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Combined Integral Transform - Adomain Decomposition Methods For Solving Nonlinear Differential Equations

Submitted by:

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Declaration

I declare that the Master Thesis entitled **Combined Integral Transform - Adomain Decomposition Methods For Solving Nonlinear Differential Equations** is my original work, and hereby certify that unless stated, all work contained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgement is made in the text.

Besan Abueid

Signature: ____________________________ Date: ___________________

DEDICATION

To my parents,

To my brothers and sisters,

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Abstract

The Adomain decomposition method is a semi-analytical technique for solving nonlinear differential equations. In the literature, one can find that this method is combined with integral transforms such as Laplace, Natural, Sumudu, and Elzaki transforms.

This thesis presents some famous integral transforms couplied with the Adomain decomposition method. These transforms include, the natural transform, the double natural transform, Laplace transform, the double Laplace transform, Elzaki transform and Sumudu transform. These transforms are presented with their properties. Then they are combined with the Adomain decomposition method to solve nonlinear ordinary and partial differential equations.

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List of Abbreviations

ADM	Adomain Decomposition Method
NDM	Natural Decomposition Method
LDM	Laplace Decomposition Method
SDM	Sumudu Decomposition Method
EDM	Elazaki Decomposition Method

Chapter 1

Introduction

1.1 Overview

Mathematical models encountered in applied mathematics, mathematical physics and engineering systems mostly tend to be nonlinear differential equations. These nonlinear equations are difficult in finding the exact or approximation solutions caused by the nonlinear part.

There are many methods have been proposed to solve nonlinear differential equations. The *Adomain Decomposition Method* (ADM) is one of these method. This method is a semi technique based on decomposing the solution to a series of functions.

At the beginning of 1980s, Adomain proposed the ADM to solve nonlinear equations [3,5]. In this method, the given equation is decomposed in linear and nonlinear parts of the differential equation. Inverting and applying the highest order differential operator that is contained in the linear part of equation, it is possible to express the solution in terms of the rest of the equation affected by the inverse operator. At this point, the solution is proposed by means of series with terms that will be determined and that give the *Adomain polyno-*

mials, in this way and by means of a recurrence relations, it is possible to find the terms of the series that give the approximate solution of the differential equation.

The ADM and its modifications [41, 43] has been used to solve linear and nonlinear differential equations, and the theoretical treatment of the convergence of Adomain decomposition method has been considered in [1, 17]. A convergence of Adomain's technique is ensured with week hypothesis on the nonlinear operator and on the functional equation.

Combined methods of the ADM with integral transforms have been proposed to handle nonlinear problems, Several transforms have been used like Laplace [25, 32, 33], natural [37, 38], Sumudu [11, 28], Elzaki [22, 34],Aboodh [39],

This thesis is mainly concerned with the combined integral transform -Adomain decomposition methods, both nonlinear ordinary and partial diffrential equations are considerd.

This thesis consist of four chapters:

Chapter 1, is an introductory chapter. The basics of the Adomain decomposition method is presented with convergence analysis.

In *Chapter 2*, we consider the natural transform decomposition method (NDM). This method is used to approximate the solutions of ordinary differential equations [37] and partial diferential equations [38]. In addition, we present double natural transform decomposition method.

In *Chapter3*, we consider the Laplace transform coupling with the decomposition method and named (LDM), this method is a numerical algorithm to solve nonlinear ordinary, partial differential equations, see Khuri [32, 33]. Moreover, we present the double decomposition method ; the method is combination of double Laplace transform with the ADM.

Chapter 4 is devoted to the Sumudu transform [11] and Elzaki transform [21]. These transforms are introduced and combined with the ADM to solve nonlinear differential equations.

1.2 Adomain Decomposition Method

In this section, we brifely recall of the *Adomain Decomposition Method* (ADM), which is a technique for solving algebraic equations, ordinary differential equations, partial differential equations, and integral equations, see [4, 6, 7]. For the material of this section we refer to [16].

Consider the more general nonlinear ordinary differential equation

$$Ly(x) + Ry(x) + \mathcal{N}y(x) \quad = \quad g(x).$$

where

L is the highest order derivative which is assumed to be invertible.

R is a linear differential operator of order less than L.

$\mathcal{N}y$ represents the nonlinear terms.

$g(x)$ represents the nonhomogeneous terms.

Solving for Ly, we get

$$Ly(x) \quad = \quad g(x) - Ry(x) - \mathcal{N}y(x). \tag{1.1}$$

Applying the inverse operator L^{-1} to both sides of Equation (1.1), we obtain

$$L^{-1}Ly = L^{-1}(g(x)) - L^{-1}(Ry(x)) - L^{-1}(\mathcal{N}y(x)), \tag{1.2}$$

where the integral operator L^{-1} may be regarded as definite integral from 0 to x. Here L^{-1} is the integration n times, where n is the highest order of the derivative.

$$L^{-1} = \int_0^x \int_0^x ... \int_0^x \mathrm{d}x\,\mathrm{d}x...\mathrm{d}x$$

$$L^{-1}Ly = y(x) - y(0) - xy'(0) - \frac{x^2}{2!}y''(0) - ... - \frac{x^n}{n!}y^{(n)}(0). \tag{1.3}$$

By substituting (1.3) in (1.2) we have

$$\begin{aligned} y(x) - y(0) - xy'(0) - \frac{x^2}{2!}y''(0) - ... - \frac{x^n}{n!}y^{(n)}(0) \\ = L^{-1}(g(x)) - L^{-1}(Ry(x)) - L^{-1}(\mathcal{N}y(x)). \end{aligned}$$

Then,

$$\begin{aligned} y(x) = L^{-1}(g(x)) + y(0) + xy'(0) + \frac{x^2}{2!}y''(0) \\ + ... + \frac{x^n}{n!}y^n(0) - L^{-1}(Ry(x)) - L^{-1}(\mathcal{N}y(x)). \end{aligned} \tag{1.4}$$

Now, replacing the unknown function y by an infinite series of y_n

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{1.5}$$

The nonlinear terms $\mathcal{N}y$ is decomposed as an infinite series of the *Adomain Polynomials*,

$A_n's$ given by

$$\mathcal{N}y = \sum_{n=0}^{\infty} A_n(y_0, y_1, y_2, ...). \tag{1.6}$$

To compute A_n, take $\mathcal{N}y = f(y)$ to be a function in y, where $y = y(x)$, then the Taylor series expansion of $f(y)$ around y_0 is given by:

$$f(y) = f(y_0) + f'(y_0)(y - y_0) + \frac{1}{2!}f''(y_0)(y - y_0)^2 + \frac{1}{3!}f'''(y_0)(y - y_0)^3 + ...$$

But $y = y_0 + y_1 + y_2 + ...$, then

$$\begin{aligned} f(y) &= f(y_0) + f'(y_0)(y_1 + y_2 + y_3 + ...) + \frac{1}{2!}f''(y_0)(y_1 + y_2 + y_3 + ...)^2 \\ &\quad + \frac{1}{3!}f'''(y_0)(y_1 + y_2 + y_3 + ...)^3 + ... \\ &= f(y_0) + f'(y_0)y_1 + f'(y_0)y_2 + ... + \frac{1}{2!}f''(y_0)y_1^2 + \frac{2}{2!}f''(y_0)y_1y_2 + \frac{1}{2!}f''(y_0)y_1y_3 \\ &\quad + ... + \frac{1}{3!}f'''(y_0)y_1^3 + \frac{3}{3!}f'''(y_0)y_1^2y_2 + \frac{1}{3!}f'''(y_0)y_1^2y_3 + ... \end{aligned}$$

Now, let $(l)(i)$ be the order of y_l^i and $(l)(i)+(m)(j)$ be the order of $y_l^i y_m^j$. Then A_n consists of all terms of order n, we have

$$A_0 = f(y_0)$$

$$A_1 = f'(y_0)y_1$$

$$A_2 = f'(y_0)y_2 + \frac{1}{2!}f''(y_0)y_1^2$$

$$A_3 = f'(y_0)y_3 + \frac{2}{2!}f''(y_0)y_1y_2 + \frac{1}{3!}f'''(y_0)y_1^3$$

$$A_4 = f'(y_0)y_4 + \frac{1}{2!}f''(y_0)(2y_1y_3 + y_2^2) + \frac{3}{3!}f'''(y_0)y_1^2y_2 + \frac{1}{4!}f''''(y_0)y_1^4$$

$$\vdots$$

Or

$$A_0 = \mathcal{N}(y_0)$$

$$A_1 = \mathcal{N}'(y_0)y_1$$

$$A_2 = \mathcal{N}'(y_0)y_2 + \frac{1}{2!}\mathcal{N}''(y_0)y_1^2$$

$$A_3 = \mathcal{N}'(y_0)y_3 + \frac{2}{2!}\mathcal{N}''(y_0)y_1y_2 + \frac{1}{3!}\mathcal{N}'''(y_0)y_1^3$$

$$A_4 = \mathcal{N}'(y_0)y_4 + \frac{1}{2!}\mathcal{N}''(y_0)(2y_1y_3 + y_2^2) + \frac{3}{3!}\mathcal{N}'''(y_0)y_1^2y_2 + \frac{1}{4!}\mathcal{N}''''(y_0)y_1^4$$

$$\vdots$$

Hence,

$$A_n = A_n(y_o, y_1, ..., y_n) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\mathcal{N}\left[\sum_{m=0}^{\infty}\lambda^m y_m\right]_{\lambda=0}. \tag{1.7}$$

Now, substituting (1.5) and (1.6) in (1.4), and solve it for y to get

$$\sum_{n=0}^{\infty} y_n = L^{-1}g(x) + \phi_0 - L^{-1}(Ry(x)) - L^{-1}(\mathcal{N}y(x)).$$

Where

$$\phi_0 = \begin{cases} y(0) - xy'(0) & \text{if} \quad L = \dfrac{d}{dx} \\ y(0) - xy'(0) - \dfrac{x^2}{2!}y''(0) & \text{if} \quad L = \dfrac{d^2}{dx^2} \\ \vdots & \\ y(0) - xy'(0) - \dfrac{x^2}{2!}y''(0) - ... - \dfrac{x^n}{n!}y^n(0) & \text{if} \quad L = \dfrac{d^{n+1}}{dx^{n+1}} \end{cases}$$

Therefore

$$\begin{aligned} y_0 &= \phi_0 + L^{-1}g(x), \\ y_{n+1} &= -L^{-1}(Ry_n) - L^{-1}(A_n) \end{aligned}$$

Example 1.1. Consider the nonlinear differential equation

$$y'(x) + y^2(x) = -1, \tag{1.8}$$

with initial condition

$$y(0) = 0. \tag{1.9}$$

Solution:

We can write the equation as

$$y'(x) = -1 - y^2(x)$$

Let $L = \dfrac{d}{dx}$, then

$$Ly = -1 - y^2(x) \tag{1.10}$$

The Adomain polynomials are

$$A_0 = y_0^2$$

$$A_1 = 2y_0y_1$$

$$A_2 = 2y_0y_2 + \frac{1}{2!}2y_1^2$$

$$A_3 = 2y_0y_3 + 2y_1y_2$$

$$\vdots$$

Take the

$$L^{-1} = \int_0^x dx$$

of (1.10) we get

$$y(x) = L^{-1}(-1) - L^{-1}(y^2(x))$$

or

$$\sum_{n=0}^{\infty} y_n(x) = -x - L^{-1}(\sum_{n=0}^{\infty} A_n)$$

Hence

$$\begin{aligned}
y_0 &= -x \\
y_1 &= -L^{-1}(A_0) = -\int_0^x (-x)^2 dx = \frac{-x^3}{3} \\
y_2 &= -L^{-1}(A_1) = -\int_0^x 2x\frac{x^3}{3} dx = -\int_0^x 2\frac{x^4}{3} dx = -2\frac{x^5}{15} \\
y_3 &= -L^{-1}(A_2) = -\int_0^x 2(-x)\frac{-2x^5}{15} + \left(\frac{x^3}{3}\right)^2 dx = -\left[\frac{-14x^7}{105} + \frac{x^7}{63}\right] = \frac{-17x^7}{315} \\
&\vdots
\end{aligned}$$

Then the solution is

$$\begin{aligned} y(x) &= -x - \frac{x^3}{3} - \frac{2x^5}{15} - \frac{17x^7}{315} - ... \\ &= -x - \frac{2x^3}{3!} - \frac{16x^5}{5!} - \frac{272x^7}{7!} - ... \end{aligned}$$

The computed terms in this series coincide with Maclaurine series for the function

$$y(x) = -\tan(x).$$

In fact $y(x) = -\tan(x)$ is the exact solution of (1.8) with condition (1.9).

1.3 Convergence of Adomain Decomposition Method

In this section, a general proof of convergence for the ADM is introduced. This technique was proposed by Cherruault et al [18]. They also proved some results about on speed of convergence for this method.

Consider the following a general functional equation:

$$y - \mathcal{N}(y) = f, \tag{1.11}$$

where $\mathcal{N} : H \to H$ is the nonlinear operator and H is a Hilbert space, and $f = L^{-1}g$ is a given function in H.

Assuming that y is the solution of (1.11) and the nonlinear operator $N(y)$ are decomposed into infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x),$$

and

$$\mathcal{N}(y) = \sum_{n=0}^{\infty} A_n,$$

where A_n's are Adomain polynomials.

Now, substituting these decomposition series in (1.11), we get

$$\sum_{n=0}^{\infty} y_n(x) - \sum_{n=0}^{\infty} A_n(x) \quad = \quad f.$$

Then the recursive terms can be written as

$$\begin{aligned} y_0 &= f, \\ y_{n+1} &= A_n(y_0, y_1, ..., y_n). \end{aligned}$$

Let

$$S_n \quad = \quad y_1 + y_2 + ... + y_n.$$

Then the Adomain decomposition method is equivalent to

$$\begin{aligned} S_0(x) &= 0, \\ S_{n+1} &= \mathcal{N}(y_0 + S_n), \end{aligned}$$

where

$$\mathcal{N}(y_0 + S_n) = \sum_{n=0}^{\infty} A_n(x).$$

$S = \lim_{n\to\infty} S_n$ if the limit exist in the Hilbert space H, then S is a solution of a fixed point equation,

$$S = \mathcal{N}(y_0 + S) \quad in\ H \tag{1.12}$$

Theorem 1.1. Let N be nonlinear operator on a Hilbert space H. The decomposition series $\sum_0^\infty y_n$ of y converges to y when

$$\exists 0 < \alpha < 1 \ \textbf{such that} \ ||y_{n+1}|| \leq \alpha ||y_n|| \ \textbf{for} \ n = 0, 1, 2, ...$$

Proof. We have

$$\begin{aligned} S_0 &= 0 \\ S_1 &= y_1 \\ S_2 &= y_1 + y_2 \\ &\vdots \\ S_n &= y_1 + y_2 + ... + y_n \end{aligned}$$

We want need to show that $\{S_n\}_{n=0}^\infty$ is a Cauchy sequence in the Hilbert space H.

$$||S_{n+1} - S_n|| = ||y_{n+1}|| \leq \alpha ||y_n|| \leq \alpha^2 ||y_{n-1}|| \leq ... \leq \alpha^{n+1} ||y_0||$$

But for any $n, m \in \mathbb{N}, n \geq m$, we have

$$\begin{aligned} ||S_n - S_m|| &= ||(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + ... + (S_{m+1} - S_m)|| \\ &\leq ||(S_n - S_{n-1})|| + ||(S_{n-1} - S_{n-2})|| + ... + ||(S_{m+1} - S_m)|| \\ &\leq \alpha^n ||y_0|| + \alpha^{n-1} ||y_0|| + ... + \alpha^{m+1} ||y_0|| \\ &= (\alpha^n + \alpha^{n-1} + ... + \alpha^{m+1}) ||y_0|| \\ &= (\alpha^{m+1} + \alpha^{m+2} + ... + \alpha^n) ||y_0|| \\ &\leq (\alpha^{m+1} + \alpha^{m+2} + ...) ||y_0|| \\ &= \frac{\alpha^{m+1}}{1-\alpha} ||y_0|| \end{aligned}$$

But $\alpha < 1$, so

$$\lim_{n,m\to\infty} ||S_n - S_m|| = 0,$$

hence $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space H and this implies that

$$\lim_{n\to\infty} S_n = S, \quad S \in H$$

i.e. $S = \sum_{n=0}^{\infty} y_n$, but solving (1.11) is equivalent to solving (1.12) and by assuming that N is a continuous operator, then

$$\begin{aligned} N(y_0 + S) &= \mathcal{N}(\lim_{n\to\infty}(y_0 + S_n)) \\ &= \lim_{n\to\infty} \mathcal{N}(y_0 + S_n) \\ &= \lim_{n\to\infty} S_{n+1} \\ &= S \end{aligned}$$

i.e. S is the solution of (1.11). □

Chapter 2

Natural Transform With Adomain Decomposition Method (NDM)

2.1 Natural Transform

Natural Transform is an integral transform similar to the Laplace transform, the Natural Transform was introduced by Khan in 2008 [31], and its properties were investigated by AL-Omari in 2013 [8]. This transform plays a role in techniques for solving ordinary differential equations.

Definition 2.1. [14] Let $f(t)$ be a function defined for $t \in (-\infty, \infty)$, then the Natural Transform of $f(t)$ denoted by $R(s, u)$ is defined as

$$N[f(t)] = R(s, u) = \int_{-\infty}^{\infty} f(ut)e^{-st}dt, \quad s, u \in (-\infty, \infty). \tag{2.1}$$

Provided the integral is convergent, here $N[f(t)]$ is called the *natural transform of time function* and the variable s and u are the *natural transform variables*.

Equation (2.1) can be written as

$$\begin{aligned}
N[f(t)] &= \int_{-\infty}^{\infty} f(ut)e^{-st}dt, \quad s,u \in (-\infty,\infty) \\
&= \left[\int_{-\infty}^{0} f(ut)e^{-st}dt, \quad s,u \in (-\infty,0)\right] + \left[\int_{0}^{\infty} f(ut)e^{-st}dt, \quad s,u \in (0,\infty)\right] \\
&= N^{-}[f(t)] + N^{+}[f(t)] \\
&= N[f(t)H(-t)] + N[f(t)H(t)] \\
&= R^{-}(s,u) + R^{+}(s,u),
\end{aligned}$$

where $H(\cdot)$ is a *Heaviside function*, i.e. $H(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t \geq 0 \end{cases}$

If the function $f(t)H(t)$ is defined on the positive real axis with $t \in \mathbb{R}$, then we define the natural transform on the set

$$A = \{f(t) : \exists M, \tau_1, \tau_2 > 0 \ such \ that \ |f(t)| < Me^{|t|/\tau_j}, \ if \ t \in (-1)^j \times [0,\infty), \ j = 1,2\}$$

by

$$N[f(t)H(t)] = N^{+}[f(t)] = R^{+}(s,u) = \int_{0}^{\infty} f(ut)e^{-st}dt, \quad s,u \in (0,\infty).$$

The Natural transform throughout the thesis of the function $f(t) > 0$ and $f(t) = 0$ for $t < 0$ is defined by

$$N^{+}[f(t)] = R(s,u) = \int_{0}^{\infty} f(ut)e^{-st}dt, \quad s > 0, u > 0 \tag{2.2}$$

Next, some examples are given.

Example 2.1. Unit step function

Let $u(t) = \begin{cases} 1 & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t \leq 0. \end{cases}$

The N-Transform of this function can be written as

$$\begin{aligned} N^{+}[u(t)] &= \int_0^{\infty} e^{-st} dt \\ &= \lim_{c \to \infty} \frac{-e^{-st}}{s} \Big|_0^c \\ &= \frac{1}{s} \end{aligned}$$

Example 2.2. Exponential function:

Let $f(t) = e^{at}$ when $t \geq 0$, where a is constant, the N-Transform of this function can be written as

$$\begin{aligned} N^{+}[f(t)] &= N[e^{at}] \\ &= \int_0^{\infty} e^{aut} e^{-st} dt \\ &= \lim_{c \to \infty} \int_0^{c} e^{-(s-au)t} dt \\ &= \lim_{c \to \infty} \frac{e^{-(s-au)t}}{-(s-au)} \Big|_0^c = \frac{1}{s-au} \end{aligned}$$

In Table 2.1, N-transform for some functions are given.

Properties of N-Transform

In this section, the main properties are presented. For detailed studies of N-transform and its properties, we refer to Belgacem and Silambarasan [12–14] and Khan [31].

Theorem 2.1. Linearity Property

$f(t)$	$R(s,u)$
1	$\frac{1}{s}$
t	$\frac{u}{s^2}$
$\sin t$	$\frac{u}{s^2+u^2}$
$\cos t$	$\frac{s}{s^2+u^2}$
$\sinh at$	$\frac{au}{s^2-a^2u^2}$
$\cosh at$	$\frac{s}{s^2-a^2u^2}$
$\frac{t^{n-1}}{(n-1)!}$	$\frac{u^{n-1}}{s^n}$

Table 2.1: Natural transform of some functions.

If a and b are any constants and $f(t)$ and $g(t)$ are functions, then

$$N^+[af(t)+bg(t)] = aN^+[f(t)] + bN^+[g(t)]$$

Proof. $N^+[f(t)] = \int_0^\infty f(ut)e^{-st}dt$ and $N^+[g(t)] = \int_0^\infty g(ut)e^{-st}dt.$

If a and b are any constants, then

$$\begin{aligned} N^+[af(t)+bg(t)] &= \int_0^\infty [af(ut)+bg(ut)]e^{-st}dt \\ &= a\int_0^\infty f(ut)e^{-st}dt + b\int_0^\infty g(ut)e^{-st}dt \\ &= aN^+[f(t)] + bN^+[g(t)] \end{aligned}$$

□

Theorem 2.2. First Translation or Shifting Property

Let $f(t)$ be a continuous function and $t \geq 0$. Then

$$N[e^{at}f(t)] = \frac{s}{s-au}R\left[\frac{su}{s-au}\right]$$

Proof. The N-transform of $e^{at}f(t)$ is given by

$$N[e^{at}f(t)] = \int_0^\infty f(ut)e^{-(s-au)t}dt.$$

Therefore, by change of variable $w = \dfrac{s-au}{s}t$, we get

$$\begin{aligned} N[e^{at}f(t)] &= \frac{s}{s-au}\int_0^\infty f\left(\frac{usw}{s-au}\right)e^{-sw}dw \\ &= \frac{s}{s-au}R\left[\frac{su}{s-au}\right] \end{aligned}$$

□

Theorem 2.3. Scaling Property

Let $N^+[f(t)] = R(s,u)$. Then

$$N^+[f(at)] = \frac{1}{a}R\left[\frac{s}{a},u\right].$$

Proof.

$$\begin{aligned} N^+[f(at)] &= \int_0^\infty f(aut)\ e^{-st}\ dt \qquad\qquad let\ \ p = at \\ &= \int_a^\infty f(up)\ e^{\frac{-s}{a}p}\ \frac{dp}{a} \\ &= \frac{1}{a}\int_a^\infty\ f(up)\ e^{\frac{-s}{a}p}\ dp \\ &= \frac{1}{a}R\left[\frac{s}{a},u\right] \end{aligned}$$

□

N-transform of Derivatives

Theorem 2.4. If $N^{+}[f(t)] = R(s,u)$, then

$$N^{+}[f'(t)] = R_1(s,u) = \int_0^\infty f'(t)e^{-st}dt = \frac{s}{u}R(s,u) - \frac{f(0)}{u}$$

Proof.

$$\begin{aligned}
N^{+}[f'(t)] &= \int_0^\infty f'(ut)e^{-st}dt \\
&= \lim_{c\to\infty}\int_0^c f'(ut)e^{-st}dt \qquad Integration\ by\ parts \\
&= \lim_{c\to\infty}\frac{f(ut)e^{-st}}{u}\Big|_0^c + \frac{s}{u}\int_0^c f'(ut)e^{-st}dt \\
&= \lim_{c\to\infty}\left[\frac{f(uc)e^{-sc}}{u} - \frac{f(0)}{u}\right] + \frac{s}{u}\int_0^c f(ut)e^{-st}dt \\
&= \frac{s}{u}R(s,u) - \frac{f(0)}{u}
\end{aligned}$$

□

Theorem 2.5. If $N[f(t)] = R(s,u)$, then

$$N^{+}[f''(t)] = R_2(s,u) = \frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}f(0) - \frac{f'(0)}{u}$$

Proof. $N[G'(t)] = \frac{s}{u}N[G(t)] - \frac{f(0)}{u}$. Let $G(t) = f'(t)$, then

$$\begin{aligned} N[f''(t)] &= \frac{s}{u}N[f'(t)] - \frac{f'(0)}{u} \\ &= \frac{s}{u}\left\{\frac{s}{u}N[f(t)] - \frac{f(0)}{u}\right\} - \frac{f'(0)}{u} \\ &= \frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}f(0) - \frac{f'(0)}{u} \end{aligned}$$

□

Theorem 2.6. If $N[f(t)] = R(s,u)$, then

$$N[f^n(t)] = R_n(s,u) = \frac{s^n}{u^n}R(s,u) - \sum_{k=0}^{n-1}\frac{s^{n-(k+1)}}{u^{n-k}}f^k(0) \tag{2.3}$$

Proof. By mathematical induction.

For $n = 1$ and 2 gives the Natural transform of first and second derivatives of $f(t)$ respectively.

$$\begin{aligned} N[f'(t)] &= \frac{s}{u}R_1(s,u) - \frac{f(0)}{u} \\ N[f''(t)] &= \frac{s^2}{u^2}R_2(s,u) - \frac{s}{u^2}f(0) - \frac{f'(0)}{u} \end{aligned} \tag{2.4}$$

To proceed the induction process, assuming equation (2.3) true for n and prove it for $n+1$, using equation (2.4),

$$\begin{aligned}
N[f^{n+1}(t)] &= N[f'^{n}(t)] = R_{n+1}(s,u) \\
&= \frac{s}{u}R_n(s,u) - \frac{f^n(0)}{u} \\
&= \frac{s}{u}\left[\frac{s^n}{u^n}R(s,u) - \sum_{k=0}^{n-1}\frac{s^{n-(k+1)}}{u^{n-k}}f^k(0)\right] - \frac{f^n(0)}{u} \\
&= \frac{s^{n+1}}{u^{n+1}}R(s,u) - \sum_{k=0}^{n}\frac{s^{n-k}}{u^{(n-k)+1}}f^k(0)
\end{aligned}$$

Which is true for $n+1$. Hence the result (2.3) follows. □

2.2 NDM For Ordinary Differential Equation

Consider the general ordinary differential equation of the form

$$Ly(x) + R(y(x)) + \mathcal{N}(y(x)) = g(x), \tag{2.5}$$

subject to initial condition

$$y(0) = h(x), \tag{2.6}$$

where

$L = \frac{d^n}{dx^n}$ is the operator of highest order.

R is a remainder of the differential operator.

$g(x)$ is a nonhomogeneous term.

$\mathcal{N}(y)$ is a non linear term.

Suppose L is the differential operator of the first order i.e., $L = \frac{d}{dx}$.

Applying the Natural transform of Equation (2.5) we have

$$\frac{s}{u}Y(s,u) - \frac{y(0)}{u} + N^+[R(y)] + N^+[\mathcal{N}(y)] = N^+[g(x)]. \tag{2.7}$$

By substituting (2.6)in (2.7) we obtain

$$Y(s,u) = \frac{h(x)}{s} + \frac{u}{s}N^+[g(x)] - \frac{u}{s}N^+[R(y)] - \frac{u}{s}N^+[\mathcal{N}(y)]. \tag{2.8}$$

Taking the inverse of the Natural transform for equation we obtain

$$y(x) \;=\; \phi(x) - N^{-1}\left[\frac{u}{s}N^+[R(y) + \mathcal{N}(y)]\right]. \tag{2.9}$$

where $\phi(x)$ is a source term.

Rewrite $y(x)$ as an infinite series of y_n, i.e.

$$y(x) \;=\; \sum_{n=0}^{\infty} y_n(x). \tag{2.10}$$

Also the nonlinear term $N(y)$ can be written as an infinite series of an Adomain polynomials, i.e.

$$\mathcal{N}(y) \;=\; \sum_{n=0}^{\infty} A_n, \tag{2.11}$$

where $A_{n's}$ are the polynomials of $y_0, y_1, \ldots, y_n$, which can be calculated by the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\mathcal{N}\left[\sum_{i=0}^{\infty}\lambda^i y_i\right]_{\lambda=0}.$$

Substituting (2.10) and (2.11) into (2.9) to get

$$\sum_{n=0}^{\infty} y_n = \phi(x) - N^{-1}\left[\frac{u}{s}N^{+}[R\sum_{n=0}^{\infty} y_n] + \frac{u}{s}N^{+}[\sum_{n=0}^{\infty} A_n]\right] \tag{2.12}$$

Now by comparing both sides of equation (2.12) we conclude that

$$\begin{aligned} y_0(x) &= \phi(x) \\ y_1(x) &= -N^{-1}\left[\frac{u}{s}N^{+}[R(y_0(x))] + \frac{u}{s}N^{+}[A_0(x)]\right] \\ y_2(x) &= -N^{-1}\left[\frac{u}{s}N^{+}[R(y_1(x))] + \frac{u}{s}N^{+}[A_1(x)]\right] \\ &\vdots \end{aligned}$$

Continuing in this manner we get the general recursive relation

$$y_{n+1}(t) = -N^{-1}\left[\frac{u}{s}N^{+}[R(y_n(x)] + \frac{u}{s}N^{-1}[A_n(x)]\right], \quad n = 0, 1, 2, ... \tag{2.13}$$

Hence from the general recursive relation in equation (2.13), we can easily compute the remaining components of $y(x)$ as $y_1(x)$, $y_2(x)$,..., where $y_0(x)$ is the given initial condition.

Finally, the exact solution is given by

$$y(x) = \sum_{n=0}^{\infty} y_n(x).$$

Example 2.3. Consider the nonlinear ordinary differential equation [40]

$$y' = y^2 - y, \tag{2.14}$$

subject to the initial condition

$$y(0) = -1.$$

Solution:

We solve this problem by the Natural Transform, taking the Natural transform to both sides of Equation (2.14), we have

$$\frac{s}{u}Y(s,u) - \frac{y(0)}{u} = N^+[y^2(x)] - Y(s,u) \tag{2.15}$$

By substituting $y(0) = -1$ we obtain

$$\begin{aligned} \left(\frac{s}{u}+1\right) Y(s,u) &= N^+[y^2(x)] - \frac{1}{u}, \\ Y(s,u) &= \frac{u}{s+u}N^+[y^2(x)] - \frac{1}{s+u} \end{aligned} \tag{2.16}$$

Then by taking the inverse of the Natural transform of the Equation (2.16) we get

$$y(x) = -e^{-x} + (N^+)^{-1}\left[\frac{u}{s+u}N^+\left[y^2(x)\right]\right]. \tag{2.17}$$

Rewrite $y(x)$ as infinite series of y_n

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{2.18}$$

Also decomposing the nonlinear term as

$$y^2 = \sum_{n=0}^{\infty} A_n. \tag{2.19}$$

The Adomain polynomials are

$$\begin{aligned} A_0 &= y_0^2(x) \\ A_1 &= 2y_0(x)y_1(x) \\ A_2 &= 2y_0(x)y_2(x) + y_1^2(x) \\ A_3 &= 2y_0y_3 + 2y_1y_2 \\ &\vdots \end{aligned}$$

By using (2.18) and (2.19) we can write (2.17) as

$$\sum_{n=0}^{\infty} y_n = -e^{-x} + (N^+)^{-1}\left[\frac{u}{s+u}N^+\left[\sum_{n=0}^{\infty} A_n(x)\right]\right]. \tag{2.20}$$

Then by comparing both side of equation (2.20) we obtain

$$y_0(x) = -e^{-x}$$

We can easily compute the remaining components of the unknown function $y(x)$ as follow

$$\begin{aligned} y_1(x) &= (N^+)^{-1}\left[\frac{u}{s+u}N^+[A_0(x)]\right] \\ &= (N^+)^{-1}\left[\frac{u}{s+u}N^+[y_0^2(x)]\right] \\ &= (N^+)^{-1}\left[\frac{u}{s+u}N^+[e^{-2x}]\right] \\ &= (N^+)^{-1}\left[\frac{u}{s+u}\frac{1}{s+2u}\right] \\ &= (N^+)^{-1}\left[\frac{1}{s+u} - \frac{1}{s+2u}\right] \\ &= e^{-x} - e^{-2x} \end{aligned}$$

$$\begin{aligned}
y_2(x) &= (N^+)^{-1}\left[\frac{u}{s+u}N^+[A_1(x)]\right] \\
&= (N^+)^{-1}\left[\frac{u}{s+u}N^+[2y_0(x)y_1(x)]\right] \\
&= (N^+)^{-1}\left[\frac{u}{s+u}N^+[-2e^{-x}(e^{-x}-e^{-2x})]\right] \\
&= -e^{-x}+2e^{-2x}-e^{-3x}
\end{aligned}$$

$$\begin{aligned}
y_3(x) &= (N^+)^{-1}\left[\frac{u}{s+u}N^+[A_2(x)]\right] \\
&= (N^+)^{-1}\left[\frac{u}{s+u}N^+[2y_0(x)y_2(x)+y_1^2(x)]\right] \\
&= (N^+)^{-1}\left[\frac{u}{s+u}N^+[-2e^{-x}(-e^{-x}+2e^{-2x}-e^{-3x})+(e^{-x}-e^{-2x})^2]\right] \\
&= -e^{-4x}+3e^{-3x}-3e^{-2x}+e^{-x}
\end{aligned}$$

Then the approximate solution is given by

$$y(x) \approx \sum_{n=0}^{\infty} y_n(x) = y_0(x)+y_1(x)+y_2(x)+...$$

Leads to the exact solution of the form

$$y(x) = \frac{1}{1-2e^x}$$

For validation, we draw the approximate solution versus the exact solution, see Figure 2.1.

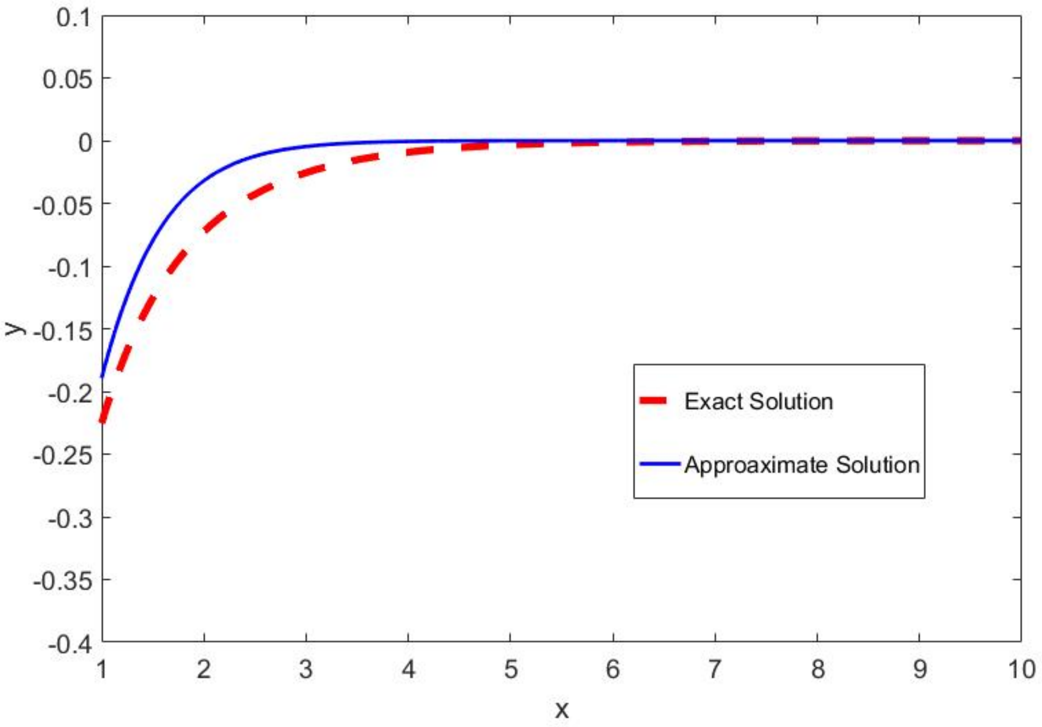

Figure 2.1: comparison between the exact solution and the solution obtained by the NDM, Example 2.3.

2.3 NDM for partial differential equations

The NDM can easily be used to solve a wide class of nonlinear partial equations and obtain an exact or analytical solution.

Example 2.4. Consider the nonlinear partial differential equation [38]

$$u_t(x,t) + u(x,t)u_x(x,t) = 0, \tag{2.21}$$

subject to initial condition

$$u(x,0) = u_0(x) = x. \tag{2.22}$$

Solution:

We solve (2.21) with (2.22) by the NDM, taking the Natural transform with respect to t for both sides of equation (2.21) , we have

$$\frac{s}{u}N^{+}[u(x,t)] - \frac{1}{u}u(x,0) = -N^{+}[u(x,t)u_x(x,t)]. \tag{2.23}$$

By using (2.22) in (2.23) we obtain

$$N^{+}[u(x,t)] \;=\; \frac{x}{s} - \frac{u}{s}N^{+}[u(x,t)u_x(x,t)]. \tag{2.24}$$

Then by taking the inverse of the Natural transform of the equation (2.24) we have

$$u(x,t) \;=\; x - N^{-1}\left[\frac{u}{s}N^{+}[u(x,t)u_x(x,t)]\right]. \tag{2.25}$$

Now, rewrite $u(x,t)$ is an infinite series of the form

$$u(x,t) \;=\; \sum_{n=0}^{\infty} u_n(x,t). \tag{2.26}$$

Also assuming Adomain polynomial for the nonlinear term as

$$uu_x \;=\; \sum_{n=0}^{\infty} A_n. \tag{2.27}$$

Where

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}N\left[\sum_{i=0}^{\infty}\lambda^i y_i\right]_{\lambda=0}, \quad n \geq 0$$

Then,

$$\begin{aligned}
A_0 &= u_0(u_0)_x \\
A_1 &= u_1(u_0)_x + u_0(u_1)_x \\
A_2 &= u_0(u_2)_x + u_2(u_0)_x + u_1(u_1)_x \\
A_3 &= u_3(u_0)_x + u_0(u_3)_x + u_2(u_1)_x + u_1(u_2)_x \\
&\vdots
\end{aligned}$$

By using (2.26) and (2.27) we can write Equation (2.25) as

$$\sum_{n=0}^{\infty} u_n(x,t) = x - N^{-1}\left[\frac{u}{s}N\left[\sum_{n=0}^{\infty} A_n(u)\right]\right] \tag{2.28}$$

Now,

$$\begin{aligned}
&u_0(x,t) = x \\
&\sum_{n=0}^{\infty} u_{n+1}(x,t) = -N^{-1}\left[\frac{u}{s}N\left[\sum_{n=0}^{\infty} A_n(u)\right]\right].
\end{aligned}$$

So, we can obtain first components of (2.28) as follow

$$\begin{aligned}
u_1(x,t) &= -N^{-1}\left[\frac{u}{s}N^{+}[A_0(u)]\right] \\
&= -N^{-1}\left[\frac{u}{s}N^{+}[u_0(u_o)_x]\right] \\
&= -N^{-1}\left[\frac{u}{s}N^{+}[x]\right] \\
&= -xN^{-1}\left[\frac{u}{s^2}\right] \\
&= -xt
\end{aligned}$$

$$\begin{aligned}
u_2(x,t) &= -N^{-1}\left[\frac{u}{s}N^{+}[A_1(u)]\right] \\
&= -N^{-1}\left[\frac{u}{s}N^{+}[u_1(u_0)_x + u_0(u_1)_x]\right] \\
&= -N^{-1}\left[\frac{u}{s}N^{+}[-2xt]\right] \\
&= -xN^{-1}\left[\frac{2u^2}{s^3}\right] \\
&= xt^2 \\
u_3(x,t) &= -N^{-1}\left[\frac{u}{s}N^{+}[A_2(u)]\right] \\
&= -N^{-1}\left[\frac{u}{s}N^{+}[u_0(u_2)_x + u_2(u_0)_x + u_1(u_1)_x]\right] \\
&= -N^{-1}\left[\frac{u}{s}N^{+}[xt^2 + xt^2 - xt^2]\right] \\
&= -N^{-1}\left[\frac{u}{s}N^{+}[-3xt^2]\right] \\
&= -xt^3. \\
&\vdots
\end{aligned}$$

In this manner, three components of the decomposition series were obtained of which $u(x,t)$ was evaluated to have the following expansion

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = x - xt + xt^2 - xt^3 + ...$$

The exact solution of (2.21) given (2.22) is

$$u(x,t) = \frac{x}{1+t} \ ,$$

or

$$u(x,t) = x - xt + xt^2 - xt^3 + ... \quad for\ |t| < 1 \tag{2.29}$$

It is clear that the computed components coincide with the corresponding terms in (2.29). For more examples see [37].

2.4 Double Natural Decomposition Method

In this section, a combined form of the double natural transform method with the Adomain decomposition method is developed for an analytical solution of the linear and nonlinear singular one dimensional Boussinesq equations. For this subject we refer to [36]. Examples are provided to illustrate the reliability of this method.

Definition 2.2. Let $f(x,t)$ be a function and $x,\, t \in \mathbb{R}$. Then the double natural transform of $f(x,t)$ denoted by $R(p,s,u,v)$ is defined as

$$N^{+}_{x,t}[f(x,t)] = R(p,s,u,v) = \frac{1}{uv}\int_0^\infty \int_0^\infty e^{\left(\frac{-p}{u}x-\frac{s}{v}t\right)} f(x,t)dtdx$$

we can write the equation in another form as

$$N^{+}_{x,t}[f(x,t)] = R(p,s,u,v) = \int_0^\infty \int_0^\infty e^{(-px+st)} f(ux,vt)dtdx \ ,$$

Provided the integral exists

where

$$\text{Re}(s), \text{Re}(p) > 0, \text{Re}(u), \text{Re}(v) > 0.$$

Next, several examples are given.

Example 2.5. Let $f(x,t) = 1$, $x, t > 0$. Then

$$\begin{aligned}
N^{+}_{x,t}[1] &= \frac{1}{uv}\int_0^\infty \int_0^\infty e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}dtdx \\
&= \frac{1}{uv}\int_0^\infty \lim_{c\to\infty}\frac{-v}{s}\; e^{\left(\frac{-p}{u}x-\frac{s}{v}t\right)}\Big|_0^c dx \\
&= \frac{1}{u}\int_0^\infty \frac{1}{s}\; e^{\left(\frac{-p}{u}x\right)}dx \\
&= \lim_{c\to\infty}\frac{-1}{sp}\; e^{\left(\frac{-p}{u}x\right)}\Big|_0^c \\
&= \frac{1}{sp}
\end{aligned}$$

Example 2.6. Let $f(x,t) = e^{(ax+bt)}$, where a and b are constants. Then the double natural transform of the function can be written as

$$N^{+}_{x,t}[e^{ax+bt}] = \frac{1}{(s-bv)(p-au)},$$

Where $\frac{p}{u} > a$ and $\frac{s}{v} > b$

Proof.

$$\begin{aligned}
N^{+}_{x,t}[e^{ax+bt}] &= \frac{1}{uv}\int_0^\infty \int_0^\infty \; e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}e^{ax+bt}dtdx \\
&= \frac{1}{uv}\int_0^\infty \int_0^\infty \; e^{-\left(\frac{p}{u}-a\right)x-\left(\frac{s}{v}-b\right)t}dtdx \\
&= \frac{1}{u}\int_0^\infty \left[\lim_{c\to\infty}\frac{-1}{s-bv}\;\; e^{-\left(\frac{p}{u}-a\right)x-\left(\frac{s}{v}-b\right)t}\Big|_0^c\right] dx \\
&= \frac{1}{u}\int_0^\infty \frac{1}{s-bv}\;\; e^{-\left(\frac{p}{u}-a\right)x}dx \\
&= \lim_{c\to\infty}\frac{-1}{(s-bv)(p-au)}\;\; e^{-\left(\frac{p}{u}-a\right)x}\Big|_0^c \\
&= \frac{1}{(s-bv)(p-au)}.
\end{aligned}$$

□

Example 2.7. Let $f(x,t) = e^{i(ax+bt)}$, where a and b are constants. Then the double natural transform of the function can be written as

$$N_{x,t}^{+}[e^{i(ax+bt)}] = \frac{1}{(s-bvi)(p-aui)}$$

Where $\frac{p}{u} > 0$ and $\frac{s}{v} > 0$

Proof.

$$\begin{aligned}
N_{x,t}^{+}[e^{i(ax+bt)}] &= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}e^{i(ax+bt)}dtdx \\
&= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{p}{u}-ai\right)x-\left(\frac{s}{v}-bi\right)t}dtdx \\
&= \frac{1}{u}\int_0^\infty \left[\lim_{c\to\infty}\frac{-1}{s-bvi}\ e^{-\left(\frac{p}{u}-ai\right)x-\left(\frac{s}{v}-bi\right)t}\Big|_0^c\right]dx \\
&= \frac{1}{u}\int_0^\infty \frac{1}{s-bvi}\ e^{-\left(\frac{p}{u}-ai\right)x}dx \\
&= \lim_{c\to\infty}\frac{-1}{(s-bvi)(p-aui)}\ e^{-\left(\frac{p}{u}-ai\right)x}\Big|_0^c \\
&= \frac{1}{(s-bvi)(p-aui)}.
\end{aligned}$$

□

Example 2.8. Let $f(x,t) = \cos(ax+bt)$, where a and b are constants. Then the double natural transform of the function can be written as

$$N_{x,t}^{+}[\cos(ax+bt)] = \frac{ps-abuv}{(p^2+a^2u^2)(s^2+b^2v^2)}$$

Proof.

$$\begin{aligned}
N^+_{x,t}[\cos(ax+bt)] &= N^+_{x,t}\left[\frac{e^{i(ax+bt)}+e^{-i(ax+bt)}}{2}\right] \\
&= \frac{1}{2}\left[N^+_{x,t}\left[e^{i(ax+bt)}\right]+N^+_{x,t}\left[e^{-i(ax+bt)}\right]\right] \\
&= \frac{1}{2}\left[\frac{1}{(s-bvi)(p-aui)}+\frac{1}{(s+bvi)(p+aui)}\right] \\
&= \frac{1}{2}\left[\frac{ps-abuv+(aus+pbv)i}{(p^2+a^2u^2)(s^2+b^2v^2)}+\frac{ps-abuv-(aus+pbv)i}{(p^2+a^2u^2)(s^2+b^2v^2)}\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\left[\frac{2(ps-abuv)}{(p^2+a^2u^2)(s^2+b^2v^2)}\right] \\
&= \frac{ps-abuv}{(p^2+a^2u^2)(s^2+b^2v^2)}.
\end{aligned}$$

□

Example 2.9. Let $f(x,t)=\sin(ax+bt)$, where a and b are constants. Then the double natural transform of the function can be written as

$$N^+_{x,t}[\sin(ax+bt)] = \frac{aus-pbv}{(p^2+a^2u^2)(s^2+b^2v^2)}$$

Proof.

$$\begin{aligned}
N^+_{x,t}[\sin(ax+bt)] &= N^+_{x,t}\left[\frac{e^{i(ax+bt)}-e^{-i(ax+bt)}}{2i}\right] \\
&= \frac{1}{2i}\left[N^+_{x,t}\left[e^{i(ax+bt)}\right]-N^+_{x,t}\left[e^{-i(ax+bt)}\right]\right] \\
&= \frac{1}{2i}\left[\frac{1}{(s-bvi)(p-aui)}-\frac{1}{(s+bvi)(p+aui)}\right] \\
&= \frac{1}{2i}\left[\frac{ps-abuv+(aus+pbv)i}{(p^2+a^2u^2)(s^2+b^2v^2)}-\frac{ps-abuv-(aus+pbv)i}{(p^2+a^2u^2)(s^2+b^2v^2)}\right]
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{2i}\left[\frac{2(aus-pbv)i}{(p^2+a^2u^2)(s^2+b^2v^2)}\right] \\ &= \frac{aus-pbv}{(p^2+a^2u^2)(s^2+b^2v^2)}. \end{aligned}$$

□

Example 2.10. The double natural transform of $f(x,t) = x^a t^b$, if $a > -1$ and $b > -1$, is given as

$$N^{+}_{x,t}[x^a t^b] = \frac{u^a\ v^b}{p^{a+1}\ s^{b+1}}\Gamma(a+1)\Gamma(b+1),$$

Proof.

$$\begin{aligned} N^{+}_{x,t}[x^a t^b] &= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}x^a t^b dt dx \\ &= \frac{1}{u}\int_0^\infty e^{\frac{-p}{u}x}x^a \left(\frac{1}{v}\int_0^\infty e^{\frac{-s}{v}t}t^b dt\right)dx \end{aligned}$$

by substituting $\dfrac{p}{u}x = r$ and $\dfrac{s}{v}t = q$, we get

$$\begin{aligned} N^{+}_{x,t}[x^a t^b] &= \frac{1}{u}\int_0^\infty e^{-r}\left(\frac{ur}{p}\right)^a\frac{u}{p}\left(\frac{1}{v}\int_0^\infty e^{-q}\left(\frac{vq}{s}\right)^b\frac{v}{s}dq\right)dr \\ &= \frac{u^a\ v^b}{p^{a+1}\ s^{b+1}}\int_0^\infty\int_0^\infty e^{-r}r^a e^{-q}q^b dr dq \\ &= \frac{u^a\ v^b}{p^{a+1}\ s^{b+1}}\Gamma(a+1)\Gamma(b+1). \end{aligned}$$

Where gamma function of a defined by

$$\Gamma(a) = \int_0^\infty e^{-r}r^{a-1}dx, \quad a > 0,$$

and gamma function of b is

$$\Gamma(b) = \int_0^\infty e^{-q} q^{b-1} dt, \quad b > 0$$

Note that if a is natural number, then $\Gamma(a+1) = a!$ $\square$

Lemma 2.1. The double natural transform of $f(x,t) = (xt)^n$ is given by

$$N^{+}_{x,t}[(xt)^n] = \frac{(n!)^2\, u^n\, v^n}{p^{n+1}\, s^{n+1}}.$$

where $n \in \mathbb{N}$

Existence condition for the double natural transform

A function $f(x,t)$ is an exponential of a and b as $x \to \infty$, $t \to \infty$, if there exist a positive constant k such that

$$|f(x,t)| \leq k\ e^{ax+bx}$$

$\forall x > X$ and $t > T$ and it is easy to get

$$\lim_{\substack{x\to\infty \\ t\to\infty}} e^{\left(\frac{-\alpha}{u}x - \frac{\beta}{v}t\right)}\ |f(x,t)| \leq k \lim_{\substack{x\to\infty \\ t\to\infty}} e^{-\left(\frac{\alpha}{u}-a\right)x - \left(\frac{\beta}{v}-b\right)t} = 0.$$

where $\dfrac{\alpha}{u} > a$ and $\dfrac{\beta}{v} > b$.

The function $f(x,t)$ is called exponential order as $x \to \infty$, $t \to \infty$, it does not grow faster than $k\ e^{ax+bx}$ as $x \to \infty$, $t \to \infty$.

Theorem 2.7. If $f(x,t)$ is a continuous function in every finite interval $(0,X)$ and $(0,T)$ and of exponential order $e^{(ax+bt)}$, then the double natural transform of $f(x,t)$ which is defined by $N^{+}_{x,t}[f(x,t)]$ exists for all $p > \alpha$, $s > \beta$ and $u \neq 0$, $v \neq 0$

Proof.

$$\begin{aligned}
|N^{+}_{x,t}[f(x,t)]| &= \left|\frac{1}{uv}\int_0^{\infty}\int_0^{\infty} e^{\left(\frac{-p}{u}x-\frac{s}{v}t\right)} f(x,t)dtdx\right| \\
&\leq k\left|\frac{1}{uv}\int_0^{\infty}\int_0^{\infty} e^{\left(\frac{-p}{u}x-\frac{s}{v}t\right)} e^{(ax+bt)}dtdx\right| \\
&\leq k\left|\frac{1}{uv}\int_0^{\infty}\int_0^{\infty} e^{-(\frac{p}{u}-a)x-(\frac{s}{v}-v)t}dtdx\right| \\
&= \frac{k}{(p-au)(s-bv)}.
\end{aligned}$$

□

Double natural transform of partial derivatives

If double natural transform of the function $f(x,t)$ is given by $N^{+}_{x,t}[f(x,t)] = R(p,s,u,v)$, then the double natural transforms of $\frac{\partial f(x,t)}{\partial x}$, $\frac{\partial^2 f(x,t)}{\partial x^2}$, $\frac{\partial f(x,t)}{\partial t}$, $\frac{\partial^2 f(x,t)}{\partial t^2}$ are given by

$$\begin{aligned}
i)\ N^{+}_{x,t}\left[\frac{\partial f(x,t)}{\partial x}\right] &= \frac{p}{u}R(p,s,u,v) - \frac{1}{u}N^{+}_{t}f(0,t) \\
ii)\ N^{+}_{x,t}\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] &= \frac{p^2}{u^2}R(p,s,u,v) - \frac{p}{u^2}N^{+}_{t}f(0,t) - \frac{1}{u}N^{+}_{t}\left[\frac{\partial f(0,t)}{\partial x}\right] \\
iii)\ N^{+}_{x,t}\left[\frac{\partial f(x,t)}{\partial t}\right] &= \frac{s}{v}R(p,s,u,v) - \frac{1}{v}N^{+}_{x}f(x,0) \\
iv)\ N^{+}_{x,t}\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] &= \frac{s^2}{v^2}R(p,s,u,v) - \frac{s}{v^2}N^{+}_{x}f(x,0) - \frac{1}{v}N^{+}_{x}\frac{\partial f(x,0)}{\partial t}
\end{aligned}$$

Proof.

$$
\begin{aligned}
i)\ N_{x,t}^{+}\left[\frac{\partial f(x,t)}{\partial x}\right] &= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}\frac{\partial f(x,t)}{\partial x}dxdt \\
&= \frac{1}{uv}\left[\int_0^\infty \lim_{c\to\infty} e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}\ f(x,t)\Big|_0^c dt+\frac{p}{u}\int_0^\infty\int_0^\infty e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}\ f(x,t)dxdt\right] \\
&= \frac{1}{uv}\int_0^\infty -e^{-\left(\frac{s}{v}t\right)}f(0,t)dt+\frac{p}{u}R(p,s,u,v) \\
&= \frac{p}{u}R(p,s,u,v)-\frac{1}{u}N_t^{+}[f(0,t)]
\end{aligned}
$$

$$
\begin{aligned}
ii)\ N_{x,t}^{+}\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] &= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}\frac{\partial^2 f(x,t)}{\partial x^2}dxdt \\
&= \frac{1}{uv}\left[\int_0^\infty \lim_{c\to\infty} e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}\ \frac{\partial f(x,t)}{\partial x}\Big|_0^c dt+\frac{p}{u}\int_0^\infty\int_0^\infty e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}\ \frac{\partial f(x,t)}{\partial x}dxdt\right] \\
&= \frac{1}{u}\left[\frac{1}{v}\int_0^\infty -e^{-\frac{s}{v}t}\ \frac{\partial f(0,t)}{\partial x}dt\right]+\frac{p}{u}\left[\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}\frac{\partial f(x,t)}{\partial x}dxdt\right] \\
&= \frac{-1}{u}N_t^{+}\left[\frac{\partial f(0,t)}{\partial x}\right]+\frac{p}{u}\left[\frac{-1}{u}N_t^{+}[f(0,t)]+\frac{p}{u}R(p,s,u,v)\right] \\
&= \frac{p^2}{u^2}R(p,s,u,v))-\frac{p}{u^2}N_t^{+}[f(0,t)]-\frac{1}{u}N_t^{+}\left[\frac{\partial f(0,t)}{\partial x}\right].
\end{aligned}
$$

The proof of iii and iv similar to that in i and ii.

□

Theorem 2.8. The double natural transform of $x^n\ \dfrac{\partial f(x,t)}{\partial t}$ is given by

$$
N_{x,t}^{+}\left[x^n\frac{\partial f(x,t)}{\partial t}\right]=(-u)^n\frac{d^n}{dp^n}\,N_{x,t}^{+}\left[\frac{\partial f(x,t)}{\partial t}\right],\quad where\ n=1,2,3,...
$$

Proof. Using the definition of double natural transform for the first order partial derivative, we get

$$N^+_{x,t}\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}\frac{\partial f(x,t)}{\partial t}dtdx \tag{2.30}$$

By taking the n^{th} derivative with respect to p for both sides of Equation (2.30), we have

$$\begin{aligned}\frac{d^n}{dp^n}N^+_{x,t}\left[\frac{\partial f(x,t)}{\partial t}\right] &= \frac{1}{uv}\int_0^\infty\int_0^\infty \frac{d^n}{dp^n}\left[e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}\frac{\partial f(x,t)}{\partial t}dtdx\right]\\ &= \frac{(-1)^n}{uv}\int_0^\infty\int_0^\infty \left(\frac{x}{u}\right)^n e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}\frac{\partial f(x,t)}{\partial t}dtdx\\ &= \frac{(-1)^n}{u^n}\frac{1}{uv}\int_0^\infty\int_0^\infty x^n\ e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}\frac{\partial f(x,t)}{\partial t}dtdx\\ &= \frac{(-1)^n}{u^n}N^+_{x,t}\left[x^n\frac{\partial(f(x,t)}{\partial t}\right],\end{aligned}$$

We obtain

$$N^+_{x,t}\left[x^n\frac{\partial(f(x,t)}{\partial t}\right] = (-u)^n\frac{d^n}{dp^n}\,N^+_{x,t}\left[\frac{\partial(f(x,t)}{\partial t}\right].$$

□

Theorem 2.9. The double natural transform of $x^n\ \dfrac{\partial^2 f(x,t)}{\partial t^2}$ is given by

$$N^+_{x,t}\left[x^n\frac{\partial^2 f(x,t)}{\partial t^2}\right] = (-u)^n\frac{d^n}{dp^n}\,N^+_{x,t}\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right],\quad where\ n=1,2,3,...$$

Proof. Using the definition of double natural transform of the second order partial derivative, we get

$$N^+_{x,t}\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}\frac{\partial^2 f(x,t)}{\partial t^2}dtdx \tag{2.31}$$

By taking the n^{th} derivative with respect to p for both sides of Equation (2.31), we have

$$\begin{aligned}
\frac{d^n}{dp^n}N^{+}_{x,t}\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] &= \frac{1}{uv}\int_0^\infty\int_0^\infty \frac{d^n}{dp^n}\left[e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}\frac{\partial^2 f(x,t)}{\partial t^2}dtdx\right]\\
&= \frac{(-1)^n}{uv}\int_0^\infty\int_0^\infty \left(\frac{x}{u}\right)^n e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}\frac{\partial^2 (f(x,t)}{\partial t^2}dtdx\\
&= \frac{(-1)^n}{u^n}\frac{1}{uv}\int_0^\infty\int_0^\infty x^n\ e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}\frac{\partial^2 (f(x,t)}{\partial t^2}dtdx\\
&= \frac{(-1)^n}{u^n}N^{+}_{x,t}\left[x^n\frac{\partial^2 f(x,t)}{\partial t^2}\right],
\end{aligned}$$

we obtain

$$N^{+}_{x,t}\left[x^n\frac{\partial^2 f(x,t)}{\partial t^2}\right] = (-u)^n\frac{d^n}{dp^n}N^{+}_{x,t}\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right].$$

□

Theorem 2.10. The double natural transform of $x^n\ g(x,t)$ is given by

$$N^{+}_{x,t}\left[x^n g(x,t)\right] = (-u)^n\frac{d^n}{dp^n}N^{+}_{x,t}\left[g(x,t)\right]\quad where\ n=1,2,3,...$$

Proof. The proof is similar to that in Theorem (2.8) and Theorem (2.9) and therfore is omitted. □

Consider the following general form of the nonlinear singular one dimensional Boussinesq equation

$$\begin{aligned}
x\psi_{tt}-\frac{\partial}{\partial x}(x\psi_x)+xa(x)\psi_{xxxx}-xb(x)\psi_{xxtt}+xc(x)\psi_t\psi_{xx}&\\
+xd(x)\psi_x\psi_{xt} = xg(x,t),&\quad (2.32)
\end{aligned}$$

subject to condition

$$\psi(x,0) \;=\; g_1(x),\;\; \frac{\partial\psi(x,0)}{\partial t} = g_2(x), \tag{2.33}$$

where $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are arbitrary functions.

Solution:

Applying double natural transform to (2.32), we have

$$N^+_{x,t}\left[x\psi_{tt} - \frac{\partial}{\partial x}(x\psi_x) + xa(x)\psi_{xxxx} - xb(x)\psi_{xxtt} + xc(x)\psi_t\psi_{xx} + xd(x)\psi_x\psi_{xt}\right] = N^+_{x,t}\left[xg(x,t)\right].$$

Using the differential property of double natural transform

$$N^+_{x,t}\left[x^n\frac{\partial f(x,t)}{\partial t}\right] = (-u)^n\frac{d^n}{dp^n}\,N^+_{x,t}\left[\frac{\partial f(x,t)}{\partial t}\right],$$

and initial condition in (2.33), we get

$$\begin{aligned}\frac{d}{dp}\left[R(p,s,u,v)\right] = \frac{1}{s}\frac{d}{dp}N^+_x(\psi(x,0)) + \frac{v}{s^2}\frac{d}{dp}N^+_x(\psi_t(x,0)) - \frac{v^2}{us^2}N^+_{x,t}\left[\phi\right]\\ + \frac{v^2}{s^2}\frac{d}{dp}g(p,s,u,v),\end{aligned} \tag{2.34}$$

where
$\phi = \frac{\partial}{\partial x}(x\psi_x) - xa(x)\psi_{xxxx} + xb(x)\psi_{xxtt} - xc(x)\psi_t\psi_{xx} - xd(x)\psi_x\psi_{xt}$.
By integrating both sides of (2.34) from 0 to p, we have

$$R(p,s,u,v) \;=\; \frac{1}{s}N^+_x(g_1(x)) + \frac{v}{s^2}N^+_x(g_2(x)) - \frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[\phi\right]dp + \frac{v^2}{s^2}g(p,s,u,v).$$

Using double inverse natural transform, we obtain

$$\psi(x,t) = g_1(x) + tg_2(x) + N^{-1}_{p,s,u,v}\left[\frac{v^2}{s^2}g(p,s,u,v)\right] - N^{-1}_{p,s,u,v}\left[\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[\phi\right]dp\right].$$

Note that

$$\begin{aligned} N^+_{x,t}[g_1(x)] &= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}g_1(x)dtdx \\ &= \frac{1}{uv}\int_0^\infty \lim_{c\to\infty}\frac{-v}{s}\ e^{\left(\frac{-p}{u}x-\frac{s}{v}t\right)}g_1(x)\Big|_0^c dx \\ &= \frac{1}{uv}\int_0^\infty \frac{v}{s}\ e^{\left(\frac{-p}{u}x\right)}g_1(x)dx \\ &= \frac{1}{s}\left[\frac{1}{u}\int_0^\infty e^{\left(\frac{-p}{u}x\right)}g_1(x)dx\right] \\ &= \frac{1}{s}N^+_x[g_1(x)]. \end{aligned}$$

So

$$N^{-1}_{p,s,u,v}\left[\frac{1}{s}N^+_x[g_1(x)]\right] = N^{-1}_{p,s,u,v}\left[N^+_{x,t}[g_1(x)]\right] = g_1(x)$$

and

$$\begin{aligned} N^+_{x,t}[tg_2(x)] &= \frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{p}{u}x+\frac{s}{v}t\right)}tg_2(x)dtdx \\ &= \frac{1}{uv}\int_0^\infty \lim_{c\to\infty}\left[\frac{-tv}{s}\ e^{\left(\frac{-p}{u}x-\frac{s}{v}t\right)} - \frac{v^2}{s^2}\ e^{\left(\frac{-p}{u}x-\frac{s}{v}t\right)}\Big|_0^c g_2(x)\right]dx \\ &= \frac{1}{uv}\int_0^\infty \frac{v^2}{s^2}\ e^{\left(\frac{-p}{u}x\right)}g_2(x)dx \end{aligned}$$

$$\begin{aligned} &= \frac{v}{s^2}\left[\frac{1}{u}\int_0^\infty e^{\left(\frac{-p}{u}x\right)}g_2(x)dx\right] \\ &= \frac{v}{s^2}N^+_x[g_2(x)]. \end{aligned}$$

So

$$N^{-1}_{p,s,u,v}\left[\frac{v}{s^2}N^+_x[g_2(x)]\right] = N^{-1}_{p,s,u,v}\left[N^+_{x,t}[tg_2(x)]\right] = tg_2(x).$$

Rewrite $\psi(x,t)$ as an infinite series $\psi_n(x,t)$

$$\psi(x,t) = \sum_{n=0}^{\infty}\psi_n(x,t), \quad n = 0,1,2,... \tag{2.35}$$

Also the nonlinear terms $\psi_t\psi_{xx}$ and $\psi_x\psi_{xt}$ can be written as an infinite series of an Adomain polynomials

$$\psi_t\psi_{xx} = \mathcal{N}_1 = \sum_{n=0}^{\infty}A_n \tag{2.36}$$
$$\psi_x\psi_{xt} = \mathcal{N}_2 = \sum_{n=0}^{\infty}B_n,$$

where $A'_n s$ and $B'_n s$ are the polynomials that are given by

$$\begin{aligned} A_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\mathcal{N}_1\left[\sum_{i=1}^{\infty}\lambda^i\psi_i\right]_{\lambda=0} \\ B_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\mathcal{N}_2\left[\sum_{i=1}^{\infty}\lambda^i\psi_i\right]_{\lambda=0}. \end{aligned}$$

By substituting (2.36) and (2.35), we get

$$\begin{aligned} \psi_n(x,t) = g_1(x) + tg_2(x) + N^{-1}_{p,s,u,v}\left[\frac{v^2}{s^2}g(p,s,u,v)\right] \\ + N^{-1}_{p,s,u,v}\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right]dp \\ + N^{-1}_{p,s,u,v}\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[xa(x)\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)_{xxxx} - xb(x)\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)_{xxtt}\right]dp \\ + N^{-1}_{p,s,u,v}\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[xc(x)\left(\sum_{n=0}^{\infty}A_n\right) + xd(x)\left(\sum_{n=0}^{\infty}B_n\right)\right]dp, \end{aligned} \tag{2.37}$$

where A_n and B_n are given by

$$\begin{aligned}
A_0 &= \psi_{0t}\psi_{0xx}\\
A_1 &= \psi_{0t}\psi_{1xx}+\psi_{1t}\psi_{0xx}\\
A_2 &= \psi_{0t}\psi_{2xx}+\psi_{1t}\psi_{1xx}+\psi_{2t}\psi_{0xx}\\
A_3 &= \psi_{0t}\psi_{3xx}+\psi_{1t}\psi_{2xx}+\psi_{2t}\psi_{1xx}+\psi_{3t}\psi_{0xx}\\
&\vdots
\end{aligned}$$

and

$$\begin{aligned}
B_0 &= \psi_{0x}\psi_{0xt}\\
B_1 &= \psi_{0x}\psi_{1xt}+\psi_{1x}\psi_{0xt}\\
B_2 &= \psi_{0x}\psi_{2xt}+\psi_{1x}\psi_{1xt}+\psi_{2x}\psi_{0xt}\\
B_3 &= \psi_{0x}\psi_{3xt}+\psi_{1x}\psi_{2xt}+\psi_{2x}\psi_{1xt}+\psi_{3x}\psi_{0xt}\\
&\vdots
\end{aligned}$$

Now, by comparing both sides of (2.37), we conclude that

$$\psi_0(x,t)=g_1(x)+tg_2(x)+N^{-1}_{p,s,u,v}\left[\frac{v^2}{s^2}g(p,s,u,v)\right],$$

$$\begin{aligned}
\psi_1(x,t) = -N^{-1}_{p,s,u,v}\frac{v^2}{us^2}\int_0^p N^{+}_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\psi_0(x,t)\right)\right]dp\\
+N^{-1}_{p,s,u,v}\frac{v^2}{us^2}\int_0^p N^{+}_{x,t}\left[xa(x)\left(\psi_0(x,t)\right)_{xxxx}-xb(x)\left(\psi_0(x,t)\right)_{xxtt}\right]dp\\
+N^{-1}_{p,s,u,v}\frac{v^2}{us^2}\int_0^p N^{+}_{x,t}\left[xc(x)\left(A_0\right)+xd(x)\left(B_0\right)\right]dp,
\end{aligned}$$

$$\begin{aligned}\psi_2(x,t) = -N^{-1}_{p,s,u,v}\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\psi_1(x,t)\right)\right]dp\\ + N^{-1}_{p,s,u,v}\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[xa(x)\left(\psi_1(x,t)\right)_{xxxx} - xb(x)\left(\psi_1(x,t)\right)_{xxtt}\right]dp\\ + N^{-1}_{p,s,u,v}\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[xc(x)\left(A_1\right) + xd(x)\left(B_1\right)\right]dp,\end{aligned}$$

and

$$\begin{aligned}\psi_{n+1}(x,t) = -N^{-1}_{p,s,u,v}\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right]dp\\ + N^{-1}_{p,s,u,v}\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[xa(x)\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)_{xxxx} - xb(x)\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)_{xxtt}\right]dp\\ + N^{-1}_{p,s,u,v}\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[xc(x)\left(\sum_{n=0}^{\infty}A_n\right) + xd(x)\left(\sum_{n=0}^{\infty}B_n\right)\right]dp, \quad (2.38)\end{aligned}$$

Hence from the general relation in (2.38), we can compute the remaining components of $\psi(x,t)$ as $\psi_3(x,t), \psi_4(x,t)$, where $\psi_n(x,t)$ is always the initial given condition.

Example 2.11. Consider nonlinear singular one dimensional Boussinesq equation [23]

$$\psi_{tt} - \frac{1}{x}\frac{\partial}{\partial x}(x\psi_x) + \psi_{xxxx} - \psi_{xxtt} - 4\psi_t\psi_{xx} + 2\psi_x\psi_{xt} = -4t, \quad (2.39)$$

Subject to initial condition

$$\psi(x,0) = 0, \quad \psi_t(x,0) = x^2. \quad (2.40)$$

solution:

Multiplying both sides of (2.39) by x and applying double Natural transform, we have

$$N^{+}_{x,t}[x\psi_{tt} - \frac{\partial(x\psi_x)}{\partial x} + x\psi_{xxxx} - x\psi_{xxtt} - 4x\psi_t\psi_{xx} + 2x\psi_x\psi_{xt}] = N^{+}_{x,t}[-4xt]. \quad (2.41)$$

Using the differentiation property of double Natural transform and initial condition given in (2.40) we get

$$\frac{d}{dp}R(p,s,u,v) \quad = \quad \frac{v}{s^2}\frac{d}{dp}N_x(\psi(x,0) \;-\; \frac{v^2}{us^2}N^{+}_{x,t}[-4xt] \;-\; \frac{v^2}{us^2}N^{+}_{x,t}[\phi], \quad (2.42)$$

where

$$\phi = \frac{\partial}{\partial x}(x\psi_x) - x\psi_{xxxx} + x\psi_{xxtt} + 4x\psi_t\psi_{xx} - 2x\psi_x\psi_{xt}.$$

Then by integrating both sides of (2.42) from 0 to p with respect to p, we have

$$R(p,s,u,v) \quad = \quad \frac{v}{s^2}N_x(x^2) \;+\; \frac{v^2}{us^2}\int_0^p N^{+}_{x,t}\left[-4xt\right]dp \;-\; \frac{v^2}{us^2}\int_0^p N^{+}_{x,t}\left[\phi\right]dp. \quad (2.43)$$

Using double inverse natural transform for (2.43), we obtain

$$\psi(x,t) = x^2t - \frac{2}{3}t^3 - \frac{v^2}{us^2}\int_0^p N^{+}_{x,t}\left[\phi\right]dp \quad (2.44)$$

Rewrite $\psi(x,t)$ as an infinite series $\psi_n(x,t)$

$$\psi(x,t) = \sum_{n=0}^{\infty}\psi_n(x,t), \quad n = 0,1,2,... \quad (2.45)$$

Also the nonlinear terms $\psi_t\psi_{xx}$ and $\psi_x\psi_{xt}$ can be written as an infinite series of an Adomain polynomials

$$\psi_t\psi_{xx} = \mathcal{N}_1 = \sum_{n=0}^{\infty}A_n, \quad \psi_x\psi_{xt} = \mathcal{N}_2 = \sum_{n=0}^{\infty}B_n, \quad (2.46)$$

where $A_n's$ and $B_n's$ are the polynomials that are given by

$$\begin{aligned} A_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\mathcal{N}_1\left[\sum_{i=1}^{\infty}\lambda^i\psi_i\right]_{\lambda=0} \\ B_n &= \frac{1}{n!}\frac{d^n}{d\lambda^n}\mathcal{N}_2\left[\sum_{i=1}^{\infty}\lambda^i\psi_i\right]_{\lambda=0}. \end{aligned}$$

where A_n and B_n are given by

$$\begin{aligned} A_0 &= \psi_{0t}\psi_{0xx} \\ A_1 &= \psi_{0t}\psi_{1xx}+\psi_{1t}\psi_{0xx} \\ A_2 &= \psi_{0t}\psi_{2xx}+\psi_{1t}\psi_{1xx}+\psi_{2t}\psi_{0xx} \\ A_3 &= \psi_{0t}\psi_{3xx}+\psi_{1t}\psi_{2xx}+\psi_{2t}\psi_{1xx}+\psi_{3t}\psi_{0xx} \\ &\vdots \end{aligned}$$

and

$$\begin{aligned} B_0 &= \psi_{0x}\psi_{0xt} \\ B_1 &= \psi_{0x}\psi_{1xt}+\psi_{1x}\psi_{0xt} \\ B_2 &= \psi_{0x}\psi_{2xt}+\psi_{1x}\psi_{1xt}+\psi_{2x}\psi_{0xt} \\ B_3 &= \psi_{0x}\psi_{3xt}+\psi_{1x}\psi_{2xt}+\psi_{2x}\psi_{1xt}+\psi_{3x}\psi_{0xt} \\ &\vdots \end{aligned}$$

The double natural decomposition method leads to the following

$$\psi_0(x,t) = x^2t - \frac{2}{3}t^3$$

and

$$\begin{aligned}\psi_{n+1}(x,t) = &-N^{-1}_{p,s,u,v}\left[\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right]dp\right]\\ &+N^{-1}_{p,s,u,v}\left[\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[x\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)_{xxxx}-x\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)_{xxtt}\right]dp\right]\\ &-N^{-1}_{p,s,u,v}\left[\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[4x\left(\sum_{n=0}^{\infty}A_n\right)-2x\left(\sum_{n=0}^{\infty}B_n\right)\right]dp\right],\end{aligned}$$

The first iteration is given by

$$\begin{aligned}\psi_1(x,t) = &-N^{-1}_{p,s,u,v}\left[\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[4xt\right]dp\right]\\ &+N^{-1}_{p,s,u,v}\left[\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[0-0\right]dp\right]\\ &-N^{-1}_{p,s,u,v}\left[\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[16xt^3\right]dp\right],\end{aligned}$$

$$\psi_1(x,t) = \frac{2}{3}t^3 - \frac{4}{5}t^5$$

In similar manner,

$$\begin{aligned}\psi_2(x,t) = &-N^{-1}_{p,s,u,v}\left[\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[16xt^3-32xt^5\right]dp\right]\\ &-N^{-1}_{p,s,u,v}\left[\frac{v^2}{us^2}\int_0^p\left[16\frac{3!uv^3}{p^2s^4}-32\frac{5!uv^5}{p^2s^6}\right]dp\right]\\ &=-N^{-1}_{p,s,u,v}\left[16\frac{3!v^3}{ps^6}-32\frac{5!uv^7}{ps^8}\right],\end{aligned}$$

$$\psi_2(x,t) = \frac{4}{5}t^5 - \frac{16}{21}t^7$$

Similarly,

$$\begin{aligned}\psi_3(x,t) = -N^{-1}_{p,s,u,v}\left[\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\psi_2\right)\right]dp\right]\\ - N^{-1}_{p,s,u,v}\left[\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[x\left(\psi_2\right)_{xxxx} - x\left(\psi_2\right)_{xxtt}\right]dp\right]\\ - N^{-1}_{p,s,u,v}\left[\frac{v^2}{us^2}\int_0^p N^+_{x,t}\left[4xA_2 - 2xB_2\right]dp\right],\end{aligned}$$

Therefore, $A_2 = \psi_{0t}\psi_{2xx} + \psi_{1t}\psi_{1xx} + \psi_{2t}\psi_{0xx} = 8t^5 - \frac{32}{3}t^7$ and $B_2 = 0$ Then we have

$$\psi_3(x,t) = \frac{16}{21}t^7 - \frac{16}{27}t^9,$$

The series solution are therefore is given by

$$\sum_{n=0}^{\infty}\psi_n(x,t) = \psi_1 + \psi_2 + \psi_3 + ... = x^2t - \frac{2}{3}t^3 + \frac{2}{3}t^3 - \frac{4}{5}t^5 + \frac{4}{5}t^5 + ... = x^2t.$$

Chapter 3

Laplace Decomposition Method (LDM)

3.1 Laplace Transform

Definition 3.1. Let $f(t)$ be a function defined for all real numbers $t \geq 0$. Then the Laplace transform of $f(t)$ denoted by $F(s) = \mathcal{L}\{f(t)\}$ is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt, \quad \mathrm{Re}(s) > 0$$

Laplace Transform of Derivatives

If the Laplace of the function $f(t)$ is given by $\mathcal{L}\{f(t)\} = F(s)$, then the Laplace transforms of $f'(t)$, $f''(t)$, $f^n(t)$, are given by [9]

$$\begin{aligned} i)\ \mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0) \\ ii)\ \mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \\ iii)\ \mathcal{L}\{f^n(t)\} &= s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - ... - f^{(n-1)}(0) \end{aligned}$$

The following table gives the Laplace transform of some functions calculated by Definition

(3.1).

$f(t)$	$\mathcal{L}\{f(t)\}$	conditions
1	$\frac{1}{s}$	$s>0$
t	$\frac{1}{s^2}$	$s>0$
t^n	$\frac{n!}{s^{n+1}}$	$n \in \mathbb{Z} \geq 0$
t^a	$\frac{\Gamma(a+1)}{s^{a+1}}$	Re(a) > -1
e^{at}	$\frac{1}{s-a}$	$s>a$
$\sin wt$	$\frac{w}{s^2+w^2}$	$s >$ Im(w)
$\cos wt$	$\frac{s}{s^2+w^2}$	$w \in \mathbb{R}$
$\sinh wt$	$\frac{w}{s^2-w^2}$	$s >$ Im(w)
$\cosh wt$	$\frac{s}{s^2-w^2}$	$s >$ Re(w)

Table 3.1: Laplace transform of some functions

3.2 Laplace decomposition method (LDM)

In this section, we present the Laplace decomposition method for solving nonlinear partial differential equations, this method joint the Laplace transform to ADM. This method provides the solution in the form of rapidly convergent series. An illustrative example is given. For this section we refer to [29]

Consider the second order nonlinear partial differential equation

$$Lu(x,t)+Ru(x,t)+Nu(x,t) \quad = \quad h(x,t), \tag{3.1}$$

subject to initial condition

$$u(x,0) = f(x), \;\; u_t(x,0) = g(x), \tag{3.2}$$

where

$L = \dfrac{\partial^2}{\partial x^2}$ is a differential operator.

R is a remaining Linear differential of order less than$L$.

Nu is a general non linear differential operator.

$h(x,t)$ is a source term.

Suppose L is the differential operator of second order, so $L = \dfrac{\partial^2}{\partial t^2}$.

Applying the Laplace transform with respect to t for (3.1), we get

$$\begin{multline} s^2\mathcal{L}\{u(x,t)\} - su(x,0) - u_t(x,0) + \mathcal{L}\{Ru(x,t)\} \\ + \mathcal{L}\{Nu(x,t)\} = \mathcal{L}\{h(x,t)\}. \tag{3.3} \end{multline}$$

By using (3.2) in (3.3) we obtain

$$s^2\mathcal{L}\{u(x,t)\} - sf(x) - g(x) + \mathcal{L}\{Ru(x,t)\} + \mathcal{L}\{Nu(x,t)\} \;\; = \;\; \mathcal{L}\{h(x,t)\},$$

or

$$\mathcal{L}\{u(x,t)\} = \frac{f(x)}{s} + \frac{g(x)}{s^2} - \frac{1}{s^2}\mathcal{L}\{Ru(x,t)\} - \frac{1}{s^2}\mathcal{L}\{Nu(x,t)\} + \frac{1}{s^2}\mathcal{L}\{h(x,t)\}. \tag{3.4}$$

Taking the inverse of the Laplace transform for (3.4) we get

$$\begin{aligned} u(x,t) &= \mathcal{L}^{-1}\left\{\frac{f(x)}{s}+\frac{g(x)}{s^2}+\frac{1}{s^2}\mathcal{L}[h(x,t)]\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}[Ru(x,t)]\right\}]-\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}[Nu(x,t)]\right\} \\ &= k(x,t)-\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}[Ru(x,t)]\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}[Nu(x,t)]\right\}, \qquad (3.5) \end{aligned}$$

where $k(x,t)$ represents the terms arising from source term and prescribed initial condition. i.e.

$$k(x,t)=\mathcal{L}^{-1}\left\{\frac{f(x)}{s}+\frac{g(x)}{s^2}+\frac{1}{s^2}\mathcal{L}\{h(x,t)\}\right\}.$$

We represent the solution as an infinite series given below

$$u(x,t) = \sum_{n=0}^{\infty}u_n(x,t). \qquad (3.6)$$

The nonlinear operator is decomposed as

$$\mathcal{N}u(x,t) = \sum_{n=0}^{\infty}A_n. \qquad (3.7)$$

Where A_n are Adomain polynomials of $u_0, u_1, u_2, ..., u_n$ they can be calculated by the following formula

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\mathcal{N}\left[\sum_{i=0}^{\infty}\lambda^i u_i\right]_{\lambda=0}, \quad n\geq 0.$$

Using (3.6) and (3.7) in (3.5) we get

$$\sum_{n=0}^{\infty}u_n(x,t) = k(x,t)-\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{Ru(x,t)\}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\left[\sum_{n=0}^{\infty}A_n\right]\right\} \qquad (3.8)$$

Now, by comparing both sides of (3.8) we have

$$\begin{aligned}
u_0(x,t) &= k(x,t) = \mathcal{L}^{-1}\left\{\frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}\mathcal{L}\{h(x,t)\}\right\} \\
u_1(x,t) &= -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{Ru_o(x,t)\}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}[A_0]\right\} \\
u_2(x,t) &= -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{Ru_1(x,t)\}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}[A_1]\right\} \\
u_3(x,t) &= -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{Ru_2(x,t)\}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}[A_2]\right\} \\
&\vdots
\end{aligned}$$

In general, the recursive relation is given by

$$u_{n+1}(x,t) = -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{Ru_n(x,t)\}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\mathcal{L}\{A_n\}\right\}, \quad n \geq 0.$$

given

$$u_0(x,t) = k(x,t) = \mathcal{L}^{-1}\left\{\frac{f(x)}{s} + \frac{g(x)}{s^2} + \frac{1}{s^2}\mathcal{L}\{h(x,t)\}\right\}.$$

Example 3.1. Consider the nonlinear partial differential equation

$$u_{tt}(x,t) + u(x,t)u_x(x,t) = -cost, \tag{3.9}$$

subject to initial condition

$$u(x,0) = 1, \quad u_t(x,0) = 0. \tag{3.10}$$

Solution: By using the recursive equation, we get:

$$\begin{aligned} u_0(x,t) &= cost \\ \sum_{n=0}^{\infty} u_{n+1}(x,t) &= -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}\left[\sum_{n=0}^{\infty} A_n(u)\right]\right] \end{aligned}$$

So, we can obtain first components of equation, as follow

$$\begin{aligned} u_1(x,t) &= -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[A_0(u)]\right] \\ &= -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[u_0(u_o)_x]\right] \\ &= -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[0]\right] \\ &= 0 \\ u_2(x,t) &= -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[A_1(u)]\right] \\ &= -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[u_1(u_0)_x + u_0(u_1)_x]\right] \\ &= -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[0]\right] \\ &= 0 \\ u_3(x,t) &= -\mathcal{L}^{-1}\left[\frac{u}{s}\mathcal{L}[A_2(u)]\right] \\ &= -\mathcal{L}^{-1}\left[\frac{u}{s}\mathcal{L}[u_0(u_2)_x + u_2(u_0)_x + u_1(u_1)_x]\right] \\ &= -\mathcal{L}^{-1}\left[\frac{u}{s}\mathcal{L}[0]\right] \\ &= 0 \\ &\vdots \end{aligned}$$

Then

$$\sum_{n=0}^{3} u_n(x,t) = \ cost,$$

The exact solution of (3.9) give (3.10) is

$$u(x,t) = cost.$$

Example 3.2. Consider the nonlinear partial differential equation [42]

$$u_t(x,t) + u(x,t)u_x(x,t) \ = \ x + xt^2, \tag{3.11}$$

subject to initial condition

$$u(x,0) = 0. \tag{3.12}$$

Solution

Apply the Laplace transform to (3.11), we have

$$s\mathcal{L}\{u(x,t)\} - u(x,0) \ = \ \mathcal{L}\{x + xt^2\} - \mathcal{L}\{u(x,t)u_x(x,t)\}. \tag{3.13}$$

By using (3.12) in (3.13) we obtain

$$\mathcal{L}\{u(x,t)\} \ = \ \frac{x}{s^2} + \frac{2x}{s^4} - \frac{1}{s}\mathcal{L}\{u(x,t)u_x(x,t)\}. \tag{3.14}$$

Then by applying the inverse of the laplace transform of (3.14) we have

$$u(x,t) \ = \ xt + \frac{xt^3}{3} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}L\{u(x,t)u_x(x,t)\}\right\}. \tag{3.15}$$

Now, we decompose the solution as an infinite sum given by

$$u(x,t) \;=\; \sum_{n=0}^{\infty} u_n(x,t). \tag{3.16}$$

Also the nonlinear term can be written as an infinite series of Adomain polynomials

$$uu_x \;=\; \sum_{n=0}^{\infty} A_n, \tag{3.17}$$

where

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\mathcal{N}\left[\sum_{i=0}^{\infty}\lambda^i y_i\right]_{\lambda=0}, \quad n \geq 0.$$

Then.

$$\begin{aligned}
A_0 &= u_0(u_0)_x \\
A_1 &= u_1(u_0)_x + u_0(u_1)_x \\
A_2 &= u_0(u_2)_x + u_2(u_0)_x + u_1(u_1)_x \\
A_3 &= u_3(u_0)_x + u_0(u_3)_x + u_2(u_1)_x + u_1(u_2)_x \\
&\vdots
\end{aligned}$$

By using (3.16) and (3.17) we can write (3.15) as

$$\sum_{n=0}^{\infty} u_n(x,t) \;=\; xt + \frac{xt^3}{3} - \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{uu_x\right\}\right\}$$

Now,

$$\begin{aligned} u_0(x,t) &= xt \\ u_1(x,t) &= \frac{xt^3}{3} - \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\sum_{n=0}^{\infty} A_0(u)\right\}\right\} \\ \sum_{n=0}^{\infty} u_{n+1}(x,t) &= -\mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{\sum_{n=0}^{\infty} A_n(u)\right\}\right\} \end{aligned}$$

So, we obtain first components of equation as follow

$$\begin{aligned} u_1(x,t) &= \frac{xt^3}{3} - \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{u(u_0)_x\right\}\right\} \\ &= \frac{xt^3}{3} - \mathcal{L}^{-1}\left\{\frac{1}{s}\mathcal{L}\left\{(xt)(t)\right\}\right\} \\ &= \frac{xt^3}{3} - x\mathcal{L}^{-1}\left\{\frac{2!}{s^4}\right\} \\ &= \frac{xt^3}{3} - \frac{2!}{3!}x\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} \\ &= \frac{xt^3}{3} - \frac{xt^3}{3} \\ &= 0 \end{aligned}$$

So

$$u_{n+1}(x,t) = 0, \quad n \geq 0.$$

In view of above modified recursive relation we get exact solution

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = xt.$$

3.3 Double Laplace Decomposition Method

In this section, the Adomain decomposition methods and the double Laplace transform method are combined in double Laplace decomposition methods see [26].

Definition 3.2. Let $f(x,t)$ be a function where $x, t > 0$. Then the double Laplace transform of $f(x,t)$ denoted by $F(p,s)$ is defined as

$$\mathcal{L}_x\mathcal{L}_t\{f(x,t)\} = F(p,s) = \int_0^\infty e^{-px}\int_0^\infty e^{-st}f(x,t)dtdx,$$

Provided the integral exists. Here p and s are complex numbers.

Next, some examples are given [19].

Example 3.3. If $f(x,t) = 1$ for $x > 0$ and $t > 0$, then

$$\mathcal{L}_x\mathcal{L}_t\{1\} = \frac{1}{ps},$$

Example 3.4. If $f(x,t) = e^{ax+bt}$ for all x and t, then the double Laplace transform of the function can be written as

$$\mathcal{L}_x\mathcal{L}_t\left\{e^{ax+bt}\right\} = \frac{1}{(p-a)(s-b)},$$

Where $p > a$ and $s > b$

Example 3.5. If $f(x,t) = e^{i(ax+bt)}$ for all x and t, then the double Laplace transform of the function can be written as

$$\mathcal{L}_x\mathcal{L}_t\left\{e^{i(ax+bt)}\right\} = \frac{1}{(p-ia)(s-ib)},$$

Where $p > 0$ and $s > 0$

Example 3.6. If $f(x,t) = \cos(ax + bt)$ where a and b are constants, then the double Laplace transform of the function can be written as

$$\mathcal{L}_x\mathcal{L}_t\left\{cos(ax+bt)\right\} = \frac{ps-ab}{(p^2+a^2)(s^2+b^2)},$$

Example 3.7. If $f(x,t) = \sin(ax + bt)$ where a and b are constants, then the double Laplace transform of the function can be written as

$$\mathcal{L}_x\mathcal{L}_t\left\{sin(ax+bt)\right\} = \frac{as+pb}{(p^2+a^2)(s^2+b^2)},$$

Example 3.8. If $f(x,t) = x^a \quad t^b$ if $a > -1$ and $b > -1$ are real numbers, then the double Laplace transform of the function is given as

$$\mathcal{L}_x\mathcal{L}_t\left\{x^a \quad t^b\right\} = \frac{\Gamma(a+1)}{p^{a+1}} \quad \frac{\Gamma(b+1)}{s^{b+1}},$$

where $\Gamma(a)$ is the Euler gamma function defined by the uniformly convergent integral.

$$\Gamma(a) = \int_0^\infty s^{a-1} \quad e^{-s} ds$$

Remark 3.1. The double Laplace transform of $(xt)^n$ is given as

$$\begin{aligned}
\mathcal{L}_x\mathcal{L}_t\left\{(xt)^n\right\} &= \int_0^\infty e^{-px}\ x^n\ dx \int_0^\infty e^{-st}\ t^n\ dt \\
&= \frac{n!}{p^{n+1}}\ \frac{n!}{s^{n+1}} \\
&= \frac{(n!)^2}{(ps)^{n+1}}.
\end{aligned}$$

Double Laplace transform of partial derivatives

Theorem 3.1. [27] If $\mathcal{L}_x\mathcal{L}_t\left\{f(x,t)\right\} = F(p,s)$, then the double Laplace of $\dfrac{\partial f(x.t)}{\partial x}$ is given by

$$\mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial f(x,t)}{\partial x}\right\} = pF(p,s) - L_t\left\{f(0,t)\right\}$$

Proof.

$$\begin{aligned}
\mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial f(x,t)}{\partial x}\right\} &= \int_0^\infty e^{-px}\int_0^\infty \ e^{-st} \ \frac{\partial f(x,t)}{\partial x}\ dxdt \\
&= \int_0^\infty\int_0^\infty e^{-px-st} \ \frac{\partial f(x,t)}{\partial x}\ dxdt \\
&= \int_0^\infty \left[\lim_{c\to\infty} e^{-px-st} \ \ f(x,t)\Big|_0^c\right] dt + \int_0^\infty\int_0^\infty p\ e^{-px-st} \ \ f(x,t)\ dxdt \\
&= -\int_0^\infty e^{-st} f(0,t)\ dt + pF(p,s) \\
&= pF(p,s) - \mathcal{L}_t\left\{f(0,t).\right\}
\end{aligned}$$

□

Theorem 3.2. [27] If $\mathcal{L}_x\mathcal{L}_t\left\{f(x,t)\right\} = F(p,s)$, then the double Laplace of $\dfrac{\partial^2 f(x,t)}{\partial x^2}$ is given by

$$\mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial^2 f(x,t)}{\partial x^2}\right\} = p^2F(p,s) - p\mathcal{L}_t\left\{f(0,t)\right\} - \mathcal{L}_t\left\{\frac{\partial f(0,t)}{\partial x}\right\}.$$

Proof.

$$\begin{aligned}
\mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial^2 f(x,t)}{\partial x^2}\right\} &= \int_0^\infty e^{-px}\int_0^\infty \ e^{-st}\ \frac{\partial^2 f(x,t)}{\partial x^2}\ dxdt \\
&= \int_0^\infty\left[\lim_{b\to\infty} e^{-px-st}\ \left.\frac{\partial f(x,t)}{\partial x}\right|_0^b\right]dt + p\int_0^\infty\int_0^\infty e^{-px-st}\ \frac{\partial f(x,t)}{\partial x}\ dxdt \\
&= -\int_0^\infty e^{-st}\frac{\partial f(0,t)}{\partial x}\ dt + p\mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial f(x,t)}{\partial x}\right\} \\
&= -\mathcal{L}_t\left\{\frac{\partial f(0,t)}{\partial x}\right\} + p\left[pF(p,s) - \mathcal{L}_t\left\{f(0,t)\right\}\right] \\
&= p^2F(p,s) - p\mathcal{L}_t\left\{f(0,t)\right\} - \mathcal{L}_t\left\{\frac{\partial f(0,t)}{\partial x}\right\}.
\end{aligned}$$

□

Theorem 3.3. [24] If $\mathcal{L}_x\mathcal{L}_t\left\{f(x,t)\right\} = F(p,s)$, then the double Laplace of $\frac{\partial f(x.t)}{\partial t}$ is given by

$$\mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial f(x,t)}{\partial t}\right\} = sF(p,s) - \mathcal{L}_x\left\{f(x,0)\right\} \tag{3.18}$$

Proof.

$$\begin{aligned}
\mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial f(x,t)}{\partial t}\right\} &= \int_0^\infty e^{-px}\int_0^\infty \ e^{-st}\ \frac{\partial f(x,t)}{\partial t}\ dtdx \\
&= \int_0^\infty\int_0^\infty e^{-px-st}\ \frac{\partial f(x,t)}{\partial t}\ dtdx \\
&= \int_0^\infty\left[\lim_{b\to\infty} e^{-px-st}\ \left. f(x,t)\right|_0^b\right]dx + \int_0^\infty\int_0^\infty s\ e^{-px-st}\ f(x,t)\ dxdt \\
&= -\int_0^\infty e^{-st}f(x,0)\ dx + sF(p,s) \\
&= sF(p,s) - \mathcal{L}_x\left\{f(x,0)\right\}.
\end{aligned}$$

□

Theorem 3.4. [26] If $\mathcal{L}_x\mathcal{L}_t\left\{f(x,t)\right\} = F(p,s)$, and $\mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial f(x,t)}{\partial t}\right\} = sF(p,s) - \mathcal{L}_t\left\{f(x,0)\right\}$ then the double Laplace of $\frac{\partial^2 f(x,t)}{\partial t^2}$ is given by

$$\mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} = s^2F(p,s) - s\mathcal{L}_x\left\{f(x,0)\right\} - \mathcal{L}_x\left\{\frac{\partial f(x,0)}{\partial t}\right\}$$

Proof. The proof is similar to the previous theorem. □

Lemma 3.1. Double Laplace transform of the non constant coefficient second order partial derivative $x^r\frac{\partial^2 f(x,t)}{\partial t^2}$ is given as

$$\mathcal{L}_x\mathcal{L}_t\left\{x^r\frac{\partial^2 f(x,t)}{\partial t^2}\right\} = (-1)^r\frac{d^r}{dp^r}\left[s^2F(p,s) - s\mathcal{L}_x\left\{f(x,0)\right\} - \mathcal{L}_x\left\{\frac{\partial f(x,0)}{\partial t}\right\}\right]$$

Proof. By taking the r^{th} derivative with respect to p for both sides of equation, we have

$$\begin{aligned}
\frac{d^r}{dp^r}\mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} &= \frac{d^r}{dp^r}\int_0^\infty e^{-px}\int_0^\infty e^{-st}\ \frac{\partial^2 f(x,t)}{\partial t^2}\ dtdx \\
&= \int_0^\infty\int_0^\infty \frac{d^r}{dp^r}\ e^{-px-st}\frac{\partial^2 f(x,t)}{\partial t^2}\ dtdx \\
&= \int_0^\infty\int_0^\infty (-x)^r\ e^{-px-st}\frac{\partial^2 f(x,t)}{\partial t^2}\ dtdx \\
&= (-1)^r\int_0^\infty\int_0^\infty e^{-px-st}\left(x^r\frac{\partial^2 f(x,t)}{\partial t^2}\right)\ dtdx.
\end{aligned} \tag{3.19}$$

So,

$$(-1)^r\frac{d^r}{dp^r}\mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial^2 f(x,t)}{\partial t^2}\right\} = \mathcal{L}_x\mathcal{L}_t\left\{x^r\frac{\partial^2 f(x,t)}{\partial t^2}\right\}.$$

□

Lemma 3.2. [20] Double Laplace transform of the function $x^r f(x,t)$ is given as

$$\mathcal{L}_x\mathcal{L}_t\left\{x^r f(x,t)\right\} = (-1)^r \frac{d^r}{dp^r}\left[\mathcal{L}_x\mathcal{L}_t\left\{f(x,t)\right\}\right].$$

Proof. The proof is similar to the previous lemma and therefore is omitted. □

Example 3.9. Consider the singular nonlinear one dimensional of hypolic equation [27]

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{1}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial u}{\partial x}\right) - a(x)u\frac{\partial u}{\partial x} + u^2 = f(x,t), \quad (3.20)$$

subject to initial condition

$$u(x,0) = f_1(x), \quad \frac{\partial u(x,0)}{\partial t} = f_2(x), \quad (3.21)$$

where $\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)$ is Bessel operators, $f(x,t)$ and $a(x)$ are known functions.

Solution

Solving this problem by Laplace double transfom, Multiplying (3.20) by x and applying the Laplace double transform for (3.20), we have

$$\mathcal{L}_x\mathcal{L}_t\left\{x\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial u}{\partial x}\right) - xa(x)u\frac{\partial u}{\partial x} + xu^2\right\} = \mathcal{L}_x\mathcal{L}_t\left\{xf(x,t)\right\}.$$

Using the differential property of double Laplace transform

$$\mathcal{L}_x\mathcal{L}_t\left\{x^r\frac{\partial^2 u}{\partial t^2}\right\} = (-1)^r\frac{d^r}{dp^r}\left[\mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial^2 u}{\partial t^2}\right\}\right],$$

and using definition of the double Laplace transform of partial derivative for (3.18) and single Laplace transform for initial condition, we get

$$\begin{aligned}\frac{d}{dp}U(p,s) = \frac{1}{s}\frac{d}{dp}f_1(p) + \frac{1}{s^2}\frac{d}{dp}f_2(p)\\ -\frac{1}{s^2}\mathcal{L}_x\mathcal{L}_t\left\{-\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial u}{\partial x}\right) - xa(x)u\frac{\partial u}{\partial x} + xu^2\right\}\\ +\frac{1}{s^2}\frac{d}{dp}F(p,s)\end{aligned} \quad (3.22)$$

By integrating both sides of (3.22) from 0 to p, we have

$$\begin{aligned}U(p,s) = \frac{1}{s}f_1(p) + \frac{1}{s^2}f_2(p)\\ -\frac{1}{s^2}\int_0^p \mathcal{L}_x\mathcal{L}_t\left\{-\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial u}{\partial x}\right) - xa(x)u\frac{\partial u}{\partial x} + xu^2\right\}dp\\ +\frac{1}{s^2}\int_0^p \frac{d}{dp}F(p,s)dp.\end{aligned}$$

Using the double inverse Laplace transform, we obtain

$$\begin{aligned}u(x,t) = f_1(x) + tf_2(x)\\ -\mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \mathcal{L}_x\mathcal{L}_t\left\{-\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) - \frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial u}{\partial x}\right) - xa(x)u\frac{\partial u}{\partial x} + xu^2\right\}dp\right\}\\ +\mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \frac{d}{dp}F(p,s)dp\right\}.\end{aligned} \quad (3.23)$$

Note that

$$\begin{aligned}\mathcal{L}_x\mathcal{L}_t\{f_1(x)\} &= \int_0^\infty\int_0^\infty e^{-px-st}\ f_1(x)\ dtdx\\ &= \int_0^\infty\left[\lim_{b\to\infty}\frac{-1}{s}\ e^{-px-st}\ f_1(x)\Big|_0^b\right]dx\\ &= \frac{1}{s}\int_0^\infty e^{-px}f_1(x)\ dx\\ &= \frac{1}{s}\mathcal{L}_x\{f_1(x)\}\end{aligned}$$

So,

$$\mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s}\mathcal{L}_x\left\{f_1(x)\right\}\right\} = \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\mathcal{L}_x\mathcal{L}_t\left\{f_1(x)\right\}\right\} = f_1(x).$$

and,

$$\begin{aligned}
\mathcal{L}_x\mathcal{L}_t\left\{tf_2(x)\right\} &= \int_0^\infty\int_0^\infty \; e^{-px-st} \;\; tf_2(x)\; dtdx \\
&= \int_0^\infty \left[\lim_{b\to\infty}\frac{-t}{s}\; e^{-px-st} - \frac{1}{s^2}\; e^{-px-st} \;\; f_2(x)\Big|_0^b\right] dx \\
&= \frac{1}{s^2}\int_0^\infty e^{-px}f_2(x)\; dx \\
&= \frac{1}{s^2}\mathcal{L}_x\left\{f_2(x)\right\}
\end{aligned}$$

So,

$$\mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\mathcal{L}_x\left\{f_2(x)\right\}\right\} = \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\mathcal{L}_x\mathcal{L}_t\left\{tf_2(x)\right\}\right\} = tf_2(x).$$

Rewrite $u(x,t)$ as an infinite series $u_n(x,t)$

$$u(x,t) \;=\; \sum_{n=0}^{\infty} u_n(x,t), \quad n = 0,1,2,... \tag{3.24}$$

Also the nonlinear terms can be defined as follows

$$uu_x = \mathcal{N}_1 = \sum_{n=0}^{\infty} A_n, \quad u^2 = \mathcal{N}_2 = \sum_{n=0}^{\infty} B_n.$$

Where A_n and B_n are denoted by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\mathcal{N}_1\left[\sum_{i=0}^{\infty}\lambda^i u_i\right]_{\lambda=0}. \tag{3.25}$$

$$B_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\mathcal{N}_2\left[\sum_{i=0}^{\infty}\lambda^i u_i\right]_{\lambda=0}.$$

Then,

$$\begin{aligned}
A_0 &= u_0 u_{0x} \\
A_1 &= u_0 u_{1x} + u_1 u_{0x} \\
A_2 &= u_0 u_{2x} + u_1 u_{1x} + u_2 u_{0x} \\
A_3 &= u_0 u_{3x} + u_1 u_{2x} + u_2 u_{1x} + u_3 u_{0x} \\
&\vdots
\end{aligned}$$

and,

$$\begin{aligned}
B_0 &= u_0^2 \\
B_1 &= 2u_1 u_0 \\
B_2 &= 2u_2 u_0 + u_1^2
\end{aligned}$$

$$\begin{aligned}
B_3 &= 2u_3 u_0 + 2u_2 u_1 \\
&\vdots
\end{aligned}$$

By substituting (3.24) and (3.25) into (3.23), we obtain

$$\begin{aligned}
u(x,t) = f_1(x) + tf_2(x) &- \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)\sum_{n=0}^{\infty}u_n\right\}dp\right\} \\
&- \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}\right)\sum_{n=0}^{\infty}u_n\right\}dp\right\} \\
&- \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \mathcal{L}_x\mathcal{L}_t\left\{a(x)\left(\sum_{n=0}^{\infty}A_n\right)\right\}dp\right\} \\
&- \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \mathcal{L}_x\mathcal{L}_t\left\{x\left(\sum_{n=0}^{\infty}B_n\right)\right\}dp\right\} \\
&- \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \frac{d}{dp}F(p,s)dp\right\}.
\end{aligned}$$

In particular,

$$u_0 = f_1(x) + tf_2(x) + \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \frac{d}{dp}F(p,s)dp\right\}$$

and

$$\begin{aligned}
u_1(x,t) = &-\mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)u_0\right\}dp\right\} \\
&- \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}\right)u_0\right\}dp\right\} \\
&- \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \mathcal{L}_x\mathcal{L}_t\left\{a(x)\left(A_0\right)\right\}dp\right\} \\
&- \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p L_x\mathcal{L}_t\left\{x\left(B_0\right)\right\}dp\right\} \\
&- \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \frac{d}{dp}F(p,s)dp\right\}
\end{aligned}$$

In general, we have

$$\begin{aligned}
u_{n+1}(x,t) = &-\mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)\sum_{n=0}^{\infty}u_n\right\}dp\right\}\\
&-\mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \mathcal{L}_x\mathcal{L}_t\left\{\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}\right)\sum_{n=0}^{\infty}u_n\right\}dp\right\}\\
&-\mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \mathcal{L}_x\mathcal{L}_t\left\{a(x)\left(\sum_{n=0}^{\infty}A_n\right)\right\}dp\right\}\\
&-\mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p L_x\mathcal{L}_t\left\{x\left(\sum_{n=0}^{\infty}B_n\right)\right\}dp\right\}\\
&-\mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s^2}\int_0^p \frac{d}{dp}F(p,s)dp\right\}
\end{aligned}$$

By calculating the terms $u_0, u_1, ...$, we obtain the solution as

$$u(x,t) = u_0 + u_1 + ...$$

Example 3.10. Consider the following nonlinear partial differential equation [24]

$$\frac{4}{x^2}u_t - \frac{1}{x}\left(xu_x\right)_x - \frac{1}{2}xuu_x + u^2 \quad = \quad 0, \tag{3.26}$$

subject to initial condition

$$u(x,0) = x^2. \tag{3.27}$$

Solution: Multiplying (3.26) by $\frac{x^2}{4}$ and applying the double Laplace transform, we have

$$\mathcal{L}_x\mathcal{L}_t\left\{u_t - \frac{x}{4}\left(xu_x\right)_x - \frac{x^3}{8}uu_x + \frac{x^2}{4}u^2\right\} \quad = \quad 0,$$

Using the definition of the double Laplace transform for partial derivative and single

Laplace transform for initial condition, we get

$$U(p,s) \;=\; \frac{1}{s}\mathcal{L}_x\left\{u(x,0)\right\}+\frac{1}{s}\mathcal{L}_x\mathcal{L}_t\left\{\frac{x}{4}\left(xu_x\right)_x+\frac{x^3}{8}uu_x-\frac{x^2}{4}u^2\right\}. \tag{3.28}$$

Applying the inverse double Laplace transform, we obtain

$$u(x,t)=x^2+\mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s}\mathcal{L}_x\mathcal{L}_t\left\{\frac{x}{4}\left(xu_x\right)_x+\frac{x^3}{8}uu_x-\frac{x^2}{4}u^2\right\}\right\}. \tag{3.29}$$

Using the decomposition series for $u(x,t)$ which defined by

$$u(x,t) \;=\; \sum_{n=0}^{\infty}u_n(x,t), \quad n=0,1,2,... \tag{3.30}$$

Also the nonlinear term can be defined as follows

$$uu_x=\mathcal{N}_1=\sum_{n=0}^{\infty}A_n, \quad u^2=\mathcal{N}_2=\sum_{n=0}^{\infty}B_n. \tag{3.31}$$

Using (3.30) and (3.31) into (3.29), we get

$$\begin{aligned}\sum_{n=0}^{\infty}u_n(x,t)=x^2-\mathcal{L}_p^{-1}\mathcal{L}_s^{-1}&\left\{\frac{1}{s}\mathcal{L}_xL_t\left\{\frac{x}{4}\left(x\left(\sum_{n=0}^{\infty}u_n\right)_x\right)_x\right\}\right\}\\ &+\mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s}\mathcal{L}_x\mathcal{L}_t\left\{\frac{x^3}{8}\sum_{n=0}^{\infty}A_n\right\}\right\}\\ &\qquad-\mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s}\mathcal{L}_x\mathcal{L}_t\left\{\frac{x^2}{4}\sum_{n=0}^{\infty}B_n\right\}\right\},\end{aligned}$$

the other terms are given by

$$\begin{aligned}
u_0 &= x^2, \\
u_{n+1}(x,t) &= \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s}L_x\mathcal{L}_t\left\{\frac{x}{4}\left(x\left(\sum_{n=0}^{\infty}u_n\right)_x\right)_x\right\}\right\} \\
&\quad + \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s}\mathcal{L}_x\mathcal{L}_t\left\{\frac{x^3}{8}\sum_{n=0}^{\infty}A_n\right\}\right\} \\
&\qquad - \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s}\mathcal{L}_x\mathcal{L}_t\left\{\frac{x^2}{4}\sum_{n=0}^{\infty}B_n\right\}\right\}.
\end{aligned} \tag{3.32}$$

where A_n and B_n are given by

$$\begin{aligned}
A_0 &= u_0u_{0x} \\
A_1 &= u_0u_{1x} + u_1u_{0x} \\
A_2 &= u_0u_{2x} + u_1u_{1x} + u_2u_{0x} \\
A_3 &= u_0u_{3x} + u_1u_{2x} + u_2u_{1x} + u_3u_{0x} \\
&\vdots
\end{aligned}$$

and,

$$\begin{aligned}
B_0 &= u_o^2 \\
B_1 &= 2u_0u_1 \\
B_2 &= 2u_0u_2 + u_1^2 \\
B_3 &= 2u_3u_0 + 2u_2u_1 \\
&\vdots
\end{aligned}$$

The other components of the solution can be found using (3.32) as follow

$$\begin{aligned}
u_1 &= \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s}\mathcal{L}_x\mathcal{L}_t\left\{\frac{x}{4}\left(xu_{0x}\right)_x+\frac{x^3}{8}A_0-\frac{x^2}{4}B_0\right\}\right\}\\
&= \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s}\mathcal{L}_x\mathcal{L}_t\left\{x^2\right\}\right\}\\
&= x^2t\\
u_2 &= \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s}\mathcal{L}_x\mathcal{L}_t\left\{\frac{x}{4}\left(xu_{1x}\right)_x+\frac{x^3}{8}A_1-\frac{x^2}{4}B_1\right\}\right\}\\
&= \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s}\mathcal{L}_x\mathcal{L}_t\left\{\frac{x}{4}\left(xu_{1x}\right)_x+\frac{x^3}{8}(4x^3t)-\frac{x^2}{4}(2x^4t)\right\}\right\}\\
&= \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s}\mathcal{L}_x\mathcal{L}_t\left\{x^2t\right\}\right\}\\
&= \frac{x^2t^2}{2}\\
u_3 &= \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s}\mathcal{L}_x\mathcal{L}_t\left\{\frac{x}{4}\left(xu_{2x}\right)_x+\frac{x^3}{8}A_2-\frac{x^2}{4}B_2\right\}\right\}\\
&= \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s}\mathcal{L}_x\mathcal{L}_t\left\{\frac{x}{4}\left(xu_{2x}\right)_x+\frac{x^3}{8}(4x^3t^2)-\frac{x^2}{4}(2x^4t^2)\right\}\right\}\\
&= \mathcal{L}_p^{-1}\mathcal{L}_s^{-1}\left\{\frac{1}{s}\mathcal{L}_x\mathcal{L}_t\left\{\frac{x^2t^2}{2}\right\}\right\}\\
&= \frac{x^2t^3}{6}.
\end{aligned}$$

and so on.

The series solution is given by

$$\begin{aligned} u(x,t) &= u_0 + u_1 + u_2 + u_3 + ... \\ &= x^2 \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + ...\right) \end{aligned}$$

In fact, the exact solution is

$$\begin{aligned} u(x,t) &= x^2 e^t \\ &= x^2(1 + t + \frac{t^2}{2} + \frac{t^3}{6} + ...). \end{aligned}$$

Chapter 4

Other Transforms

In literature one can find many transforms, like Aboodh transform [2], Wavelet transform [35]. In this chapter we present the famous transform Sumudu and Elzaki .

4.1 Sumudu Transform

Sumudu transform is an integral transform which is applied to find the solution of ordinary and partial differential equations. It has many applications in science and engineering. The Sumudu transform was introduced by Wamgula in 1933 [30].

Definition 4.1. The Sumudu transform of a function $f(t)$ denoted by $G(u)$ over the set A

$$A = \{f(t) : \exists M, \tau_1 > 0, and/or \;\; \tau_2 > 0,\;\; such\ that\ |f(t)| < Me^{|t|/\tau_j}\ if\ t \in (-1)^j \times [0,\infty)\;\; j = 1,2\}$$

is defined by

$$G(u) = S[f(t)] = \int_0^\infty f(ut)\ e^{-t}\ dt, \quad u \in (-\tau, \tau).$$

We can write the above equation in other form:

$$G(u) = S[f(t)] = \frac{1}{u} \int_0^\infty f(t)\ e^{-\frac{t}{u}}\ dt, \quad u \in (-\tau, \tau).$$

In Table 4.1, Sumudu transform for some functions are given.

$f(t)$	$G(u)$
1	1
t	u
$\frac{t^{n-1}}{(n-1)!}$	u^{n-1}
e^{at}	$\frac{1}{1-au}$
$\frac{t^{n-1}e^{at}}{(n-1)!}$	$\frac{u^{n-1}}{(1-au)^n}$
$\sin at$	$\frac{au}{1+a^2u^2}$
$\cos at$	$\frac{1}{1+a^2u^2}$
$\sinh at$	$\frac{au}{1-a^2u^2}$
$\cosh at$	$\frac{1}{1-a^2u^2}$

Table 4.1: Sumudu transform of some functions.
.1

Properties of Sumudu Transform

In next theorems, we presented the main properties of the Sumudu transform, see [10].

Theorem 4.1. Linearity property

If a and b are any real and $f(t)$ and $g(t)$ are functions in A, then

$$S[af(t)+bg(t)] = aS[f(t)] + bS[g(t)]$$

Proof. If a and b are any constants, then

$$\begin{aligned} S[af(t)+bg(t)] &= \int_0^\infty [af(ut)+bg(ut)]e^{-t}dt \\ &= a\int_0^\infty f(ut)e^{-t}dt + b\int_0^\infty g(ut)e^{-t}dt \\ &= aS[f(t)]+bS[g(t)]. \end{aligned}$$

□

Theorem 4.2. First Scale Preserving Property

Let the Sumudu transform of $f(t) \in A$ is $G(u)$, then

$$S[f(at)] = G(au)$$

Proof.

$$\begin{aligned} S[f(at)] &= \int_0^\infty f(aut)\ e^{-t}dt \\ &= G(au). \end{aligned}$$

□

Theorem 4.3. First Shifting Property

Let the Sumudu transform of $f(t) \in A$ is $G(u)$, then

$$S[e^{at}f(t)] = \frac{1}{1-au}\ G\left[\frac{u}{1-au}\right].$$

Proof. The Sumudu transform of $e^{at}f(t)$ is given by

$$S[e^{at}f(t)] = \int_0^\infty f(ut)\ e^{aut}\ e^{-t}\ dt = \int_0^\infty f(ut)\ e^{-(1-au)t}\ dt$$

Let $w = t(1-au)$ and $t = \dfrac{w}{1-au}$, then we obtain

$$\begin{aligned} S[e^{at}f(t)] &= \int_0^\infty f\left(\frac{uw}{1-au}\right)e^{-w}\frac{dw}{1-au} \\ &= \frac{1}{1-au}\int_0^\infty f\left(\frac{uw}{1-au}\right)e^{-w}dw \\ &= \frac{1}{1-au}\,G\left[\frac{u}{1-au}\right]. \end{aligned}$$

□

Sumudu Transform of Derivatives

let$f(t)$ be acontinoues function having exponential order, if $G(u)$ is Sumudu transform of $f(t)$, then Sumudu Transforms of derivatives of that fuction are given as follows:

Theorem 4.4. If $S[f(t)] = G(u)$, then

$$S[f'(t)] = \frac{G(u)}{u} - \frac{f(0)}{u}.$$

Proof.

$$\begin{aligned} S[f'(t)] &= \int_0^\infty f'(ut)e^{-t}dt \\ &= \lim_{c\to\infty}\left[\frac{f(ut)e^{-t}}{u}\Big|_0^c\right] + \frac{1}{u}\int_0^\infty f(ut)e^{-t}dt \\ &= \frac{-f(0)}{u} + \frac{1}{u}G(u) \\ &= \frac{G(u)}{u} - \frac{f(0)}{u} \end{aligned}$$

□

Theorem 4.5. If $S[f(t)] = G(u)$, then

$$S[f''(t)] = \frac{G(u)}{u^2} - \frac{f(0)}{u^2} - \frac{f'(0)}{u}.$$

Proof.

$$\begin{aligned} S[f''(t)] &= \int_0^\infty f''(ut)e^{-t}dt \\ &= \lim_{c\to\infty}\left[\frac{f'(ut)e^{-t}}{u}\Big|_0^c\right] + \frac{1}{u}\int_0^\infty f'(ut)e^{-t}dt \\ &= \frac{-f'(0)}{u} + \frac{1}{u}\left[\frac{G(u)}{u} - \frac{f(0)}{u}\right] \\ &= \frac{G(u)}{u^2} - \frac{f(0)}{u^2} - \frac{f'(0)}{u} \end{aligned}$$

□

Theorem 4.6. If $S[f(t)] = G(u)$, then

$$S[f^{(n)}(t)] = \frac{G(u)}{u^n} - \sum_{k=0}^{n-1} \frac{1}{u^{n-k}}\, f^{(k)}(0).$$

Sumudu decomposition method (SDM)

The Sumudu Decomposition Method (SDM), is a combination of Sumudu Transform Method and Adomain decomposition method.

The nonlinear term can easily be handled by the use of Adomain polynomials. The technique is described and illustrated in the next examples.

Example 4.1. Consider the nonlinear partial differential equation [15]

$$y_t + yy_x = y_{xx}, \tag{4.1}$$

with the initial condition

$$y(x,0) \;=\; 2x, \quad t > 0. \tag{4.2}$$

Solution:

Taking the Sumudu transform to both sides of (4.1), we have

$$\frac{S[y(x,t)]}{u} - \frac{y(x,0)}{u} = -S[yy_x] + S[y_{xx}].$$

By substituting $y(x,0) = 2x$ we obtain

$$S[y(x,t)] \;=\; 2x - uS[yy_x] + uS[y_{xx}]. \tag{4.3}$$

Then by taking the inverse of the Sumudu transform of the (4.3) we have

$$y(x,t) \;=\; 2x - S^{-1}\left[uS[yy_x]\right] + S^{-1}\left[uS[y_{xx}]\right]. \tag{4.4}$$

Rewrite $y(x,t)$ as infinite series of $y_n(x,t)$

$$y(x,t) \;=\; \sum_{n=0}^{\infty} y_n(x,t), \quad n = 0,1,2,... \tag{4.5}$$

The nonlinear term can be written by

$$yy_x = \sum_{n=0}^{\infty} A_n. \tag{4.6}$$

The Adomain polynomials are

$$\begin{aligned}
A_0 &= y_0 y_{0x} \\
A_1 &= y_0 y_{1x} + y_1 y_{0x} \\
A_2 &= y_0 y_{2x} + y_1 y_{1x} + y_2 y_{0x} \\
A_3 &= y_0 y_{3x} + y_1 y_{2x} + y_2 y_{1x} + y_3 y_{0x} \\
&\vdots
\end{aligned}$$

By using (4.5) and (4.6) we can write (4.4) as

$$\sum_{n=0}^{\infty} y_n = 2x - S^{-1}\left[uS\left[\sum_{n=0}^{\infty} A_n\right]\right] + S^{-1}\left[uS\left[\sum_{n=0}^{\infty} y_n\right]_{xx}\right].$$

So we get the iterations as follows

$$\begin{aligned}
y_0(x,t) &= 2x \\
y_1(x,t) &= -S^{-1}\left[uS\left[A_0\right]\right] + S^{-1}\left[uS\left[y_0\right]_{xx}\right] \\
&= -S^{-1}\left[uS[4x]\right] \\
&= -4xt
\end{aligned}$$

$$\begin{aligned}
y_2(x,t) &= -S^{-1}\left[uS\left[A_1\right]\right] + S^{-1}\left[uS\left[y_1\right]_{xx}\right] \\
&= -S^{-1}\left[uS[-16xt]\right] \\
&= 8xt^2
\end{aligned}$$

$$
\begin{aligned}
y_3(x,t) &= -S^{-1}\left[uS\left[A_2\right]\right] + S^{-1}\left[uS\left[y_2\right]_{xx}\right] \\
&= -S^{-1}\left[uS[48xt^2]\right] \\
&= -16xt^3.
\end{aligned}
$$

Thus, summing the above iterations we obtain

$$
\sum_{n=0}^{3} y_n(x,t) = 2x\left(1 - 2t + (2t)^2 - (2t)^3\right).
$$

The exact solution is

$$
\begin{aligned}
y(x,t) &= \frac{2x}{1+2t} \\
&= 2x(1-(2t)+(2t)^2-(2t)^3+...),\ |t| < \frac{1}{2}
\end{aligned}
$$

The computed terms coincide with the first terms in the exact solution.

4.2 Elzaki Transform

Tarig Elzaki introduced an integral transform named the Elzaki transform in 2011 [21]. This transform is applied to the solve of ordinary and partial differential equations.

Definition 4.2. The Elzaki transform of a function $f(t)$ over the set A of functions given by

$$
A = \{f(t) : \exists M, k_1, k_2 > 0 \ such\ that\ |f(t)| < Me^{|t|/k_j},\ if\ t \in (-1)^j \times [0,\infty)\}
$$

is defined by

$$E[f(t)] = T(v) = v \int_0^\infty f(t)\ e^{-\frac{t}{v}}\ dt, \quad v \in (-k_1, k_2),$$

we can write the equation in other form

$$E[f(t)] = T(v) = v^2 \int_0^\infty f(vt)\ e^{-t}\ dt, \quad v \in (-k_1, k_2).$$

The following table Elzaki transform for some functions are given.

Special function	Elzaki transform
$f(t)$	$E[f(t)] = T(v)$
1	v^2
t	v^3
$t^n,\ n = 0, 1, 2, ...$	$n!v^{n+2}$
e^{at}	$\frac{v^2}{1 - av}$
$\frac{t^{n-1}e^{at}}{(n-1)!},\ n = 1, 2, ...$	$\frac{v^{n+1}}{(1 - av)^n}$
$\sin at$	$\frac{av^3}{1 + a^2v^2}$
$\cos at$	$\frac{v^2}{1 + a^2u^3}$

Table 4.2: Elzaki transform of some functions.

Elzaki transform of derivatives

If the Elzaki transform of the function $f(t)$ is given by $T(v)$, then Elzaki Transforms of derivatives of that fuction are given as follows:

Theorem 4.7. If $E[f(t)] = T(v)$, then

$$E[f'(t)] = \frac{T(v)}{v} - vf(0).$$

Proof.

$$\begin{aligned} E[f'(t)] &= v\int_0^\infty f'(t)e^{-\frac{t}{v}}dt \\ &= \lim_{c\to\infty}\left[vf(t)e^{-\frac{t}{v}}\Big|_0^c\right] + \int_0^\infty f(t)e^{-\frac{t}{v}}dt \\ &= \frac{T(v)}{v} - vf(0). \end{aligned}$$

□

Theorem 4.8. If $E[f(t)] = T(v)$, then

$$E[f''(t)] = \frac{T(v)}{v^2} - f(0) - vf'(0).$$

Proof.

$$E[f''(t)] = v\int_0^\infty f''(t)e^{-\frac{t}{v}}dt \tag{4.7}$$

let

$$g(t) = f'(t)$$

then

$$E[g'(t)] = \frac{E[g(t)]}{v} - vg(0)$$

we find that by using previous theorem we get

$$E[f''(t)] = \frac{T(v)}{v^2} - f(0) - vf'(0).$$

□

Theorem 4.9. If $E[f(t)] = T(v)$, then

$$E[f^{(n)}(t)] = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k}\ f^k(0),$$

Elzaki decomposition method (EDM)

We conclude this section by introducing Elzaki decomposition method. This method is a combination of Elzaki transform and the Adomain decomposition method, it is used to solve linear and nonlinear partial differential equations [22].

Example 4.2. Consider the nonlinear partial differential equation [44]

$$u_t + uu_x - u_{xx} = 0, \tag{4.8}$$

with the initial condition

$$u(x,0) \ = \ x. \tag{4.9}$$

Solution:

Applying the Elzaki transform coupled with the ADM to (4.8), we have

$$\frac{T(x,v)}{v} - vu(x,0) + E[uu_x] - E[uxx] = 0$$

By substituting $u(x,0)=x$ we obtain

$$T(x,v) \quad = \quad v^2x - vE[uu_x] + vE[u_{xx}], \tag{4.10}$$

Then by taking the inverse of the Elzaki transform of the equation (4.10) we get

$$u(x,t) \quad = \quad x - E^{-1}\left[vE[uux] - [uxx]\right]. \tag{4.11}$$

Rewrite $u(x,t)$ as an infinite series of $u_n(x,t)$

$$u(x,t) \quad = \quad \sum_{n=0}^{\infty} u_n(x,t), \quad n = 0,1,2,... \tag{4.12}$$

Also the nonlinear term can be written by

$$\mathcal{N} = uu_x = \sum_{n=0}^{\infty} A_n. \tag{4.13}$$

The Adomain polynomials are

$$\begin{aligned} A_0 &= u_0u_{0x} \\ A_1 &= u_0u_{1x} + u_1u_{0x} \\ A_2 &= u_0u_{2x} + u_1u_{1x} + u_2u_{0x} \\ A_3 &= u_0u_{3x} + u_1u_{2x} + u_2u_{1x} + u_3u_{0x} \\ &\vdots \end{aligned}$$

By using (4.12) and (4.13) we can write (4.11) as

$$\sum_{n=0}^{\infty} u_n(x,t) \quad = \quad x - E^{-1}\left[vE\left[\sum_{n=0}^{\infty} A_n(u) - (\sum_{n=0}^{\infty} u_n)_{xx}\right]\right].$$

We now express few components as follow

$$
\begin{aligned}
u_0(x,t) &= x \\
u_1(x,t) &= -E^{-1}\left[vEA_0(u) - (u_0)_{xx}(x,t)\right] \\
&= -E^{-1}\left[vE(x-0)\right] \\
&= -xt \\
u_2(x,t) &= -E^{-1}\left[vE\left[A_1(u) - (u_1)_{xx}(x,t)\right]\right] \\
&= -E^{-1}\left[vE(-2xt-0)\right] \\
&= xt^2 \\
u_3(x,t) &= E^{-1}\left[vE\left[A_2(u) - (u_2)_{xx}(x,t)\right]\right] \\
&= -E^{-1}\left[vE(xt^2-0)\right] \\
&= -xt^3
\end{aligned}
$$

The first four terms of the decomposition series solution for Equation (4.8) is given by

$$u(x,t) = x - xt + xt^2 - xt^3 + ...$$

The exact solution is

$$
\begin{aligned}
u(x,t) &= \frac{x}{1+t}, \quad |t| < 1 \\
&= x(1 - t + t^2 - t^3 + ...).
\end{aligned}
$$

Chapter 5

Conclusion

In this thesis, a general review of the integral transforms combind with the Adomain decomposition method were presented . Started with the Natural decomposition method followed by the Laplace decomposition method, Sumudu decomposition method, and finally Elzaki decomposition method.These methods were applied for several nonlinear ordinary and partial differential equations. In addition, we employed the double Natural decomposition method and double Laplace decomposition method to solve nonlinear Bossinesq equation.

All the above methods are semi-analytical techniques, based on decomposing the solution to aseries of functions. The terms of the solution are obtained by a recurrence relation.

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