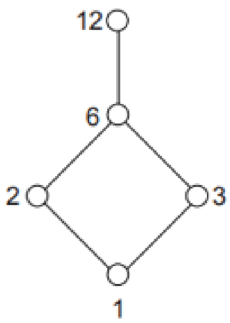

Figure 1.5:

1.2.4 Ideals and Filters

The dual concepts of ideal and filter are of fundamental importance in algebra and have a variety of applications. Thus in this subsection we introduce definitions, some theorems, and examples.

Definition 1.2.6. [2] Let L be a lattice.

1 . A nonempty subset I of L is called an **ideal** of L if

(a) $x \in L$,$a \in I$ and $x \leq_P a \Rightarrow x \in I$.

(a) $a, b \in I \Rightarrow a \vee b \in I$.

A **proper ideal**, that is, an ideal $I \neq L$. The set of all ideals of L is denoted by $\mathcal{J}(L)$.

2 . Dually, a nonempty subset F of L is called a **filter** of L if

(a) $x \in L, a \in F$ and $x \geq_P a \Rightarrow x \in F$.

(a) $a, b \in F \Rightarrow a \wedge b \in F$.

A **proper filter**, that is, a filter $F \neq L$. The set of all filters of L is denoted by $\mathcal{F}(L)$.

Example 1.2.9.

1 . *The lattice L itself is both a filter and an ideal.*

2 . *The family of all finite subsets of any set X is an ideal of the lattice $(P(X), \subseteq)$.*

Any intersection $I \cap J$ of two ideals I and J of a lattice L is not empty, since for $i \in I$ and $j \in J$, we have $i \wedge j \leq_P i, j$ and so $i \wedge j \in I \cap J$. Furthermore, $I \cap J$ is an ideal of L (The verification is simple).

Remark 1.2.1. Every ideal I of lattice L is a sublattice, since $a \wedge b \leq_P a$ for all $a, b \in I$ and thus, $a \wedge b \in I$. Dually, every filter F of L is a sublattice.

Definition 1.2.7. [30] Let A be a nonempty subset of a lattice L.

1) The ideal generated by A, denoted by $(A\,]$, is the smallest ideal of L containing A.

2) The filter generated by A, denoted by $[A)$, is the smallest filter of L containing A.

Definition 1.2.8. [17] if $A = \{a\}$, then we write $(a\,]$ or $a^{\downarrow}$ instead of $(\{a\}\,]$ such that

$$(a\,] = a^{\downarrow} = \{x \in L | x \leq_P a\}$$

is known as the **principal ideal** generated by a. Dually,

$$[a) = a^{\uparrow} = \{x \in L | x \geq_P a\}$$

is known as the **principal filter** generated by a.

Example 1.2.10. *Consider the lattice $L = \{1, a, b, c, d, 0\}$ with Hasse diagram given in Figure 1.6. The ideal generated by $\{0, a, b, c, d\}$ is $(\{0, a, b, c, d\}\,] = \{0, a, b, c, d\} = a^{\downarrow}$, and the principal ideal generated by d is $d^{\downarrow} = \{0, d\}$. Of course, $1^{\downarrow} = L$.*

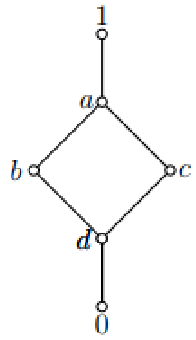

Figure 1.6:

Theorem 1.2.5. *[6] If L is a lattice then, ordered by set inclusion, the set $\mathcal{J}(L)$ of ideals of L form a lattice in which the lattice operations are given by*

$$J \wedge K = J \cap K;$$
$$J \vee K = \{x \in L | x \leq_P j \vee k, for some j \in J, k \in K\}.$$

Proof. We have to show that every pair of ideals of L has an infimum and a supremum in $\mathcal{J}(L)$. It is clear that if J and K are ideals of L, then so $J \cap K$, and that this is the biggest ideal of L that is contained in both J and K. Hence, $J \wedge K \in \mathcal{J}(L)$.

Now, any ideal that contains both J and K must clearly contain all the elements x such that $x \leq_P j \vee k$ where $j \in J$ and $k \in K$. Conversely, the set of all such x clearly contains both J and K, and is contained in every ideal of L that contains both J and K.

Moreover, this set is also an ideal of L: if $x \in J \vee K$ and $r \in L$ such that $r \leq_P x \leq_P j \vee k$ for some $j \in J$ and $k \in K$, then (by transitivity)

$$r \leq_P j \vee k \Rightarrow r \in J \vee K.$$

Also, if $x, y \in J \vee K$, then

$$x \leq_P j \vee k \qquad \text{and} \qquad y \leq_P j_1 \vee k_1 \qquad \text{for some } j, j_1 \in J \text{ and } k, k_1 \in K.$$

Hence

$$x \vee y \leq_P (j \vee k) \vee (j_1 \vee k_1) = (j \vee j_1) \vee (k \vee k_1),$$

Where $(j \vee j_1) \in J$ and $(k \vee k_1) \in K$ since J and K are ideals. Therefore $x \vee y \in J \vee K$. Thus we see that $J \vee K$ is in $\mathcal{J}(L)$. □

Chapter 2

Skew Lattices

In this chapter we introduce the concept of skew lattice. We present the algebraic structure of a skew lattice. Also we introduce the three Greens equivalence relations on a skew lattice. In addition we discuss some properties of a skew lattice. Furthermore we present the order structure of a skew lattice and the diagram representing the skew lattice graphically.

2.1 Algebraic structure

As in a lattice there exist two ways to present a skew lattice: one as algebraic structure satisfying certain axiomatic identities and one as order structure satisfying certain axioms. In this section we present an algebraic structure of skew lattice and some examples.

To explain an algebraic structure we first illustrate that

$$(b \wedge a) \vee a = a \quad \text{and} \quad (b \vee a) \wedge a = a$$

are absorption laws. Furthermore, there are four other absorption laws that will be described later.

Definition 2.1.1. [10] A **skew lattice** is an algebraic structure $S = (S; \vee, \wedge)$ where $\vee$ and $\wedge$ are associative, idempotent binary operations satisfying the absorption identities

$$x \wedge (x \vee y) = x = (y \vee x) \wedge x \quad and \quad x \vee (x \wedge y) = x = (y \wedge x) \vee x. \tag{2.1}$$

Given that $\vee$ and $\wedge$ are associative, and idempotent binary operations, 2.1 is equivalent to the following identities:

$$x \wedge y = x \;\; iff \;\; x \vee y = y \quad and \quad x \wedge y = y \;\; iff \;\; x \vee y = x. \tag{2.2}$$

Remark 2.1.1. In a skew lattice, if one of the two operations $\wedge$ and $\vee$ is commutative then so is the other, and thus the commutativity axiom is satisfied. Satisfying the commutative axiom along with the three associative, idempotent and absorption axioms gives a lattice. In other words, the skew lattice is a non-commutative generalization of lattice.

Example 2.1.1. *Any lattice is a skew lattice. Thus, the lattices* $(\mathbb{N}, lcm, gcd)$ *and* $(P(X), \cup, \cap)$ *are skew lattices.*

Conversely, not all skew lattices are lattices.

Example 2.1.2. *Let* $S = \{a, b\}$ *with* $\wedge$ *and* $\vee$ *operations defined by the following Cayley tables:*

$\wedge$	a	b
a	a	b
b	a	b

$\vee$	a	b
a	a	a
b	b	b

Then $(S; \vee, \wedge)$ *is a skew lattice but not a lattice since* $a \wedge b \neq b \wedge a$.

Definition 2.1.2. [19] A **partial function** f from A to B is a function in which $f(a)$ is not defined for every $a \in A$.

Example 2.1.3. *Let* $f(x) = 1/(x+1)^3$. *Then* f *is a partial function from* $\mathbb{R}$ *to* $\mathbb{R}$ *since* $f(-1)$ *is not defined.*

Definition 2.1.3. [25] Let X, Y, A be sets, $A \subset X$, and $f : X \longrightarrow Y$ be a function. The function $g : A \longrightarrow Y$, $g(x) := f(x)$ is called the **restriction** of f to A, and we use the notation $f|_A := g$.

Example 2.1.4. *Let* $P(A, B) = \{f : F \longrightarrow B \text{ where } F \subseteq A\}$ *is the set of all partial functions from a set* A *to a set* B. *Given* $f : F \longrightarrow B$ *and* $g : G \longrightarrow B$ *are in* $P(A, B)$ *where,* F, $G \subseteq A$. *We define* $\wedge$ *and* $\vee$ *as follows:*

$$f \wedge g = f|_{F\cap G} \quad \text{and} \quad f \vee g = g \cup f|_{F-G}.$$

It is clear that $f \wedge g$ *and* $f \vee g$ *are partial functions from* A *to* B. *Now,*

1. $f \wedge f = f \quad$ *and* $\quad f \vee f = f$.
2. $(f \wedge g) \wedge h = f|_{F\cap G\cap H} = f \wedge (g \wedge h)$ *and*
   $(f \vee g) \vee h = h \cup g|_{G-H} \cup f|_{F-G\cup H} = f \vee (g \vee h)$.
3. $f \wedge (f \vee g) = f|_{F\cap(F\cup G)} = f = (f \cup g|_{G-F})|_{(G\cup F)\cap F} = (g \vee f) \wedge f$ *and*
   $f \vee (f \wedge g) = f|_{F\cap G} \cup f|_{F-F\cap G} = f = f \cup g|_{G\cap F-F} = (g \wedge f) \vee f$.

Thus $\vee$ *and* $\wedge$ *are idempotent, associative binary operations satisfying the absorption identities, and* $(P(A, B); \vee, \wedge)$ *is a skew lattice. But it is not a lattice since*

$$f \wedge g = f|_{F\cap G} \neq g|_{G\cap F} = g \wedge f \quad \text{and} \quad f \vee g = g \cup f|_{F-G} \neq f \cup g|_{G-F} = g \vee f,$$

i.e. $\vee$ *and* $\wedge$ *are not commutative.*

Remark 2.1.2. The algebraic structure of a skew lattice enriched with the following absorption laws is the same as the algebraic structure of a lattice.

1. $(x \wedge y) \vee x = x$
2. $x \wedge (y \vee x) = x$
3. $(x \vee y) \wedge x = x$
4. $x \vee (y \wedge x) = x$

Proof. If we admit all the absorption laws we get that:
$x \wedge y = (x \wedge y) \wedge (x \vee (x \wedge y)) = x \wedge y \wedge x$,
$y \wedge x = ((y \wedge x) \vee x) \wedge (y \wedge x) = x \wedge y \wedge x$,
$x \vee y = (x \vee y) \vee (x \wedge (x \vee y)) = x \vee y \vee x$,
$y \vee x = ((y \vee x) \wedge x) \vee (y \vee x) = x \vee y \vee x$.
Thus the two operations $\wedge$ and $\vee$ are commutative, and we have a lattice. □

Definition 2.1.4. [16] Every skew lattice is *regular*, i.e., it satisfies the identities

$$x \wedge y \wedge x \wedge z \wedge x = x \wedge y \wedge z \wedge x \quad \text{and} \quad x \vee y \vee x \vee z \vee x = x \vee y \vee z \vee x.$$

Remark 2.1.3. A **sub-skew lattice** $(T; \vee, \wedge)$ of the skew lattice $(S; \vee, \wedge)$ is a nonempty subset T of S that is closed under $\vee$ and $\wedge$ operations.

Example 2.1.5. *Any one-element subset of a skew lattice is a sub-skew lattice.*

Example 2.1.6. $(\mathbb{N}, max, min)$ *is a sub-skew lattice of the skew lattice* $(\mathbb{Z}, max, min)$.

2.2 Green's Relations on Skew Lattices

Given an algebraic structure $(A, +, *)$ where $+$ and $*$ denote two arbitrary binary operations, an equivalence relation θ on A is **congruence relation** if for all $a, b, c, d \in A$,

$$a\theta b \quad \text{and} \quad c\theta d \qquad \text{imply} \qquad a + c \; \theta \; b + d \quad \text{and} \quad a * c \; \theta \; b * d.$$

The equivalence classes under a congruence relation θ are called **congruence classes**.And the congruence class containing the element $a \in A$ is denoted by $[a]_\theta$. The set of all congruence classes is denoted by A/θ, it forms an algebraic structure of the same type as A under the operations

$$[a]_\theta +_\theta [b]_\theta = [a + b]_\theta \quad \text{and} \quad [a]_\theta *_\theta [b]_\theta = [a * b]_\theta.$$

In this section we discuss three congruence relations on skew lattice, which have an important role in the further development of skew lattice. They are called Green's relations.

Definition 2.2.1. [8] On a skew lattice $(S; \vee, \wedge)$ the three canonical **Green's equivalence relations** $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{D}$ on S are defined by the equivalences

$$x\mathcal{R}y \Leftrightarrow (x \wedge y = y \quad \text{and} \quad y \wedge x = x) \Leftrightarrow (x \vee y = x \quad \text{and} \quad y \vee x = y)$$
$$x\mathcal{L}y \Leftrightarrow (x \wedge y = x \quad \text{and} \quad y \wedge x = y) \Leftrightarrow (x \vee y = y \quad \text{and} \quad y \vee x = x)$$

and by the equivalences

$$x\mathcal{D}y \Leftrightarrow (x \wedge y \wedge x = x \quad \text{and} \quad y \wedge x \wedge y = y)$$
$$\Leftrightarrow (x \vee y \vee x = x \quad \text{and} \quad y \vee x \vee y = y),$$

for any points x, y in S.

The congruence classes of Green's relations are called $\mathcal{R}$-classes, $\mathcal{L}$-classes and $\mathcal{D}$-classes. An $\mathcal{R}$-, $\mathcal{L}$- and $\mathcal{D}$-class containing $x \in S$ is denoted by $\mathcal{R}_x$, $\mathcal{L}_x$ and $\mathcal{D}_x$, respectively.

Remark 2.2.1. In a skew lattice S, all congruence classes ($\mathcal{D}$-classes,$\mathcal{R}$-classes and $\mathcal{L}$-classes) are sub-skew lattices of S.

Theorem 2.2.1. *[15] For any elements x, y of a skew lattice S, $x\mathcal{D}y$ if and only if $x \vee y = y \wedge x$.*

Proof. If $x\mathcal{D}y$, then

$$\begin{aligned} x \vee y &= [y \vee (x \vee y)] \wedge (x \vee y) && \text{(by absorption law)} \\ &= y \wedge (x \vee y). \end{aligned}$$

Thus

$$\begin{aligned} y \wedge x &= y \wedge x \wedge (x \vee y) && \text{(by absorption law)} \\ &= y \wedge x \wedge [y \wedge (x \vee y)] \\ &= y \wedge (x \vee y) = x \vee y \end{aligned}$$

Conversely, if $x \vee y = y \wedge x$, then

$$x \wedge y \wedge x = x \wedge (x \vee y) = x$$

and

$$y \wedge x \wedge y = (x \vee y) \wedge y = y$$

Thus $x \ \mathcal{D} \ y$. □

Remark 2.2.2. No two distinct elements are commutative under $\wedge$ and $\vee$ operations in each $\mathcal{D}$-class, i.e. given $x \ \mathcal{D} \ y$, then

$$x \wedge y = y \wedge x \text{ iff } x = y$$

and

$$x \vee y = y \vee x \text{ iff } x = y.$$

Proof. Let $x, y \in S$. Given $x\mathcal{D}y$ and $x \wedge y = y \wedge x$. Then,

$$x = x \wedge y \wedge x = y \wedge x \wedge x = y \wedge x$$

and

$$y = y \wedge x \wedge y = y \wedge y \wedge x = y \wedge x.$$

So, $x = y$. The proof for $\vee$ is similar. The converse direction is clear. □

Theorem 2.2.2. *[15] In every skew lattice, the identities*

$$(x \wedge y) \vee (y \wedge x) = y \wedge x \wedge y$$

and

$$(x \vee y) \wedge (y \vee x) = y \vee x \vee y$$

hold for all x, y.

Proof. since $x \wedge y \ \mathcal{D} \ y \wedge x$, Theorem 2.2.1. gives

$$(x \wedge y) \vee (y \wedge x) = (y \wedge x) \wedge (x \wedge y) = y \wedge x \wedge y$$

and

$$(x \vee y) \wedge (y \vee x) = (y \vee x) \vee (x \vee y) = y \vee x \vee y.$$

□

Lemma 2.2.1. *[15] $\mathcal{D}$ is a congruence and $S/\mathcal{D}$ is a lattice. Given any congruence $\mathcal{C}$ on S such that $S/\mathcal{C}$ is a lattice, $\mathcal{D} \subseteq \mathcal{C}$. Thus $S/\mathcal{D}$ is the maximal lattice image of S.*

Proof. Given $x\mathcal{D}y$ and $u \in S$,

$$\begin{aligned}(u \vee x) \vee (u \vee y) \vee (u \vee x) &= (u \vee x \vee y \vee u) \vee x \\ &= u \vee (x \vee y \vee x \vee u \vee x) \\ &= u \vee x \vee u \vee x = u \vee x,\end{aligned}$$

and likewise,

$$(u \vee y) \vee (u \vee x) \vee (u \vee y) = u \vee y.$$

So that $u \vee x \ \mathcal{D} \ u \vee y$. Similarly, $u \wedge x \ \mathcal{D} \ u \wedge y$, $x \vee u \ \mathcal{D} \ y \vee u$ and $x \wedge u \ \mathcal{D} \ y \wedge u$ so that $\mathcal{D}$ is indeed a congruence.

Since both $x \wedge y \ \mathcal{D} \ y \wedge x$ and $x \vee y \ \mathcal{D} \ y \vee x$ for all $x, y \in \mathcal{S}$,

$$\mathcal{D}_x \wedge_{\mathcal{D}} \mathcal{D}_y = \mathcal{D}_{x \wedge y} = \mathcal{D}_{y \wedge x} = \mathcal{D}_y \wedge_{\mathcal{D}} D_x$$

and

$$\mathcal{D}_x \vee_{\mathcal{D}} \mathcal{D}_y = \mathcal{D}_{x \vee y} = \mathcal{D}_{y \vee x} = \mathcal{D}_y \vee_{\mathcal{D}} \mathcal{D}_x.$$

So $S/\mathcal{D}$ is commutative, and thus is a lattice.

Suppose now that $\mathcal{C}$ is a congruence such that $S/\mathcal{C}$ is a lattice, and suppose that $x\mathcal{D}y$. Then

$$\begin{aligned}x \vee y \ \mathcal{C} \ y \vee x &\Rightarrow (x \vee y) \vee x \ \mathcal{C} \ (y \vee x) \vee x \\ &\Rightarrow x \ \mathcal{C} \ y \vee x.\end{aligned}$$

Again,

$$\begin{aligned}x \vee y \ \mathcal{C} \ y \vee x &\Rightarrow y \vee (x \vee y) \ \mathcal{C} \ y \vee (y \vee x) \\ &\Rightarrow y \ \mathcal{C} \ y \vee x.\end{aligned}$$

Thus $x \mathcal{C} y \vee x$ and $y \vee x \mathcal{C} y$ imply $x \mathcal{C} y$. Hence $D \subseteq C$ and so $S/\mathcal{D}$ is the maximal lattice image of S. □

Thus, $\mathcal{D}$ is the smallest congruence for which the quotient $S/\mathcal{D}$ satisfies the property of commutativity, and thus is a lattice.

2.3 Some Properties of Skew Lattices

In this section, we present some properties of a skew lattice. Also examples and theorems are presented to illustrate these properties.

Definition 2.3.1. [10] A skew lattice is **rectangular** if $x \wedge y \wedge x = x$, or equivalently, $x \vee y \vee x = x$, or also equivalently, $x \wedge y = y \vee x$ holds.

Remark 2.3.1. A skew lattice S is rectangular if and only if it consists of a single $\mathcal{D}$-class. So by Remark 2.2.2 there is no two distinct elements that commutes under $\vee$ and $\wedge$ operations in rectangular skew lattices. Therefore, there is no lattice that can be regarded as rectangular skew lattice.

Example 2.3.1. *All $\mathcal{D}$-classes are rectangular sub-skew lattices.*

Definition 2.3.2. If R is a binary relation on a set A and $B \subseteq A$, then the **restriction** of R to B is the relation $R|_B = R \cap (B \times B) = \{(x, y) | (x, y) \in R \textit{ and } x, y \in B\}$.

Remark 2.3.2. If T is a sub-skew lattice of a skew lattice S, then the $\mathcal{D}$ relation on T is the restriction of $\mathcal{D}$ in S to T. Thus if T itself is a rectangular sub-skew lattice of S, it follows that T consists of a single $\mathcal{D}$-class which implies T is entirely contained in a $\mathcal{D}$-class of S. Hence each $\mathcal{D}$-class is a maximal rectangular sub-skew lattice of S.

Example 2.3.2. *Let T be a rectangular sub-skew lattice of $P(A, B)$, then T is the set of partial functions with common domain. Given $f, g \in T$ with domains F and G, respectively. Since T is rectangular, f and g are contained in one $\mathcal{D}$-class of T and thus we get*

$$f \wedge g = f \vee g \Rightarrow f|_{F \cap G} = g \cup f|_{F-G} \Rightarrow F = G.$$

Theorem 2.3.1. *[26] **(The First Decomposition Theorem)**. In any skew lattice S each $\mathcal{D}$-congruence class is a maximal rectangular subalgebra of S and $S/\mathcal{D}$ is the maximal lattice image of S.*

Proof. By Lemma 2.2.1 and Remark 2.3.2. □

Definition 2.3.3. [10] A skew lattice is **right-handed** [respectively, **left-handed**] if it satisfies the identities

$$x \wedge y \wedge x = y \wedge x \quad and \quad x \vee y \vee x = x \vee y. \tag{2.3}$$

$$[x \wedge y \wedge x = x \wedge y \quad and \quad x \vee y \vee x = y \vee x]. \tag{2.4}$$

Remark 2.3.3. A skew lattice is right-handed (respectively, left-handed) if $\mathcal{R} = \mathcal{D}$ ($\mathcal{L} = \mathcal{D}$).

Remark 2.3.4. The identities for right-handed [respectively, left-handed] skew lattices (2.3) [(2.4)] necessarily assert that both $x \wedge y = y$ and $x \vee y = x$ [$x \wedge y = x$ and $x \vee y = y$] hold in each $\mathcal{D}$-class.

Proof. Assume x and y are any elements in a right-handed skew lattice S such that $x\mathcal{D}y$, then

$$y \wedge x \wedge y = y \Rightarrow x \wedge y \wedge x \wedge y = y \Rightarrow x \wedge y = y,$$

and

$$x \vee y \vee x = x \Rightarrow x \vee y = x.$$

The proof for left-handed skew lattice is similar. □

Example 2.3.3. *For any lattice, using the commutativity of $\wedge$ and $\vee$, we get that the identities of right-handed and left-handed skew lattices are satisfied.*

Example 2.3.4. *Given any skew lattice S, then $S/\mathcal{R}$ and $S/\mathcal{L}$ are left-handed and right-handed skew lattices respectively.*

Example 2.3.5. *Given f and g are any two partial functions in the skew lattice $(P(A, B); \vee, \wedge)$ with their domains F and G respectively, then*

$$f \wedge g \wedge f = f|_{F\cap G} = f \wedge g \quad \textit{and} \quad f \vee g \vee f = f \cup g|_{G-F} = g \vee f.$$

So, $(P(A, B); \vee, \wedge)$ is a left-handed skew lattice.

Definition 2.3.4. Given three sets E, F, G and two mappings $\varphi : E \longrightarrow G$, $\psi : F \longrightarrow G$, the **fiber product** $E \times_G F$ (relative to φ and ψ) is the subset of the Cartesian product $E \times F$ consisting of the pairs (x, y) such that $\varphi(x) = \psi(y)$.

Note that $\mathcal{L}$ ($\mathcal{R}$) is the smallest congruence making $S/\mathcal{L}$ ($S/\mathcal{R}$) a right-handed (left-handed) skew lattice. This is illustrated in the following theorem:

Theorem 2.3.2. *[26]* ***(The second Decomposition Theorem).*** *Given any skew lattice S, $S/\mathcal{R}$ and $S/\mathcal{L}$ are its respective maximal left and right-handed images, with S being isomorphic to the fibered product, $S/\mathcal{R} \times_{S/\mathcal{D}} S/\mathcal{L}$, of both over their common maximal lattice image under the map $x \longmapsto (\mathcal{R}_x, \mathcal{L}_x)$.*

Definition 2.3.5. [12] The maximal left-handed image $S/\mathcal{R}$ is called the **left factor** of S and the maximal right-handed image $S/\mathcal{L}$ is called the **right factor** of S.

Definition 2.3.6. [14] A skew lattice S is **symmetric** if given x and y in S, $x \wedge y = y \wedge x$ if and only if $x \vee y = y \vee x$.

Remark 2.3.5. A skew lattice is **upper symmetric** if $x \wedge y = y \wedge x$ implies $x \vee y = y \vee x$. Dually, it is **lower symmetric** if $x \vee y = y \vee x$ implies $x \wedge y = y \wedge x$.

Example 2.3.6. *The skew lattice $(P(A,B);\vee,\wedge)$ is symmetric since for any two partial functions f and g with their domains F and G respectively, when $f \wedge g = g \wedge f$,*

$$f|_{F\cap G} = g|_{G\cap F} \Rightarrow g \cup f|_{F-G} = f \cup g|_{G-F} \Rightarrow f \vee g = g \vee f.$$

Conversely, $f \vee g = g \vee f$ implies $f \wedge g = g \wedge f$.

Definition 2.3.7. [13] A skew lattice is called **left strongly symmetric** if it satisfies the identity

$$(x \wedge y) \vee x = x \wedge (y \vee x) \tag{2.5}$$

and is called **right strongly symmetric** if it satisfies the identity

$$(x \vee y) \wedge x = x \vee (y \wedge x). \tag{2.6}$$

A skew lattice is **strongly symmetric** if it is both left and right strongly symmetric.

Example 2.3.7. *Any lattice* $(L; \vee, \wedge)$ *is a symmetric skew lattice due to the commutative axiom. Using the absorption identities in addition to the commutativity of* $\wedge$ *and* $\vee$ *we get that:*

$$(x \wedge y) \vee x = (y \wedge x) \vee x = x = x \wedge (x \vee y) = x \wedge (y \vee x).$$

Thus a lattice L is a left strongly symmetric skew lattice. by the same way, L is a right strongly symmetric skew lattice ,which implies that a lattice L is a strongly symmetric skew lattice also.

Lemma 2.3.1. *[13] Any right-handed skew lattice is right strongly symmetric. Dually, any left-handed skew lattice is left strongly symmetric.*

Proof. Let S is a right-handed skew lattice. Given x and y in S, then

$$\begin{aligned} (x \vee y) \wedge x &= (x \vee y \vee x) \wedge x \\ &= x. && \text{(by absorption law)} \end{aligned}$$

$$\begin{aligned} x \vee (y \wedge x) &= x \vee (x \wedge y \wedge x) \\ &= x. && \text{(by absorption law)} \end{aligned}$$

Hence, S is right strongly symmetric. The proof for left-handed skew lattice is similar. □

Definition 2.3.8. [4] A skew lattice S is called **normal** if $x \wedge y \wedge z \wedge w = x \wedge z \wedge y \wedge w$ and it is called **conormal** if $x \vee y \vee z \vee w = x \vee z \vee y \vee w$ for all $x, y, z, w \in S$.

Skew lattices that are simultaneously normal and conormal, are called **binormal**.

Example 2.3.8. *Any lattice is binormal skew lattice due to the commutative axiom.*

Example 2.3.9. *Given f, g, h and i are partial functions in the skew lattice $(P(A,B);\vee,\wedge)$ with their domains F, G, H and I respectively, then*

$$f \wedge g \wedge h \wedge i = f|_{F\cap G\cap H\cap I} = f|_{F\cap H\cap G\cap I} = f \wedge h \wedge g \wedge i$$

So, $(P(A,B);\vee,\wedge)$ is a normal skew lattice.

Definition 2.3.9. [11] A skew lattice $(S,\vee,\wedge)$ is **distributive**, if the operations are distributive from both sides at the same time:

$$x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x), \tag{2.7}$$

$$x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x). \tag{2.8}$$

Example 2.3.10. *Every rectangular skew lattice is distributive. Given x,y and z are elements in a rectangular skew lattice S, then*

$$\begin{aligned} x \wedge (y \vee z) \wedge x &= x \wedge (z \wedge y) \wedge x \\ &= x \wedge z \wedge x \wedge y \wedge x \\ &= (x \wedge z \wedge x) \wedge (x \wedge y \wedge x) \\ &= (x \wedge y \wedge x) \vee (x \wedge z \wedge x). \end{aligned}$$

Similarly, $x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x)$.

Definition 2.3.10. [5] A skew lattice is called **strongly distributive** if it satisfies the identities

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \tag{2.9}$$

$$(y \vee z) \wedge x = (y \wedge x) \vee (z \wedge x). \tag{2.10}$$

Example 2.3.11. *Given f, g and h are partial functions in the skew lattice $(P(A, B); \vee, \wedge)$ with domains F, G and H respectively, then*

$$f \wedge (g \vee h) = f|_{F \cap (G \cup H)} = f|_{F \cap H} \cup f|_{(F \cap G) - (F \cap H)} = (f \wedge g) \vee (f \wedge h)$$

and

$$(g \vee h) \wedge f = (h \cup g|_{G-H})|_{(G \cup H) \cap F} = h|_{H \cap F} \cup g|_{(G-H) \cap F} = (g \wedge f) \vee (h \wedge f).$$

So, $(P(A, B); \vee, \wedge)$ is a strongly distributive skew lattice.

Definition 2.3.11. [15] A skew lattice S is **quasi-distributive** if its lattice image $S/\mathcal{D}$ is distributive.
It is clear that a distributive skew lattice S implies a distributive lattice image $S/\mathcal{D}$, which implies a quasi-distributive skew lattice S.

Definition 2.3.12. [11] A skew lattice is **cancellative** if

$$x \vee z = y \vee z \quad and \quad x \wedge z = y \wedge z \text{ } imply \text{ } x = y \tag{2.11}$$

and

$$x \vee y = x \vee z \quad and \quad x \wedge y = x \wedge z \text{ } imply \text{ } y = z. \tag{2.12}$$

Let S is a skew lattice. We say S has a **zero element** if there exists $0 \in S$ such that $x \wedge 0 = 0 = 0 \wedge x$ for all $x \in S$. Dually, S has a **one element** if there exists $1 \in S$ such that $x \vee 1 = 1 = 1 \vee x$ for all $x \in S$. Note that the zero and one elements in a skew lattice are also called a **bottom** and **top** respectively and they are unique if they exists.

2.4 Order Structure

The order structure has an important role in the study of a skew lattice. In this section we present the concept of order structure. Theorems are presented to illustrate this concept. Also we introduce the admissible Hasse diagram to represent the order structure of a skew lattice.

Definition 2.4.1. [27] The **natural partial order** $\leq$ on a skew lattice S is defined by

$$x \leq y \text{ if and only if } x \wedge y = y \wedge x = x, \text{ or equivalently, } x \vee y = y \vee x = y.$$

Definition 2.4.2. [13] On a skew lattice S the **natural preorder** is defined by

$$x \preceq y \Leftrightarrow x \wedge y \wedge x = x \Leftrightarrow y \vee x \vee y = y.$$

Obviously, $x\mathcal{D}y$ if and only if $x \preceq y$ and $y \preceq x$.Furthermore, for all $x, y \in S$, $x \leq y$ implies $x \preceq y$ but not conversely, while if the commutativity is satisfied the converse is also true, i.e. for lattices, $x \leq y$ if and only if $x \preceq y$.

It is clear that the natural partial order $\leq$ is reflexive, antisymmetric and transitive, thus it is a partial order relation on a skew lattice S. For the natural preorder $\preceq$, clearly it is reflexive. Given $x \preceq y \preceq z$, by regularity of $\wedge$ we get:

$$\begin{aligned}
x \wedge z \wedge x &= x \wedge y \wedge x \wedge z \wedge x \wedge y \wedge x \\
&= x \wedge y \wedge (x \wedge z \wedge y \wedge x) \\
&= x \wedge (y \wedge x \wedge y \wedge z \wedge y) \wedge x \\
&= x \wedge y \wedge z \wedge y \wedge x \\
&= x \wedge y \wedge x = x.
\end{aligned}$$

Hence, $x \preceq z$ and thus $\preceq$ is transitive, so $\preceq$ is a preorder relation on S.

Remark 2.4.1. [2] A skew lattice S is **totally preordered** if for all $x, y \in S$, either $x \preceq y$ or $y \preceq x$.

Example 2.4.1. *Let S be a rectangular skew lattice. If $x, y \in S$, then $x \wedge y \wedge x = x$, or equivalently, $x \vee y \vee x = x$ which implies $x \preceq y$ or $y \preceq x$. So S is totally preordered.*

Theorem 2.4.1. *[15] In any skew lattice S,*

$$x, y \succeq z \text{ implies } x \vee z \vee y = x \vee y, \tag{2.13}$$

$$x, y \preceq z \text{ implies } x \wedge z \wedge y = x \wedge y. \tag{2.14}$$

Proof. Let S be any skew lattice. Given x, y and z in S. For (2.13), using regularity of $\vee$ and the fact that $x, y \succeq z$ we obtain:

$$\begin{aligned} x \vee z \vee y &= (x \vee z \vee x) \vee z \vee (y \vee z \vee y) \\ &= (x \vee z \vee x) \vee (y \vee z \vee y) \\ &= x \vee y. \end{aligned}$$

For (2.14), using regularity of $\wedge$ and the fact that $x, y \preceq z$ we obtain:

$$\begin{aligned} x \wedge z \wedge y &= (x \wedge z \wedge x) \wedge z \wedge (y \wedge z \wedge y) \\ &= (x \wedge z \wedge x) \wedge (y \wedge z \wedge y) \\ &= x \wedge y. \end{aligned}$$

□

The following result will be useful for understanding the lemmas and theorems ahead.

Theorem 2.4.2. *[29] Let S be a skew lattice and $x, y, z \in S$. Then,* *(1)* $x \wedge y \preceq x, y$ *and* $x, y \preceq x \vee y$;

(2) $x \wedge y \wedge x \leq x \leq x \vee y \vee x$;

(3) $x \wedge y \leq y \vee x$.

Proof. The proving of (1) and (2) is simple. For (3), let $x, y \in S$. Then, by absorption laws, we get:

$$x \wedge y \wedge (y \vee x) = x \wedge y \quad \text{and} \quad (y \vee x) \wedge x \wedge y = x \wedge y.$$

Hence, $x \wedge y \leq y \vee x$. □

The next lemma present a characterization for the natural partial order by an identity.

Lemma 2.4.1. *[23] Let* S *be a skew lattice and* $x, y \in S$. *Then* $x \geq y$ *iff* $y = x \wedge y \wedge x$ *or, dually,* $x = y \vee x \vee y$.

Proof. Let $x, y \in S$. If $x \geq y$ then

$$x \wedge y \wedge x = y \wedge x = y.$$

Conversely, by Theorem 2.4.2

$$y = x \wedge y \wedge x \leq x \quad \text{or} \quad x = y \vee x \vee y \geq y.$$

□

Thus we can derive the following characterization for right-handed and left-handed skew lattices.

Theorem 2.4.3. *[23] Let* S *be a skew lattice.* S *is right-handed iff for all* $x, y \in S$, $y \wedge x \leq x$ *and* $x \leq x \vee y$. *Analogously,* S *is left-handed iff for all* $x, y \in S$, $x \wedge y \leq x$ *and* $x \leq y \vee x$.

Proof. First, we have to prove for right-handed case. Let $x, y \in S$. By Lemma 2.4.1,

$$y \wedge x = x \wedge y \wedge x \text{ is equivalent to } y \wedge x \leq x$$

as well as

$$x \vee y = x \vee y \vee x \text{ is equivalent to } x \leq x \vee y.$$

The left-handed case is analogous. □

Theorem 2.4.4. *[26] For left-handed skew lattices, the following identities hold:*

$$x \wedge (y \vee x) = x = (x \wedge y) \vee x \tag{2.15}$$

$$(x \vee (y \wedge x)) \wedge x = x \vee (y \wedge x) \tag{2.16}$$

$$(x \vee (y \wedge x)) \wedge y = y \wedge x. \tag{2.17}$$

Proof. Let S is a left-handed skew lattice. Given $x, y \in S$. For (2.15)

$$x \wedge (y \vee x) = x \wedge (x \vee y \vee x) = x.$$

Similarly,

$$(x \wedge y) \vee x = x.$$

For (2.16), we know that $y \wedge x \preceq x$. So

$$x \vee (y \wedge x) \vee x = x.$$

Thus by (2.2) $x \vee (y \wedge x) \vee x = x$ is equivalent to

$$(x \vee (y \wedge x)) \wedge x = x \vee (y \wedge x).$$

For (2.17), using (2.16), left-handed identity (2.4) and the absorption identities we get

$$\begin{aligned}(x \vee (y \wedge x)) \wedge y &= (x \vee (y \wedge x)) \wedge x \wedge y \\ &= (x \vee (y \wedge x)) \wedge x \wedge y \wedge x \\ &= (x \vee (y \wedge x)) \wedge y \wedge x = y \wedge x.\end{aligned}$$

□

Theorem 2.4.5. *[23] Let S be a skew lattice and $x, y \in S$. Then $x \geq y$ and $x\mathcal{D}y$ implies $y = x$.*

Proof. If $x \geq y$ and $x\mathcal{D}y$ then $y = x \wedge y \wedge x = x$. $\square$

Remark 2.4.2. Let S be a skew lattice with comparable $\mathcal{D}$-classes A and B. Then, $B \leq A$ in the lattice $S/\mathcal{D}$ if and only if there exist $x \in A$ and $y \in B$ such that $y \preceq x$.

Proof. Let $x \in A$ and $y \in B$ such that A and B are denoted by $\mathcal{D}_x$ and $\mathcal{D}_y$ respectively. Then,

$$\begin{aligned}
\mathcal{D}_y \leq \mathcal{D}_x \quad &\Rightarrow \quad \mathcal{D}_y \preceq \mathcal{D}_x \\
&\Rightarrow \quad \mathcal{D}_y \wedge_{\mathcal{D}} \mathcal{D}_x \wedge_{\mathcal{D}} \mathcal{D}_y = \mathcal{D}_y \\
&\Rightarrow \quad \mathcal{D}_{y \wedge x \wedge y} = \mathcal{D}_y \\
&\Rightarrow \quad y \wedge (y \wedge x \wedge y) \wedge y = y \wedge x \wedge y = y \\
&\Rightarrow \quad y \preceq x.
\end{aligned}$$

Now, let there exist $x \in A$ and $y \in B$ such that $y \preceq x$. Then,
$\mathcal{D}_y \wedge_{\mathcal{D}} \mathcal{D}_x \wedge_{\mathcal{D}} \mathcal{D}_y = \mathcal{D}_{y \wedge x \wedge y} = \mathcal{D}_y$ which implies $B \preceq A$. Hence $B \leq A$ since $S/\mathcal{D}$ is a lattice. $\square$

Theorem 2.4.6. *[29] Let A and B be comparable $\mathcal{D}$-classes in a skew lattice S such that $A \geq B$. Then, for each $a \in A$, there exists $b \in B$ such that $a \geq b$, and dually, for each $b \in B$, there exists $a \in A$ such that $a \geq b$.*

Proof. Given $a \in A$. Take any $y \in B$ and set $b \in S$ as $b = a \wedge y \wedge a$. By Remark 2.4.2 we get $y \preceq a$ and so $y \preceq a \wedge y$. Hence $y \preceq a \wedge y$ and $a \wedge y \preceq y$ implies $a \wedge y \; \mathcal{D} \; y$. Using the idempotency of $\wedge$ and the fact $a \wedge y \; \mathcal{D} \; y \wedge a$ we obtain

$$b = (a \wedge y) \wedge (y \wedge a) \in B$$

and

$$a \wedge b \wedge a = a \wedge (a \wedge y \wedge a) \wedge a = a \wedge y \wedge a = b,$$

which implies $a \geq b$. The proof of the dually case is the same. □

The skew lattices can graphically represented as follows:

Definition 2.4.3. [13] An **admissible Hasse diagram** of (a subset of) a skew lattice is a Hasse diagram for the natural partial order $\leq$ (usually indicated by full down edges) with all $\mathcal{D}$-relationships indicated (usually by horizontal dashed edges).

Example 2.4.2. *Figure 2.1 shows the admissible Hasse diagram of the right-handed skew lattice determined by the following Cayley tables:*

$\wedge$	0	2	3	1
0	0	0	0	0
2	0	2	3	2
3	0	2	3	3
1	0	2	3	1

$\vee$	0	2	3	1
0	0	2	3	1
2	2	2	2	1
3	3	3	3	1
1	1	1	1	1

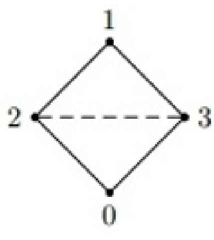

Figure 2.1:

Note that 0 and 1 in the skew lattice above are singleton $\mathcal{D}-$classes.

However, unlike what happens in the lattice case, the skew lattice operations are not uniquely defined by the natural partial order i.e., in the case of lattice, we have $x \leq_P y$ if and only if $x \wedge y = x$ if and only if $x \vee y = y$, while, the equivalence in Lemma 2.4.1 does not fully describe the skew lattice operations.

Hence, the admissible Hasse diagram expresses partial information of the skew lattice. For instance, Figure 2.2 shows the admissible Hasse diagram of both the right and left-handed skew lattices determined by the following *Cayley tables*:

The right-handed skew lattice:

$\wedge$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	b	0	a
b	0	a	b	0	b
c	0	0	0	c	c
1	0	a	b	c	1

$\vee$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	a	1	1
b	b	b	b	1	1
c	c	1	1	c	1
1	1	1	1	1	1

The left-handed skew lattice:

$\wedge$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	0	a
b	0	b	b	0	b
c	0	0	0	c	c
1	0	a	b	c	1

$\vee$	0	a	b	c	1
1	0	a	b	c	1
a	a	a	b	1	1
b	b	a	b	1	1
c	c	1	1	c	1
1	1	1	1	1	1

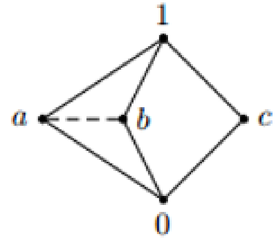

Figure 2.2:

Chapter 3

Ideals On Skew Lattices

In this chapter we introduce the concept of ideal and filter in skew lattices. In addition we present the concepts of skew ideal and principal ideal. Also we discuss some theorems and examples to illustrate these concepts.

3.1 Classical Ideals and Filters

The concept of ideal is of fundamental importance in the study of algebra. Filters, the order duals of lattice ideals have a variety of applications. In this section we introduce the concept of ideal and filter in a skew lattice. We discuss some theorems illustrating this concept. Also we provide the relation between the ideals and the D-classes of a skew lattice.

Definition 3.1.1. [29] A nonempty subset I of a skew lattice S closed under $\vee$ is an **ideal** of S if, for all $x \in S$ and $y \in I$, $x \preceq y$ implies $x \in I$.

Theorem 3.1.1. *[29] Let S be a skew lattice and I a subset of S closed under $\vee$. The following statements equivalently define an ideal:*

(*i*) *for all $x \in S$ and $y \in I$, $x \preceq y$ implies $x \in I$;*

(*ii*) *for all $x \in S$ and $y \in I$, $y \wedge x$, $x \wedge y \in I$;*

(*iii*) *for all $x \in S$ and $y \in I$, $x \wedge y \wedge x \in I$.*

Proof. Let us show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. Let $x \in S$ and $y \in I$. If (i) holds, Theorem 2.4.2. implies that $y \wedge x$, $x \wedge y \preceq y$ so that $y \wedge x$, $x \wedge y \in I$.
Now, assume that (ii) is hold, then due to $x \wedge y \in S$ and $y \wedge x \in I$ we get

$$x \wedge y \wedge x = (x \wedge y) \wedge (y \wedge x) \in I$$

Finally, assuming (iii) we get $x = x \wedge y \wedge x \in I$. □

In the same way, we can define the filter of a skew lattice.

Definition 3.1.2. [29] A nonempty subset F closed under $\wedge$ of a skew lattice S is a **filter** of S if one of the following equivalent statements holds:

(*i*) for all $x \in F$ and $y \in S$, $x \preceq y$ implies $y \in F$;

(*ii*) for all $x \in F$ and $y \in S$, $y \vee x$, $x \vee y \in F$;

(*iii*) for all $x \in F$ and $y \in S$, $y \vee x \vee y \in F$.

Example 3.1.1. *The sets $I = \{ 0, a, b \}$ and $F = \{ 1, a, b \}$ in Figure 2.2 are an ideal and filter respectively.*

The next theorem present a characterization for ideal and filter.

Theorem 3.1.2. *[29] Let S be a skew lattice and I a nonempty subset of S. Then I is an ideal iff the following equivalence holds for all $a, b \in S$*

$$a, b \in I \Leftrightarrow a \vee b \vee a \in I.$$

Analogously, a nonempty subset F of S is a filter iff for all $a, b \in S$,

$$a, b \in F \Leftrightarrow a \wedge b \wedge a \in F.$$

Proof. Let us suppose that I is an ideal of S. If $a, b \in I$, then $a \vee b \vee a \in I$ since I is closed under $\vee$. Let $a, b \in S$ such that $a \vee b \vee a \in I$. As $a, b \preceq a \vee b \vee a$ then $a, b \in I$. Conversely suppose that the equivalence holds. Thus, I is closed under $\vee$. Let $a \in I$ and $b \in S$ such that $b \preceq a$. Then $a \vee b \vee a = a \in I$ and therefore by the equivalence $b \in I$.
Now, Let us suppose that F is an filter of S. If $a, b \in F$, then $a \wedge b \wedge a \in F$ since F is closed under $\wedge$. Let $a, b \in S$ such that $a \wedge b \wedge a \in F$. As $a \wedge b \wedge a \preceq a, b$ then $a, b \in F$. Conversely suppose that the equivalence holds. Thus, F is closed under $\wedge$. Let $a \in F$ and $b \in S$ such that $a \preceq b$. Then $a \wedge b \wedge a = a \in F$ and therefore by the equivalence $b \in F$. □

Corollary 3.1.1. *[29] All ideals and filters in a skew lattice are sub skew lattices.*

Proof. Let I be an ideal of a skew lattice S. I is a subset of S closed under the operation $\vee$, by definition. On the other hand, if $x, y \in I$ then $x \wedge y \preceq x$ implies $x \wedge y \in I$. The proof regarding filters is similar. □

Note that the ideal [filter] of a skew lattice covers the $\mathcal{D}$-classes it intersects. This is illustrated in the following theorem:

Theorem 3.1.3. *[29] Let S be a skew lattice, I an ideal of S and F a filter of S. If $x \in S$, $y \in I$ and $x\mathcal{D}y$, then $x \in I$. Similarly, if $x \in S$, $y \in F$ and $x\mathcal{D}y$, then $x \in F$.*

Proof. If I is an ideal and $x \in S$, whenever $y \in I$ is such that $x\mathcal{D}y$, then $x \preceq y$ implying that $x \in I$. The proof for filter is analogous. □

Remark 3.1.1. The ideal in a skew lattice S is the union of $\mathcal{D}$-classes which constitute the elements of the lattice ideal of $S/\mathcal{D}$.

Proof. Let I be an ideal of S and $y \in I$. Then, by Theorem 3.1.3, for any $s \in S$ such that $s\mathcal{D}y$, we get $s \in I$. Thus $\mathcal{D}_y$ is contained in I and so I is a union of some $\mathcal{D}$-classes of S.

Now, let $\mathcal{D}_y$ and $\mathcal{D}_x$ are contained in I. Then, $\mathcal{D}_y \vee_{\mathcal{D}} \mathcal{D}_x = \mathcal{D}_{y \vee x}$ is contained in I since $y \vee x \in I$. If $\mathcal{D}_x \in S/\mathcal{D}$ and $\mathcal{D}_y$ is contained in I such that $\mathcal{D}_x \leq \mathcal{D}_y$. Thus $x \preceq y$, which implies $x \in I$ since I is an ideal and so $\mathcal{D}_x$ is contained in I. $\square$

Corollary 3.1.2. *[29] Let S be a skew lattice and $I, F \subseteq S$ being unions of $\mathcal{D}$-classes of S. Then, $I/\mathcal{D}$ is a lattice ideal of $S/\mathcal{D}$ iff I is an ideal of S; dually, $F/\mathcal{D}$ is a lattice filter of $S/\mathcal{D}$ iff F is a filter of S.*

Proof. Let I be a union of $\mathcal{D}$-classes. If I is an ideal of S then by Remark 3.1.1, we obtain $I/\mathcal{D}$ is a lattice ideal of $S/\mathcal{D}$. Conversely, if $x, y \in I$ then by the fact that $\mathcal{D}_x$,$\mathcal{D}_y \in I/\mathcal{D}$ and the close property of $I/\mathcal{D}$ under $\vee_{\mathcal{D}}$ operation, $D_{x \vee y \vee x} \in I/\mathcal{D}$ which implies $x \vee y \vee x \in I$. Let $x, y \in S$ such that $x \vee y \vee x \in I$. As $x, y \preceq x \vee y \vee x$ then $\mathcal{D}_x, \mathcal{D}_y \leq \mathcal{D}_{x \vee y \vee x}$ which implies $\mathcal{D}_x, \mathcal{D}_y \in I/\mathcal{D}$ and thus $x, y \in I$. Since the equivalence $x, y \in I \Leftrightarrow x \vee y \vee x \in I$ is hold, I is an ideal of S. $\square$

3.2 Skew Ideals and Filters

In this section we introduce the concept of a skew ideal and skew filter in a skew lattice. Theorems and examples are presented to illustrate these concepts.

Definition 3.2.1. [29] A nonempty subset I of S closed under $\vee$ is a **skew ideal** of S if, for all $x \in S$ and $y \in I$, $x \leq y$ implies $x \in I$.

Definition 3.2.2. [29] A nonempty subset F of S closed under $\wedge$ is a **skew filter** of S if, for all $x \in S$ and $y \in F$, $x \geq y$ implies $x \in F$.

Example 3.2.1. *Let S be the skew lattice given in Example 2.4.2 The subsets $I = \{\ 0, 2, 3\ \}$ and $F = \{\ 1, 2, 3\ \}$ of S are a skew ideal and a skew filter respectively.*

Remark 3.2.1. All ideals of a skew lattice S are skew ideals.

Proof. Let S be a skew lattice and I an ideal of S. By definition, I is closed under $\vee$. Now, let $x \in S$ and $y \in I$ be such that $x \leq y$. Then $x \preceq y$ implies $x \in I$, and thus I is a skew ideal. □

However, not all skew ideals of S are ideals. We can see this in the following example:

Example 3.2.2. *The set $I = \{\ 0,\ a\ \}$ in Figure2.2 is a skew ideal. On the other hand, I is not an ideal, since $b \preceq a$, but $b \notin I$.*

Theorem 3.2.1. *[29] All skew ideals [filters] are sub skew lattices.*

Proof. Let S be a skew lattice and I a skew ideal of S. By definition, I is closed under $\vee$. We shall see that I is also closed under $\wedge$. Let $x, y \in I$. As $x \wedge y \wedge x \leq x$ and $y \wedge x \wedge y \leq y$, both $x \wedge y \wedge x$ and $y \wedge x \wedge y$ are in I. But

$$x \wedge y = (x \wedge y \wedge x) \wedge (y \wedge x \wedge y) = (y \wedge x \wedge y) \vee (x \wedge y \wedge x)$$

where the second equality follows from the fact that $(x \wedge y \wedge x)\mathcal{D}(y \wedge x \wedge y)$. Hence, $x \wedge y \in I$ as required. The proof for $y \wedge x \in I$ is similar. The case of skew filters is analogous. □

Depending on the characterization for the natural partial order described in Lemma 2.4.1 we characterize skew ideals and skew filters of a skew lattice as follows:

Theorem 3.2.2. *[29] A nonempty subset I of S is a skew ideal of S iff*

(*i*) *for all $x, y \in I$, $x \vee y \in I$;*

(*ii*) *for all $x \in S$ and $y \in I$, $y \wedge x \wedge y \in I$.*

Proof. Let I is a skew ideal of S. Then (i) is hold. Let $x \in S$ and $y \in I$. As $y \wedge x \wedge y \leq y$ it follows that $y \wedge x \wedge y \in I$.
Conversely, let $x \in S$ and $y \in I$ be such that $x \leq y$. Then $x = y \wedge x \wedge y \in I$. □

Theorem 3.2.3. *[29] A nonempty subset F of S is a skew filter of S iff*

(*i*) *for all $x, y \in I$, $x \wedge y \in F$;*

(*ii*) *for all $x \in S$ and $y \in F$, $y \vee x \vee y \in F$.*

Proof. The similar proof of Theorem 3.2.2. □

Remark 3.2.2. In general, the skew ideal does not cover the $\mathcal{D}$-classes it intersects, i.e., $x\mathcal{D}y$ with $x \in S$ and $y \in I$ does not necessarily imply that $x \in I$.

Example 3.2.3. *The set $I = \{\ 0,\ b\ \}$ in Figure2.2 is a skew ideal and $a\mathcal{D}b$, but $a \notin I$.*

3.3 Principal Ideals and Filters

In this section we present the concept of a principle ideal and principle filter in a skew lattice. Also we discuss some theorems and examples to illustrate these concepts.

Like other algebraic structures. We now define the ideal generated by a subset of a skew lattice.

Definition 3.3.1. [29] Let X be a nonempty subset of a skew lattice S. Let Y be the set of all [skew] ideals of S containing X. The intersection M of all elements in Y is also a [skew] ideal of S that contains X is called the [skew] ideal generated by X, denoted by $X^{\downarrow}$ $\left[X^{\downarrow *}\right]$. If X is a singleton and $x \in X$, then M is said to be a principal [skew] ideal generated by x, written $x^{\downarrow}$ $\left[x^{\downarrow *}\right]$. Principal [skew] filters have an analogous definition and are denoted by $x^{\uparrow}$ $\left[x^{\uparrow *}\right]$.

Theorem 3.3.1. *[29] Let S be a skew lattice and $x \in S$. Then,*

(i) $x^{\downarrow} = S \wedge x \wedge S = \{y \in S \colon y \preceq x\}$ *and* $x^{\uparrow} = S \vee x \vee S = \{y \in S \colon x \preceq y\}$.

(ii) $x^{\downarrow *} = x \wedge S \wedge x = \{y \in S \colon y \leq x\}$ *and* $x^{\uparrow *} = x \vee S \vee x = \{y \in S \colon x \leq y\}$.

Proof. (*i*) Let us first show that $S \wedge x \wedge S = \{y \in S \colon y \preceq x\}$. Fix $a \in S$. Thus, $a \wedge x \wedge b \preceq x$ due to regularity:

$$a \wedge x \wedge b \wedge x \wedge a \wedge x \wedge b = a \wedge x \wedge b \wedge a \wedge x \wedge b = a \wedge x \wedge b.$$

Conversely, if $a \preceq x$ then

$$a = a \wedge x \wedge a \in S \wedge x \wedge S.$$

The equality $S \vee x \vee S = \{y \in S \colon x \preceq y\}$ has an analogous proof.

Now, we will show that $x^{\downarrow} = S \wedge x \wedge S$. Let $y, z \in S \wedge x \wedge S$, that is, $y, z \preceq x$. Then

$$x \vee y \vee x = x \qquad \text{and} \qquad x \vee z \vee x = x$$

so that

$$x = x \vee x = x \vee y \vee x \vee x \vee z \vee x = x \vee y \vee x \vee z \vee x = x \vee y \vee z \vee x$$

due to regularity. Similarly, $x = x \vee z \vee y \vee x$. Hence, $y \vee z$,$z \vee y \preceq x$ and therefore $y \vee z$,$z \vee y \in S \wedge x \wedge S$.

Let $y \in S$ and $z \in S \wedge x \wedge S$ such that $y \preceq z$. Fix $a, b \in S$ such that $z = a \wedge x \wedge b$. Thus, $y \preceq a \wedge x \wedge b$ so that

$$y = y \wedge a \wedge x \wedge b \wedge y \in S \wedge x \wedge S.$$

By idempotency,

$$x = x \wedge x \wedge x \in S \wedge x \wedge S.$$

Thus $S \wedge x \wedge S$ is an ideal of S containing x.

Let I be an ideal of S such that $x \in I$. Let $a, b \in S$. Then, $a \wedge x \wedge b \preceq x \in I$ so that $a \wedge x \wedge b \in I$ and, therefore, $S \wedge x \wedge S \subseteq I$. The proof regarding principal filters is analogous.

(ii) Let us first show that $x \wedge S \wedge x = \{y \in S \colon y \leq x\}$. Fix $a \in S$. Thus, $x \wedge a \wedge x \leq x$. Conversely, let $a \in S$ such that $a \leq x$. Then $a = x \wedge a \wedge x \in x \wedge S \wedge x$. The proof of $x \vee S \vee x = \{y \in S \colon x \leq y\}$ is analogous.

Now, we will show that $x^{\downarrow *} = x \wedge S \wedge x$. Let $y, z \in x \wedge S \wedge x$. Let $a, b \in S$ such that

$$y = x \wedge a \wedge x \text{ and } z = x \wedge b \wedge x.$$

By absorption,

$$y \vee z \vee x = (x \wedge a \wedge x) \vee (x \wedge b \wedge x) \vee x = (x \wedge a \wedge x) \vee x = x.$$

Similarly $x \vee y \vee z = x$. Hence $y \vee z \leq x$ and its an analogous proof to show that $z \vee y \leq x$. Therefore $y \vee z$,$z \vee y \in x \wedge S \wedge x$.

Now let $a, b \in S$ such that $a \leq x \wedge b \wedge x$. Then

$$a = x \wedge b \wedge x \wedge a \wedge x \wedge b \wedge x \in x \wedge S \wedge x.$$

By idempotency,

$$x = x \wedge x \wedge x \in x \wedge S \wedge x.$$

Thus $x \wedge S \wedge x$ is an ideal of S containing x.

Let I be an ideal of S such that $x \in I$. Let $a \in S$. As $x \wedge a \wedge x \leq x \in I$ then $x \wedge a \wedge x \in I$ so that $x \wedge A \wedge x \subseteq I$. The proof regarding principal skew filters is analogous.

□

Corollary 3.3.1. *[29] Let S be a skew lattice and $x \in S$. Then, $x \wedge S \wedge x \subseteq S \wedge x \wedge S$ and, dually, $x \vee S \vee x \subseteq S \vee x \vee S$.*

Proof. Let $a \in S$. Then $x \wedge a \wedge x \preceq x$ so that $x \wedge a \wedge x \in S \wedge x \wedge S$. The duality have the similar proof. □

Example 3.3.1. *Let S be the skew lattice given in Example 2.4.2 The ideal generated by the set $X = \{ 0, 2, 3 \}$ is $X^{\downarrow} = \{ 0, 2, 3 \} = 2^{\downarrow}$. The skew ideal generated by the set $y = \{ 0, 1, 2 \}$ is $y^{\downarrow *} = \{ 0, 1, 2, 3 \} = S$ and the principle skew ideal generated by the element 2 is $2^{\downarrow *} = \{ 0, 2 \}$. Since 1 is the top element in S, $x \leq 1$ for all $x \in S$ and so $1^{\downarrow *} = S$.*

Theorem 3.3.2. *[29] Let S be a skew lattice X a nonempty subset of S and $x, y \in S$. Then,*

(i) $X^{\downarrow} = \bigcup\{\mathcal{D}_x^{\downarrow} \colon x \in X\}$ *and* $X^{\uparrow} = \bigcup\{\mathcal{D}_x^{\uparrow} \colon x \in X\}$.

(ii) $x^{\downarrow} = y^{\downarrow} \Leftrightarrow \mathcal{D}_x = \mathcal{D}_y$ *and* $x^{\uparrow} = y^{\uparrow} \Leftrightarrow \mathcal{D}_x = \mathcal{D}_y$.

Proof. (*i*) This is direct consequence of Remark 3.1.1: as the ideals in S are just the unions of the $\mathcal{D}$-classes that constitute the elements of the ideals of $S/\mathcal{D}$, then the principal ideals are just unions of blocks constituting the corespondent principal lattice ideal.

(*ii*) Let $x^{\downarrow} = y^{\downarrow}$. Then $x \preceq y$ and $y \preceq x$, by the definition. Thus $x\mathcal{D}y$ which implies $\mathcal{D}_x = \mathcal{D}_y$. Conversely, by Theorem 3.1.3, $x\mathcal{D}y$ implies $x \in y^{\downarrow}$ and $y \in x^{\downarrow}$ implies $x^{\downarrow} \subseteq y^{\downarrow}$ and $y^{\downarrow} \subseteq x^{\downarrow}$.

□

Theorem 3.3.3. *[29] Let S be a skew lattice X a nonempty subset of S and $x, y \in S$. Then,*

(*i*) $x^{\downarrow *} \cap \mathcal{D}_x = \{x\}$ *and* $x^{\uparrow *} \cap \mathcal{D}_x = \{x\}$.

(*ii*) $x^{\downarrow *} = y^{\downarrow *} \Leftrightarrow x = y$ *and* $x^{\uparrow *} = y^{\uparrow *} \Leftrightarrow x = y$.

Proof. (*i*) Let $a \in x^{\downarrow *} \cap \mathcal{D}_x$. Then, $a \leq x$ and $a\mathcal{D}x$ so that $a = x$.

(*ii*) Suppose that $x^{\downarrow *} = y^{\downarrow *}$. Then $x \leq y$ and $y \leq x$ which implies $x\mathcal{D}y$, and thus by Theorem 2.4.5, $x = y$.

□

Theorem 3.3.4. *[29] Let S be a skew lattice. For all $x \in S$, the principal ideal $x^{\downarrow}$ is the union of all principal skew ideals $y^{\downarrow *}$ such that $y \in \mathcal{D}_x$.*

Proof. Let $x \in S$. Suppose that $a \in x^{\downarrow}$ and $a \notin \mathcal{D}_x$. Then by Remark 2.4.2, $\mathcal{D}_a \leq \mathcal{D}_x$. Theorem 2.4.6 implies that there exist $y \in \mathcal{D}_x$ such that $a \leq y$. Thus, $a \in y^{\downarrow *}$. On the other hand, if $a \in y^{\downarrow *}$ with $y\mathcal{D}x$, then $a \leq y \preceq x$. Thus $a \preceq x$ and so $a \in x^{\downarrow}$. □

Theorem 3.3.5. *[29] Let S be a skew lattice and $x, y \in S$ such that $x\mathcal{D}y$. Then,* $| x^{\downarrow *} | = | y^{\downarrow *} |$.

Proof. Consider the maps $\phi\colon x^{\downarrow *} \to y^{\downarrow *}$ and $\varphi\colon y^{\downarrow *} \to x^{\downarrow *}$ defined by $\phi(a) = y \wedge a \wedge y$ and $\varphi(b) = x \wedge b \wedge x$, for every $a \in x^{\downarrow *}$ and $b \in y^{\downarrow *}$. Both of these maps are clearly well defined. Now, we have to show that they are the inverse of each other. Let $c \in y^{\downarrow *}$. Then,

$$\phi \circ \varphi(c) = y \wedge x \wedge c \wedge x \wedge y = y \wedge x \wedge y \wedge c \wedge y \wedge x \wedge y = y \wedge c \wedge y = c$$

due to regularity and the assumption that $x\mathcal{D}y$. Similarly, $\varphi \circ \phi(c) = c$. Thus the inverse functions φ and ϕ are one to one and so $| x^{\downarrow *} | = | y^{\downarrow *} |$. □

Corollary 3.3.2. *[29] Let S be a skew lattice and $x \in S$. Then,*

$$| x^{\downarrow} | \leq | \mathcal{D}_x | \cdot | x^{\downarrow *} |.$$

Proof. Let $x \in S$. By Theorem 3.3.4,

$$x^{\downarrow} = \bigcup_{y \in \mathcal{D}_x} y^{\downarrow *}.$$

Then,

$$\begin{aligned} | x^{\downarrow} | &= | \bigcup_{y \in \mathcal{D}_x} y^{\downarrow *} | \\ &\leq \Sigma_{y \in \mathcal{D}_x} | y^{\downarrow *} | . \end{aligned}$$

By Theorem 3.3.5,

$$\begin{aligned} &= \Sigma_{y \in \mathcal{D}_x} | x^{\downarrow *} | \\ &= | \mathcal{D}_x | \cdot | x^{\downarrow *} | . \end{aligned}$$

□

Example 3.3.2. *Let S be a skew lattice given in Example 2.4.2. Then,* $2^{\downarrow} = \{ 0, 2, 3 \}$, $2^{\downarrow *} = \{ 0, 2 \}$ *and* $\mathcal{D}_2 = \{ 2, 3 \}$. *Thus*

$$| 2^{\downarrow} | \leq | \mathcal{D}_2 | \cdot | 2^{\downarrow *} |$$

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