Deanship of the Graduate Studies and Scientific Research

Palestine Polytechnic University

Bounded and Absolutely Summing Operators Between Sequence Spaces

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Master of Science Thesis

Hebron - Palestine

Aug., 2009

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Supervisor: Dr. Ibrahim Almasri

A Thesis submitted to the Department of Mathematics of Palestinian Polytechnic University as a partial fulfillment of the requirements for the degree of master of science.

The program of graduated studies / Department of Mathematics

Deanship of the Graduate Studies

Bounded and Absolutely Summing Operators Between Sequence Spaces

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Master thesis submitted and accepted, Date: 6/9/2009.

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Palestine Polytechnic University

2009

Declaration

I certify that this thesis, submitted for the degree of master, is the result of my own research except where otherwise acknowledged, and that this thesis (or any part of the same) has not been submitted for a higher degree to any other university or institution.

Signed:..........Ala..

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Signed:..

'Ala Ata Abdelhady Talahmeh

Date: 6 / 9 / 2009.

Dedication

To my parents;

my fiancée 'Nejood' ,

my teachers

and

my brothers

I dedicate this work.

Acknowledgment

I gratefully extend my thanks to those people who helped in completing this work, specially and personally to my supervisor, Dr. Ibrahim Almasri for their help and advice throughout the period for the study. And I would like to thank Dr. Mohammad Saleh from Birzeit University for his valuable help. Also, I would like to thank all doctors in my faculty those helped me in completing this research.

Abstract

In this thesis, we consider the diagonal operators d_α ; $\alpha = (\alpha_n)$ between the sequence ℓ^p spaces, and we give the necessary and sufficient conditions for this diagonal operator to be bounded. Then we give necessary and sufficient conditions for general operators between ℓ^p – spaces to be bounded.

Finally, we give a different variation of the statement of Garling's Theorem, which gives a complete characterization for diagonal operators d_α between ℓ^p – spaces to be p-absolutely summing, and we give an alternative proof for this fundamental theorem.

Contents

Index of Special Notation

R	The set of all real numbers.
C	Complex plane or the field of complex numbers.
$C[a,b]$	Space of continuous functions.
$D(T)$	Domain of an operator T.
$\dim X$	Dimension of space X.
I	Identity operator.
inf	Infimum (greatest lower bound).
$N(T)$	Null space of an operator.
0	Zero operator.
$R(T)$	Range of an operator T.
sup	Supremum (least upper bound).
$\|T\|$	Norm of an operator T.
X'	Dual space of a normed space X.
$\|x\|$	Norm of x.
$< x,y >$	Inner product of x and y.
R^n	Euclidean n-space.
$\Pi_p(X,Y)$	The space of absolutely p-summing operators.
$\sigma(T)$	Hilbert – Schmidt norm of T.
d_α	Diagonal operator.
min	minimum.
ℓ^p	A sequence space.

Introduction

The subject of operator theory came into focus rapidly after 1900. Reisz studied the algebra of bounded operators on the Hilbert space ℓ^2. Between 1929-1932, von Neumann introduced the unbounded operators and he also used the concept of infinite matrices that had been a popular way to understand operators.

In the last three decades of the twentieth century , a considerable interest was paid to the question: When does an operator A take ℓ^p – space to an ℓ^q – space ?

Crone (see [15]) characterized matrices from ℓ^2 to ℓ^2. In ([16]) there is a complete characterization to matrices from ℓ^1 to ℓ^1 and from ℓ^∞ to ℓ^∞. Then ([16]) gave a sufficient condition for A to map ℓ^p into ℓ^p; $1 < p < \infty$. In ([16]) another sufficient condition for A to map ℓ^p into ℓ^p was given.

Bennett in ([11]) gave necessary conditions for A to map ℓ^p into ℓ^q .Then ([13]) and ([19]) gave complete characterization for certain operators (matrices) to map ℓ^p into ℓ^q.

In chapter two of this thesis, I give a complete characterizations for the diagonal operator d_α to map ℓ^p into ℓ^q.

In chapter three I talk about the absolutely summing operators between sequence spaces. The summing of linear operators is a very basic concept in Banach space theory . In 1933, Orlicz's studied when every unconditionally convergent series $\sum_n x_n$ in L^p (where $1 \le p \le 2$) the $\sum_n \|x_n\|^2$ is convergent. In 1947, Macphail showed that in ℓ^1, such a series may have $\sum_n \|x_n\|$ divergent.

Dvoretzky and Rogers then proved that the same applies in every infinite dimensional Banach space. After this came the definition of absolutely summing operator to one for which $\sum_n \|Tx_n\|$ is convergent for every unconditionally convergent series $\sum_n x_n$. The summing is an example of " operator ideal " norms.

The theory of absolutely summing operators was one of the most profound developments in Banach space theory between 1950 and 1970. It originates in a fundamental paper of Grothendieck (which actually appeared in 1956). It was not until the 1968 paper of Lindenstrauss and Pelczynski that Grothendieck`s ideas become widely known. Since 1968, the theory of absolutely summing operators has become an important aspect of modern Banach space theory . At the end of sixties , Pietsch promoted the notion of p-summing operators between Banach spaces, which extends to all values of $p \in [1,\infty)$ .Then Lindenstrauss and Pelczynski gave the important result in *p*-summing that *every operator from* ℓ^1 to ℓ^2 *is 1- summing* ; another formulation of this is the famous Grothendieck's inequality .

Chapter One
Bounded and Continuous linear Operators.

1.1 Linear Operators.

1.1.1 Definition. Let X and Y be normed spaces, then $T: D(T) \subseteq X \to Y$ is said to be a linear operator if

(i) The domain $D(T)$ of T is a subspace of X.

(ii) for all $x, y \in D(T)$ and scalars α,

$$T(x+y) = T(x) + T(y).$$
$$T(\alpha\, x) = \alpha\, T(x).$$

1.1.2 Definition.

(i) The null space, denoted by $N(T)$ is defined by

$N(T) = \{x \in D(T) : Tx = 0\}$.

(ii) The range of T, denoted by $R(T)$ is defined by

$$R(T) = \{\, y : y = Tx, \text{for some } x \in D(T)\,\}.$$

1.2 Examples.

1.2.1 Identity Operator. The identity operator $I: X \to X$ is defined by $I(x) = x$ for all $x \in X$.

1.2.2 Zero Operator. The zero operator $0: X \to Y$ is defined by $0(x) = x$ for all $x \in X$.

1.2.3 Differentiation. Let X be the vector space of all polynomials on $[a,b]$. We may define a linear operator T on X by setting $Tx(t) = x'(t)$ for every $x \in X$. Where the prime denotes differentiation with respect to t. This operator T maps X onto itself.

1.2.4 Integration. A linear operator T from $C[a,b]$ into itself can be defined by $Tx(t) = \int_a^t x(\tau)d\tau \quad , \quad t \in [a,b]$.

1.2.5 Matrices. A real matrix $A = (\alpha_{jk})_{r\times n}$ defines an operator $T: R^n \to R^r$ by means of $y = Ax$ where $x = (\zeta_j)_{n\times 1}$ and $y = (\xi_j)_{r\times 1}$, writing $y = Ax$ out, we have

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \xi_n \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdot & \cdot & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdot & \cdot & \alpha_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{r1} & \alpha_{r2} & \cdot & \cdot & \alpha_{rn} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \cdot \\ \cdot \\ \zeta_n \end{bmatrix}$$

In these examples we can easily verify that the ranges and null spaces of the linear operators are vector spaces , and this is the content of the following theorem.

1.3 Theorem (Range and Null space).

Let $T: D(T) \subset X \to Y$ be a linear operator, where X and Y are normed spaces. Then:

a) $T(0) = 0$.

b) $R(T)$ is a subspace of Y.

c) If $\dim(D(T)) = n < \infty$, then $\dim((R(T)) \leq n$.

d) The null space $N(T)$ is a subspace of X.

Proof:

a) $T(0) = T(x - x) = Tx - Tx = 0.$

b) We take any $y_1, y_2 \in R(T)$ and show that $\alpha y_1 + \beta y_2 \in R(T)$ for any scalars α, β. Since $y_1, y_2 \in R(T)$; we have $y_1 = Tx_1$, $y_2 = Tx_2$ for some $x_1, x_2 \in D(T)$; and $\alpha x_1 + \beta x_2 \in D(T)$ because $D(T)$ is a vector space. The linearity of T yields $T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2 = \alpha y_1 + \beta y_2$. Hence $\alpha y_1 + \beta y_2 \in R(T)$.

c) Notice that the image of any spanning set of $D(T)$ is a spanning set for $R(T)$. So, let $y_1, y_2, \ldots\ldots, y_n, y_{n+1} \in R(T)$, then $\exists$ $x_1, x_2, \ldots\ldots x_n, x_{n+1} \in D(T)$ such that $Tx_i = y_i \quad , i = 1, 2, \ldots\ldots, n+1$.

Since $\dim(D(T)) = n$, so $x_1, x_2, \ldots\ldots, x_{n+1}$ are linearly dependent.

Hence $\exists$ $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ not all zero such that $\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_{n+1} x_{n+1} = 0$.

Since T is linear and $T(0) = 0$, then

$T(\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_{n+1} x_{n+1}) = \alpha_1 y_1 + \alpha_2 y_2 + \ldots + \alpha_{n+1} y_{n+1} = 0$.

This shows that $\{y_1, y_2, \ldots, y_{n+1}\}$ is a linearly dependent set because the α_j's are not all zero. Therefore $\dim R(T) < n+1$, i.e. $\dim R(T) \le n$.

d) We take any $x_1, x_2 \in N(T)$. Then $Tx_1 = Tx_2 = 0$, since T is linear, for any scalars α, β we have $T(\alpha x_1 + \beta x_2) = \alpha Tx_1 + \beta Tx_2 = 0$.

This shows that $\alpha x_1 + \beta x_2 \in N(T)$. Hence $N(T)$ is a vector space.

Q.E.D.

1.4 Bounded linear operator.

1.4.1 Definition. Let X and Y be normed spaces and $T : D(T) \to Y$ a linear operator, where $D(T) \subset X$. The operator T is said to be bounded if there is a real number $c > 0$ such that for all $x \in D(T)$,

$$\|Tx\| \le c\,\|x\|. \qquad (1)$$

1.4.2 Remarks:

(i) $\|Tx\| \le c\,\|x\|$ shows that bounded sets in $D(T)$ are mapped onto bounded sets in Y.

(ii) The term bounded here is different from bounded in calculus, where a bounded function is one whose range is a bounded set.

The question here is "what is the smallest possible c such that (1) still holds for all nonzero $x \in D(T)$? [we can leave out $x = 0$ since $Tx = 0$ for $x = 0$].

By division, $\dfrac{\|Tx\|}{\|x\|} \le c \quad (x \ne 0). \Rightarrow \sup\left\{\dfrac{\|Tx\|}{\|x\|}, x \ne 0\right\} \le c.$ Hence the answer to our question is that the smallest possible c in (1) is that supremum.

So, we define the norm of T as $\|T\| = \sup\left\{\dfrac{\|Tx\|}{\|x\|}, x \ne 0, x \in D(T)\right\}$.

If $D(T) = \{0\}$, we define $\|T\| = 0$. Note that (1) with $c = \|T\|$ is $\|Tx\| \le \|T\|\,\|x\|$.

1.4.3 Lemma. Let T be a bounded linear operator. Then

a) An alternative formula for the norm of T is $\|T\| = \sup\{\|Tx\|, x \in D(T), \|x\| = 1\}$.

b) $\|T\|$ satisfies the properties of the norm.

Proof: (a) we write $\|x\| = a$ and set $y = \frac{1}{a}x$, where $x \neq 0$. Then $\|y\| = \frac{\|x\|}{a} = 1$ and

since T is linear, $\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|}, x \neq 0\right\} = \sup\left\{\left\|T\left(\frac{x}{\|x\|}\right)\right\|, x \neq 0\right\} = \sup\{\|Ty\|, \|y\| = 1\}$

(b) $\|T\| \geq 0$ is obvious and so is $\|0\| = 0$. From $\|T\| = 0$, we have $Tx = 0\ \forall x \in D(T)$ so that $T = 0$. Furthermore,

$\sup\{\|\alpha Tx\|,\ \|x\| = 1\} = \sup\{|\alpha|\,\|Tx\|, \|x\| = 1\} = |\alpha| \sup\{\|Tx\|, \|x\| = 1\}$ where $x \in D(T)$.

Finally,

$\sup\{\|(T_1 + T_2)(x)\|, \|x\| = 1\} = \sup\{\|T_1x + T_2x\|, \|x\| = 1\} \leq \sup\{\|T_1x\|, \|x\| = 1\} + \sup\{\|T_2x\|, \|x\| = 1\}$
where $x \in D(T)$. Q.E.D.

1.5 Examples:

1.5.1 Identity Operator. The identity operator $I: X \to X$ on a normed space $X \neq \{0\}$ is bounded and has norm $\|I\| = 1$

1.5.2 Differentiation Operator. Let X be the normed space of all polynomials on $J = [0,1]$ with norm given $\|x\| = \max|x(t)|,\ t \in J$.

A differentiation operator T is defined on X by $Tx(t) = x'(t)$. This operator is linear but not bounded. Indeed, take $x_n(t) = t^n$, then $\|x_n\| = \max\{|x_n(t)|, 0 \leq t \leq 1\} = \max\{t^n, 0 \leq t \leq 1\} = 1$.

but $Tx_n(t) = x_n'(t) = nt^{n-1}$, so $\|Tx_n\| = \max\{|nt^{n-1}|, 0 \leq t \leq 1\} = n$ and $\frac{\|Tx_n\|}{\|x_n\|} = n$. Since $n \in N$ is arbitrary this shows that there is no fixed number c such that $\frac{\|Tx_n\|}{\|x_n\|} \leq c$. Hence T is not bounded. Q.E.D.

1.5.3 Integral Operator.

We can define an integral operator $T : C[0,1] \to C[0,1]$ by $y = Tx$ where $y(t) = \int_0^1 k(t,\tau)x(\tau)d\tau$. Here k is a given function, which is called the kernel of T and is assumed to be continuous on the closed square $J \times J = G$ in the $t\tau - plane$, where $J = [0,1]$. This operator is linear and bounded.

To prove that T is bounded, we first note that the continuity of k on the closed square implies that k is bounded, say, $|k(t,\tau)| \le k_0$ for all $(t,\tau) \in G$, where k_0 is a real number. Furthermore, $|x(t)| \le \max\{|x(t)|, t \in J\} = \|x\|$.

Hence $\|y\| = \|Tx\| = \max\left\{\left|\int_0^1 k(t,\tau)x(\tau)d\tau\right|,\ t \in J\right\} \le \max\left\{\int_0^1 |k(t,\tau)||x(\tau)|d\tau, t \in J\right\} \le k_0\|x\|$.

Hence T is bounded. Q.E.D.

1.5.4 The matrix operator.

We can define a matrix operator $T : R^n \to R^m$ by $y = Tx = Ax$. where A is the real matrix $(\alpha_{jk})_{mxn}$ and $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$. This operator is linear and bounded. We'll prove that T is bounded,

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdot & \cdot & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdot & \cdot & \alpha_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{j1} & \alpha_{j2} & \cdot & \cdot & \alpha_{jn} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{m1} & \alpha_{m2} & \cdot & \cdot & \alpha_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_j \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_j \\ \cdot \\ \cdot \\ y_m \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \sum_{k=1}^{n} \alpha_{1k}x_k \\ \cdot \\ \cdot \\ \sum_{k=1}^{n} \alpha_{mk}x_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ y_m \end{bmatrix}$$

$$\Rightarrow \|Tx\|^2 = \sum_{j=1}^{m}\left|\sum_{k=1}^{n}\alpha_{jk}x_k\right|^2$$

Now from the Cauchy – Schwarz inequality, we obtain,

$$\|Tx\|^2 \le \sum_{j=1}^{m}\left[\left(\sum_{k=1}^{n}|\alpha_{jk}|^2\right)\left(\sum_{k=1}^{n}|x_k|^2\right)\right] = \sum_{k=1}^{n}|x_k|^2\sum_{j=1}^{m}\sum_{k=1}^{n}|\alpha_{jk}|^2 = \|x\|^2\left(\sum_{j=1}^{m}\sum_{k=1}^{n}|\alpha_{jk}|^2\right)$$

$\therefore \|Tx\| \le \left(\sum_{j=1}^{m}\sum_{k=1}^{n}|\alpha_{jk}|^2\right)^{\frac{1}{2}}\|x\|$ which is of the form $\|Tx\| \le c\|x\|$ with $c = \left(\sum_{j=1}^{m}\sum_{k=1}^{n}|\alpha_{jk}|^2\right)^{\frac{1}{2}}$,

i.e T is bounded. Q.E.D.

1.6 Continuity and Boundedness.

Operators are mappings, so that the definition of continuity applies to them. It is a fundamental fact that for a linear operator, continuity and boundedness become equivalent concepts as follows.

1.6.1 Theorem. Let $T: D(T) \to Y$ be a linear operator, where $D(T) \subset X$ and X, Y are normed spaces. Then.

a) T is continuous iff T is bounded.

b) If T is continuous at a single point, it is continuous.

Proof:

a) For $T = 0$ the statement is trivial. So let $T \ne 0$. Then $\|T\| \ne 0$. We assume T to be bounded and consider any $x_0 \in D(T)$. Let any $\varepsilon > 0$ be given. Then, since T is linear, for every $x \in D(T)$ such that $\|x - x_0\| < \delta$ where $\delta = \frac{\varepsilon}{\|T\|}$, we obtain $\|Tx - Tx_0\| = \|T(x - x_0)\| \le \|T\|\ \|x - x_0\| \le \|T\|\delta = \varepsilon$. Since $x_0 \in D(T)$ was arbitrary, this shows that T is continuous.

Conversely, assume that T is continuous at an arbitrary $x_0 \in D(T)$. Then, given any $\varepsilon > 0$, there is a $\delta > 0$. such that $\|Tx - Tx_0\| \le \varepsilon\ \forall x \in D(T)$ satisfying

$\|x - x_0\| \le \delta$ we now take any $y \neq 0$ in $D(T)$ and set $x = x_0 + \frac{\delta}{\|y\|} y$. Then

$x - x_0 = \frac{\delta}{\|y\|} y$. Hence $\|x - x_0\| = \delta$ so,

$$\varepsilon \ge \|Tx - Tx_0\| = \|T(x - x_0)\| = \left\| T\left(\frac{\delta\, y}{\|y\|}\right) \right\| = \frac{\delta}{\|y\|}\|Ty\|$$

i.e., $\frac{\delta}{\|y\|}\|Ty\| \le \varepsilon \Rightarrow \|Ty\| \le \frac{\varepsilon}{\delta}\|y\|$. This can be written as $\|Ty\| \le c\|y\|$

where $c = \frac{\varepsilon}{\delta}$; which shows that T is bounded.

b) Continuity of T at a point implies boundedness of T by the second part of the proof of (a), which in turn implies continuity of T by (a). Q.E.D

1.7 Linear Functional.

1.7.1 Definition : A linear functional f is a linear operator with domain in a normed space and range in the scalar field R or C .

1.7.2 Definition :

(i) A functional $f : X \to R$ is linear if $f(ax + by) = af(x) + bf(y)$, where $x, y \in X$, and $a, b \in R$

(ii) A linear functional f is bounded if $\exists c \in R$ such that $|f(x)| \le c\|x\|$.

(iii) The norm of f is denoted by $\|f\|$ and defined as $\|f\| = \sup\{|f(x)|, \|x\| = 1, x \in D(f)\}$.

1.7.3 Examples.

(1) The norm $\|.\| : X \to R$ is not a linear functional .

(2) Let $f_a : R^3 \to R$ be defined as $f_a(x) = a.x$ where a is some fixed vector in R^3 . Then f_a is linear and bounded with $\|f_a\| = \| a\|$.

(3) Definite Integral.

Let $x \in C[a,b]$ and define $f(x) = \int_a^b x(t)\,dt$, i.e., $f : C[a,b] \to R$. Then f is a bounded linear functional with $\|f\| = b - a$.

1.8 Isomorphism.

1.8.1 Definition (Isomorphism).

(i) An isomorphism T of a vector space X onto a vector space Y over the same field is a bijective mapping which preserves the two algebraic operations of vector space; thus, for all $x, y \in X$ and scalars α, $T(x+y) = T(x) + T(y)$, $T(\alpha x) = \alpha T(x)$, i.e, $T : X \to Y$ is a bijective linear operator, Y is then called isomorphic with X, and X and Y are called isomorphic vector spaces.

(ii) An isomorphism of a normed space X onto a normed space Y is a bijective linear operator $T : X \to Y$ which preserves the norm, i.e, for all $x \in X$, $\|Tx\| = \|x\|$. X is then called isomorphic with Y, and X and Y are called isomorphic normed spaces. From an abstract point of view, X and Y are then identical, the isomorphism merely a mounting to renaming of the elements.

1.9 Dual Space.

1.9.1 Definition : Let X be a vector space, the algebraic dual space of X, denoted by X^*, is the set of all linear functionals on X.

1.9.2 Definition : Let X be a normed space. Then the set of all bounded linear functionals on X constitutes a normed space with norm defined by $\|f\| = \sup\{|f(x)|, \|x\| = 1, x \in X\}$ which is called the *dual space* of X and is denoted by X'.

1.9.3 Remarks.

(1) X^* is itself a vector space where the vector addition and scalar multiplication are defined as :

$(f_1 + f_2)(x) = f_1(x) + f_2(x)$, $(\alpha f_1)(x) = \alpha f_1(x)$.

(2) Since X^* is a vector space, so we may consider its dual X^{**} which is called the second dual of X.

(3) We can obtain a $g \in X^{**}$ by choosing a fixed $x \in X$ and setting $g_x(f) = f(x), x \in X$ is fixed, $f \in X^*$ is variable. Note that g_x is linear functional . Hence $g_x \in X^{**}$. To each $x \in X$, these corresponds a $g_x \in X^{**}$. This defines the so called canonical mapping $C: X \to X^{**}$ where $C(x) = g_x$.

1.9.4 Definition: Let $p \geq 1$ be a fixed real number. Each element in the space ℓ^p is a sequence $x = (\xi_j)$ of numbers such that $\sum_{j=1}^{\infty} |\xi_j|^p$ converges i.e.,

$\sum_{j=1}^{\infty} |\xi_j|^p < \infty$, and the norm of x is defined by $\|x\| = \left(\sum_{j=1}^{\infty} |\xi_j|^p \right)^{\frac{1}{p}}$.

1.9.5 *Example* : The dual space of ℓ^p is ℓ^q; here $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: Every $x \in \ell^p$ has a unique representation

(1) $$x = \sum_{k=1}^{\infty} \xi_k e_k \text{ where } e_k = (\delta_{kj}).$$

We consider any $f \in \ell^{p'}$, where $\ell^{p'}$ is the dual space of ℓ^p. Since f is linear and bounded ,

(2) $$f(x) = \sum_{k=1}^{\infty} \xi_k \gamma_k \text{ where } \gamma_k = f(e_k).$$

Let q be the conjugate of p and consider $x_n = (\xi_k^{(n)})$ with

(3) $$\xi_k^{(n)} = \begin{cases} \dfrac{|\gamma_k|^q}{\gamma_k} & \text{if } k \leq n \text{ and } \gamma_k \neq 0, \\ 0 & \text{if } k > n \text{ or } \gamma_k = 0. \end{cases}$$

By substituting this into (2) we obtain $f(x_n) = \sum_{k=1}^{\infty} \xi_k^{(n)} \gamma_k = \sum_{k=1}^{n} |\gamma_k|^q$.

We also have , using (3) and $(q-1)p = q$,

$$|f(x_n)| \leq \|f\| \|x_n\| = \|f\| \left(\sum_{k=1}^{n} |\xi_k^{(n)}|^p \right)^{\frac{1}{p}} = \|f\| \left(\sum_{k=1}^{n} |\gamma_k|^{(q-1)p} \right)^{\frac{1}{p}} = \|f\| \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{\frac{1}{p}}.$$

So, $|f(x_n)| = \sum_{k=1}^{n} |\gamma_k|^q \le \|f\| \left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{\frac{1}{p}}$.

Dividing by the last factor, we get $\left(\sum_{k=1}^{n} |\gamma_k|^q \right)^{\frac{1}{q}} \le \|f\|$.

Since n is arbitrary, letting $n \to \infty$, we obtain

(4) $$\left(\sum_{k=1}^{\infty} |\gamma_k|^q \right)^{\frac{1}{q}} \le \|f\|.$$

Hence $(\gamma_k) \in \ell^q$.

Conversely, for any $b = (\beta_k) \in \ell^q$ we can get a corresponding bounded linear functional g on ℓ^p. In fact , we may define g on ℓ^p by setting $g(x) = \sum_{k=1}^{\infty} \xi_k \beta_k$ where $x = (\xi_k) \in \ell^p$. The linearity and boundedness of g follows from the Holder's inequality . Hence $g \in \ell^{p'}$.

We finally prove that the norm of f is the norm on the space ℓ^q.

From (2) and the Holder inequality we have $|f(x)| = \left| \sum_{k=1}^{\infty} \xi_k \gamma_k \right| \le \left(\sum_{k=1}^{\infty} |\xi_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |\gamma_k|^q \right)^{\frac{1}{q}} = \|x\| \left(\sum_{k=1}^{\infty} |\gamma_k|^q \right)^{\frac{1}{q}}$; hence by taking the supremum over all x of norm 1 we obtain $\|f\| \le \left(\sum_{k=1}^{\infty} |\gamma_k|^q \right)^{\frac{1}{q}}$.

From (4) we see that the equality sign must hold, i.e,

(5) $$\|f\| = \left(\sum_{k=1}^{\infty} |\gamma_k|^q \right)^{\frac{1}{q}}.$$

This can be written $\|f\| = \|c\|_q$, where $c = (\gamma_k) \in \ell^q$ and $\gamma_k = f(e_k)$.The mapping of $\ell^{p'}$ onto ℓ^q defined by $f \mapsto c$ is linear and bijective , and from (5) we see that it is norm preserving , so that it is an isomorphism. Q.E.D.

1.10 Banach Spaces and Hilbert Spaces.

1.10.1 Definition : The normed space X is said to be complete if every Cauchy sequence in X converges.

1.10.2 Definition : A Banach space is a complete normed space .

1.10.3 Definition : An inner product space is a vector space X with an inner product defined on X where an inner product on X is a mapping of $X \times X$ into the scalar field K of X ; i.e. ,

with every pair of vectors x and y there is associated a scalar which is written $<x,y>$ and is called the inner product of x and y, such that for all vectors x, y, z and scalars α we have :

(1) $<x+y,z> = <x,z>+<y,z>$

(2) $<\alpha x,y>=\alpha<x,y>$

(3) $<x,y>=\overline{<y,x>}$

(4) $<x,x> \geq 0$.

$<x,x> = 0 \Leftrightarrow x=0$

1.10.4 Definition : A Hilbert space is a complete inner product space .

1.10.5 Definition: An orthogonal set M in an inner product space X is a subset $M \subset X$ whose elements are pair wise orthogonal, that is $<x,y>=0$ for all distinct $x, y \in M$.

1.10.6 Definition: An orthonormal set $M \subset X$ is an orthogonal set in X whose elements have norm 1 , i.e, for all $x, y \in M$, $<x,y> = \begin{cases} 0 & if\ x \neq y \\ 1 & if\ x = y \end{cases}$. If an orthogonal or orthonormal set M is countable , we can arrange it in a sequence (x_n) and call it an orthogonal or orthonormal sequence respectively . More generally , an indexed set, or family , (x_α), $\alpha \in I$, is called orthogonal if $<x_\alpha, x_\beta>=0$ for all $\alpha, \beta \in I$, $\alpha \neq \beta$.The family is called orthonormal if it is orthogonal and all x_α have norm 1 , so that for all $\alpha, \beta \in I$ we have

$$<x_\alpha , x_\beta> = \begin{cases} 0 & if \ \alpha \neq \beta \\ 1 & if \ \alpha = \beta \end{cases} .$$

Chapter Two

Bounded Operators Between Sequence Spaces

2.1 Diagonal Operators.

2.1.1 Definition: Let $\alpha = (\alpha_n)$ be a sequence of complex numbers . Define d_α by $d_\alpha(x) = (\alpha_n x_n)$, where $x = (x_n)$ then d_α is called a diagonal operator.

In our research we will be interested in the diagonal operators from one ℓ^p space to another. In general, this operator need not be bounded, for example let $d_\alpha : \ell^3 \to \ell^1$ be defined as $d_\alpha(x) = (\alpha_n x_n)$ where $\alpha_n = \frac{1}{\sqrt[3]{n}}$, $x_n = \frac{1}{\sqrt[3]{n^2}}$.

Now $x = (x_n) \in \ell^3$. On the other hand, $\|d_\alpha(x)\|_1 = \|(\alpha_n x_n)\|_1 = \|(\frac{1}{n})\|_1 \to \infty$

$\therefore$ d_α is not bounded.

The following example shows that the diagonal operator could be bounded , let $d_\alpha : \ell^2 \to \ell^1$ be defined as $d_\alpha(x) = (\alpha_n x_n)$ where $\alpha_n = \frac{1}{n}$, now

$\| d_\alpha(x) \|_1 = \|(\alpha_n x_n)\|_1 = \sum_{n=1}^{\infty} |\alpha_n||x_n| \le \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} \|x\|_2$, it is of the form $\| d_\alpha(x) \|_1 \le k \|x\|_2$

where $k = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}} < \infty$ $\therefore$ d_α is bounded.

So, the question arises: what are the conditions to be put on the sequence (α_n) to make the corresponding diagonal operator bounded?

The answer of this question follows from the following theorems.

2.2 Bounded Diagonal Operators.

2.2.1 Theorem : $d_\alpha : \ell^p \to \ell^q$ is bounded iff $\alpha \in \ell^{\frac{pq}{p-q}}$ where $p > q \ge 1$.

Furthermore, $\|d_\alpha\| = \|\alpha\|_s$ where $s = \frac{pq}{p-q}$.

Proof: Let $\alpha \in \ell^{\frac{pq}{p-q}}$ and $p > q \geq 1$, then

$$\|d_\alpha(x)\|_q = \|(\alpha_n x_n)\|_q = \left(\sum_{n=1}^{\infty}|\alpha_n|^q |x_n|^q\right)^{\frac{1}{q}} \leq \left(\sum_{n=1}^{\infty}|\alpha_n|^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} \left(\sum_{n=1}^{\infty}|x_n|^{q.\frac{p}{q}}\right)^{\frac{1}{\frac{p}{q}.q}}$$

(by Holder's inequality)

$$\Rightarrow \quad \|d_\alpha(x)\|_q \leq \left(\sum_{n=1}^{\infty}|\alpha_n|^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} \left(\sum_{n=1}^{\infty}|x_n|^p\right)^{\frac{1}{p}}$$

or $\|d_\alpha(x)\|_q \leq k\,\|x\|_p$ where $k = \left(\sum_{n=1}^{\infty}|\alpha_n|^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} < \infty$ (since $\alpha \in \ell^{\frac{pq}{p-q}}$)

$\therefore$ d_α is bounded, and $\|d_\alpha\| \leq \left(\sum_{n=1}^{\infty}|\alpha_n|^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}}$ ....(1)

Conversely, suppose that d_α is bounded, consider the sequence

$$x_n = \left(\xi_k^{(n)}\right) \text{ with } \xi_k^{(n)} = \begin{cases} |\alpha_k|^{\frac{q}{p-q}} & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

Now, $\| d_\alpha(x_n)\|_q \leq \| d_\alpha\| \|x_n\|_p = \| d_\alpha\| \left(\sum_{k=1}^{n}|\xi_k^{(n)}|^p\right)^{\frac{1}{p}} = \| d_\alpha\| \left(\sum_{k=1}^{n}|\alpha_k|^{\frac{pq}{p-q}}\right)^{\frac{1}{p}}$

$$\Rightarrow \| d_\alpha(x_n)\|_q = \left(\sum_{k=1}^{n}|\alpha_k|^{\frac{pq}{p-q}}\right)^{\frac{1}{q}} \leq \| d_\alpha\| \left(\sum_{k=1}^{n}|\alpha_k|^{\frac{pq}{p-q}}\right)^{\frac{1}{p}}.$$

Dividing by the last factor we get,

$$\left(\sum_{k=1}^{n}|\alpha_k|^{\frac{pq}{p-q}}\right)^{\frac{1}{q}-\frac{1}{p}} \leq \| d_\alpha \| \text{ or } \left(\sum_{k=1}^{n}|\alpha_k|^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} \leq \| d_\alpha \|.$$

Since n is arbitrary, letting $n \to \infty$, we obtain

$$\left(\sum_{k=1}^{\infty}|\alpha_k|^{\frac{pq}{p-q}}\right)^{\frac{p-q}{pq}} \leq \| d_\alpha \| \quad \text{...................(2)}$$

Hence $\alpha \in \ell^{\frac{pq}{p-q}}$. Furthermore from (1) and (2) we get $\|d_\alpha\| = \|\alpha\|_{\frac{pq}{p-q}}$. Q.E.D.

2.2.2 Corollary: $d_\alpha : \ell^p \to \ell^1$ is bounded iff $\alpha \in \ell^{p'}$ where $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof: Follows directly from Theorem 2.2.1 by taking $q = 1$. Q.E.D

2.2.3 Theorem : $d_\alpha : \ell^p \to \ell^q$ is bounded iff $\alpha \in \ell^\infty$ where $1 \le p \le q \le \infty$. Furthermore, $\|d_\alpha\| = \|\alpha\|_\infty$.

Proof: Suppose that d_α is bounded. Let e_n be the sequence whose n^{th} term is 1 and all other terms are zero. Then any sequence $x \in \ell^p$ has a unique representation $x = \sum_{k=1}^{\infty} x_k e_k$ where $x = (x_k)$.

Since d_α is linear and bounded, then $d_\alpha(x) = \sum_{k=1}^{\infty} x_k \xi_k$ where $\xi_k = d_\alpha(e_k)$, where the numbers $\xi_k = d_\alpha(e_k)$ are uniquely determined by d_α . Also $\| e_k \|_p = 1$and $|\alpha_k| = \| d_\alpha(e_k)\|_q \le \| d_\alpha \| \| e_k \|_p = \| d_\alpha \| \Rightarrow \sup_k |\alpha_k| \le \| d_\alpha \| < \infty$ (since d_α is bounded) .

$\Rightarrow \|\alpha\|_\infty \le \| d_\alpha \|$............................(1)

Conversely, suppose that $\alpha \in \ell^\infty$ and let $x \in \ell^p$, so

$$\| d_\alpha(x)\|_q = \|(\alpha_n x_n)\|_q$$

$$\le \|(\alpha_n x_n)\|_p$$

$$= \left(\sum_{n=1}^{\infty} |\alpha_n|^p |x_n|^p \right)^{\frac{1}{p}}$$

$$\le \left(\sup|\alpha_n|^p \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

$$= \left(\sup|\alpha_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

$\Rightarrow \quad \| d_\alpha(x)\|_q \le \|\alpha\|_\infty \|x\|_p$ which implies that $\| d_\alpha \| \le \|\alpha\|_\infty$....................(2)

From (1) and (2) we get that d_α is bounded iff $\alpha \in \ell^\infty$ and $\|d_\alpha\| = \|\alpha\|_\infty$

Q.E.D.

2.2.4 ***Theorem***: Suppose that $1 \le q < \infty$, then $\alpha \in \ell^\infty$ if $d_\alpha : \ell^\infty \to \ell^q$ is bounded. Furthermore $\|\alpha_k\|_\infty \le \| d_\alpha \|$.

Proof: Let $1 \le q \le \infty$ and suppose that $d_\alpha : \ell^\infty \to \ell^q$ is bounded.

Let e_n be the sequence whose n^{th} term is 1 and all other terms are zero. Then any sequence $x \in \ell^\infty$ can be written as $x = \sum_{k=1}^{\infty} x_k e_k$ where $x = (x_k)$.

Since d_α is linear and bounded, then $d_\alpha(x) = \sum_{k=1}^{\infty} x_k \xi_k$ where $\xi_k = d_\alpha(e_k)$, and

$\| d_\alpha(e_k)\|_q \le \| d_\alpha \| \| e_k \|_\infty = \| d_\alpha \|$

$\Rightarrow |\alpha_k| \le \| d_\alpha \| \quad \Rightarrow \sup_k |\alpha_k| \le \| d_\alpha \| < \infty$ (since d_α is bounded)

$\Rightarrow \|\alpha_k\|_\infty \le \| d_\alpha \| < \infty$. Hence $\alpha \in \ell^\infty$. Q.E.D.

2.2.5 ***Theorem***: suppose that $1 \le q < \infty$, then $d_\alpha : \ell^\infty \to \ell^q$ is bounded if $\alpha \in \ell^q$. Furthermore $\|d_\alpha\| \le \|\alpha\|_q$.

Proof : Let $1 \le q < \infty$ and suppose that $\alpha \in \ell^q$ and let $x \in \ell^\infty$, so

$$\| d_\alpha(x)\|_q = \|(\alpha_n x_n)\|_q$$

$$\le \left(\sup|x_n|^q \sum_{n=1}^{\infty} |\alpha_n|^q \right)^{\frac{1}{q}}$$

$$= \left(\sum_{n=1}^{\infty} |\alpha_n|^q \right)^{\frac{1}{q}} \|x\|_\infty$$

which is of the form $\| d_\alpha(x)\|_q \le k \|x\|_\infty$ where $k = \left(\sum_{n=1}^{\infty} |\alpha_n|^q \right)^{\frac{1}{q}} < \infty$,

$\therefore$ d_α is bounded, and $\|d_\alpha\| \le \|\alpha\|_q$. Q.E.D.

2.3 Examples of Bounded Diagonal Operator.

2.3.1 Example: If we take $p = 4, q = 1$ in Theorem 2.2.1 , we obtain ,

$d_\alpha : \ell^4 \to \ell^1$ is bounded iff $\alpha \in \ell^{\frac{4}{3}}$. In particular if $\alpha_n = \frac{1}{n}$, then

$d_{\frac{1}{n}} : \ell^4 \to \ell^1$ is bounded since $\frac{1}{n} \in \ell^{\frac{4}{3}}$ (since $\left(\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}\right)^{\frac{3}{4}} < \infty$).

But if we take $\alpha_n = \frac{1}{\sqrt{n}}$, then $d_{\frac{1}{\sqrt{n}}} : \ell^4 \to \ell^1$ is not bounded since $\frac{1}{\sqrt{n}} \notin \ell^{\frac{4}{3}}$

$(\because \left(\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}\right)^{\frac{3}{4}} = \infty)$.

2.3.2 Example: If we take $p = 1, q = 2$ in theorem 2.2.3 we obtain that

$d_\alpha : \ell^1 \to \ell^2$ is bounded iff $\alpha \in \ell^\infty$. In particular if $\alpha_n = \frac{1}{\sqrt{n}}$, then

$d_{\frac{1}{\sqrt{n}}} : \ell^1 \to \ell^2$ is bounded since $\alpha \in \ell^\infty$.

But if we take $\alpha_n = 2^n$, then $d_{2^n} : \ell^1 \to \ell^2$ is not bounded since $2^n \notin \ell^\infty$.

2.4 Matrix transformation of ℓ^p spaces.

2.4.1 Definition: A matrix $A=(a_{jk})$ is said to map ℓ^p into ℓ^q if $Ax\in\ell^q$ whenever $x\in\ell^p$. The norm of A is given by

$$\|A\|_{p,q}=\sup\left\{\|Ax\|_q\ ,\|x\|_p\le 1\right\}=\sup\left\{\left(\sum_j\left|\sum_k a_{jk}x_k\right|^q\right)^{\frac{1}{q}},\ \|x\|_p\le 1\right\}.........(*)$$

The estimation of this matrix norm is complicated , but the general problem of necessary conditions here may be described as follows:

Given $1\le p,q,r\le\infty$, what is the smallest value of $s=s(p,q,r)$ for which

$\sum_j\left(\sum_k|a_{jk}|^r\right)^{\frac{s}{r}}<\infty$.....................(**) whenever A maps ℓ^p into ℓ^q ? The answer of this question follows from the following theorem.

2.4.2 Theorem [11, Theorem1.1]**:** Let $1\le p,q,r\le\infty$ be given and suppose that $r\ge\frac{p}{p-1}$. Then the condition (**) holds whenever A maps ℓ^p into ℓ^q provided that :

(i) $s=\infty$, in case $p\le q$;

(ii) $\frac{1}{s}\le\frac{1}{q}-\frac{1}{p}$ in case $p>q$ and $r\ge 2$;

(iii) $\frac{1}{s}\le\frac{1}{q}-\frac{1}{p}-\frac{1}{r}+\frac{1}{2}$, in case $p>q$, $r<2$ and $q\le 2$;

(iv) $\frac{1}{s}\le\min\left\{\frac{1}{q}-\frac{1}{p},\frac{2}{q}\left(1-\frac{1}{p}-\frac{1}{r}\right)\right\}$ in case $p>q>2>r$ and $r\ne\frac{2p}{p+q-2}$;

(v) $\frac{1}{s}\le\frac{1}{q}-\frac{1}{p}$, in case $\infty>p>q>2>r=\frac{2p}{p+q-2}$.

The proof of this theorem is long and complicated and will not be given here. Part (ii) of this theorem has interesting results so that we only mention some special cases of part(ii) in the following corollaries.

2.4.3 Corollary: $\sum_j\left(\sum_k|a_{jk}|^2\right)^{\frac{1}{2}}<\infty$ whenever A maps ℓ^∞ into ℓ^1.

Proof: Setting $p=\infty$ and $q=1$ in Theorem 2.4.2 (ii) we obtain $s\geq 1$ and $r\geq 2$ so that the smallest value of s and r are respectively 1 and 2 . Now setting $s=1$ and $r=2$ in (**) we get $\sum_j\left(\sum_k|a_{jk}|^2\right)^{\frac{1}{2}}<\infty$. Q.E.D.

2.4.4 Corollary: If A maps ℓ^2 into ℓ^1, then $\sum_j\sum_k|a_{jk}|^2<\infty$.

Proof: Setting $p=2$ and $q=1$ in Theorem 2.4.2 (ii) we obtain $s\geq 2$ and $r\geq 2$ so that the smallest value of s and r are 2 . Now setting $s=r=2$ in (**) we get $\sum_j\left(\sum_k|a_{jk}|^2\right)^{\frac{2}{2}}<\infty$ or $\sum_j\sum_k|a_{jk}|^2<\infty$. Q.E.D.

2.4.5 Corollary: If A maps $\ell^{p'}$ into ℓ^1 ,then $\sum_j\left(\sum_k|a_{jk}|^2\right)^{\frac{p}{2}}<\infty$,

where $\frac{1}{p}+\frac{1}{p'}=1$.

Proof: Setting $p=p'$ and $q=1$ in Theorem 2.4.2 (ii) we obtain $s\geq p$ and $r\geq 2$ so that the smallest value of s and r are respectively p and 2. Now setting $s=p$ and $r=2$ in (**) we get $\sum_j\left(\sum_k|a_{jk}|^2\right)^{\frac{p}{2}}<\infty$. Q.E.D.

2.4.6 Corollary: If the diagonal matrix $\alpha=(\alpha_n)$ maps ℓ^p into ℓ^q, then $\alpha\in\ell^{\frac{pq}{p-q}}$.

Proof: From Theorem 2.4.2(ii) we get the smallest value of s and r are respectively $\frac{pq}{p-q}$ and 2 . Now since α maps ℓ^p into ℓ^q, then condition (**) is satisfied , i.e., $\sum_j\left(\sum_k|\alpha_{jk}|^2\right)^{\frac{pq}{2(p-q)}}<\infty$

or $\sum_{j}\left(\left|\alpha_{j1}\right|^{2}+\left|\alpha_{j2}\right|^{2}+\ldots\ldots+\left|\alpha_{jj}\right|^{2}+\ldots\ldots\right)^{\frac{pq}{2(p-q)}}<\infty$

or $\sum_{j}\left(\left|\alpha_{jj}\right|^{2}\right)^{\frac{pq}{2(p-q)}}<\infty$ (since α is a diagonal matrix)

or $\sum_{j}\left|\alpha_{j}\right|^{\frac{pq}{p-q}}<\infty$. Hence $\alpha \in \ell^{\frac{pq}{p-q}}$, which is the same result we got in Theorem 2.2.1 but in a different way. Q.E.D.

2.5 Matrix operators on ℓ^p (Sufficient Conditions).

2.5.1 Definition: Let $B(\ell^p)$ be the normed linear space of all bounded linear operators on ℓ^p into ℓ^p; so that $A \in B(\ell^p)$ iff $\forall x \in \ell^p, Ax \in \ell^p$ where $(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k$ for $n = 1,2,$.........................

The norm $\|A\|$ of a matrix A in $B(\ell^p)$ is given by $\|A\| = \sup \left\{ \|Ax\|_p \ , \|x\|_p \leq 1 \right\}$.

So , the question arises : what are the conditions to be put on the matrix A to make it in $B(\ell^p)$? The answer of this question follows from the following theorem.

2.5.2 Theorem[10,Theorem1]: If $b_{nk} > 0$ for $n,k = 1,2,......,$ and if

$$\sup_{n\geq 1} \sum_{k=1}^{\infty} |a_{nk}| (b_{nk})^{\frac{1}{p}} = M_1 < \infty$$

and $\sup_{k\geq 1} \sum_{n=1}^{\infty} |a_{nk}| (b_{nk})^{\frac{-1}{q}} = M_2 < \infty$, then $A \in B(\ell^p)$ and $\|A\| \leq M_1^{\frac{1}{q}} M_2^{\frac{1}{p}}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: Let $y_n = \sum_{k=1}^{\infty} a_{nk} x_k$ where $x = (x_k) \in \ell^p$. Now,

$$|y_n| = \left| \sum_{k=1}^{\infty} a_{nk} x_k \right|$$

$$= \left| \sum_{k=1}^{\infty} \left((a_{nk})^{\frac{1}{q}} (b_{nk})^{\frac{1}{qp}} (a_{nk})^{\frac{1}{p}} (b_{nk})^{\frac{-1}{qp}} x_k \right) \right|$$

$$\leq \left(\sum_{k=1}^{\infty} \left| (a_{nk})^{\frac{1}{q}} (b_{nk})^{\frac{1}{qp}} \right|^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{\infty} \left| (a_{nk})^{\frac{1}{p}} (b_{nk})^{\frac{-1}{qp}} (x_k) \right|^p \right)^{\frac{1}{p}} \quad \text{(by Holder's inequality)}$$

$$\leq \left(\sum_{k=1}^{\infty} |a_{nk}| (b_{nk})^{\frac{1}{p}} \right)^{\frac{1}{q}} \left(\sum_{k=1}^{\infty} |a_{nk}| (b_{nk})^{\frac{-1}{q}} |x_k|^p \right)^{\frac{1}{p}}$$

$$\Rightarrow |y_n|^p \le \left(\sum_{k=1}^{\infty} |a_{nk}|(b_{nk})^{\frac{1}{p}}\right)^{p-1} \left(\sum_{k=1}^{\infty} |a_{nk}|(b_{nk})^{\frac{-1}{q}} |x_k|^p\right) \quad (\because \frac{1}{p}+\frac{1}{q}=1)$$

$$\le M_1^{p-1}\left(\sum_{k=1}^{\infty} |a_{nk}|(b_{nk})^{\frac{-1}{q}} |x_k|^p\right), \text{ and hence}$$

$$\sum_{n=1}^{\infty} |y_n|^p \le M_1^{p-1} \sum_{k=1}^{\infty} |x_k|^p \sum_{n=1}^{\infty} |a_{nk}|(b_{nk})^{\frac{-1}{q}}$$

$$\le M_1^{p-1} M_2 \sum_{k=1}^{\infty} |x_k|^p .$$

$$\Rightarrow \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}} \le M_1^{\frac{p-1}{p}} M_2^{\frac{1}{p}} \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}}$$

$$\Rightarrow \|Ax\|_p \le M_1^{\frac{1}{q}} M_2^{\frac{1}{p}} \|x\|_p$$

$$\Rightarrow \sup\{\|Ax\|_p, \|x\|_p \le 1\} \le \sup\left\{M_1^{\frac{1}{q}} M_2^{\frac{1}{p}} \|x\|_p,\ \|x\|_p \le 1\right\}$$

$\Rightarrow \|A\| \le M_1^{\frac{1}{q}} M_2^{\frac{1}{p}}$. Since $M_1, M_2 < \infty$, then $\|A\| < \infty$

$\therefore\ A \in B(\ell^p)$. Q.E.D.

In the following special cases, there are complete nice characterizations.

2.5.3 Theorem[16, C1]: The matrix A maps ℓ^1 to ℓ^1 iff $\sup_{k\ge 0} \sum_{n=0}^{\infty} |a_{nk}| < \infty$.

2.5.4 Theorem[16, C2]: The matrix A maps ℓ^∞ to ℓ^∞ iff $\sup_{n\ge 0} \sum_{k=0}^{\infty} |a_{nk}| < \infty$.

In the same book (see[16]) , we see that if both the conditions in Theorems 2.5.3 and 2.5.4 are satisfied , then A maps ℓ^p to ℓ^p.

In ([17]) another sufficient condition is given as follows.

2.5.5 Theorem[17,C4]: For $1 < p < \infty$, A maps ℓ^p into ℓ^p if

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} |a_{nk}|^q\right)^{\frac{p}{q}} < \infty .$$

2.6 Matrix Operators on ℓ^p to ℓ^q.

2.6.1 Theorem[19,Theorem1]: Let $\infty > p \geq q > 1$. Then an infinite nonnegative matrix $A = (a_{ij})$ maps ℓ^p into ℓ^q if and only if there exists a positive constant C and a sequence $u = (u_j)_{j=1}^{\infty}$ of nonnegative numbers with the following properties :

(i) $u_j = 0$ if and only if $a_{ij} = 0$, $\forall i, j$.

(ii) $\|u\|_p \leq 1$ if $p > q$.

(iii) for each $j = 1,2,........,$

$$\sum_{i=1}^{\infty} a_{ij}\left(\sum_{k=1}^{\infty} a_{ik}u_k\right)^{q-1} \leq C u_j^{p-1}. \qquad (1)$$

The best value of C in (1) for such a sequence u can be found is $\left(\|A\|_{p,q}\right)^q$.

Proof: Assume that C and $u_k (k \geq 1)$ are positive numbers satisfying (ii) and (iii) . We will show that $\|Ax\|_q \leq C^{\frac{1}{q}}\|x\|_p$, $x \in \ell^p$. (2)

Let $y_i = \sum_{j=1}^{\infty} a_{ij}x_j$ where $x = (x_j) \in \ell^p$. Now,

$$|y_i| = \left|\sum_{j=1}^{\infty} a_{ij}x_j\right|$$

$$= \left|\sum_{j=1}^{\infty}\left((a_{ij})^{\frac{1}{q}}(u_j)^{\frac{1-q}{q}} x_j (a_{ij}u_j)^{\frac{q-1}{q}}\right)\right|$$

$$\leq \left(\sum_{j=1}^{\infty}\left|(a_{ij})^{\frac{1}{q}}(u_j)^{\frac{1-q}{q}} x_j\right|^q\right)^{\frac{1}{q}}\left(\sum_{k=1}^{\infty}\left|(a_{ik}u_k)^{\frac{q-1}{q}}\right|^{\frac{q}{q-1}}\right)^{1-\frac{1}{q}} \qquad \text{(By Holder's inequality)}$$

$$\leq \left(\sum_{j=1}^{\infty} a_{ij}u_j^{1-q}|x_j|^q\right)^{\frac{1}{q}}\left(\sum_{k=1}^{\infty} a_{ik}u_k\right)^{1-\frac{1}{q}}$$

$$\Rightarrow |y_i|^q \leq \left(\sum_{j=1}^{\infty} a_{ij}\, u_j^{1-q}|x_j|^q\right)\left(\sum_{k=1}^{\infty} a_{ik}u_k\right)^{q-1}$$

$$\Rightarrow \sum_{i=1}^{\infty} |y_i|^q \le \sum_{j=1}^{\infty} u_j^{1-q} |x_j|^q \sum_{i=1}^{\infty} a_{ij} \left(\sum_{k=1}^{\infty} a_{ik} u_k \right)^{q-1}$$

$$\le \sum_{j=1}^{\infty} u_j^{1-q} |x_j|^q \, C u_j^{p-1}$$

(By condition (iii))

$$\Rightarrow \sum_{i=1}^{\infty} |y_i|^q \le C \sum_{j=1}^{\infty} u_j^{p-q} |x_j|^q$$

$$= C \sum_{j=1}^{\infty} \left(u_j^p \right)^{\frac{p-q}{p}} \left(|x_j|^p \right)^{\frac{q}{p}}$$

(If $p > q$, a second application of Holder's inequality yields)

$$\le C \left(\sum_{j=1}^{\infty} |u_j|^p \right)^{\frac{p-q}{p}} \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{\frac{q}{p}}$$

$$= C \| u \|_p^{p-q} \| x \|_p^q$$

$$\Rightarrow \left(\sum_{i=1}^{\infty} |y_i|^q \right)^{\frac{1}{q}} \le C^{\frac{1}{q}} \| u \|_p^{\frac{p-q}{q}} \| x \|_p$$

$$\le C^{\frac{1}{q}} (1)^{\frac{p-q}{q}} \| x \|_p$$

(By condition (ii))

$\therefore$ $\quad \|Ax\|_q \le C^{\frac{1}{q}} \|x\|_p$. (For $p = q$ see [18])

Conversely, let $A = (a_{ij})$ be a nonnegative matrix taking ℓ^p into ℓ^q . Assume that A is positive and put $C = \left(\|A\|_{p,q} \right)^q$. For each $n = 1,2,........$, we can define a positive n-tuple $u^{(n)} = (u_j^{(n)})$ with $\| u^{(n)} \|_p = 1$ such that

$\sum_{i=1}^{n} a_{ij} \left(\sum_{k=1}^{n} a_{ik} u_k^{(n)} \right)^{q-1} \le C \left(u_j^{(n)} \right)^{p-1}$, $j = 1,2,........, n$ (see [12]and [12 page 223-224]).

Define , for $j = 1,2,........$

$u_j = \liminf_{n \to \infty} \left(u_j^{(n)} \right)$. Then $u = (u_j)_{j=1}^{\infty}$ is a sequence of positive numbers such that (ii) and (iii) are satisfied. Q.E.D.

Borwein and X.Gao in their paper (see[13]) improved Theorem 2.6.1 as follows :

2.6.2 Theorem[13,Theorem A]: Let $p>1$. Then a nonnegative matrix $A=(a_{ij})$ maps ℓ^p into ℓ^p if and only if there exist a positive number C and a positive sequence $u=(u_j)$ such that

$$\sum_{i=1}^{\infty} a_{ij}\left(\sum_{k=1}^{\infty} a_{ik}u_k\right)^{p-1} \le C\,u_j^{p-1}. \quad j=1,2,\ldots\ldots \quad (1)$$

and then $\|A\|_p \le C^{\frac{1}{p}}$. Further, if the nonnegative matrix $A=(a_{ij})$ maps ℓ^p into ℓ^p, then there exist a positive sequence u for which (1) holds with $C=\left(\|A\|_p\right)^p$.

2.6.3 Theorem[13,Theorem D]: Let $p>q>1$. Then a nonnegative matrix $A=(a_{ij})$ maps ℓ^p into ℓ^q if and only if there exist a positive constant C and a positive sequence $u=(u_j)$ with the following properties :

(a) $\|u\|_p \le 1$,

(b) $$\sum_{i=1}^{\infty} a_{ij}\left(\sum_{k=1}^{\infty} a_{ik}u_k\right)^{q-1} \le C\,u_j^{p-1} \quad , \quad j=1,2,\ldots\ldots \quad (2)$$

and then $\|A\|_{p,q} \le C^{\frac{1}{q}}$. Further, if the nonnegative matrix $A=(a_{ij})$ maps ℓ^p into ℓ^q, then there exist a nonnegative sequence $u=(u_j)$ with $0<\|u\|_p\le 1$ for which (2) holds with $C=\left(\|A\|_{p,q}\right)^q$, and $u_j=0$ only when $a_{ij}=0$, $\forall i,j$.

In order to prove these theorems we need the following theorem.

2.6.4 Theorem [13,Theorem 1]: Suppose that $p,q>1$, that the nonnegative matrix $A=(a_{ij})$ maps ℓ^p into ℓ^q, and that $C>\left(\|A\|_{p,q}\right)^q$. Then there exists a positive sequence $u=(u_j)$ such that $\|u\|_p \le 1$ and (2) is true .

Proof: The proof is difficult. [see 13].

Now we will give the proofs of the necessity parts of Theorems 2.6.2 and 2.6.3.

Proof of Theorems 2.6.2 and 2.6.3: Suppose that $p \geq q > 1$ and that the nonnegative matrix $A = (a_{ij})$ maps ℓ^p into ℓ^q. Let $C_n = \left(\|A\|_{p,q}\right)^p + \frac{1}{n}$ for $n \in N$.

Then , by Theorem 2.6.4 , there is a positive sequence $u^{(n)} = (u_j^{(n)})$ such that

$$\left\|u^{(n)}\right\|_p \leq 1 \text{ and } \sum_{i=1}^{\infty} a_{ij}\left(\sum_{k=1}^{\infty} a_{ik} u_k^{(n)}\right)^{q-1} \leq C_n \left(u_j^{(n)}\right)^{p-1}, \quad j = 1,2,\ldots\ldots..$$

Case 1. Let $p > q > 1$. Define $u = (u_j)$ where $u_j = \liminf_{n\to\infty} \left(u_j^{(n)}\right)$. Then $\|u\|_p \leq 1$

$$\text{and } \sum_{i=1}^{\infty} a_{ij}\left(\sum_{k=1}^{\infty} a_{ik} u_k\right)^{q-1} \leq \left(\|A\|_{p,q}\right)^p u_j^{p-1}. \quad j = 1,2,\ldots\ldots..$$

Case 2. $p = q > 1$. The proof is difficult. [see 13].

2.7 Factorable Matrix Operators on ℓ^p to ℓ^q.

2.7.1 Definition: A matrix $A=(a_{nk})_{n,k=1}^{\infty}$ of the type $a_{nk}=\begin{cases} a_n b_k, & 1\le k\le n \\ 0, & k>n \end{cases}$

is called factorable matrix.

G. Bennett in his paper (see [14]) gave the following theorem that gives different equivalent form for the factorable matrix to map ℓ^p to ℓ^q with $1<p\le q<\infty$.

2.7.2 Theorem [14,Theorem 2]: : Let $1<p\le q<\infty$, let **a** and **b** be sequences of nonnegative numbers , and let A be a factorable matrix. Then the following conditions are equivalent.

(i) A maps ℓ^p into ℓ^q ;

(ii) there exists K_1 such that , for $m=1,2,\ldots\ldots$,

$$\sum_{n=1}^{m}\left(a_n\sum_{k=1}^{n}b_k^{p^*}\right)^q \le K_1\left(\sum_{k=1}^{m}b_k^{p^*}\right)^{\frac{q}{p}};$$

(iii) there exists K_2 such that , for $m=1,2,\ldots\ldots$,

$$\left(\sum_{n=m}^{\infty}a_n^q\right)^{\frac{1}{q}}\left(\sum_{k=1}^{m}b_k^{p^*}\right)^{\frac{1}{p^*}} \le K_2 ;$$

(iv) there exists K_3 such that , for $m=1,2,\ldots\ldots$,

$$\sum_{k=m}^{\infty}\left(b_k\sum_{n=k}^{\infty}a_n^q\right)^{p^*} \le K_3\left(\sum_{n=m}^{\infty}a_n^q\right)^{\frac{p^*}{q^*}}$$

Where $p^*=\dfrac{p}{p-1}$ and $q^*=\dfrac{q}{q-1}$.

The proof is long and complicated and will not be given here.

Chapter Three

Absolutely P-summing Operators

3.1 The class of absolutely p-summing operators.

This section establishes the basic facts about the class Π_p of absolutely p-summing operators between Banach spaces. After defining the notion of an absolutely p-summing operator and giving a brief discussion regarding the viewpoint of absolutely p-summing operators being operators that increase the degree of summability of a sequence, we show that $\Pi_p(X,Y)$ is a Banach space with norm π_p and we show that Π_p is an "operator ideal" and establish the basic inclusion relationship between Π_p's of different index.

After that we define the Hilbert –Schmidt operator and we show that the class of Hilbert–Schmidt operator coincides with the absolutely 2-summing operators.

We close this chapter with some theorems , and by these theorems we can determine when the diagonal operator between ℓ^p – spaces is absolutely summing.

3.1.1 Definition[3,4] : let $1 \le p < \infty$, and let X and Y be Banach spaces, then an operator $T : X \to Y$ is said to be absolutely p-summing if there exists a constant $\rho > 0$ such that for each natural number n and for each $x_1, x_2, \ldots, x_n \in X$, we have

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{\frac{1}{p}} \le \rho \ \sup\left\{\left(\sum_{i=1}^{n} |f(x_i)|^p\right)^{\frac{1}{p}} : \|f\| \le 1 \ , \ f \in X^*\right\}. \qquad (*)$$

or $$\left(\sum_{i=1}^{n} \|Tx_i\|^p\right)^{\frac{1}{p}} \le \rho \ \sup\left\{\left(\sum_{i=1}^{n} |< x_i, f >|^p\right)^{\frac{1}{p}} : \|f\| \le 1 \ , \ f \in X'\right\}$$

3.1.2 Remarks:

(i) We denote the fact that $T:X\to Y$ is absolutely p-summing by $T\in\Pi_p(X,Y)$.

(ii) For $T\in\Pi_p(X,Y)$ we define the p-summing norm $\pi_p(T)$ by $\pi_p(T)=\inf\{\rho>0:(*)\text{holds}\}$.

(iii) Every absolutely p-summing operator is bounded. Indeed,

let $T:X\to Y$ be an absolutely p-summing , i.e., there exists a constant $\rho>0$ such that for each natural number n and for each $x_1,x_2,\ldots,x_n\in X$, we have

$\left(\sum_{i=1}^{n}\|Tx_i\|^p\right)^{\frac{1}{p}}\le\rho\ \sup\left\{\left(\sum_{i=1}^{n}|f(x_i)|^p\right)^{\frac{1}{p}}:\|f\|\le1\ ,\ f\in X^*\right\}$. Take in particular $n=1$,

then we obtain

$$\left(\|Tx\|^p\right)^{\frac{1}{p}}\le\rho\sup\left\{\left(|f(x)|^p\right)^{\frac{1}{p}}:\|f\|\le1\right\}$$

$$\Rightarrow\ \|Tx\|\le\rho\sup\{|f(x)|:\|f\|\le1\}.$$

$\Rightarrow\ \|T\|\le\rho$. Hence T is bounded.

To better understand the definition of absolutely p-summing operators we consider two classes of vector – valued sequence spaces: ℓ_p^{strong} and ℓ_p^{weak}.

3.1.3 Definition: If X is a Banach space and $1\le p<\infty$, then a sequence (x_n) in X is said to be strongly p-summable if $(\|x_n\|)\in\ell^p$; in case (x_n) <u>is strongly p-summable</u> we say $(x_n)\in\ell_p^{strong}$ and give (x_n) the norm

$$\|(x_n)\|_{\ell_p^{strong}}=\|(\|x_n\|)\|_p.$$

3.1.4 Definition: If X is a Banach space and $1\le p<\infty$, then a sequence (x_n) in X is called <u>weakly p-summable</u> if for each $f\in X^*$, $(f(x_n))\in\ell^p$, the set of all such sequences is denoted by ℓ_p^{weak}, then

$$\|(x_n)\|_{\ell_p^{weak}}=\sup\left\{\left(\sum|f(x_n)|^p\right)^{\frac{1}{p}}:f\in X^*,\|f\|\le1\right\}<\infty.$$

When p=1, we call it strongly (weakly) summable.

Now, if $T: X \to Y$ is a bounded linear operator, T induces natural operators from $\ell_p^{strong}(X)$ to $\ell_p^{strong}(Y)$ and from $\ell_p^{weak}(X)$ to $\ell_p^{weak}(Y)$, i.e., if (x_n) is strongly summable and T is bounded , then

$$\sum \| Tx_n \| \leq \sum \|T\| \, \|x_n\| = \|T\| \sum \|x_n\| < \infty$$

i.e. (Tx_n) is strongly summable . Similarly, if (x_n) is weakly summable and T is bounded, then $\sum |f(Tx_n)| = \sum |T^* f(x_n)| \leq \sum \|T^*\| |f(x_n)| = \|T\| \sum |f(x_n)| < \infty$

i.e. (Tx_n) is weakly summable. So the question arises when this operator takes $\ell_p^{weak}(X)$ into $\ell_p^{strong}(Y)$. This happens precisely when T is absolutely p-summing with the p-summing norm of T precisely equal to the operator norm of the induced operator from ℓ_p^{weak} to ℓ_p^{strong} i.e. if

$T \in \Pi_p(X,Y)$ and $(x_n) \in \ell_p^{weak}$, then $(Tx_n) \in \ell_p^{strong}$. Indeed , since $T \in \Pi_p(X,Y)$, then $\exists\ \delta > 0$ such that $\forall\, x_1, x_2, ..., x_n \in X$ we have,

$$\left(\sum_{i=1}^{n} \| Tx_i \|^p \right)^{\frac{1}{p}} \leq \rho \sup \left\{ \left(\sum_{i=1}^{n} |f(x_i)|^p \right)^{\frac{1}{p}} : \|f\| \leq 1 \right\} \qquad (**)$$

Since $(x_n) \in \ell_p^{weak} \Rightarrow (f(x_n)) \in \ell_p \quad \forall f.$ i.e. $\left(\sum_{n=1}^{\infty} |f(x_n)|^p \right)^{\frac{1}{p}} < \infty$.

This implies $\sup \left\{ \left(\sum_{i=1}^{n} |f(x_i)|^p \right)^{\frac{1}{p}} : \|f\| \leq 1 \right\} < \infty$

so, from (*) we have $\left(\sum_{i=1}^{n} \|Tx_i\|^p \right)^{\frac{1}{p}} < \infty$. i.e. $(Tx_n) \in \ell_p^{strong}$

A consequence of this is the fact that $T \in \Pi_p(X,Y)$ precisely when for finitely nonzero sequence (x_n) in X we have for some $\rho \geq 0$ that

$$\| (Tx_n) \|_{\ell_p^{strong}(Y)} \leq \rho \| (x_n) \|_{\ell_p^{weak}(X)} \qquad (***)$$

and, ρ in (***) is precisely $\pi_p(T)$.

3.1.5 Theorem [5]: $\Pi_p(X,Y)$ is a normed linear space with norm π_p.

Proof : We'll prove triangle inequality only, let $S,T \in \Pi_p(X,Y)$ and let $x_1, x_2, \dots\dots\dots, x_n \in X$, then by considering the finitely nonzero sequence $(x_1, x_2, \dots, x_n, 0, 0, \dots) \in X$ we have

$$\|(S+T)(x_k)\|_{\ell_p^{strong}} = \| \|(S+T)(x_k)\| \|_p \leq \| (\|Sx_k\|) \|_p + \| (\|Tx_k\|) \|_p$$

$$\leq \pi_p(S) \ \|(x_k)\|_{\ell_p^{weak}} + \pi_p(T) \ \|(x_k)\|_{\ell_p^{weak}}$$

$$= (\pi_p(S) \ + \pi_p(T)) \ \|(x_k)\|_{\ell_p^{weak}}$$

it follows that $S+T$ is absolutely p-summing and that $\pi_p(S+T) \leq \pi_p(S) + \pi_p(T)$ Q.E.D.

3.1.6 Theorem[5]: $\Pi_p(X,Y)$ is a Banach space with norm π_p.

Proof : Let (T_n) be a π_p- Cauchy sequence. Since $\|T\| \leq \pi_p(T)$ always holds, (T_n) is Cauchy in the classical operator norm as well. Therefore there is a bounded linear operator $T_0 : X \to Y$ such that $\lim_n \|T_0 - T_n\| = 0$

claim: $T_0 \in \Pi_p(X,Y)$ and $\lim_n \Pi_p(T - T_n) = 0$.

Proof of claim: Let $\varepsilon > 0$ be given. Choose N_ε so that whenever $m,n \geq N$ we have $\Pi_p(T_m - T_n) \leq \varepsilon$.

Then given any finitely nonzero sequence (x_k) of members of X we have $\| (T_m - T_n)(x_k) \|_{\ell_p^{strong}} \leq \varepsilon \|(x_k)\|_{\ell_p^{weak}}$ whenever $m,n \geq N_\varepsilon$. If $x_k = 0$ for $k \geq k_0$ then this translates to mean that once $m,n \geq N$,

$$\left(\sum_{k=1}^{k_0} \|T_m x_k - T_n x_k\|^p \right)^{\frac{1}{p}} \leq \varepsilon \ \|(x_x)\|_{\ell_p^{weak}} .$$

If we let $m \to \infty$, we see that $\left(\sum_{k=1}^{k_0} \|T_0 x_k - T_n x_k\|^p \right)^{\frac{1}{p}} \leq \varepsilon \ \|(x_k)\|_{\ell_p^{weak}}$.

So, $T_0 - T_n \in \Pi_p(X,Y)$ $\forall n \geq N$ and that $\lim \Pi_p(T_0 - T_n) = 0$.

Now, $T_0 = (T_0 - T_n) + T_n$ belongs to $\Pi_p(X,Y)$ as well and the completeness of $\Pi_p(X,Y)$ with the norm Π_p is established. Q.E.D.

3.1.7 Theorem[5]: Π_p is an operator ideal, i.e. if $S: X \to Y$ and $T: Y \to Z$ are bounded linear operators one of which is absolutely p-summing, then TS is absolutely p-summing.

Proof: Suppose $T \in \Pi_p$. Then for any finitely nonzero sequence (x_n) in X,

$$\left(\sum \|TSx_i\|^p\right)^{\frac{1}{p}} \leq \pi_p(T)\| Sx_i\|_{\ell_p^{weak}} \leq \pi_p(T)\|S\| \|x_i\|_{\ell_p^{weak}}$$

and so $\pi_p(ST) \leq \pi_p(T)\|S\|$.

Suppose $S \in \Pi_p$. Then for any finitely nonzero sequence (x_n) in X,

$$\left(\sum \|TSx_i\|^p\right)^{\frac{1}{p}} \leq \|T\| \left(\sum \|Sx_i\|^p\right)^{\frac{1}{p}} \leq \|T\| \pi_p(S) \|(x_i)\|_{\ell_p^{weak}}$$

and so, $\pi_p(TS) \leq \|T\| \pi_p(S)$. Q.E.D.

It follows from this theorem that we have the following:

If $R: W \to X$, $S: X \to Y$ and $T: Y \to Z$ are bounded linear operators with $S \in \Pi_p(X,Y)$, then TSR is absolutely p-summing with $\pi_p(TSR) \leq \|T\| \pi_p(S) \|R\|$.

Now, how do the classes Π_p compare for different p's? The answer is in the following theorem.

3.1.8Theorem[5]: If $1 \leq p < q < \infty$, then $\Pi_p(X,Y) \subseteq \Pi_q(X,Y)$ and the inclusion map is contractive.

Proof: Let $T \in \Pi_p(X,Y)$.Then for any finitely nonzero sequence (x_i) of vectors in X and any sequence (λ_i) of scalars,

$$\left(\sum \|\lambda_i Tx_i\|^p\right)^{\frac{1}{p}} = \left(\sum \|T(\lambda_i x_i)\|^p\right)^{\frac{1}{p}} \leq \pi_p(T) \|(\lambda_i x_i)\|_{\ell_p^{weak}}.$$

Suppose r is chosen for the purpose: $\frac{1}{r} + \frac{1}{q} = \frac{1}{p}$.

Then Holder's inequality gives

$\| (\lambda_i x_i) \|_{\ell_p^{weak}} \leq \| (\lambda_i) \|_r \|(x_i)\|_{\ell_q^{weak}}$. If we let $\lambda_i = \|Tx_i\|^{\frac{q}{r}}$, then

$$\| (\lambda_i Tx_i) \|_{\ell_p^{strong}} = \left(\sum \|\lambda_i Tx_i\|^p\right)^{\frac{1}{p}} = \left(\sum \left(\|Tx_i\|^{\frac{q}{r}} \|Tx_i\|\right)^p\right)^{\frac{1}{p}} = \left(\sum \|Tx_i\|^q\right)^{\frac{1}{p}}$$

Where $\| (\lambda_i) \|_r = \left(\sum |\lambda_i|^r\right)^{\frac{1}{r}} = \left(\sum \|Tx_i\|^q\right)^{\frac{1}{r}}$.

Since we may as well suppose (λ_i) is not entirely zeros we see that

$$\left(\sum \|Tx_i\|^q\right)^{\frac{1}{q}} = \left(\sum \|Tx_i\|^q\right)^{\frac{1}{p}} / \left(\sum \|Tx_i\|^q\right)^{\frac{1}{r}}$$

$$= \|(\lambda_i Tx_i)\|_{\ell_p^{strong}} / \| (\lambda_i) \|_r$$

$$\leq \pi_p(T) \|(\lambda_i x_i)\|_{\ell_p^{weak}} / \|\lambda_i\|_r$$

$$\leq \pi_p(T) \|(\lambda_i)\|_r \|(x_i)\|_{\ell_q^{weak}} / \| (\lambda_i) \|_r$$

$$= \pi_p(T) \|(x_i)\|_{\ell_q^{weak}}$$

$\therefore$ $T \in \Pi_q(X,Y)$ Q.E.D.

3.1.9 Definition: (Hilbert – Schmidt Operator).

Let H and K be Hilbert spaces and $T : H \to K$ be a bounded linear operator .We say T is a Hilbert – Schmidt operator if for some complete orthonormal sequence $(e_i)_{i \in I}$ in H, $\sum \|Te_i\|^2 < \infty$. The number $\sigma(T) = \left(\sum \|Te_i\|^2\right)^{\frac{1}{2}}$ is called the Hilbert – Schmidt norm of T. Every Hilbert – Schmidt operator admits a representation in the form $\sum_i \lambda_i < ., e_i > f_i$ for some $\lambda_i \in \ell^2$ and some orthonormal sequences (e_n) and (f_n) in H and K respectively. Indeed

$$Tx = T\left(\sum < x, e_i > e_i\right) = \sum < x, e_i > Te_i = \sum \|Te_i\| < x, e_i > \frac{Te_i}{\|Te_i\|} = \sum \lambda_i < x, e_i > f_i \quad \therefore$$

$$T = \sum \lambda_i < ., e_i > f_i$$

3.1.10 Example[5]: If H and K are Hilbert spaces, then $\Pi_2(H,K)$ coincides with the class of Hilbert – Schmidt operators from H to K; moreover, the Π_2 norm and the Hilbert – Schmidt norm are the same.

Proof : suppose that $T : H \to K$ is absolutely 2-summing. Let $e_1, e_2, ..., e_n$ be orthonormal vectors. Then for any $x \in H$, $\sum_i |< x, e_i >|^2 \le \|x\|^2$ by Bessel's inequality. Therefore , $\left(\sum \|Te_i\|^2\right)^{\frac{1}{2}} \le \pi_2(T)\|(e_i)\|_{l_2^{weak}}$

$$= \pi_2(T) \sup\left\{\left(\sum_{i=1}^{n} |f(e_i)|^2\right)^{\frac{1}{2}}, \|f\| \le 1\right\}$$

$$= \pi_2(T) \sup\left\{\left(\sum_{i=1}^{n} |< x, e_i >|^2\right)^{\frac{1}{2}}, \|x\| \le 1\right\}$$

$$\le \pi_2(T) \sup\left\{\left(\|x\|^2\right)^{\frac{1}{2}}, \|x\| \le 1\right\} = \pi_2(T).$$

It follows that T is a Hilbert – Schmidt operator and $\sigma(T) \le \pi_2(T)$ (*)

Now suppose T is a Hilbert – Schmidt operator and represent T in the form $Tx = \sum_n \lambda_n < x, e_n > f_n$, where $(\lambda_n) \in \ell_2$ with $\sigma(T) = \|(\lambda_n)\|_2$ and with (e_n) and (f_n) orthonormal sequences in H and K respectively. $\forall x \in H$ we have,

$$\|Tx\|^2 = < Tx, Tx > = < \sum \lambda_n < x, e_n > f_n , \sum \lambda_n < x, e_n > f_n >$$

$= \sum_n |\lambda_n|^2 |< x, e_n >|^2 |f_n|^2 = \sum_n |\lambda_n|^2 |< x, e_n >|^2$, so for $x_1, x_2, ..., x_n \in H$, we see that

$$\left(\sum_{i=1}^{k} \|Tx_i\|^2\right)^{\frac{1}{2}} = \left(\sum_n |\lambda_n|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{k} |< x_i, e_n >|^2\right)^{\frac{1}{2}} \le \sigma(T) \sup\left\{\left(\sum_{i=1}^{k} |< x_i, u >|^2\right)^{\frac{1}{2}} : \|u\| \le 1\right\}$$

$\Rightarrow \quad \|Tx\|_{\ell_p^{strong}} \le \rho \|(x_n)\|_{\ell_p^{weak}}$. Here $\rho = \sigma(T)$.

$\therefore$ $T \in \Pi_2(H;K)$ and $\pi_2(T) \le \sigma(T)$ (**)

From (*) and (**) we get $\sigma(T) = \pi_2(T)$. Q.E.D.

3.2 Absolutely P-Summing Diagonal Operators .

Diagonal operators from one ℓ^p space to another are not r-summing in general, to see this consider the diagonal operator $d_\alpha : \ell^2 \to \ell^{\frac{3}{2}}$ be defined as $d_\alpha(x) = (\alpha_n x_n)$ where $\alpha_n = \frac{1}{\sqrt{n}}$. If $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in \ell^2$, then

$$\sum_{i=1}^{n} \left\| d_\alpha x^{(i)} \right\|^{\frac{3}{2}} = \sum_{i=1}^{n} \left(\sum_{j=1}^{\infty} \left| \alpha_j x_j^{(i)} \right|^{\frac{3}{2}} \right)^{\frac{2}{3} \cdot \frac{3}{2}}$$

$$= \sum_{j=1}^{\infty} \left| \alpha_j \right|^{\frac{3}{2}} \sum_{i=1}^{n} \left| x_j^{(i)} \right|^{\frac{3}{2}}$$

$$= \sum_{j=1}^{\infty} \frac{1}{j^{\frac{3}{4}}} \sum_{i=1}^{n} \left| x_j^{(i)} \right|^{\frac{3}{2}} = \infty .$$

$\therefore$ d_α is not $\frac{3}{2}$-summing .

But if we take $\alpha_n = \frac{1}{n}$, then $d_\alpha : \ell^2 \to \ell^{\frac{3}{2}}$ is $\frac{3}{2}$-summing, indeed let $e^{(j)}$ denote the sequence with 1 in the j^{th} position, and with 0 elsewhere.

If $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in \ell^2$, then

$$\sum_{i=1}^{n} \left\| d_\alpha x^{(i)} \right\|^{\frac{3}{2}} = \sum_{i=1}^{n} \left(\sum_{j=1}^{\infty} \left| \alpha_j x_j^{(i)} \right|^{\frac{3}{2}} \right)^{\frac{2}{3} \cdot \frac{3}{2}}$$

$$= \sum_{j=1}^{\infty} \left| \alpha_j \right|^{\frac{3}{2}} \sum_{i=1}^{n} \left| x_j^{(i)} \right|^{\frac{3}{2}}$$

$$= \sum_{j=1}^{\infty} \frac{1}{j^{\frac{3}{2}}} \sum_{i=1}^{n} \left| < x_j^{(i)}, e^{(j)} > \right|^{\frac{3}{2}}$$

$$\leq \sum_{j=1}^{\infty} \frac{1}{j^{\frac{3}{2}}} \sup\left\{ \sum_{i=1}^{n} \left| < x_j^{(i)}, f > \right|^{\frac{3}{2}} : \|f\| \leq 1,\ f \in \ell^2 \right\}$$

$$\therefore \sum_{i=1}^{n} \left\| d_\alpha x^{(i)} \right\|^{\frac{3}{2}} \leq \rho \sup\left\{ \sum_{i=1}^{n} \left| < x_j^{(i)}, f > \right|^{\frac{3}{2}} : \|f\| \leq 1,\ f \in \ell^2 \right\}, \text{ where } \rho = \sum_{j=1}^{\infty} \frac{1}{j^{\frac{3}{2}}} < \infty$$

Hence d_α is $\frac{3}{2}$-summing.

So we will look for condition(s) on the sequence $\alpha = (\alpha_n)$ under which the diagonal operator d_α be p-summing .

In his paper, Garling(see [2]) proved the following theorem that gives necessary and sufficient conditions for the diagonal operator d_α to be p-summing .

3.2.2 Theorem: (Garling's Theorem)[2,Theorem 9].

The mapping d_α is r-summing $(1 \le r < \infty)$ from $\ell^{p'}$ into ℓ^q if and only if the following conditions are satisfied:

(i) If $1 \le p \le 2$ and $p < q$,

$$\alpha \in \ell_p \text{ for } 1 \le r \le p$$

$$\alpha \in \ell_r \text{ for } p \le r \le q.$$

(ii) If $1 \le p = q < 2$, $\alpha \in \ell_{p^-}$ for $1 \le r \le p$, $\alpha \in \ell_p$ for $p \le r$

(iii) If $p = q = 2$, $\alpha \in \ell_2$ for all values of r.

(iv) If $1 \le q < p \le 2$, $\alpha \in \ell_q$ for all values of r.

(v) If $1 \le q \le 2$ and $2 < p \le \infty$, $\alpha \in \ell_{\varphi(p,q)}$ for all values of r.

(vi) If $2 < q \le p < \infty$, $\alpha \in \ell_p$ for $1 \le r \le p$

(vii) If $2 < p < q \le \infty$, $\alpha \in \ell_p$ for $1 \le r \le p$, $\alpha \in \ell_r$ for $p \le r \le q$

(viii) If $2 \le q \le p = \infty$, $\alpha \in \ell_\infty$ $\forall$ values of r.

In the proof , Garling used the notion of 0-summing and some theorems depending on this notion . In this thesis we avoid the concept of 0-summing and we give an alternative proof of the previous theorem directly as the following theorems.

3.2.3 Theorem: Suppose that $1 \le r < \infty$, $p \ge 1$ and $q \ge 1$.

(i) If $r \le q$ and $\alpha \in \ell^r$ then d_α is r-summing from $\ell^{p'}$ into ℓ^q .

(ii) If $p \le r$ and d_α is r-summing from $\ell^{p'}$ into ℓ^q, then $\alpha \in \ell^r$.

(iii) If $p \le r \le q$, d_α is r-summing from $\ell^{p'}$ into ℓ^q iff $\alpha \in \ell^r$.

Proof: (i) let $e^{(j)}$ denote the sequence with 1 in the j^{th} position, and with 0 elsewhere. Suppose that $r \le q$ and that $\alpha \in \ell^r$. If $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in \ell^{p'}$,

$$\sum_{i=1}^{n} \left\| d_\alpha x^{(i)} \right\|^r = \sum_{i=1}^{n} \left(\sum_{j=1}^{\infty} \left| \alpha_j x_j^{(i)} \right|^q \right)^{\frac{r}{q}}$$

$$= \sum_{j=1}^{\infty} \left| \alpha_j \right|^r \sum_{i=1}^{n} \left| x_j^{(i)} \right|^r$$

$$= \sum_{j=1}^{\infty} \left| \alpha_j \right|^r \sum_{i=1}^{n} \left| < x^{(i)}, e^{(j)} > \right|^r \quad (\because < x^{(i)}, e^{(j)} > = x_j^{(i)})$$

$$\le \left(\sum_{j=1}^{\infty} \left| \alpha_j \right|^r \right) \sup \left\{ \sum_{i=1}^{n} \left| < x^{(i)}, f > \right|^r, \|f\| \le 1, f \in (\ell^{p'})' = \ell^p \right\} \quad (\because \left\| e^{(j)} \right\| = 1).$$

$$\therefore \quad \sum_{i=1}^{n} \left\| d_\alpha x^{(i)} \right\|^r \le \rho \ \sup \left\{ \sum_{i=1}^{n} \left| < x^{(i)}, f > \right|^r, \|f\| \le 1, f \in (\ell^{p'})' = \ell^p \right\} \quad ,$$

where $\rho = \sum_{j=1}^{\infty} \left| \alpha_j \right|^r < \infty \ (\because \alpha \in \ell^r)$.

So that d_α is r-summing.

(ii) Suppose that $p \le r$ and d_α is r-summing from $\ell^{p'}$ into ℓ^q, then there exists a constant $\rho > 0$ such that for each natural number n and for each $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in \ell^{p'}$, we have

$$\left(\sum_{i=1}^{n} \left\| d_\alpha x^{(i)} \right\|^r \right)^{\frac{1}{r}} \le \rho \ \sup \left\{ \left(\sum_{i=1}^{n} \left| < x^{(i)}, f > \right|^r \right)^{\frac{1}{r}} : \|f\| \le 1 \ , \ f \in \ell^p \right\}.$$

In particular,

$$\left(\sum_{i=1}^{n} \left\| d_\alpha e^{(i)} \right\|^r \right)^{\frac{1}{r}} \le \rho \ \sup \left\{ \left(\sum_{i=1}^{n} \left| < e^{(i)}, f > \right|^r \right)^{\frac{1}{r}} : \|f\| \le 1 \ , \ f \in \ell^p \right\}$$

$$\le \rho \ \sup \left\{ \left(\sum_{i=1}^{\infty} \left| < e^{(i)}, f > \right|^r \right)^{\frac{1}{r}} : \|f\| \le 1 \ , \ f \in \ell^p \right\}$$

$$\leq \rho \sup\left\{\left(\sum_{i=1}^{\infty}\left|< e^{(i)}, f >\right|^{p}\right)^{\frac{1}{p}} : \|f\| \leq 1\ ,\ f \in \ell^{p}\right\} \quad \text{(since } p \leq r\text{)}$$

$$= \rho \sup\left\{\left(\sum_{i=1}^{\infty}|f_i|^{p}\right)^{\frac{1}{p}} : \|f\| \leq 1\ ,\ f \in \ell^{p}\right\} \leq \rho.1$$

$\Rightarrow \left(\sum_{i=1}^{n}\|d_\alpha e^{(i)}\|^{r}\right)^{\frac{1}{r}} \leq \rho < \infty$. Since n is arbitrary, letting $n \to \infty$, we get

$$\sum_{i=1}^{\infty}\left\|d_\alpha\left(e^{(i)}\right)\right\|^{r} = \sum_{i=1}^{\infty}|\alpha_i|^{r} < \infty.$$

$\therefore\ \alpha \in \ell^{r}$.

Finally (iii) is a consequence of (i) and (ii). Q.E.D.

3.2.4 Corollary : Suppose that $1 \leq r < \infty$ and $1 \leq p = q < 2$.

(i) If $p \leq r$ and $\alpha \in \ell^{p}$ then d_α is r-summing from $\ell^{p'}$ into ℓ^{q}.

(ii) If $r \leq p$ and d_α is r-summing from $\ell^{p'}$ into ℓ^{q}, then $\alpha \in \ell^{p}$.

Proof : (i) Suppose that $1 \leq p = q < 2$ and $\alpha \in \ell^{p}$ for $p \leq r$, we want to show that d_α is r-summing. It is sufficient to show that d_α is p-summing since $p \leq r$.

Let $e^{(i)}$ denote the sequence with 1 in the j^{th} position, and with 0 elsewhere.

Suppose that $p \leq r$ and $\alpha \in \ell^{p}$. If $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in \ell^{p'}$, then

$$\sum_{i=1}^{n}\left\|d_\alpha\left(x^{(i)}\right)\right\|^{p} = \sum_{i=1}^{n}\left(\sum_{j=1}^{\infty}\left|\alpha_j x_j^{(i)}\right|^{p}\right)^{\frac{p}{p}} \quad \text{(since } p = q\text{)}.$$

$$\leq \sum_{j=1}^{\infty}|\alpha_j|^{p} \sup\left\{\sum_{i=1}^{n}\left|< x^{(i)}, f >\right|^{p}, \|f\| \leq 1, f \in (\ell^{p'})' = \ell^{p}\right\}$$

$\therefore\ d_\alpha$ is p-summing and hence it is r-summing.

(ii) The same as the proof of theorem 3.2.3 (ii).

3.2.5 Theorem: If $2 \le q \le p = \infty$, then d_α is r-summing $(1 \le r < \infty)$ from $\ell^{p'}$ into ℓ^q *iff* $\alpha \in \ell^\infty$.

Proof : Suppose that $2 \le q \le p = \infty$, d_α is r-summing $\forall r$. We want to show that $\alpha \in \ell^\infty$. Since d_α is r-summing, then it is bounded . Then by Theorem 2.2.3 $\|d_\alpha\| = \|\alpha\|_\infty$,so $\alpha \in \ell^\infty$ (since d_α is bounded).

Conversely, suppose that $\alpha \in \ell^\infty$, then the mapping $d_\alpha : \ell^2 \to \ell^q$ is bounded for $2 \le q$ (byTheorem 2.2.3). Now ,consider the inclusion mapping $i : \ell^1 \to \ell^2$, then it is 1-summing(by using the Grothendiek's result that says ' every bounded linear operator from ℓ^1 to a Hilbert space is 1-summing') , so that the mapping $d_\alpha \circ i = d_\alpha$ is 1-summing from ℓ^1 to ℓ^q (by Theorem 3.1.7)

$\therefore$ d_α is r-summing from $\ell^{p'}$ into ℓ^q for $2 \le q \le p = \infty$ Q.E.D.

3.2.6 Theorem : Suppose that $1 \le q < p < \infty$, then

(i) $d_\alpha : \ell^{p'} \to \ell^q$ is $q-$summing if $\alpha \in \ell^q$.

(ii) $\alpha \in \ell^p$ if $d_\alpha : \ell^{p'} \to \ell^q$ is $q-$summing .

Proof: (i) Let $e^{(j)}$ denote the sequence with 1 in the j^{th} position, and with 0 elsewhere. Suppose that $1 \le q < p < \infty$, and that $\alpha \in \ell^q$.

If $x^{(1)}, x^{(2)}, \ldots, x^{(n)} \in \ell^{p'}$,

$$\sum_{i=1}^{n} \left\|d_\alpha x^{(i)}\right\|^q = \sum_{i=1}^{n} \left(\sum_{j=1}^{\infty} \left|\alpha_j x_j^{(i)}\right|^q \right)^{\frac{q}{q}} = \sum_{i=1}^{n} \sum_{j=1}^{\infty} \left|\alpha_j x_j^{(i)}\right|^q$$

$$= \sum_{j=1}^{\infty} \left|\alpha_j\right|^q \sum_{i=1}^{n} \left|< x^{(i)}, e^{(j)} >\right|^q$$

$$\le \left(\sum_{j=1}^{\infty} \left|\alpha_j\right|^q \right) \sup\left\{ \sum_{i=1}^{n} \left|< x^{(i)}, f >\right|^q, \|f\| \le 1 \right\}.$$

$$\therefore \sum_{i=1}^{n} \left\|d_\alpha x^{(i)}\right\|^q \le \rho \sup\left\{ \sum_{i=1}^{n} \left|< x^{(i)}, f >\right|^q, \|f\| \le 1 \right\}, \text{ where}$$

$\rho = \sum_{j=1}^{\infty} \left|\alpha_j\right|^q < \infty$ (since $\alpha \in \ell^q$). So that d_α is q-summing.

(ii) The same as the proof of Theorem 3.2.3 (ii).

3.2.7 Corollary :

(i) If $1 \le r \le 2$ and $\alpha \in \ell^r$ then d_α is r-summing from ℓ^2 into ℓ^2.

(ii) If $r \ge 2$ and d_α is r-summing from ℓ^2 into ℓ^2, then $\alpha \in \ell^r$.

(iii) d_α is 2-summing from ℓ^2 into ℓ^2 iff $\alpha \in \ell^2$. Furthermore, $\pi_2(d_\alpha) = \|\alpha\|_2$.

Proof: Follows from Theorem 3.2.3 by taking $p = q = 2$. Q.E.D.

3.2.8 Corollary : The mapping d_α is 2-summing from ℓ^2 into ℓ^2 iff d_α is a Hilbert-Schmidt operator, Furthermore $\pi_2(d_\alpha) = \sigma(d_\alpha)$.

Proof: Follows directly from Example 3.1.10 by taking $H = K = \ell^2$. Q.E.D.

3.2.9 Corollary : The mapping d_α is a Hilbert-Schmidt operator iff $\alpha \in \ell^2$. Furthermore, $\sigma(d_\alpha) = \|\alpha\|_2$.

Proof: Follows directly from Corollaries 3.2.7 and 3.2.8. Q.E.D.

3.2.10 Corollary : The mapping d_α from ℓ^1 into ℓ^2 is bounded iff d_α is 1-summing.

Proof: Suppose that d_α is 1-summing, then d_α is bounded (by Remarks 3.1.2(iii)) . Conversely, suppose that d_α is bounded, then d_α is 1-summing (by using the Grothendiek's result that says ' every bounded linear operator from ℓ^1 to a Hilbert space is 1-summing'). Q.E.D.

3.3 Examples :

3.3.1 Example: if we take $p=2$ and $q=4$ in Theorem 3.2.3 we get the following: $d_\alpha : \ell^2 \to \ell^4$ is r-summing for $2 \le r \le 4$ iff $\alpha \in \ell^r$.

3.3.2 Example: if we take $p=q=\frac{3}{2}$ in Corollary 3.2.4 , then we get , $d_\alpha : \ell^3 \to \ell^{\frac{3}{2}}$ is r-summing for $r \ge \frac{3}{2}$ if $\alpha \in \ell^{\frac{3}{2}}$. In particular , if $\alpha_n = \frac{1}{n}$,, then $d_\alpha : \ell^3 \to \ell^{\frac{3}{2}}$ is r-summing for $r \ge \frac{3}{2}$ since $\frac{1}{n} \in \ell^{\frac{3}{2}}$.

3.3.3 Example: if we take $q=1$ and $p=3$ in Theorem 3.2.6(i) we get , $d_\alpha : \ell^{\frac{3}{2}} \to \ell^1$ is $1-$summing if $\alpha \in \ell^1$. In particular if we put $\alpha_n = 2^{-n}$, then $d_\alpha : \ell^{\frac{3}{2}} \to \ell^1$ is $1-$summing since $\alpha_n = 2^{-n} \in \ell^1$.

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