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Absolutely Summing Operators Between Hardy Spaces

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Absolutely Summing Operators Between Hardy Spaces

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Declaration

I declare that the master thesis entitled *"Absolutely summing operators between Hardy spaces"* is my own work, and hereby certify that unless stated, all work contained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgment is made in the text.

Majeda Qunnais

Signature: ——————-　　　　Date: : ——————-

Dedication

This thesis is dedicated to my family, friends, colleagues, and students for their love, endless support and encouragement.

. Majeda Qunnais

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Abstract

The Hardy space H^p , $1 \leq p \leq \infty$, is a subspace of L^p that contain of all functions with Fourier series $\sum_{n=0}^{\infty} c_n e^{int}$. If $a = (a_n)$ is a given vector, then the diagonal operator $d_a : H^p \to H^q$ is defined by $d_a(\sum_{n=0}^{\infty} c_n e^{infty}) = \sum_{n=0}^{\infty} a_n c_n e^{int}$.

The absolutely summing operator$u : X \to Y$ is a linear operator between Banach spaces. We say that u is p summing operator for $1 \leq p \leq \infty$ if there is a constant $c \geq 0$ such that regardless of the natural number m and regardless of the choice of $x_1, ..., x_m$ in X we have,

$$\left(\sum_{i=1}^{m} \|ux_i\|^p \right)^{\frac{1}{p}} \leq c. \sup \left\{ \left(\sum_{i=1}^{m} |\phi(x_i)|^p \right)^{\frac{1}{p}}, \quad \phi \in X^{'}, \|\phi\| \leq 1 \right\} \tag{1}$$

In this thesis, we consider the diagonal operator d_a between Hardy spaces H^p and H^q where $1 \leq p, q \leq \infty$ and a is the sequence (a_n).

In this thesis we find necessary and sufficient conditions for this diagonal operator to be 2 summing. We were able to prove that $d_a : H^p \to H^q$ is 2 summing if and only if $a \in l^2$. After that, we prove that this operator is 1 summing if and only if $a \in l^1$.

Contents

Introduction

The work of A. Grothendieck in the 1950s is considered the root of the theory of absolutely summing operators. In later works of Pietsch and Lindenstrauss and Pelczy nski clarified Grothendieck's insights and nowadays the idea of absolutely summing operators is a central topic of investigation. For more details on absolutely summing operators, it is recommended to have a look at the book by Diestel-Jarchow-Tonge (1995).

Several mathematicians worked on diagonal operators between l^p spaces. Researchers gave the necessary and sufficient conditions on the vector $\mathbf{a}$ so that the diagonal operator $d_a : l^p \to l^q$ is bounded. Garling (1974) proved the necessary and sufficient conditions for the the diagonal operator $d_a : l^p \to l^q$ to be r summing. Almasri (2016) considered the same problem, but used different techniques. Moreover, Almasri (2016) gave necessary and sufficient conditions for the diagonal operator from Hardy spaces H^p to sequence spaces l^q to be r-summing..

In this thesis, necessary and sufficient conditions on the sequence (a_n) were studied so that the diagonal operator $d_a : H^p \to H^q$ defined by $d_a(f) = \sum a_n c_n z^n$ where $f = \sum c_n z^n$ is absolutely 2 summing.

Furthermore, I considered the problem of finding necessary and sufficient conditions for $d_a : H^p \to H^q$ to be 2 summing and 1 summing. This thesis was divided into four chapters:

Chapter one is an introduction to functional analysis. It begins with two types of vector spaces. The first one is Normed spaces and the second is the Inner Product spaces. After that, linear operators are introduced shortly as well as when they are bounded. Also the linear functional is introduced.

Chapter two talks about the diagonal operators on sequence spaces. It starts with a definition and then theorems on them. Finally we talk about when are these diagonal operators bounded.

Chapter three introduces the Hardy spaces. They are the spaces of analytic functions on the unit disc. Then, some theorems on these spaces are discussed and used to discuss the diagonal operators on these spaces.

The last chapter is the most important chapter in this study. This chapter is about absolutely summing operators on spaces. First, these operators are introduced and some theorems on them are presented. These operators are discussed in three sections. The first one is between sequence spaces, then from Hardy spaces to sequence spaces and at the end, I discuss the absolutely summing diagonal operators between Hardy spaces.

Chapter 1

Functional Analysis

Functional analysis is a type of mathematical analysis dealing with functionals. It emerged as a distinct field in the 20th century, when it was realized that diverse mathematical processes, from arithmetic to calculus procedures, exhibit very similar properties. A functional, like a function, is a relationship between objects, but the objects may be numbers, vectors, or functions. Groupings of such objects are called spaces.

In this chapter, important concepts in functional analysis will be introduced. These concepts will be used in later chapters. The most known spaces, which are the normed spaces and inner product spaces, are discussed in this following part.
In functional analysis, we need to study also operators on spaces such as bounded operators and linear functional.

1.1 Normed and Inner product spaces

Vector spaces are a corner stone concept in mathematics. In fact, in problems we have a set X whose elements may be vectors , or sequences of numbers, or functions, and these elements can be added and multiplied by constants and the result being again an element of X. The concept of a vector space as defined below will involve a general field F, which will be $\mathbb{R}$ or $\mathbb{C}$.

Before that, we need two important concepts. The first one is the ring. Ring is a set R together with two operations additions and multiplication. R with addition is an abelian group, multiplication is associative and satisfies the distribution laws. The other important concept is field. A commutative ring in which the set of all nonzero elements forms a group with respect to multiplication is called a field. That means each none zero element has inverse. Which is denoted by $\mathbb{F}$ in this thesis. Examples, real numbers, rational numbers and complex numbers.

Definition 1.1.1. *Vector Space*
Kreyszig (1978) defines a vector space as follows:
A vector space over a field $\mathbb{F}$ is a nonempty set X of elements x, y, ... (called vectors) together with two algebraic operations. These operations are called vector addition and multiplication of vectors by scalars, that is, by elements of $\mathbb{F}$.
Let $x, y, z \in X$
Vector addition is commutative and associative , that is, for all vectors we have ,

$$x + y = y + x$$
$$(x + y) + z = x + (y + z)$$

*Furthermore, there exists a vector **0** which is the addition identity, called the zero vector, and for every vector x there exists a vector $-x$ such that for all vectors we have*

$$x + 0 = x$$

$$x + (-x) = 0$$

Multiplication by scalars is associative , that is, for all scalars α, β we have ,

$$\alpha(\beta x) = (\alpha\beta)x$$
$$1x = x$$

where 1 is the multiplication identity. Also, they satisfy the distribution laws

$$\alpha(x + y) = \alpha x + \alpha y$$

$$(\alpha + \beta)x = \alpha x + \beta x$$

Here, are some examples of vector space.

Example 1. The real line $\mathbb{R}$.
The space of all real numbers.
Let $x, y \in \mathbb{R}$ and $a \in \mathbb{F}$, then:
$(x + y) = x + y$
$a.x = a \times x$

Example 2. The Euclidean space $\mathbb{R}^n$.
This space is the set of all ordered n-tuples of real numbers.
Let $x, y \in \mathbb{R}^n$ defined by $x = (x_1, x_2, x_3, ..., x_n)$, $y = (y_1, y_2, y_3, ..., y_n)$, and $a \in \mathbb{F}$ then:
$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, ..., x_n + y_n) \in \mathbb{R}^n$
$a.x = (a.x_1, a.x_2, a.x_3, ..., a.x_n) \in \mathbb{R}^n$

Example 3. The l^∞ space.
The set of all bounded sequences of complex numbers; that is, every element of l^∞ is a complex sequence $x = (x_1, x_2, x_3, ...)$ such that there is a real number c_x with $|x_i| \leq c_x$ for all $i = 1, 2, 3, ...$
Let $x, y \in l^\infty$ such that $x = (x_n), y = (y_n)$, then:
$x + y = (x_n + y_n)$
$a.x = (a.x_n)$

Definition 1.1.2. *Space l^p, where $1 \leq p < \infty$*
Kreyszig (1978) defines a l^p space as follows:
The l^p space is the vector space of all sequences x_n that satisfy

$$\sum_{i=1}^{\infty} |(x_i)|^p < \infty$$

Definition 1.1.3. *Space L^p, where $p \geq 1$*
Kreyszig (1978) defines the L^p spaces as:
The L^p space is the vector space of all functions $f(t)$ defined on $[a, b]$ that satisfy

$$\int_a^b |f(t)|^p \, dt < \infty$$

Definition 1.1.4. *Completeness of the vector spaces*
The Completeness of the vector spaces is defined by Kreyszig (1978) as:
A vector space is said to be complete if every Cauchy sequence converges in the same space.
For example, the space of all real numbers $\mathbb{R}$ is complete.

Here, there is an example of incomplete vector space.

Example 1. The rational numbers $\mathbb{Q}$
Let the sequence $x = (x_n) = \left(1 + \frac{1}{n}\right)^n$
This sequence is of rational numbers but it converges to Euler's number e which is an irrational number.

Particularly, important spaces are obtained if we take a vector space and define on it a norm. The result is called a normed space.

Definition 1.1.5. *Norm*
Kreyszig (1978) states that a norm on a vector space X is a function

$$\|.\| : X \to \mathbb{R}^+ := [0, \infty)$$

that satisfies, for all $x, y \in X$ and $\alpha \in \mathbb{F}$

- $\|x\| = 0$ *if and only if* $x = 0$ *and* $\|x\| \geq 0$
- $\|\alpha x\| = |\alpha| \, \|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$

Definition 1.1.6. *Normed Space*
Kreyszig (1978) states that a normed space is a vector space X over a field $\mathbb{F}$ with a norm defined on X.

Definition 1.1.7. *Banach Space*
Kreyszig (1978) states that a complete normed space is called Banach space.
The real numbers $\mathbb{R}$ is a Banach space.

Other important spaces are the inner product spaces $(X, \langle , \rangle)$, where X is a vector space and $\langle , \rangle$ is the inner product defined as following definition.

Definition 1.1.8. *Inner Product*
Kreyszig (1978) defines that An inner product on a vector space X is a function

$$\langle ., . \rangle : X * X \to \mathbb{C}$$

that satisfies, for all $x, y, z \in X and\ \alpha \in \mathbb{F}$

- $\langle x, x\rangle = 0$ *if and only if* $x = 0$ *and* $\langle x, x\rangle \geq 0$
- $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$
- $\langle x, y\rangle = \overline{\langle y, x\rangle}$
- $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

where $\overline{\langle y, x\rangle}$ *is the conjugate of* $\langle y, x\rangle$
An inner product on X *defines a norm on* X *given by* $\|x\| = \sqrt{\langle x, x\rangle}$.

Definition 1.1.9. *Inner Product Space*
Kreyszig (1978) publishes that An inner product space is a vector space X *over a field* $\mathbb{F}$ *with an inner product defined on* X.

Definition 1.1.10. *Hilbert Space*
Kreyszig (1978) states that A complete inner product space is called Hilbert space.
The l^2 *space is a Hilbert space.*

By a simple straightforward calculation Kreyszig (1978) proves that a norm on an inner product space satisfies the important parallelogram equality. This equality will be defined below.

Theorem 1.1.1. *The Parallelogram Equality.*
The parallelogram equality is

$$2(\|x\|^2 + \|y\|^2) = \|x + y\|^2 + \|x - y\|^2$$

Proof. .
$\|x + y\|^2 + \|x - y\|^2 = \langle x + y, x + y\rangle + \langle x - y, x - y\rangle$
$= \langle x, x + y\rangle + \langle y, x + y\rangle + \langle x, x - y\rangle - \langle y, x - y\rangle$
$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle + \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle$
$= 2\langle x, x\rangle + 2\langle y, y\rangle$
$= 2\|x\|^2 + 2\|y\|^2$
$= 2(\|x\|^2 + \|y\|^2)$ □

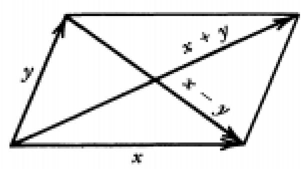

Figure 1.1: The Parallelogram equality

Now, in the following example, we will show that l^p spaces are normed spaces and Banach spaces. But l^p space is an inner product space only where $p = 2$ which is also a Hilbert space. l^p where $p \neq 2$ are not inner product spaces.

Lemma 1.1.1. *The space l^p, where $1 \leq p < \infty$*
The l^p space is a normed space with a norm defined by

$$\|x_n\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}$$

This space is complete, that is a Banach space.

Proof. .
Let (X_n) be any Cauchy sequence in the space l^p, then for every $\epsilon > 0$ there is a natural number N such that for all $n, m \geq N$,

$$\|X_n - X_m\|_p < \epsilon$$

$$\left(\sum_{i=1}^{\infty} |x_i^n - x_i^m|^p\right)^{\frac{1}{p}} < \epsilon$$

where x_i^n, x_i^m is the ith element in the sequence $(X_n), (X_m)$ respectively.
It follows that for every $i = 1, 2, 3, ..$ we have for $m, n \geq N$,

$$|x_i^n - x_i^m| < \epsilon$$

That is for each fixed i, $x_i^1, x_i^2, x_i^3, ...$ is a Cauchy sequence of numbers. It converges since $\mathbb{R}$ and $\mathbb{C}$ are complete.
Say $(x_i^n) \to x_i$ as $n \to \infty$. Using these limits, we define $x = (x_1, x_2, x_3, ...)$. Show that $x \in l^p$ and $(X_n) \to X$.

$$\left(\sum_{i=1}^{\infty} |x_i^n - x_i^m|^p\right)^{\frac{1}{p}} < \epsilon$$

$$\sum_{i=1}^{\infty} |x_i^n - x_i^m|^p < \epsilon^p$$

Let $m \to \infty$ then,

$$\sum_{i=1}^{\infty} |x_i^n - x_i|^p < \epsilon^p$$

This means, $X_n - X \in l^p$. By Minkowski inequality that we will state in the next chapter, $X \in l^p$.

$$\begin{aligned}\|X\|_p &= \|X - X_n + X_n\|_p \\ &\leq \|X_n - X\|_p + \|X_n\|_p\end{aligned}$$

which is finite, so $X \in l^p$
Furthermore, $(X_n) \to X$. Since the Cauchy sequence was arbitrary, then the space l^p is complete. □

Now, for inner product space, we have two cases:
case 1: if $p = 2$,
The space l^2 is an inner product space with inner product defined by

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i \overline{y_i}$$

Convergence of this series follows from the Cauchy-Schwarz inequality which will be discussed in the next chapter.
Hence, l^2 is a Hilbert space.

case 2: $p \neq 2$;
The space l^p where $p \neq 2$ is not an inner product space, hence not a Hilbert space. We can prove this by showing that the norm does not satisfy the parallelogram equality.
For example, let $x = (1, 1, 0, 0, ...)$ and $y = (1, -1, 0, 0, ...)$. $x, y \in l^p$ and,

$$\|x\| = \|y\| = 2^{\frac{1}{p}}$$

$$\|x + y\| = \|x - y\| = 2$$

hence

$$2(\|x\|^2 + \|y\|^2) \neq \|x + y\|^2 + \|x - y\|^2$$

so, l^p where $p \neq 2$ is not an inner product space.

Similarly, L^p spaces are normed spaces and Banach spaces. But L^p are not inner product spaces except when $p = 2$, hence L^2 space is inner product space and Hilbert space.

Lemma 1.1.2. *Space L^p, where $p \geq 1$*
The Space L^p is a normed space with norm defined by

$$\|x(t)\|_p = \left(\int_a^b |x(t)|^p \, dt \right)^{\frac{1}{p}}$$

This space is complete, so it's a Banach space.
Now, for inner product space, we have two cases:
case 1: if $p = 2$,
The space L^2 is an inner product space with inner product defined by

$$\langle x, y \rangle = \int_a^b x(t)\overline{y}(t)dt$$

L^2 is a Hilbert space by the Cauchy-Schwarz inequality.

case 2: $p \neq 2$;
The space L^p where $p \neq 2$ is not an inner product space, hence not a Hilbert space. We can prove this by showing that the norm does not satisfy the parallelogram equality.

1.2 Linear Operator

In the case of vector spaces and, in particular, normed spaces, a mapping is called an operator. The following defines the linear operators.

Definition 1.2.1. *Linear Operator*
Kreyszig (1978) states the definition of the linear operators as follows:
An operator T from a normed space X to a normed space Y is said to be linear if for all $x, y \in D(T)$ and $\alpha \in F$ then,

$$T(x + y) = Tx + Ty$$

$$T(\alpha x) = \alpha T(x)$$

Here, we will present some examples of the linear operators.

Example 1. Identity operator.
The identity operator $I : X \to X$ defined by $I(x) = x$ for all $x \in X$ is a linear operator.

Example 2. Zero operator.
The zero operator $O : X \to Y$ defined by $O(x) = 0$ for all $x \in X$ is a linear operator.

Example 3. Differentiation operator.
Let X be the vector space of all polynomials on $[a, b]$, we may define a linear operator $T : X \to X$ such that $T(x(t)) = x\prime(t)$ for all $x \in X$, $t \in [a, b]$.

Example 4. Integral operator.
Let $C\,[a, b]$ be the vector space of all continuous functions on $[a, b]$, we may define a linear operator $T : C\,[a, b] \to C[a, b]$ by $y = Tx$ where

$$y(t) = \int_a^t x(p)dp$$

For a review of some elementary concepts related to mappings, one-to-one and onto as follows,

Definition 1.2.2. *One to One Operator*
If each element of the codomain is mapped to at most one element of the domain, then the operator $T : X \to Y$ is called one to one from X to Y.
If $x_1, x_2 \in X$, $x_1 \neq x_2$ then $T(x_1) \neq T(x_2)$.
A one to one operator is called also injection.

Definition 1.2.3. *Onto Operator*
If each element of the codomain is mapped to at least one element of the domain, then the operator $T : X \to Y$ is called onto from X to Y
If for all $y \in Y$ then there exist $x \in X$ such that $y = T(x)$.
A one to one operator is called also injection.

Definition 1.2.4. *Bijective Operator*
An operator is called bijection if each element of the codomain is mapped to exactly one element of the domain. That is, the operator is both injective and surjective.

1.3 Bounded Linear Operator

In functional analysis, a bounded linear operator is a linear transformation T between normed vector spaces X and Y for which the ratio of the norm of Tx to that of x is bounded.

Definition 1.3.1. *Bounded Linear Operator*
Let X and Y be two normed spaces and T be a linear operator such that $T : D(T) \to Y$, where $D(T) \subset X$ the operator T is said to be bounded if there exist a real number c such that for all $x \in D(T)$ we have

$$\|Tx\|_Y \leq c\,\|x\|_X \tag{1.1}$$

The norm of T is given by $\|T\| = \sup\{\|Tx\|, \|x\| \leq 1\}$

Remark 1.3.1. .
The smallest possible value of c such that equation 1.1 holds for all nonzero $x \in D(T)$ is given by division

$$\frac{\|Tx\|}{\|x\|} \tag{1.2}$$

and this shows that c must be at least as big as the supermum of the above expression on $X - \{0\}$. Hence, the smallest possible c in equation 1.1 is that supremum. This quantity is denoted by $\|T\|$, thus , $\|T\| = \sup \frac{\|Tx\|}{\|x\|}$, $x \neq 0, x \in X$
$\|T\|$ is called the norm of the operator T.
From equation 1.1 with $c = \|T\|$ we get $\|Tx\| \leq \|T\| \|x\|$
An alternative formula for the norm of T is $\|T\| = \sup \|Tx\|$ where $x \in D(T)$ with $\|x\| = 1$.

Now, we will show some examples of bounded linear operators between normed spaces.

Example 1. Identity operator.
The identity operator $I : X \to X$ on a normed space $X \neq \{0\}$ is bounded and has a norm $\|I\| = 1$.

Example 2. Zero operator.
The zero operator $O : X \to Y$ on a normed space X is bounded and has norm $\|O\| = 0$.

Example 3. Integral operator
We can define an integral operator $T : C[0,1] \to C[0,1]$ by

$$Tx = \int_0^1 k(t,p)x(p)dp$$

where $k(t,p)$ is a given function, which is called the kernel of T and is $G = J \times J$ where $J = [0,1]$
If k is bounded with $|k(t,p)| \leq k_0$ then,

$$\|T\| = \max_{t \in J} \left| \int_0^1 k(t,p)x(p)dp \right| \leq \max_{t \in J} \int_0^1 |k(t,p)| \, |x(p)| \, dp \leq k_0 \max_p |x(p)| = k_0 \|x\|$$

$$\|T\| \leq k_0 \|x\|$$

then we can say that T is bounded operator.

Furthermore, there are some linear operators which are not bounded. Here is an example that is not bounded.

Example 4. Differentiation operator.
Let X be the vector space of all polynomials on $J = [0,1]$, we may define a linear operator $T : X \to X$ such that $T(x(t)) = x\prime(t)$ for all $x \in X$ and $t \in J$
For example, let $x(t) = t^n$ then $Tx = nt^{n-1}$.
Then $\|x\| = 1$ and $\|Tx\| = n$, since n is arbitrary then there is no fixed c such that $\frac{\|Tx\|}{\|x\|} \leq c$.
So, we can conclude that T is not bounded operator.

Theorem 1.3.1. .
Kreyszig (1978) proves that for the normed spaces X and Y and for a linear operator $T : X \to Y$ the following statements are equivalent.

1. T is bounded.

2. T is continuous.

3. T is continuous at 0.

Proof. .
Assume that f is continuous at an arbitrary $x_0 \in D(f)$, then for given $\epsilon > 0$ there is an $\delta > 0$ such that $\|f(x) - f(x_0)\| \leq \epsilon, \ \ x \in D(f)$ with $\|x - x_0\| \leq \delta$, now take $y \neq 0$ in $D(f)$ and let

$$x = x_0 + \frac{\delta}{\|y\|} y, \quad then, \quad x - x_0 = \frac{\delta}{\|y\|} y$$

so $\|x - x_0\| = \delta$. Since f is linear we have,

$$\|f(x) - f(x_0)\| = \|f(x - x_0)\| = \left\| f\left(\frac{\delta}{\|y\|} y\right) \right\| = \frac{\delta}{\|y\|} \|fy\|$$

$$\frac{\delta}{\|y\|} \|fy\| \leq \epsilon \quad then \quad \|fy\| \leq \frac{\epsilon}{\delta} \|y\|$$

Let $c = \epsilon/\delta$ then $\|fy\| \leq c\,\|y\|$, so f is bounded.

Conversely, assume that f is bounded. If $f = 0$ so its trivial. Let $f \neq 0$ so $\|f\| \neq 0$. Let $x_0 \in D(f)$, given $\epsilon > 0$ let $\delta = \epsilon / \|f\|$ we obtain,

$$\|fx - fx_0\| = \|f(x - x_0)\| \leq \|f\| \, \|x - x_0\| < \|f\| \, \delta = \epsilon$$

Since $x_0 \in D(f)$ is arbitrary, we can say that f is continuous.

Clearly, if T is continuous operator then it is continuous at every point, and Continuity of T at a point implies boundedness of T that implies that it is also continuous.

□

Remark 1.3.2. .
The space of bounded linear operator:
Let X and Y be two normed spaces, let $B(X, Y)$ be the vector space of all bounded linear operators T from X to Y. The space $B(X, Y)$ is itself a normed space with norm defined by

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\|=1} \|Tx\|, \quad x \in X$$

1.4 Linear Functional

A functional is an operator whose range lies in a scalar field, whether it is real line $\mathbb{R}$ or complex plane $\mathbb{C}$.

Definition 1.4.1. *Linear Functional*
Kreyszig (1978) publishes the definition of the linear functional as follows:
Let X be a normed space over a field $\mathbb{F}$, and let $T : X \to \mathbb{F}$ be a linear operator, then we call T a linear functional on X.

The set of all linear functionals on a normed space X is denoted by X^*.

Definition 1.4.2. *Bounded Linear Functional*
A bounded linear functional f is a bounded linear operator with range in the scalar field of a normed space X in which the domain lies.

Thus there exists a real number c such that for all $x \in D(f)$ then $|f(x)| \leq c\,\|x\|$. Furthermore, the norm of f is given by,

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \qquad x \in D(f),$$

Another form of $\|f\|$ is

$$\|f\| = \sup_{\|x\|=1} |f(x)| \qquad x \in D(f)$$

and this implies that $|f(x)| \leq \|f\|\,\|x\|$

Theorem 1.4.1. .
A linear functional f with domain $D(f)$ in a normed space is continuous if and only if f is bounded.

There are many examples of the bounded linear functional. Here, we will present a few of them.

Example 1. Dot Product operation
The familiar dot product with one factor kept fixed defines a functional $f : \Re^3 \to \Re$ by means of

$$f(x) = x.a = x_1a_1 + x_2a_2 + x_3a_3$$

where $a, x \in R^3$ such that $a = (a_1, a_2, a_3), x = (x_1, x_2, x_3)$ (a is fixed).
f is bounded linear functional with $|f(x)| = |a.x| \leq \|a\|\,\|x\|$ so that $\|f\| \leq \|a\|$.
On the other hand, by talking $x = a$ we obtain

$$\|f\| \geq \frac{|f(a)|}{\|a\|} = \frac{\|a\|^2}{\|a\|} = \|a\|$$

hence, the norm of f is $\|f\| = \|a\|$

Example 2. Definite Integral
The definite integral is a number if we consider it for a single function. However, the situation changes completely if we consider that integral for all functions in a certain function space. Then the integral becomes a functional on that space, call it f. As a space, let us choose $C[a, b]$. Then f is defined by

$$f(x) = \int_a^b x(t)dt \qquad x \in C[a, b]$$

f is bounded linear functional with $\|f\| = b - a$.
If $x \in C[a, b]$ where $\|x\| = \max\{x(t), t \in [a, b]\}$, now, we obtain that,

$$|f(x)| = \left|\int_a^b x(t)dt\right| \leq (b - a) \max_{t \in [a,b]} |x(t)| = (b - a)\,\|x\|$$

by talking the sup for both sides get $\|f\| \leq b - a$
On the other hand, let $x = x_0 = 1$, so $\|x_0\| = 1$ and use

$$\|f\| \geq \frac{|f(x_0)|}{\|x_0\|} = |f(x_0)| = \int_a^b 1dt = b - a$$

Example 3. Operator On l^2
Define a linear functional f on l^2 by choosing a fixed $a = (a_i) \in l^2$ and setting

$$f(x) = \sum_{i=1}^{\infty} a_i x_i$$

where $x = (x_i) \in l^2$. This series converges absolutely and f is bounded, since the Cauchy-Schwarz inequality gives

$$|f(x)| = \left|\sum a_i x_i\right| \leq \sum |a_i x_i| \leq \sqrt{\sum |x_i|^2}\sqrt{\sum |a_i|^2} = \|x\| \, \|a\|$$

The following example is of a bounded functional but not linear.

Example 4. Norm
The norm $\|.\| : X \to \mathbb{R}$ on a normed space $(X, \|.\|)$ is a functional on X which is not linear since, $\|\alpha x + \beta y\| \leq |\alpha| \, \|x\| + |\beta| \, \|y\|$.

1.5 Dual Space

It is of basic importance that the set of all linear functionals defined on a vector space X can itself be made into a vector space. This space is denoted by $X^{'}$ and is called the dual space of X.

Definition 1.5.1. *Dual Space*
Kreyszig (1978) defines that the dual space of a vector space X is the set of all bounded liner functionals on X, denoted by $X^{'}$ and it can itself be made into a vector space.

Its algebric operations of vector space are defined in a natural way as follows:
If $f_1, f_2 \in X^{'}, \alpha \in \mathbb{F}, x \in X$ then,
Sum: $f_1 + f_2 : (f_1 + f_2)(x) = f_1(x) + f_2(x)$.
Product: $\alpha f_1 : (\alpha f)(x) = \alpha f(x)$.

Definition 1.5.2. *Dual Space of a Normed Space*
Let X be a normed space. Then the set of all linear functionals on X constitutes a normed space with norn defined by

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} = \sup_{\|x\|=1} |f(x)| \qquad x \in D(f)$$

which is called the dual space of X and is denoted by X' .

Corollary 1.5.1. .
The dual space of a normed space is Banach space.

Now we will present some examples of the dual spaces.

- The dual space of $\mathbb{R}^n$ is $\mathbb{R}^n$.

- The dual space of l^1 is l^∞.
- The dual space of l^p is l^q where $1 < p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1$.

Definition 1.5.3. *Isomorphism*
Kreyszig (1978) states that an isomorphism T of a vector space X onto a vector space Y over the same field is a bijective mapping which preserves the two algebraic operations of vector space; thus, for all $x, y \in X$ and scalar α we have,

$$T(x + y) = T(x) + T(y) \qquad\qquad T(\alpha x) = \alpha T(x)$$

That is, $T : X \to Y$ is a bijective linear operator. Y is then called isomorphic with X, and X, Y are called isomorphic vector spaces.

A bijection mapping $T : X \to Y$ between normed spaces is an isomorphism if it preserves the norm.

Chapter 2

Diagonal Operators on l^p Spaces

In functional analysis and related areas of mathematics, a sequence space is a vector space whose elements are infinite sequences of real or complex numbers. Equivalently, it is a function space whose elements are functions from the natural numbers to the field K of real or complex numbers. The set of all such functions is naturally identified with the set of all possible infinite sequences with elements in K, and can be turned into a vector space under the operations of point wise addition of functions and point wise scalar multiplication. The most important sequence spaces in analysis are the l^p spaces, with the p - norm.

2.1 Sequence Spaces l^p

Definition 2.1.1. *Sequence Spaces*
We defined before the space of sequences.
Space l^∞ is the space of all bounded sequences of complex numbers that is, if $x = (x_n) \in l^\infty$, then there is c_x such that,

$$\|x_n\| \leq c_x, \quad \forall n = 1, 2, 3, ...$$

where c_x is a positive real number that depends on x only and does not depend on n.

Let $p \geq 1$ be a fixed real number. By definition, each element in the space l^p is a sequence $x = (x_1, x_2, x_3...)$ of complex numbers such that $\sum_{n=1}^{\infty} |x_n|^p$ converges.

Example 1. .
Let $x = (x_n) = (1/2)^n$,
$\sum_{n=1}^{\infty} |x_n| = 1 < \infty$, then this series converges with $x \in l^1$.

Example 2. .
Let $x = (x_n) = (1/n)$,
we have $|x_n| \leq |1/n| \leq 1 \quad \forall n \geq 1$
then, x_n is bounded sequence with $\|x_n\|_{l^\infty} < \infty$
This means that $(x_n) \in l^\infty$

Theorem 2.1.1. .
The sequence spaces l^p together with p norm are normed spaces. If $x = (x_i) \in l^p$ then the norm is

defined by

$$\|x\|_p = \begin{cases} (\sum_{i=1}^{\infty} |x_i|^p)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \sup |x_i| & \text{if } p = \infty \end{cases}$$

These spaces are complete, so l^p spaces are Banach spaces.

Theorem 2.1.2. .
The sequence space l^2 together with inner product is an inner product space. If $x = (x_i), y = (y_i) \in l^2$, where the inner product is defined by

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y_i}$$

and the norm is defined by

$$\|x\|^2 = \langle x, x \rangle = \sum_{i=1}^{\infty} |x_i|^2$$

is complete, so, l^2 is Hilbert space.

Theorem 2.1.3. .
For $1 \leq p \leq q \leq \infty$, if $x \in l^p$ then $x \in l^q$. That means, $l^p \subseteq l^q$.

Proof. .
Let $x \in l^p$, then $\sum_{i=1}^{\infty} |x_i|^p < \infty$. Therefore, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ then $|x_n| < 1$.
Since $p \leq q$, then $0 \leq q - p$ and $|x_n|^{q-p} < 1$, hence $|x_n|^q < |x_n|^p$.
Let $M = \max \left\{ |x_1|^{q-p}, |x_2|^{q-p}, |x_3|^{q-p}, ..., |x_{n_0}|^{q-p}, 1 \right\}$. Then,

$$\sum_{i=1}^{\infty} |x_i|^q = \sum_{i=1}^{\infty} |x_i|^p |x_i|^{q-p} \leq M \sum_{i=1}^{\infty} |x_i|^p < \infty$$

Thus, $x \in l^q$

The converse is not true.
In case $1 \leq p \neq q \leq \infty$ then $l^p \neq l^q$ in general.
for example, if we take the sequence $x_n = n^{-1/p}$
$\sum_{n=1}^{\infty} |x_n|^q = \sum_{n=1}^{\infty} \frac{1}{n^{q/p}} < \infty$ since $p < q$ then $q/p > 1$. Therefore, $x \in l^q$.
But $\sum_{n=1}^{\infty} |x_n|^p = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$. Therefore, $x \notin l^p$ □

2.2 Theorems on l^p spaces

In this section, We will state without proof some fundamental inequalities on l^p spaces.

Definition 2.2.1. *The Conjugate Indices*
Kreyszig (1978) publishes the definition of the conjugate indices as follows:
Let $p, q > 1$ be two real numbers. Then p and q are called conjugate indices if they satisfy the equality.

$$\frac{1}{p} + \frac{1}{q} = 1$$

Remark 2.2.1. *The Auxiliary Inequality*
Let $p > 1$, define q by

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$\frac{1}{p-1} = q - 1$$

If we take $u = t^{p-1}$ then $t = u^{q-1}$.
if α, β are two non negative numbers then we can prove the inequality:

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

Theorem 2.2.1. *The Holder's Inequality*
According to (Kreyszig, 1978, p.14),
If $x = (x_n) \in l^p$ and $y = (y_n) \in l^q$, where p and q are conjugate indices. Then $xy \in l^1$ with

$$\|xy\|_1 \leq \|x\|_p \|y\|_q$$

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |y_n|^q\right)^{\frac{1}{q}}$$

Proof. .
By the Auxiliary Inequality we have two cases:
Case 1: if x or $y = 0$, then this is travail.
Case 2 : if $x \neq 0$ and $y \neq 0$ where $A = \|x\|_p$ and $B = \|y\|_q$,
let $\alpha = \frac{|x|}{A}$ and let $\beta = \frac{|y|}{B}$
Apply the auxiliary inequality,

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

$$\frac{|xy|}{AB} \leq \frac{|x|^p}{pA^p} + \frac{|y|^q}{qB^q}$$

Talk sum for both sides to get,

$$\frac{1}{AB}\sum_{n=1}^{\infty} |x_n y_n| \leq \frac{1}{pA^p}\sum_{n=1}^{\infty} |x_n|^p + \frac{1}{qB^q}\sum_{n=1}^{\infty} |y_n|^q$$

Substitute the vales of A and B to get,

$$\frac{1}{\|x\|_p \|y\|_q}\sum_{n=1}^{\infty} |x_n y_n| \leq \frac{1}{p} + \frac{1}{q} = 1$$

Multiply both sides by $\|x\|_p \|y\|_q$ to get the result, which is,

$$\|xy\|_1 \leq \|x\|_p \|y\|_q$$

□

If we take the case where $p = q = 2$, in which p equals its conjugate q. This case will be discussed in the following the inequality.

Corollary 2.2.1. *Cauchy-Schwarz inequality. (Kreyszig, 1978)*
If $x = (x_n) \in l^2$ and $y = (y_n) \in l^2$ then, $xy \in l^1$ with

$$\|xy\|_1 \leq \|x\|_2 \|y\|_2$$

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \sqrt{\sum_{n=1}^{\infty} |x_n|^2} \sqrt{\sum_{n=1}^{\infty} |y_n|^2}$$

Theorem 2.2.2. *The Minkowski Inequality (Kreyszig, 1978)*
For $p \geq 1$. Let $x = (x_n), y = (y_n) \in l^p$. Then, $x + y \in l^p$ with

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}$$

Thus, x, y satisfy the triangle inequality, which is,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

2.3 Diagonal Operators

Definition 2.3.1. *Diagonal Operator from l^p space to l^q space*
As per Garling (1974), a diagonal operator $d_a : l^p \to l^q$ is defined by, $d_a(x_n) := (a_n.x_n)$ where (a_n) is a sequence of complex numbers, $(x_n) \in l^p$ and $(a_n x_n) \in l^q$

Now, we will show two examples of diagonal operators between sequence spaces.

Example 1.
A diagonal operator $d_a : l^2 \to l^1$, where $(a_n) = \left(\frac{1}{n}\right)$ is defined by $d_a(x) = (a_n x_n)$ for all $x = (x_n) \in l^2$, then $(a_n x_n) \in l^1$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 = \sum_{n=1}^{\infty} \left(\frac{1}{n^2}\right) = \frac{\pi^2}{6} < \infty \ \ then \ \ (a_n) \in l^2$$

Now, by Holder inequality, we get that $(a_n x_n) \in l^1$

$$\sum_{n=1}^{\infty} |a_n x_n| \leq \left(\sum_{n=1}^{\infty} | \ a_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} | \ x_n|^2\right)^{\frac{1}{2}} = \|a\|_2 \|x\|_2 < \infty$$

$$Let \ \ (x_n) = \left(\frac{1}{2}\right)^{\frac{n}{2}}, (x_n) \in l^2 \ \ since \ \ \left(\sum_{n=1}^{\infty} | \ x_n|^2\right)^{\frac{1}{2}} = \left(\sum_{n=1}^{\infty} \left| \left(\frac{1}{2}\right)^{\frac{n}{2}} \right|^2\right)^{\frac{1}{2}} = \left(\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n\right)^{\frac{1}{2}} = \sqrt{2} < \infty$$

$$(a_n x_n) = \left(\frac{1}{n}\right) \left(\frac{1}{2}\right)^{\frac{n}{2}} \in l^1$$

note that $(a_n x_n) \in l^p$ for all $p \geq 1$

Example 2.
A diagonal operator $d_a : l^3 \to l^1$, where $(a_n) = (\frac{1}{3})^{\frac{2n}{3}}$ defined by $d_a(x) = (a_n x_n)$ for all $x = (x_n) \in l^3$, then $(a_n x_n) \in l^1$

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{\frac{2n.3}{3.2}} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n} = \frac{1}{2} < \infty \ \ then \ \ (a_n) \in l^{\frac{3}{2}}$$

Now, by Holder inequality, we get that $(a_n x_n) \in l^1$

$$\sum_{n=1}^{\infty} |a_n x_n| \leq \left(\sum_{n=1}^{\infty} |\ a_n|^{\frac{3}{2}}\right)^{\frac{2}{3}} \left(\sum_{n=1}^{\infty} |\ x_n|^{3}\right)^{\frac{1}{3}} \leq \frac{1}{2} \|x\|_3 < \infty$$

$$Let \ \ (x_n) = \left(\frac{1}{n}\right)^{\frac{2}{3}}, (x_n) \in l^3 \ \ since \ \ \sum_{n=1}^{\infty} |\ x_n|^3 \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

$$(a_n x_n) = \left(\frac{1}{3}\right)^{\frac{2n}{3}} \left(\frac{1}{n}\right)^{\frac{2}{3}} \in l^1$$

note that $(a_n x_n) \in l^p$ for all $p \geq 1$

Diagonal operator can be bounded or not. We have two cases of bounded diagonal operator, $d_a : l^p \to l^q$
Case 1:
$d_a : l^p \to l^q$ is bounded where $p > q \geq 1$. And $\|d_a\| = \|a\|_s$ where $s = \frac{pq}{p-q}$.
Case 2 :
$d_a : l^p \to l^q$ is bounded where $1 \leq p \leq q \leq \infty$, where $\|d_a\| = \|a\|_{\infty}$.
This will be discussed in the following section.

2.4 Bounded Diagonal Operators

In this section, we will discuss the conditions to decide if the diagonal operator is bounded or not in the following theorems.

Theorem 2.4.1. .
Garling (1974) proves that the diagonal operator $d_a : l^p \to l^q$ is bounded if and only if $a \in l^s$ where $s = \frac{pq}{p-q}$ for $p > q \geq 1$. Furthermore, $\|d_a\| = \|a\|_s$

Proof. .
Suppose that d_a is bounded, then show that $a \in l^{\frac{pq}{p-q}}$
Consider the sequence $(x_n) = (\delta_k)$ where

$$(\delta_k) = \begin{Bmatrix} |a_k|^{\frac{q}{p-q}}, k \leq n \\ 0 \qquad k > n \end{Bmatrix}$$

Now,

$$\|d_a(x_n)\|_q \leq \|d_a\| \, \|x_n\|_p$$

$$= \|d_a\| \left(\sum_{k=1}^{n} |\delta_k|^p \right)^{\frac{1}{p}}$$

$$= \|d_a\| \left(\sum_{k=1}^{n} |a_k|^{\frac{pq}{p-q}} \right)^{\frac{1}{p}}$$

$$\|d_a(x_n)\|_q = \left(\sum_{k=1}^{n} |a_k|^{\frac{pq}{p-q}} \right)^{\frac{1}{q}} \leq \|d_a\| \left(\sum_{k=1}^{n} |a_k|^{\frac{pq}{p-q}} \right)^{\frac{1}{p}}$$

divide by $\left(\sum_{k=1}^{n} |a_k|^{\frac{pq}{p-q}}\right)^{\frac{1}{p}}$ to get

$$\left(\sum_{k=1}^{n} |a_k|^{\frac{pq}{p-q}} \right)^{\frac{p-q}{pq}} \leq \|d_a\|$$

since n is arbitrary, we can let $n \to \infty$ to get ,

$$\left(\sum_{k=1}^{\infty} |a_k|^{\frac{pq}{p-q}} \right)^{\frac{p-q}{pq}} \leq \|d_a\|$$

so,

$$\|a\|_s \leq \|d_a\|. \tag{2.1}$$

Conversely, let $a \in l^{\frac{pq}{p-q}}$ then show that d_a is bounded
By Holder inequality,

$$\|d_a(x)\|_q = \|a_n x_n\|_q = \left(\sum_{n=1}^{\infty} |a_n|^q |x_n|^q \right)^{\frac{1}{q}}$$

$$\leq \left(\sum_{n=1}^{\infty} |a_n|^{\frac{pq}{p-q}} \right)^{\frac{p-q}{pq}} \left(\sum_{n=1}^{\infty} |x_n|^{\frac{pq}{p}} \right)^{\frac{q}{pq}}$$

$$\leq \left(\sum_{n=1}^{\infty} |a_n|^{\frac{pq}{p-q}} \right)^{\frac{p-q}{pq}} \left(\sum_{n=1}^{\infty} |x_n|^{\frac{pq}{p}} \right)^{\frac{q}{pq}}$$

$$let\ k = \left(\sum_{n=1}^{\infty} |a_n|^{\frac{pq}{p-q}} \right)^{\frac{p-q}{pq}} \quad then,\ k < \infty, \quad since\ a \in l^{\frac{pq}{p-q}}$$

so, we have , $\|d_a(x)\|_q \leq k\,\|x\|_p$. That means, d_a is bounded with

$$\|d_a\|_q \leq \left(\sum_{n=1}^{\infty} |a_n|^{\frac{pq}{p-q}} \right)^{\frac{p-q}{pq}} = \|a\|_s$$

so,

$$\|d_a\|_q \leq \|a\|_s \tag{2.2}$$

By equations 2.1 and 2.2, we get that $\|d_a\|_q = \|a\|_s$ □

Theorem 2.4.2. .
Garling (1974) also proves that the diagonal operator $d_a : l^p \to l^q$ *is bounded if and only if* $a \in l^\infty$ *for* $1 \le p \le q \le \infty$*, and* $\|d_a\| = \|a\|_\infty$

Proof. .
Suppose that d_a is bounded, then show that $a \in l^\infty$.
Let the sequence (e_n) in which all terms are zeros except in the nth term we have 1. Any sequence $x = (x_k) \in l^p$ can be represented uniquely by e_n with $x = \sum_{k=1}^{\infty} x_k e_k$
Since d_a is bounded linear operator, then $d_a(x) = \sum_{k=1}^{\infty} x_k \zeta_k$ where $(\zeta_k) = d_a(e_k)$ and $\|e_k\|_p = 1$

$$|a_k| = \|d_a(e_k)\|_q \le \|d_a\| \, \|e_k\|_p = \|d_a\|$$

take supermum for both sides to get $\sup_k |a_k| \le \|d_a\| < \infty$. Then,

$$\|a\|_\infty \le \|d_a\| \tag{2.3}$$

Conversely, suppose that $s \in l^\infty$ then show that d_a is bounded,

$$\begin{aligned}
\|d_a(x)\|_q &= \|a_n x_n\|_q \\
&\le \|a_n x_n\|_p \\
&= \left(\sum_{n=1}^{\infty} |a_n|^p \, |x_n|^p \right)^{\frac{1}{p}} \\
&\le \left(\sup |a_n|^p \sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \\
&= (\sup |a_n|^p)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \\
\|d_a(x)\|_q &\le \|a\|_\infty \, \|x\|_p \qquad \textit{so,}
\end{aligned}$$

$$\|d_a\|_q \le \|a\|_\infty \tag{2.4}$$

From equations 2.3 and 2.4, we get that $\|d_a\|_q = \|a\|_\infty$ ☐

We can summarize theorem 2.4.1 and theorem 2.4.2 that the multiplier (diagonal operator) $d_a : l^p \to l^q$ is bounded if and only if

$$\|d_a\| = \begin{cases} \|a\|_\infty & \text{if } p \le q \\ \|a\|_s & \text{if } p \ge q \end{cases} \qquad \textit{where} \qquad \frac{1}{s} = \frac{1}{q} - \frac{1}{p}$$

The following example presents a bounded diagonal operator between sequence spaces.

Example 1. .
Let the operator $d_a : l^2 \to l^1$ and let $(x_n) = (1/n)$, then

$$\|x_n\|_{l^2} = \left(\sum_{n=1}^{\infty} |x_n|^2 \right)^{\frac{1}{2}} = \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} < \infty$$

That means $(x_n) \in l^2$
let $(a_n) = (\frac{1}{n})$ then,

$$d_a(x_n) = \frac{1}{n}.\frac{1}{n} = \frac{1}{n^2}$$

$$\|d_a(x_n)\| = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

Thus, d_a is bounded operator.

The following example presents an unbounded diagonal operator between sequence spaces.

Example 2. .
Let $d_a : l^1 \to l^1$ and $(x_n) = (\frac{1}{n^2})$, then

$$\|x_n\|_{l^1} = \sum_{n=1}^{\infty} |x_n| = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

That means, $(x_n) \in l^1$
let $(a_n) = (n)$ then,

$$d_a(x_n) = n.\frac{1}{n^2} = \frac{1}{n}$$

$$\|d_a(x_n)\| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Thus, d_a is unbounded operator.

Chapter 3

The Hardy Spaces H^p

This chapter begins with introduction about the space of all analytic functions in the unit disc. The second section represents theorems of the space. After this comes a brief discussion of the operators on this space.

3.1 Hardy Spaces

In mathematics, a complex function is said to be analytic on a region R if it is complex differentiable at every point in R. A real function is said to be analytic on a region R if it is real differentiable at every point in R. There exist both real analytic functions and complex analytic functions.

Definition 3.1.1. *The Hardy Spaces*
Duren (2000) says that the definition of the Hardy spaces as:
The Hardy spaces $H^p, 1 \leq p \leq \infty$ *are Banach spaces consisting of analytic functions in the unit disc* $D = \{z \in \mathbb{C} : |z| < 1\}$ *whose boundary values are in* $L^p(T)$ *where* $T = \{z \in \mathbb{C} : |z| = 1\}$ *is the unit circle. Thus* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *for* $|z| < 1$

For analytic functions in a disk, the integral means are

$$M_p(r, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta \right\}^{\frac{1}{p}},$$

$$M_\infty(r, f) = \max_{0 \leq \theta < 2\pi} \left| f(re^{i\theta}) \right|$$

A function $f(z)$ analytic in the unit disk $|z| < 1$ is said to be of class $H^p, 1 \leq p < \infty$ if $M_p(r, f)$ remains bounded as $r \to 1$. H^∞ is the class of all bounded analytic functions in the unit disk, where $M_\infty(r, f)$ remains bounded as $r \to 1$.

Recall that, functions of period 2π have Fourier series, that is,

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}, \quad \textit{where}$$

$$(c_n) = \frac{1}{2\pi}\int_0^{2\pi} g(t)e^{-int}dt$$

If $g \in L^p(0,2\pi)$ for some $1 \leq p \leq \infty$, then $g \in L^1(0,2\pi)$ and

$$|c_n| \leq \frac{1}{2\pi}\int_0^{2\pi} |g(t)|\left|e^{-int}\right| dt = \|g\|_{L^1}$$

So, the sequence (c_n) is bounded. Now if $c_n = 0$ for all $n < 0$, we can define

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

and it converges for $|z| < 1$

Let $f \in L^p[0,2\pi]$ then f has a Fourier series $\sum_{n=-\infty}^{\infty} c_n e^{int}$ where $t \in [0,2\pi]$ and

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$$

The Hardy space consists of functions with Fourier series $\sum_{n=0}^{\infty} c_n e^{int}$. So, H^p is subspace of L^p space.
Note that H^p is the same of L^p where $c_n = 0$ if $n < 0$

Remark 3.1.1. .
The Hardy space together with norm is normed space. If $f(z) \in H^p$, the norm is defined by

$$\|f\|_p = \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}}, \qquad 1 \leq p < \infty$$

$$\|f\|_\infty = \max_{0\leq\theta<2\pi} \left|f(re^{i\theta})\right|, \qquad p = \infty$$

Note that the Hardy spaces are Banach spaces and this is immediate since it is a closed subspace of L^p which is complete.

Proof. .
For $f, g \in H^p$ and $\alpha \in F$, where F is the field of H^p
Clearly, $\|f\| \geq 0$ and $\|f\| = 0$ iff $f = 0$
$\|\alpha f\| = |\alpha| \|f\|$

$$\|\alpha f\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |\alpha f|^p d\theta\right)^{\frac{1}{p}} = (|\alpha|^p)^{\frac{1}{p}}\left(\frac{1}{2\pi}\int_0^{2\pi} |f|^p d\theta\right)^{\frac{1}{p}} = |\alpha| \|f\|$$

$\|f+g\| \leq \|f\| + \|g\|$

$$\|f+g\| = \left(\frac{1}{2\pi}\int_0^{2\pi} |f+g|^p d\theta\right)^{\frac{1}{p}} \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f|^p d\theta\right)^{\frac{1}{p}} + \left(\frac{1}{2\pi}\int_0^{2\pi} |g|^p d\theta\right)^{\frac{1}{p}} = \|f\| + \|g\|$$

□

Remark 3.1.2. .
The Hardy space H^2 together with inner product is inner product space. If $f(z), g(z) \in H^2$, the inner product is defined by

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta,$$

or equivalently, if $f, g \in H^2$ where $f = \sum_{n=0}^{\infty} a_n z^n$ and $g = \sum_{n=0}^{\infty} b_n z^n$ then

$$\langle f, g\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

H^2 is Hilbert space since it is a closed subspace of the complete inner product space L^2

Proof. .
Let $f = \sum_{n=0}^{\infty} a_n z^n$, $g = \sum_{n=0}^{\infty} b_n z^n$, $h = \sum_{n=0}^{\infty} c_n z^n \in H^2$ and $\alpha \in F$ where F is the field of scalars.
$\langle f, f\rangle = \sum_{n=0}^{\infty} a_n \overline{a_n} = \sum_{n=0}^{\infty} |a_n|^2 \geq 0$ and clearly $\langle f, f\rangle = 0$ iff $f = 0$
$\langle f, g\rangle = \overline{\langle g, f\rangle}$

$$\overline{\langle g, f\rangle} = \overline{\sum_{n=0}^{\infty} b_n \overline{a_n}} = \sum_{n=0}^{\infty} \overline{b_n} a_n = \sum_{n=0}^{\infty} a_n \overline{b_n} = \langle f, g\rangle$$

$\langle \alpha f, g\rangle = \alpha \langle f, g\rangle$

$$\langle \alpha f, g\rangle = \sum_{n=0}^{\infty} \alpha a_n \overline{b_n} = \alpha \sum_{n=0}^{\infty} a_n \overline{b_n} = \alpha \langle f, g\rangle$$

$\langle f + g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$

$$\langle f + g, h\rangle = \sum_{n=0}^{\infty}(a_n + b_n)\overline{c_n} = \sum_{n=0}^{\infty} a_n \overline{c_n} + \sum_{n=0}^{\infty} b_n \overline{c_n} = \langle f, h\rangle + \langle g, h\rangle$$

□

The following example gives the inner product between two functions in H^2.

Example 1. .
Consider $(c_n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$ and let $(b_n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$
Let $f = \sum_{n=0}^{\infty} c_n z^n$ and $g = \sum_{n=0}^{\infty} b_n z^n$ then:

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta,$$

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} e^{-in\theta} d\theta,$$

$$= \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Theorem 3.1.1. *.*
Let $1 \leq p \leq q \leq \infty$, if $f \in H^q$ then $f \in H^p$.

Proof. .
We have three cases

Case 1: $\left|f(re^{i\theta})\right| < 1$

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta \leq 1(2\pi)/2\pi = 1 < \infty$$

$$\|f\| = \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \leq 1 < \infty$$

Case 2: $\left|f(re^{i\theta})\right| \geq 1$
since $f \in H^q$ with $q < \infty$ then

$$\|f\| = \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^q d\theta\right\}^{\frac{1}{q}} < \infty$$

that means;

$$\|f\| = \frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^q d\theta < \infty$$

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^q d\theta < \infty$$

$$\|f\| = \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} < \infty$$

Case 3: when $q = \infty$:
Since $f \in H^\infty$ then, $\max_\theta \left|f(re^{i\theta})\right| < \infty$,
Let $\max_\theta \left|f(re^{i\theta})\right| = M$

$$\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} max_\theta \left|f(re^{i\theta})\right|^p d\theta \leq \frac{M^p}{2\pi}(2\pi - 0) = M^p < \infty$$

$$\|f\| = \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta\right\}^{\frac{1}{p}} \leq (M^p)^{\frac{1}{p}} = M < \infty$$

In these cases we have $\|f\| < \infty$ so, $f \in H^p$.
So, $H^q \subset H^p$ for $1 \leq p \leq q \leq \infty$ □

3.2 Theorems in Hardy Spaces

In this section, We will state without proof some fundamental inequalities on H^p spaces.

Theorem 3.2.1. *The Holder's Inequality*
Duren (2000) proves that if $f \in H^p$ and $g \in H^q$, where p and q are conjugate indices then , $fg \in H^1$ with

$$\|fg\|_{H^1} \leq \|f\|_{H^p} \|g\|_{H^q}$$

Proof. .
By the Auxiliary Inequality we have two cases:
Case 1: if f or $g = 0$, then this is travail.
Case 2 : if $f \neq 0$ and $g \neq 0$ where $A = \|f\|_p$ and $B = \|g\|_q$,
let $\alpha = \frac{|f|}{A}$ and let $\beta = \frac{|g|}{B}$
Apply the auxiliary inequality,

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

$$\frac{|fg|}{AB} \leq \frac{|f|^p}{pA^p} + \frac{|g|^q}{qB^q}$$

Integrate both sides to get,

$$\frac{1}{AB}\int |fg|\, du \leq \frac{1}{pA^p}\int |f|^p\, du + \frac{1}{qB^q}\int |g|^q\, du$$

Substitute the vales of A and B to get,

$$\frac{1}{\|f\|_p \|g\|_q}\int |fg|\, du \leq \frac{1}{p} + \frac{1}{q} = 1$$

Multiply both sides by $\|f\|_p \|g\|_q$ to get the result, which is,

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

□

Special case, where $p = q = 2$ then we have,

$$\|fg\|_{H^1} \leq \|f\|_{H^2} \|g\|_{H^2}$$

Now, we have two questions concerning the relation between (c_n) and f where $f = \sum c_n z^n$
If $f = \sum c_n z^n$ belongs to H^p space, then what can be said about the sequence (c_n)?
If we know the sequence (c_n), to which space of the Hardy does f belong to ?

Duren (2000) summarizes that the Hausdorff-Young theorem states that if a function in L^p for $1 \leq p \leq 2$, then its sequence of Fourier coefficients is in l^q where $(1/p + 1/q = 1)$ and the norm of the sequence is less than or equal to the norm of the function. Conversely, every l^p sequence of complex numbers $(1 \leq p \leq 2)$ is the sequence of Fourier coefficients of some function in L^q where $(1/p + 1/q = 1)$ and the norm of the function is less than or equal to the norm of the sequence.

This result may be also be expressed in H^p spaces as follows.

Theorem 3.2.2. *The Hausdorff-Young Theorem.*
Duren (2000) writes that Hausdorff-Young arrives that if $f(z) = \sum_{n=0}^{\infty} c_n z^n \in H^p, \quad (1 \leq p \leq 2)$ *then,*

$$(c_n) \in l^q \qquad \frac{1}{p} + \frac{1}{q} = 1$$

with $\|c_n\|_q \leq \|f\|_p$
Conversely, if c_n *in any* l^p *sequence of complex numbers* $(1 \leq p \leq 2)$ *, then* $f(z) = \sum c_n z^n \in H^q$ *with* $(1/p + 1/q = 1)$ *and* $\|f\|_q \leq \|c_n\|_p$

The following theorem provides further information about the coefficients of H^p functions.

Theorem 3.2.3. *The Hardy Littlewood Theorem. (Duren, 2000)*
If $f(z) = \sum_{n=0}^{\infty} c_n z^n \in H^p, \quad (0 < p \leq 2)$ *then,* $\sum n^{p-2} |c_n|^p < \infty$ *and*

$$\left(\sum_{n=0}^{\infty} (n+1)^{p-2} |c_n|^p \right)^{\frac{1}{p}} \leq C_p \|f\|_p$$

where C_p *depends only on* p.

Theorem 3.2.4. *(Duren, 2000)*
Let (c_n) *be a sequence of complex numbers such that* $\sum n^{q-2} |c_n|^q < \infty$ *for some* q *with* $2 \leq q < \infty$. *Then the function* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *is in* H^q *and*

$$\|f\|_q \leq C_q \left(\sum_{n=0}^{\infty} (n+1)^{q-2} |c_n|^q \right)^{\frac{1}{q}}$$

where C_q *depends only on* q

Remark 3.2.1. .
Let $f = \sum c_n z^n \in H^2$ *then,* $(c_n) \in l^2$ *with* $\|f\|_{H^2} = \|c_n\|_{l^2}$.

Proof. .
Because of $f = \sum c_n z^n \in H^2$. Then, by the Hausdorff-Young inequality we get,
$(c_n) \in l^2$ with $\|c_n\|_{l^2} \leq \|f\|_{H^2}$
Now, since $(c_n) \in l^2$, then by the Hausdorff-Young inequality we get,
$f = \sum c_n z^n$ -and this is given- with $\|f\|_{H^2} \leq \|c_n\|_{l^2}$
this means that $\|f\|_{H^2} = \|c_n\|_{l^2}$ □

3.3 Diagonal Operators on Hardy Spaces

A complex sequence a_n is said to be multiplier of H^p into the sequence space l^q if $(a_n c_n) \in l^q$ whenever $\sum c_n z^n \in H^p$. Similarly, (a_n) is said to be multiplier of H^p into H^q if $\sum c_n z^n \in H^p$ implies $\sum a_n c_n z^n \in H^q$.
One way to give information about the coefficients of H^p functions is to identify the multipliers of H^p.
The space of all multipliers from H^p to H^q is denoted by (H^p, H^q).

A diagonal operator from a Hardy space to a sequence space.

Definition 3.3.1. .
A diagonal operator $d_a : H^p \to l^q$ *is defined by* $d_a(f) = (a_n c_n) \in l^q$, *where* $f = \sum_{n=0}^{\infty} c_n z^n \in H^p$

A diagonal operator from a Hardy space to another Hardy space.

Definition 3.3.2. .
Duren (2000) defines the diagonal operators between Hardy spaces as follows:
A diagonal operator $d_a : H^p \to H^q$ *is defined by* $d_a(f) = \sum_{n=0}^{\infty} a_n c_n z^n \in H^q$, *where*
$f = \sum_{n=0}^{\infty} c_n z^n \in H^p$

Now, there are two examples about the operators on the Hardy spaces. The first example from a Hardy space to a sequence space. The other one between Hardy spaces.

Example 1.
Let the sequence $(a_n) = (1/n)$, define the diagonal operator $d_a : H^2 \to l^1$ by $d_a(f) = (a_n c_n) = (c_n/n)$, now, $(a_n) \in l^2$ since $\|a\| = \left(\sum_{n=1}^{\infty} |1/n|^2\right)^{1/2} \leq \sum_{n=1}^{\infty} 1/n^2 < \infty$
By Holder inequality,

$$\sum_{n=1}^{\infty} |a_n c_n| \leq \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} |c_n|^2\right)^{\frac{1}{2}} = \|a\|_2 \|c\|_2 < \infty$$

so, $(a_n c_n) \in l^1$

Example 2.
Let the sequence $(a_n) = (1/2)^{n/2}$, define the diagonal operator $d_a : H^2 \to H^\infty$ by

$$d_a(f) = \sum a_n c_n z^n = \sum \frac{c_n z^n}{2^{\frac{n}{2}}}$$

Now, $(a_n) \in l^2$ since

$$\|a\|_2 = \left(\sum \left|\frac{1}{2}\right|^{\frac{2n}{2}}\right)^{\frac{1}{2}} = \left(\sum \frac{1}{2^n}\right)^{\frac{1}{2}} = \sqrt{2} < \infty$$

By Holder inequality,

$$\sum |a_n c_n| \leq \left(\sum |a_n|^2\right)^{\frac{1}{2}} \left(\sum |c_n|^2\right)^{\frac{1}{2}} = \|a\|_2 \|c\|_2 < \infty$$

so, $(a_n c_n) \in l^1$, then by Hausdorff-Young inequality $d_a(f) = \sum a_n c_n z^n \in H^\infty$

Note that $(a_n c_n) \in l^1$ so $(a_n c_n) \in l^p$ where $p \geq 1$ and also we can conclude $d_a(f) = \sum a_n c_n z^n \in H^\infty$ and $d_a(f) = \sum a_n c_n z^n \in H^q$ where $1 \leq q \leq \infty$ and p, q will be conjugate indices by the Housdroff-Young inequality.

Chapter 4

Absolutely Summing Operators

In this chapter, the main topic of this study present: p summing operators. This topic may define in some ways. In section one, we will define the most common definition of these operators. After that, we will check the conditions on diagonal operators that make them summing. First, between sequence spaces. Then, between Hardy spaces.

4.1 Definition

Definition 4.1.1. *Absolutely summing operators*
Diestel (1995) states the definition of absolutely summing operators as :
Let $1 \leq p \leq \infty$ and $u : X \to Y$ be a linear operator between Banach spaces. We say that u is p summing operator if there is a constant $c \geq 0$ such that regardless of the natural number m and regardless of the choice of $x_1, ..., x_m$ in X we have,

$$\left(\sum_{i=1}^{m} \|ux_i\|^p \right)^{\frac{1}{p}} \leq c. \sup \left\{ \left(\sum_{i=1}^{m} |\phi(x_i)|^p \right)^{\frac{1}{p}}, \quad \phi \in X^{'}, \|\phi\| \leq 1 \right\} \tag{4.1}$$

where $X^{'}$ is the dual space of X.

The least c for which inequality 4.1 always holds is denoted by $\pi_p(u)$ and called the p summing norm.
The right hand side of the above definition could be replaced by :

$$c. \sup \left\{ \left\| \sum \epsilon_i x_i \right\|_X, \|\epsilon_i\|_{p'} \leq 1 \right\}$$

where $p^{'}$ is the conjugate of p.
Let $\prod_p(X, Y)$ denote the set of all p summing operators from X to Y.

Theorem 4.1.1. .
Every absolutely $p-$ summing operator is bounded operator.

Proof. .
Let u be an absolutely p - summing operator so it satisfies inequality 4.1. By definition,

if $m = 1$ in inequality 4.1 then we get

$$(\|ux\|^p)^{\frac{1}{p}} \le \pi_p(u). \sup\left\{(|\phi(x)|^p)^{\frac{1}{p}}, \quad \phi \in X^*, \|\phi\| \le 1\right\}$$

$$\|ux\| \le \pi_p(u). \sup\{|\phi(x)|, \quad \phi \in X^*, \|\phi\| \le 1\}$$

$$\|ux\| \le \pi_p(u)$$

Take the sup for both sides on $\|x\| \le 1$

$$\sup\{\|ux\|, \|x\| \le 1\} \le \sup\{\pi_p(u), \|x\| \le 1\}$$

$$\|u\| \le \pi_p(u)$$

So we conclude that u is bounded operator.

□

We can say that $\prod_p(X,Y)$ is subspace of $B(X,Y)$ the space of all bounded linear operators from X to Y, and $\pi_p(u)$ defines a norm on $\prod_p(X,Y)$ with $\|u\| \le \pi_p(u)$ for all $u \in \prod_p(X,Y)$.

The converse is in general false, to see this consider the following example:
Let the operator $T : l^2 \to l^2$ defined by $T(x_1, x_2, ...) = (\frac{x_1}{\sqrt{1}}, \frac{x_2}{\sqrt{2}}, ...)$
This operator is bounded but not 2 summing.
Let $x \in l^2$

$$\|T\| = \sup\{\|Tx\|, \|x\| \le 1\}$$

$$\|T\| = \sup\left\{\left(\sum_i \left(\frac{x_i}{\sqrt{i}}\right)^2\right)^{\frac{1}{2}}, \|x\| \le 1\right\}$$

$$\|T\| = \sup\left\{\left(\sum_i \frac{x_i^2}{i}\right)^{\frac{1}{2}}, \|x\| \le 1\right\}$$

$$\|T\| \le \sup\left\{\left(\sum_i x_i^2\right)^{\frac{1}{2}}, \|x\| \le 1\right\}$$

$$\|T\| \le \sup\{\|x\|, \|x\| \le 1\}$$

Then, this operator is bounded.
Now, let (x^j) be sequences in l^2 where $1 \le j \le \infty$, and $T(x^j) = \left(\frac{x_n^j}{\sqrt{n}}\right) \in l^2$ then,

$$\sum_j \|Tx^j\|^2 = \sum_j \left\|\frac{x_n^j}{\sqrt{n}}\right\|^2$$

$$= \sum_j \left(\sum_n \frac{x_n^{j2}}{n}\right)^{\frac{1.2}{2}}$$

$$= \sum_j \sum_n \frac{x_n^{j2}}{n}$$

$$= \sum_n \frac{1}{n} \sum_j x_n^{j2} = \infty$$

That means, this operator is not 2 summing operator.

Theorem 4.1.2. .
$\prod_p(X,Y)$ is normed space with norm π_p
I will show that by definition. Let $(x_i) \in X$
Let $T, U \in \prod_p(X,Y)$ and let $\alpha \in F$ where F is the field of X then we have

$$\left(\sum_{i=1}^{m} \|Tx_i\|^p\right)^{\frac{1}{p}} \leq \pi_p(T).\sup\left\{\left(\sum_{i=1}^{m} |\phi(x_i)|^p\right)^{\frac{1}{p}}, \quad \phi \in X^*, \|\phi\| \leq 1\right\}$$

$$\left(\sum_{i=1}^{m} \|Ux_i\|^p\right)^{\frac{1}{p}} \leq \pi_p(U).\sup\left\{\left(\sum_{i=1}^{m} |\phi(x_i)|^p\right)^{\frac{1}{p}}, \quad \phi \in X^*, \|\phi\| \leq 1\right\}$$

- *$\pi_p(T) \geq 0$ and $\pi_p(T) = 0$ if and only if $T = 0$ this is clear by definition*
- *$\pi_p(\alpha T) = |\alpha| \pi_p(T)$*

$$\left(\sum_{i=1}^{m} \|\alpha Tx_i\|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{m} |\alpha|^p \|Tx_i\|^p\right)^{\frac{1}{p}} = (|\alpha|^p)^{\frac{1}{p}} \left(\sum_{i=1}^{m} \|Ux_i\|^p\right)^{\frac{1}{p}}$$

$$\left(\sum_{i=1}^{m} \|\alpha Tx_i\|^p\right)^{\frac{1}{p}} \leq \alpha\pi_p(T).\sup\left\{\left(\sum_{i=1}^{m} |\phi(x_i)|^p\right)^{\frac{1}{p}}, \quad \phi \in X^*, \|\phi\| \leq 1\right\}$$

$$\pi_p(\alpha T) = |\alpha| \pi_p(T)$$

- *$\pi_p(T+U) \leq \pi_p(T) + \pi_p(U)$*

$$\left(\sum_{i=1}^{m} \|(T+U)(x_i)\|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{m} \|T(x_i)\|^p + \|U(x_i)\|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{m} \|Tx_i\|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{m} \|Ux_i\|^p\right)^{\frac{1}{p}}$$

$$\left(\sum_{i=1}^{m} \|(T+U)(x_i)\|^p\right)^{\frac{1}{p}} \leq (\pi_p(T) + \pi_p(U)).\sup\left\{\left(\sum_{i=1}^{m} |\phi(x_i)|^p\right)^{\frac{1}{p}}, \quad \phi \in X^*, \|\phi\| \leq 1\right\}$$

$$\pi_p(T+U) \leq \pi_p(T) + \pi_p(U)$$

We conclude that $\prod_p(X,Y)$ is normed space with p norm π_p

.

Remark 4.1.1. .
If T is p summing operator and U, V are bounded operators between normed spaces W, X, Y, Z. Then the following composition operator is p summing operator.

$$A : W \xrightarrow{U} X \xrightarrow{T} Y \xrightarrow{V} Z$$

Proof. .
Since $\|(V \circ T \circ U)(x_i)\| \leq \|V\| \, \|T\| \, \|U\|$

$$\left(\sum_{i=1}^{m} \|(V \circ T \circ U)(x_i)\|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{m} (\|V(x_i)\| \, \|T(x_i)\| \, \|U(x_i)\|)^p\right)^{\frac{1}{p}} = \|V\| \, \|U\| \left(\sum_{i=1}^{m} \|T(x_i)\|^p\right)^{\frac{1}{p}}$$

$$\leq c. \|V\| \, \|U\| \sup \left\{ \left(\sum_{i=1}^{m} |\phi(x_i)|^p\right)^{\frac{1}{p}}, \quad \phi \in X^*, \|\phi\| \leq 1 \right\}$$

$$\leq k \sup \left\{ \left(\sum_{i=1}^{m} |\phi(x_i)|^p\right)^{\frac{1}{p}}, \quad \phi \in X^*, \|\phi\| \leq 1 \right\}$$

□

Remark 4.1.2. .
For $1 \leq p \leq q \leq \infty$. If T is p summing operator, then T is q summing operator.

4.2 Absolutely Summing Operators from l^p to l^q spaces.

In this section, we will present a summary of some cases of the operators to be summing operator between sequence spaces.

Now we will state Garling's theorem that determins the conditions on the vector **a**, so that the diagonal operator between the l^p sequence spaces become absoluetly summing operator.

Theorem 4.2.1. *Garling's Theorem. (Garling, 1974)*
Let $d_a : l^p \to l^q$ be the diagonal operator, d_a is absolutely r - summing operator if and only if :

1. *if $1 \leq p < q \leq 2$ with,*
   $a \in l^p$ for $1 \leq r \leq p$. or
   $a \in l^r$ for $p \leq r \leq q$.
2. *if $1 \leq p = q < 2$ with,*
   $a \in l^{p'}$ for $1 \leq r \leq p$, or
   $a \in l^p$ for $p \leq r$.
3. *if $p = q = 2$ with,*
   $a \in l^2$ for all values of r.
4. *if $1 \leq q < p \leq 2$ with,*
   $a \in l^q$ for all values of r.
5. *if $1 \leq q \leq 2 < p \leq \infty$ with,*
   $a \in l^\theta$ for all values of r, where θ depends on p, q.
6. *if $2 < q \leq p < \infty$ with,*
   $a \in l^p$ for $1 \leq r \leq p$.

7. *if $2 < p < q \leq \infty$ with,*
   $a \in l^p$ for $1 \leq r \leq p$. or
   $a \in l^r$ for $p \leq r \leq q$.

8. *if $2 \leq q \leq p = \infty$ with,*
   $a \in l^\infty$ for all values of r.

Here, we will show some fundamental theorems about the diagonal operators $d_a : l^p \to l^q$ which is defined by : $d_a(c_n) = (a_n c_n) \in l^q$ where $(c_n) \in l^p$. There are two cases, if $(a_n) \in l^2$ and $(a_n) \in l^1$.

The following result will be used more than once in the proofs.

" If T is an operator from a normed space X into a Hilbert space H, then $\pi_2^2(T) \leq \sum_j \|T^* e_j\|^2$ where T^* is the adjoint operator and (e_j) is an orthonormal basis for H." (Jameson, 1987)

The following theorem is a spacial case from l^2 to l^2. After that we will discuss more general cases.

Theorem 4.2.2. .
Let $d_a : l^2 \to l^2$ be the diagonal operator defined by $d_a(c_n) = (a_n c_n)$, if $(a_n) \in l^2$ then d_a is 2 - summing with $\pi_2(d_a) \leq \|a\|_2$

Proof. .
By Holder inequality , since $(a_n), (c_n) \in l^2$ then $a.c \in l^1$ with $\|a.c\| \leq \|a\| \|c\|$
since $l^1 \subset l^2$ then $a.c \in l^2$ and $\|d_a(c_n)\| = \|a.c\|$ finite.

And $d_a^* : (l^2)^* \to \left(l^2\right)^*$ is defined by $(d_a^* x)(c_n) = (x_n a_n c_n)$ with $(d_a^* x)(c_n) = x(d_a c_n)$ and $a \in l^2$ then

$$\begin{aligned} \|d_a^* e_n\| &= \sup\{|d_a^* e_n(c_n)|, \|c_n\|_{l^2} \leq 1\} \\ &= \sup\{|e_n(d_a c_n)|, \|c_n\|_{l^2} \leq 1\} \\ &= \sup\{|a_n c_n|, \|c_n\|_{l^2} \leq 1\} \\ &\leq |a_n| \end{aligned}$$

$$\sum_{n=1}^{\infty} \|d_a^* e_n\|^2 \leq \sum_{n=1}^{\infty} |a_n|^2$$

by the quoted result before if $d_a : l^2 \to l^2$ then,

$$\pi_2^2(d_a) \leq \sum_{n=1}^{\infty} \|d_a^* e_n\|^2 \leq \sum_{n=1}^{\infty} |a_n|^2$$

take square root for both sides to get

$$\pi_2(d_a) \leq \left(\sum_{n=1}^{\infty} |a_j|^2\right)^{\frac{1}{2}} = \|a\|_2$$

$$\pi_2(d_a) \leq \|a\|_2$$

□

Theorem 4.2.3. .
Let $d_a : l^2 \to l^1$ be defined by $d_a(c_n) = (a_n c_n) \in l^1$ where $(c_n) \in l^2$. If $(a_n) \in l^2$, the operator is 2-summing with $\pi_2(d_a) \le \|a\|_2$

Proof. .
Since $(a_n), (c_n) \in l^2$ then by Holder inequality, we get $a.c \in l^1$.
$\|d_a\| = \|a.c\| \le \|a\| \|c\|$ which is finite.

And $d_a^* : (l^1)^* \to \left(l^2\right)^*$ is defined by $\left(d_a^* x\right)(c_n) = (x_n a_n c_n)$ with $\left(d_a^* x\right)(c_n) = x(d_a c_n)$ and $a \in l^2$ then

$$\begin{aligned}\|d_a^* e_n\| &= \sup\left\{\left|d_a^* e_n(c_n)\right|, \|c_n\|_{l^2} \le 1\right\}\\ &= \sup\left\{\left|e_n(d_a c_n)\right|, \|c_n\|_{l^2} \le 1\right\}\\ &= \sup\left\{\left|a_n c_n\right|, \|c_n\|_{l^2} \le 1\right\}\\ &\le |a_n|\end{aligned}$$

so we get

$$\sum_{n=1}^{\infty} \|d_a^* e_n\|^2 \le \sum_{n=1}^{\infty} |a_n|^2$$

by the quoted result before if $d_a : l^2 \to l^1$ then,

$$\pi_2^2(d_a) \le \sum_{n=1}^{\infty} \|d_a^* e_n\|^2 \le \sum_{n=1}^{\infty} |a_n|^2$$

take square root for both sides to get

$$\pi_2(d_a) \le \left(\sum_{n=1}^{\infty} |a_j|^2\right)^{\frac{1}{2}} = \|a\|_2$$

$$\pi_2(d_a) \le \|a\|_2$$

□

Corollary 4.2.1. .
For $1 \le q \le \infty$ Let $d_a : l^2 \to l^q$ defined by $d_a(c_n) = (a_n c_n) \in l^q$ where $(c_n) \in l^2$. If $(a_n) \in l^2$, then the operator d_a is 2- summing with $\pi_2(d_a) \le \|a\|_2$

Proof. .
Consider the composition :

$$d_a : l^2 \xrightarrow{c_a} l^1 \xrightarrow{i} l^q$$

where $c_a(c_n) = a_n c_n$ which is defined in the previous theorem with $\pi_2(c_a) \le \|a\|_2$

and i is the inclusion map defined by $i(x) = x$ as an element in l^q since $l^1 \subset l^q$ with $\|i\| = 1$

$$\pi_2(d_a) = \|i \circ c_a\| \le \pi_2(c_a)(1) \le \|a\|_2$$

$$\pi_2(d_a) \le \|a\|_2$$

□

Theorem 4.2.4. .
Let $d_a : l^\infty \to l^1$ *be defined by* $d_a(c_n) = (a_n c_n) \in l^1$ *where* $(c_n) \in l^\infty$. *If* $(a_n) \in l^1$, *the operator is 2-summing with* $\pi_2(d_a) \leq \|a\|_2$

Proof. .
Since $(a_1) \in l^1$ and $(c_n) \in l^\infty$ then by Holder inequality $a.c \in l^1$ with $\|a.c\| \leq \|a\| \, \|c\|$.

And $d_a^* : (l^1)^* \to (l^\infty)^*$ is defined by $(d_a^* x)(c_n) = x_n a_n c_n$ with $(d_a^* x)(c_n) = x(d_a c_n)$ and $a \in l^1$ then

$$\begin{aligned} \|d_a^* e_n\| &= \sup \{|d_a^* e_n(c_n)|, \|c_n\|_{l^\infty} \leq 1\} \\ &= \sup \{|e_n(d_a c_n)|, \|c_n\|_{l^\infty} \leq 1\} \\ &= \sup \{|a_n c_n|, \|c_n\|_{l^\infty} \leq 1\} \\ &\leq |a_n| \end{aligned}$$

so we get

$$\sum_{n=1}^{\infty} \|d_a^* e_n\|^2 \leq \sum_{n=1}^{\infty} |a_n|^2$$

by the quoted text before if $d_a : l^\infty \to l^1$ then,

$$\pi_2^2(d_a) \leq \sum_{n=1}^{\infty} \|d_a^* e_n\|^2 \leq \sum_{n=1}^{\infty} |a_n|^2$$

take square root for both sides to get

$$\pi_2(d_a) \leq \left(\sum_{n=1}^{\infty} |a_j|^2 \right)^{\frac{1}{2}} = \|a\|_2 \leq \|a\|_1 < \infty$$

$$\pi_2(d_a) \leq \|a\|_2$$

□

Corollary 4.2.2. .
For $1 \leq p, q \leq \infty$, *let* $d_a : l^p \to l^q$ *defined by* $d_a(c_n) = (a_n c_n) \in l^q$ *where* $(c_n) \in l^p$. *If* $(a_n) \in l^1$, *then the operator is 2- summing with* $\pi_2(d_a) \leq \|a\|_2$

Proof. .
Consider the composition :

$$d_a : l^p \xrightarrow{i} l^\infty \xrightarrow{c_a} l^1 \xrightarrow{i} l^q$$

Where i is the inclusion map defined by $i(x) = x$ as an element in l^∞ since $l^p \subset l^\infty$ where $\|i\| = 1$
c_a is the diagonal operator defined in the previous theorem with $\pi_2(c_a) \leq \|a\|_2$
and j is the inclusion map defined by $j(x) = x$ as an element in l^q since $l^1 \subset l^q$ with $\|j\| = 1$

$$\pi_2(d_a) = \|i \circ c_a\| \leq \pi_2(c_a)(1) \leq \|a\|_2$$

$$\pi_2(d_a) \leq \|a\|_2$$

□

4.3 Absolutely Summing Operators from H^p to l^q spaces.

Acording to Almasri (1999) study which was about the absoluetly summing operators from a Hardy space to a sequence space, the summary is,

The diagonal operator $d_a : H^p \to l^q$ is absolutely $2-$summing operator.
$d_a \in \prod_2(H^p, l^q)$

- Part one: For $1 \le p, q \le 2$. $a \in l^q$
  . $k \; \|a\|_q \le \pi_2(d_a) \le \|a\|_q$, k is constant.

- Part two: For $1 \le p \le 2, 2 \le q \le \infty, a \in l^2$
  . $\pi_2(d_a) = \|a\|_2$.

- Part three: For $2 \le p \le \infty, 1 \le q \le 2, a \in l^{\phi(p,q)}$
  . $k_1 \|a\|_{\phi(p,q)} \le \pi_2(d_a) \le k_2 \|a\|_q$, $k_1, k_2 > 0$.

- Part four: For $2 \le p, q \le \infty$
  . $\max\left\{\|a\|_p, \; k \left\|n^{1/p-1/2} a_n\right\|_{p'}\right\} \le \pi_2(d_a) \le \|a\|_2$.

4.4 Absolutely Summing Operators from H^p to H^q spaces.

In this section we will talk about conditions on the vector **a**, so that the diagonal operator from a Hardy space to another one is bounded and also the conditions to make the operator absolutely summing operator.

But first we will talk about the adjoint operator since it will help us in some theorems.

The Adjoint Operator

In mathematics, specifically in functional analysis, each bounded linear operator on a complex Hilbert space has a corresponding adjoint operator. If one thinks of operators on a complex Hilbert space as "generalized complex numbers", then the adjoint of an operator plays the role of the complex conjugate of a complex number.

In a similar sense there can be defined an adjoint operator for linear operators between Banach spaces.

The adjoint of an operator A may also be called the Hermitian adjoint, Hermitian conjugate or Hermitian transpose of A and is denoted by A^*.

Consider a linear operator $A : H_1 \to H_2$, where H_1, H_2 are two Hilbert spaces. The adjoint operaor is the unique linear operator $A^* : H_2 \to H_1$ that satisfies :

$$\langle Ah_1, h_2 \rangle_{H_2} = \langle h_1, A^* h_2 \rangle_{H_1}$$

where $h_i \in H_i$ and $\langle ., . \rangle_{H_i}$ is the inner product in the Hilbert space H_i.

We can define the adjoint of an operator $A : E \to F$, where E, F are Banach spaces. The adjoint operator is defined as $A^* : F^* \to E^*$ with

$$(A^* f)(u) = f(Au), \qquad f \in F^*, u \in E$$

We defined before a multiplier $d_a : H^p \to H^q$ by $d_a(f) = \sum a_n c_n z^n$ where $f = \sum c_n z^n \in H^p$. $(a_n) \in (H^p, H^q)$ if $\sum a_n c_n z^n \in H^q$ for all $\sum c_n z^n \in H^p$

The diagonal operator from H^p to H^q

We will take three parts in this section. In part one, we will show that for $1 \le p, q \le \infty$ the diagonal operator $d_a : H^p \to H^q$ is absolutely 2- summing operator if and only if $a \in l^2$ with $\pi_2(d_a) = \|a\|_2$. In part two, for $1 \le p, q \le \infty$, the diagonal operator $d_a : H^p \to H^q$ is absolutely 2-summing operator if $a \in l^1$ with $\pi_2(d_a) \le \|a\|_2$. In part three, we will prove that for $1 \le p, q \le \infty$ the diagonal operator $d_a : H^p \to H^q$ is absolutely 1- summing operator if and only is $a \in l^1$ with $\pi_1(d_a) = \|a\|_1$. First of all, we have one main theorem that we will use in others.

Theorem 4.4.1. *.*
For $1 \le p \le 2$. If $a \in l^2$ then the diagonal operator $d_a : H^p \to H^2$ is absolutely 2- summing with,

$$\pi_2(d_a) \le \|a\|_2$$

Proof. .
The following result is helpful in proving this theorem:
" If T is an operator from a normed space X into a Hilbert space H, then $\pi_2^2(T) \le \sum_j \|T^* e_j\|^2$ where T^* is the adjoint operator and e_j is an orthonormal basis for H." (Jameson, 1987)

Let $d_a^* : (H^q)^* \to (H^p)^*$ be defined by $(d_a^* x)(f) = x_n a_n c_n$ with $(d_a^* x)(f) = x(d_a f)$ where $x \in H^{q^*}$ and $a \in l^2$ then

$$\begin{aligned}\|d_a^* e_n\| &= \sup\{|d_a^* e_n(f)|, \|f\|_{H^p} \le 1\} \\ &= \sup\{|e_n(d_a f)|, \|f\|_{H^p} \le 1\} \\ &= \sup\{|a_n c_n|, \|f\|_{H^p} \le 1\}\end{aligned}$$

Since $f = \sum c_n z^n \in H^p$ then by Hausdorff-Young, $(c_n) \in l^{p'}$ with $\|c\| \le \|f\|$. So,

$$\|d_a^* e_n\| \le |a_n|$$

so we get

$$\sum_{n=1}^{\infty} \|d_a^* e_n\|^2 \le \sum_{n=1}^{\infty} |a_n|^2$$

by the quoted result if $d_a : H^p \to H^2$ then,

$$\pi_2^2(d_a) \le \sum_{n=1}^{\infty} \|d_a^* e_n\|^2 \le \sum_{n=1}^{\infty} |a_n|^2$$

take square root of both sides to get

$$\pi_2(d_a) \le \left(\sum_{n=1}^{\infty} |a_j|^2\right)^{\frac{1}{2}} = \|a\|_2$$

$$\pi_2(d_a) \le \|a\|_2$$

□

Part one: In this part, we will show that for $1 \leq p, q \leq \infty$ the diagonal operator $d_a : H^p \to H^q$ is absolutely 2- summing operator iff $a \in l^2$ with $\pi_2(d_a) = \|a\|_2$.

Corollary 4.4.1. .
For $1 \leq q \leq 2$, if $(a_n) \in l^2$, then, the diagonal operator $d_a : H^2 \to H^q$ is 2-summing with $\pi_2(d_a) \leq \|a\|_2$.

Proof. .
Consider the composition

$$d_a : H^2 \xrightarrow{c_a} H^2 \xrightarrow{i} H^q$$

where i is the inclusion map $i : H^2 \to H^q$ is defined by $i(f) = f$ but as an element in H^q since $H^2 \subset H^q$ with $\|i\| = 1$,

and c_a is the multiplier $c_a : H^2 \to H^2$ which is defined by $c_a(f) = \sum a_n c_n z^n \in H^2$ where $\sum c_n z^n \in H^2$ with $\pi_2(c_a) \leq \|a\|_2$ by theorem 4.4.1. Now ,

$$\pi_2(d_a) = \|i \; o \; c_a\| \leq \|i\| \, \|c_a\| \leq \|a\|_2 \, .$$

Which proves the result. □

Theorem 4.4.2. .
Let $d_a : H^2 \to H^\infty$ be a diagonal operator defined by $d_a(f) = \sum a_n c_n z^n$, where $(a_n) \in l^2$ and $f = \sum c_n z^n \in H^2$. Then d_a is absolutely 2- summing operator with $\pi_2(d_a) \leq \|a\|_2$.

Proof. .
Consider the composition:

$$d_a : H^2 \xrightarrow{j} l^2 \xrightarrow{c_a} l^1 \xrightarrow{i} H^\infty$$

where j is the natural isometry map that is defined by $j(f) = (c_n) \in l^2$ where $f = \sum c_n z^n \in H^2$
$c_a : l^2 \to l^1$ is defined by $c_a(c_n) = (a_n c_n) \in l^1$ by Holder inequality
We show before that $\pi_2(c_a) \leq \|a_n\|_{l^2}$
and i is the natural isometry map defined by $i(a_n c_n) = \sum a_n c_n z^n \in H^\infty$

$$\pi_2(d_a) = \|i \circ c_a \circ j\| \leq \|i\| \, \|c_a\| \, \|j\| \leq \|a_n\|_2$$

$$\pi_2(d_a) \leq \|a\|_2$$

□

Corollary 4.4.2. .
For $2 \leq q \leq \infty$, let $(a_n) \in l^2$. Then the diagonal operator $d_a : H^2 \to H^q$ which is defined before is 2-summing operator with $\pi_2(d_a) \leq \|a\|_2$.

Proof. .
Consider the composition

$$d_a : H^2 \xrightarrow{c_a} H^\infty \xrightarrow{i} H^q$$

where i is the inclusion map, $i : H^\infty \to H^q$ is defined by $i(f) = f$ but as an element in H^q since $H^\infty \subset H^q$ with $\|i\| = 1$, and c_a is the multiplier $c_a : H^2 \to H^\infty$ is defined by $c_a(f) = \sum a_n c_n z^n \in H^\infty$ where $\sum c_n z^n \in H^2$ with $\pi_2(c_a) \leq \|a\|_2$ by theorem 4.4.2. Now ,

$$\pi_2(d_a) = \|i \ o \ c_a\| \leq \|i\| \, \|c_a\| \leq \|a\|_2 \, .$$

Which proves the result. □

Theorem 4.4.3. .
For $2 \leq p \leq \infty$ and $1 \leq q \leq \infty$ the diagonal operator $d_a : H^p \to H^q$ is absolutely 2- summing operator if $a \in l^2$ with $\pi_2(d_a) \leq \|a\|_2$.

Proof. .
Let $(a_n) \in l^2$, the operator $d_a : H^2 \to H^q$ defined before is 2-summing operator,
by corollary 4.4.1 for $1 \leq q \leq 2$ and by corollary 4.4.2 for $2 \leq q \leq \infty$
We can say that the operator $d_a : H^2 \to H^q$ is absolutely 2 summing for $1 \leq q \leq \infty$ with $\pi_2(d_a) \leq \|a\|_2$.
Now define the composite operator :

$$c_a : H^p \xrightarrow{i} H^2 \xrightarrow{d_a} H^q$$

where i is the inclusion map $i : H^p \to H^2$ is defined by $i(f) = f$ but as an element in H^2 since $H^p \subset H^2$ with $\|i\| = 1$.
Now ,

$$\pi_2(c_a) = \|d_a \ o \ i\| \leq \|i\| \, \|d_a\| \leq \|a\|_2 \, .$$

Which proves the result. □

Theorem 4.4.4. .
For $1 \leq p \leq 2$ and $1 \leq q \leq \infty$ if $a \in l^2$ then the diagonal operator $d_a : H^p \to H^q$ is absolutely summing with $\pi_2(d_a) \leq \|a\|_2$

Proof. .
Consider the composition :

$$d_a : H^p \xrightarrow{i} H^1 \xrightarrow{c_a} H^\infty \xrightarrow{j} H^q$$

where i is the inclusion map defined by $i(f) = f$ as an element in H^1 since $H^p \subset H^1$
c_a is the multiplier which is defined in the following theorem 4.4.8 with $\pi_2(c_a) \leq \|a\|_2$
and j is the inclusion map defined by $j(f) = f$ as an element in H^q since $H^\infty \subset H^q$

$$\pi_2(d_a) \leq (1)(\pi_2(c_a))(1) \leq \|a\|_2$$

Which proves the result. □

Corollary 4.4.3. .
For $1 \leq p, q \leq \infty$ the diagonal operator $d_a : H^p \to H^q$ is absolutely 2- summing operator if $a \in l^2$ with $\pi_2(d_a) \leq \|a\|_2$.

Proof. .
This follows from theorem 4.4.3 and theorem 4.4.4. □

Theorem 4.4.5. .
For $1 \leq p \leq 2$ and $1 \leq q \leq \infty$. If the diagonal operator $d_a : H^p \to H^q$ is $2-$summing operator, then, $a \in l^2$ with $\|a\|_2 \leq \pi_2(d_a)$.

Proof. .
For any $f_1, f_2, ..., f_N \in H^p$, since $d_a \in \prod_2(H^p, H^q)$. Then by definition ,

$$\left(\sum_{n=1}^{N} \|d_a f_n\|_q^2\right)^{\frac{1}{2}} \leq \pi_2(d_a) \sup\left\{\left\|\sum_{n=1}^{N} \epsilon_n f_n\right\|_{H^p}, \quad \|\epsilon\|_2 \leq 1\right\}$$

Since $H^2 \subset H^p$, then, $\|g\|_{H^p} \leq \|g\|_{H^2}$, so

$$\left(\sum_{n=1}^{N} \|d_a f_n\|_q^2\right)^{\frac{1}{2}} \leq \pi_2(d_a) \sup\left\{\left\|\sum_{n=1}^{N} \epsilon_n f_n\right\|_{H^2}, \quad \|\epsilon\|_2 \leq 1\right\} \tag{4.2}$$

Let $f_n(t) = e^{int}$, then the Fourier coefficients of these functions are

$$c_n(m) = \begin{cases} 1 & \text{if } n = m \\ 0 & otherwise \end{cases}$$

$$\left(\sum_{n=1}^{N} \left\|\sum_m a_m c_m e^{imt}\right\|_q^2\right)^{\frac{1}{2}} \leq \pi_2(d_a) \sup\left\{\left\|\sum_{n=1}^{N} \epsilon_n \sum_m c_m e^{imt}\right\|_{H^2}, \quad \|\epsilon\|_2 \leq 1\right\} \tag{4.3}$$

$$\left(\sum_{n=1}^{N} |a_n|^2\right)^{\frac{1}{2}} \leq \pi_2(d_a) \sup\left\{\left(\sum_{n=1}^{N} |\epsilon_n|^2\right)^{\frac{1}{2}}, \quad \|\epsilon\|_2 \leq 1\right\}$$

Let $N \to \infty$ then,

$$\left(\sum_{n=1}^{\infty} |a_n|^2\right)^{\frac{1}{2}} \leq \pi_2(d_a) \sup\left\{\left(\sum_{n=1}^{\infty} |\epsilon_n|^2\right)^{\frac{1}{2}}, \quad \|\epsilon\|_2 \leq 1\right\}$$

$$\|a\|_2 \leq \pi_2(d_a)$$

□

Theorem 4.4.6. .
For $2 \leq p \leq \infty$ and $1 \leq q \leq \infty$. If the diagonal operator $d_a : H^p \to l^q$ is $2-$summing operator, then, $(a_n) \in l^2$ with $\|a\|_2 \leq \pi_2(d_a)$.

Proof. .
For any $f_1, ..., f_N \in H^p$ since $d_a \in \prod_2(H^p, l^q)$, then by definition

$$\left(\sum_{n=1}^{N} \|d_a f_n\|_q^2\right)^{\frac{1}{2}} \leq \pi_2(d_a) \sup\left\{\left\|\sum_{n=1}^{N} \epsilon_n f_n\right\|_{H^p}, \quad \|\epsilon\|_2 \leq 1\right\} \tag{4.4}$$

Let $f_n(t) = (\lambda_n e^{int})$ where $(\lambda_n) \in \mathbb{C}$. Then Fourier coefficients of these functions are

$$c_n(m) = \begin{cases} \lambda_n & \text{if } n = m \\ 0 & otherwise \end{cases}$$

By Hausdorff-Young theorem,

$$\left(\sum_{n=1}^{N}|a_n\lambda_n|^2\right)^{\frac{1}{2}} \le \pi_2(d_a)\sup\left\{\left\|\sum_{n=1}^{N}\epsilon_n\lambda_n e^{int}\right\|_{H^p}, \quad \|\epsilon\|_2 \le 1\right\}$$

$$\le \pi_2(d_a)\sup\left\{\left(\sum_{n=1}^{N}|\epsilon_n\lambda_n|^{p'}\right)^{\frac{1}{p'}}, \quad \|\epsilon\|_2 \le 1\right\}$$

By Holder's inequality

$$\left(\sum_{n=1}^{N}|a_n\lambda_n|^2\right)^{\frac{1}{2}} \le \pi_2(d_a)\sup\left(\sum_{n=1}^{N}|\lambda_n|^{2p/p-2}\right)^{\frac{p-2}{2p}}$$

Since λ_n are any complex number, let $|\lambda_n|^2 = |a_n|^{p-2}$

$$\left(\sum_{n=1}^{N}|a_n|^2|a_n|^{p-2}\right)^{\frac{1}{2}} \le \pi_2(d_a)\sup\left(\sum_{n=1}^{N}|a_n|^p\right)^{\frac{p-2}{2p}}$$

$$\left(\sum_{n=1}^{N}|a_n|^p\right)^{\frac{1}{2}} \le \pi_2(d_a)\sup\left(\sum_{n=1}^{N}|a_n|^p\right)^{\frac{p-2}{2p}}$$

$$\textit{divide by} \quad \left(\sum_{n=1}^{N}|a_n|^p\right)^{\frac{p-2}{2p}}$$

$$\left(\sum_{n=1}^{N}|a_n|^p\right)^{\frac{1}{2}-\frac{p-2}{2p}} \le \pi_2(d_a)$$

Let $N \to \infty$ then,

$$\left(\sum_{n=1}^{\infty}|a_n|^p\right)^{\frac{1}{p}} \le \pi_2(d_a)$$

$$\|a\|_p \le \pi_2(d_a)$$

since $l^2 \subset l^p$ for $p \ge 2$ then $\|a\|_2 \le \|a\|_p$, we conclude that

$$\|a\|_2 \le \pi_2(d_a)$$

□

Corollary 4.4.4. .
For $1 \le p, q \le \infty$, if the diagonal operator $d_a : H^p \to H^q$ is $2-$summing operator then $a \in l^2$ and $\|a\|_2 \le \pi_2(d_a)$

Proof. .
This follows from theorem 4.4.5 and theorem 4.4.6. □

Theorem 4.4.7. .
For $1 \leq p, q \leq \infty$ the diagonal operator $d_a : H^p \to H^q$ is absolutely 2- summing operator iff $a \in l^2$ with $\pi_2(d_a) = \|a\|_2$.

Proof. .

By corollary 4.4.3 we have,
For $1 \leq p, q \leq \infty$ the diagonal operator $d_a : H^p \to H^q$ is absolutely 2- summing operator if $a \in l^2$ with $\pi_2(d_a) \leq \|a\|_2$.
By corollary 4.4.4 we have,
For $1 \leq p, q \leq \infty$. If the diagonal operator $d_a : H^p \to H^q$ is $2-$summing operator. Then, $a \in l^2$ with $\|a\|_2 \leq \pi_2(d_a)$.

The result is found and $\pi_2(d_a) = \|a\|_2$. □

Part two: In this part, we will prove that for $1 \leq p, q \leq \infty$, the diagonal operator $d_a : H^p \to H^q$ is absolutely 2- summing operator if $a \in l^1$ with $\pi_2(d_a) \leq \|a\|_2$.

Corollary 4.4.5. .
For $1 \leq q \leq 2$ let $(a_n) \in l^1$. Then the diagonal operator $d_a : H^1 \to H^q$ which is defined before is 2 - summing operator with $\pi_2(d_a) \leq \|a\|_2$.

Proof. .
Consider the composition

$$d_a : H^1 \xrightarrow{c_a} H^2 \xrightarrow{i} H^q$$

where i is the inclusion map $i : H^2 \to H^q$ is defined by $i(f) = f$ but as an element in H^q since $H^2 \subset H^q$ with $\|i\| = 1$.

and c_a is the multiplier $c_a : H^1 \to H^2$ is defined by $c_a(f) = \sum a_n c_n z^n \in H^2$ where $\sum c_n z^n \in H^1$ with $\pi_2(c_a) \leq \|a\|_2$ from theorem 4.4.1. Now ,

$$\pi_2(d_a) = \|i \; o \; c_a\| \leq \|i\| \, \|c_a\| \leq \|a\|_2 \, .$$

Which proves the result. □

Theorem 4.4.8. .
Let $d_a : H^1 \to H^\infty$ be a diagonal operator defined by $d_a(f) = \sum a_n c_n z^n$. where $(a_n) \in l^1$, $f = \sum c_n z^n \in H^1$. Then d_a is absolutely 2- summing operator with $\pi_2(d_a) \leq \|a\|_2$.

Proof. .
Consider the composition :

$$d_a : H^1 \xrightarrow{j} l^\infty \xrightarrow{c_a} l^1 \xrightarrow{i} H^\infty$$

where j is the natural isometry map that defined by $j(f) = (c_n) \in l^\infty$ with $f = \sum c_n z^n \in H^1$
$c_a : l^\infty \to l^1$ defined by $c_a(c_n) = (a_n c_n) \in l^1$ by Holder inequality
we showed before that $\pi_2(c_a) \leq \|a_n\|_{l^2}$
and i is the natural isometry map defined by $i(a_n c_n) = \sum a_n c_n z^n \in H^\infty$

$$\pi_2(d_a) = \|i \circ c_a \circ j\| \leq \|i\| \, \|c_a\| \, \|j\| \leq \|a_n\|_2$$

We can conclude the result with

$$\pi_2(d_a) \le \|a\|_2$$

□

Corollary 4.4.6. .
For $2 \le q \le \infty$ let $(a_n) \in l^1$. Then the diagonal operator $d_a : H^1 \to H^q$ which is defined before is 2 - summing operator with $\pi_2(d_a) \le \|a\|_2$.

Proof. .
Consider the composition

$$d_a : H^1 \xrightarrow{c_a} H^\infty \xrightarrow{i} H^q$$

where i is the inclusion map $i : H^\infty \to H^q$ is defined by $i(f) = f$ but as an element in H^q since $H^\infty \subset H^q$ where $2 \le q \le \infty$ with $\|i\| = 1$.

and c_a is the multiplier $c_a : H^1 \to H^\infty$ is defined by $c_a(f) = \sum a_n c_n z^n \in H^\infty$ where $\sum c_n z^n \in H^1$ with $\pi_2(c_a) \le \|a\|_2$ by theorem 4.4.8. Now ,

$$\pi_2(d_a) = \|i \ o \ c_a\| \le \|i\| \, \|c_a\| \le \|a\|_2 \, .$$

Which proves the result. □

Theorem 4.4.9. .
For $1 \le p, q \le \infty$, the diagonal operator $d_a : H^p \to H^q$ is absolutely 2- summing operator if $a \in l^1$ with $\pi_2(d_a) \le \|a\|_2$.

Proof. .
Let $(a_n) \in l^1$. The diagonal operator $d_a : H^1 \to H^q$ is 2 - summing operator with $\pi_2(d_a) \le \|a\|_2$, by corollary 4.4.5 for $1 \le q \le 2$ and by corollary 4.4.6 for $2 \le q \le \infty$ .We can say that the operator $d_a : H^1 \to H^q$ if absolutely 2 summing for $1 \le q \le \infty$ with $\pi_2(d_a) \le \|a\|_2$.
Now define the composite operator :

$$c_a : H^p \xrightarrow{i} H^1 \xrightarrow{d_a} H^q$$

where i is the inclusion map $i : H^p \to H^1$ is defined by $i(f) = f$ but as an element in H^1 since $H^p \subset H^1$ with $\|i\| = 1$.
Now ,

$$\pi_2(c_a) = \|d_a \ o \ i\| \le \|i\| \, \|d_a\| \le \|a\|_2 \, .$$

Which proves the result. □

Part three: For $1 \le p, q \le \infty$, the diagonal operator $d_a : H^p \to H^q$ which is defined by $d_a(f) = \sum a_n c_n z^n$ is absolutely 1- summing operator if and only if $(a_n) \in l^1$, with $\pi_1(d_a) = \|a\|_1$.

Theorem 4.4.10. .
Let $d_a : H^1 \to H^\infty$ be a diagonal operator defined by $d_a(f) = \sum a_n c_n z^n$, if $(a_n) \in l^1$ and $f = \sum c_n z^n \in H^1$. Then d_a is absolutely 1- summing operator with $\pi_1(d_a) \le \|a\|_1$.

Proof. .
Since $f = \sum c_n z^n \in H^1$ then by Hausdorff-Young inequality; $(c_n) \in l^\infty$ with $\|c_n\|_{l^\infty} \leq \|f\|_{H^1}$
Because of $(a_n) \in l^1$ and $(c_n) \in l^\infty$ then by Holder inequality,
$a.c \in l^1$ with $\|ac\|_1 \leq \|a\|_1 \|c\|_\infty$
Because of $ac \in l^1$ then by Hausdorff-Young inequality; $\sum a_n c_n z^n \in H^\infty$ with

$$\left\|\sum a_n c_n z^n\right\|_{H^\infty} \leq \|a_n c_n\|_{l^1} = \sum |a_n c_n| \tag{4.5}$$

Now, define $\phi_n(f) = (a_n c_n)$. Then ,$|\phi_n(f)| = |a_n c_n|$ .By equation 4.5 we have,

$$\|d_a(f)\| = \left\|\sum a_n c_n z^n\right\|_{H^\infty} \leq \sum |a_n c_n| = \sum |\phi_n(f)|$$

$$\|d_a(f)\| \leq \sum |\phi_n(f)|$$

now, by definition of norm of ϕ we have,

$$\|\phi_n\| = \sup \{|\phi_n(f)|, \ \ \|f\|_{H^1} \leq 1\}$$

$$\|\phi_n\| = \sup \{| \ a_n c_n|, \ \ \|f\|_{H^1} \leq 1\}$$

By Hausdorff Young inequality, $\|c_n\| \leq \|f\| \leq 1$. Then ,

$$\|\phi_n\| \leq | \ a_n|$$

"let $p \geq 1$, suppose that there are functionals $f_1, f_2, ...$ such that $\|Tx\|^p \leq \sum_i |f_i(x)|^p, \forall x$. Then,

$$\pi_p(T) \leq \left(\sum_i \|f_i\|^p\right)^{\frac{1}{p}} \quad "$$

Let $T(x) = d_a(f)$ and $f_n = \phi_n$ then we have,

$$\|d_a\|_{H^\infty} \leq \sum |a_n c_n|$$

$$\|d_a\| \leq \sum |\phi_n(f)| \quad \forall f \in H^p$$

$$\pi_1(d_a) \leq \left(\sum \|\phi_n\|\right) \leq \sum |a_n| = \|a\|_1$$

This means that,

$$\pi_1(d_a) \leq \|a\|_1$$

we conclude that d_a is absolutely 1 summing with $\pi_1(d_a) \leq \|a\|_1$ □

Corollary 4.4.7. .
For $1 \leq p, q \leq \infty$, let $d_a : H^p \to H^q$ be a diagonal operator defined by $d_a(f) = \sum a_n c_n z^n \in H^q$, where $f = \sum c_n z^n \in H^p$. If $(a_n) \in l^1$ then, d_a is absolutely 1- summing operator with $\pi_1(d_a) \leq \|a\|_1$.

Proof. .
Consider the composition

$$d_a : H^p \xrightarrow{i} H^1 \xrightarrow{c_a} H^\infty \xrightarrow{j} H^q$$

where i and j are inclusion maps,
and c_a is the multiplier defined on theorem 4.4.10 with $\pi_1(c_a) \leq \|a\|_1$

$$\pi_2(d_a) = \|j \circ c_a \circ i\| \leq \|i\| \|c_a\| \|j\|$$

Which proves the result. □

Theorem 4.4.11. .
For $1 \le p, q \le \infty$. If the diagonal operator $d_a : H^p \to H^q$ is $1-$summing operator. Then, $(a_n) \in l^1$ with $\|a\|_1 \le \pi_1(d_a)$.

Proof. .
For any $f_1, f_2, ..., f_N \in H^p$, since $d_a \in \prod_1(H^p, H^q)$. Then by definition,

$$\sum_{n=1}^{N} \|d_a f_n\|_q \le \pi_1(d_a) \sup \left\{ \left\| \sum_{n=1}^{N} \epsilon_n f_n \right\|_{H^p} , \quad \|\epsilon\|_{l^\infty} \le 1 \right\}$$

Since $H^\infty \subset H^p$ then, $\|f\|_{H^p} \le \|f\|_{H^\infty}$ and,

$$\sum_{n=1}^{N} \|d_a f_n\|_q \le \pi_1(d_a) \sup \left\{ \left\| \sum_{n=1}^{N} \epsilon_n f_n \right\|_{H^\infty} , \quad \|\epsilon\|_{l^\infty} \le 1 \right\} \tag{4.6}$$

Let $f_n(t) = (e^{int})$, then the Fourier coefficients of these functions are

$$c_n(m) = \begin{cases} 1 & \text{if } n = m \\ 0 & otherwise \end{cases}$$

$$\sum_{n=1}^{N} \left\| \sum_{m} a_m c_m e^{imt} \right\|_q \le \pi_1(d_a) \sup \left\{ \left\| \sum_{n=1}^{N} \epsilon_n \sum_{m} c_m e^{imt} \right\|_{H^\infty} , \quad \|\epsilon\|_{l^\infty} \le 1 \right\} \tag{4.7}$$

$$\sum_{n=1}^{N} |a_n| \le \pi_1(d_a) \sup \left\{ \max_{1 \le n \le N} |\epsilon_n| , \quad \|\epsilon\|_{l^\infty} \le 1 \right\}$$

Let $N \to \infty$ then,

$$\sum_{n=1}^{\infty} |a_n| \le \pi_1(d_a) \sup \left\{ \max_{1 \le n \le \infty} |\epsilon_n| , \quad \|\epsilon\|_{l^\infty} \le 1 \right\}$$

$$\|a\|_1 \le \pi_1(d_a)$$

□

Corollary 4.4.8. .
For $1 \le p, q \le \infty$. The diagonal operator $d_a : H^p \to H^q$ is 1 summing operator if and only if $a \in l^1$ with $\pi_1(d_a) = \|a\|_1$.

Proof. .

By corollary 4.4.7 we have,
The diagonal operator $d_a : H^p \to H^q$ is absolutely 1- summing operator if $(a_n) \in l^1$ with $\pi_1(d_a) \le \|a\|_1$.

By theorem 4.4.11 we have,
If the diagonal operator $d_a : H^p \to H^q$ is absolutely 1- summing operator, then $(a_n) \in l^1$ with $\|a\|_1 \le \pi_1(d_a)$

The result is:

The diagonal operator $d_a : H^p \to H^q$ is 1 summing operator if and only if $a \in l^1$ with $\pi_1(d_a) = \|a\|_1$. □

Theorem 4.4.12. .
For $1 \leq p, q \leq \infty$. The diagonal operator $d_a : H^p \to H^q$ is r summing operators if and only if $(a_n) \in l^1$ for all $r \geq 1$.

Proof. .
By corollary 4.4.8, this operator is 1 summing with $\pi_1(d_a) = \|a\|_1$.
By remark 4.1.2 we get the result. □

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