Palestine Polytechnic University
Deanship of Graduate Studies and Scientific Research
Master Program of Mathematics

Fractional Differentiation

Prepared by

Wisam Fakhouri

M.Sc. Thesis

Hebron - Palestine

2017

Fractional Differentiation

Prepared by

Wisam Fakhouri

Supervisor

Dr. Ahmed Khamayseh

M.Sc. Thesis
Hebron - Palestine

Submitted to the Department of Mathematics at Palestine Polytechnic University as a partial fulfilment of the requirement for the degree of Master of Science.

Fractional Differentiation

Prepared by

Wisam Fakhouri

Supervisor

Dr. Ahmed Khamayseh

Master thesis submitted an accepted, **Date September, 2017.**

The name and signature of the examining committee members

Dr. Ahmed Khamayseh	Head of committee	signature
Dr. Ibrahim Alghrouz	External Examiner	signature
Dr. Ali Zein	Internal Examiner	signature

Palestine Polytechnic University

Declaration

I declare that the master thesis entitled "*Fractional Differentiation* " is my own work, and hereby certify that unless stated, all work contained within this thesis is my own independent research and has not been submitted for the award of any other degree at any institution, except where due acknowledgment is made in the text.

Wisam Fakhouri

Signature: ______________ Date: ______________

Statement of Permission to Use

In presenting this thesis in partial fulfillment of the requirements for the master degree in mathematics at Palestine Polytechnic University, I agree that the library shall make it available to borrowers under rules of library. Brief quotations from this thesis are allowable without special permission, provided that accurate acknowledgment of the source is made.

Permission for the extensive quotation from, reproduction, or publication of this thesis may be granted by my main supervisor, or in his absence, by the Dean of Graduate Studies and Scientific Research when, in the opinion either, the proposed use of the material is scholarly purpose. Any copying or use of the material in this thesis for financial gain shall not be allowed without may written permission.

Wisam Fakhouri

Signature: ______________ Date: ______________

Dedications

This work is dedicated to my parents

Wisam

Acknowledgements

Foremost, I would like to express my sincere gratitude to my supervisor Dr. Ahmed Khamayseh for his continuous support, encouragement and inspiring guidance throughout my thesis .

I also would like to express the deepest appreciation to the committee members Dr. Ibrahim Alghrouz and Dr. Ali Zein for their valuable remarks.

Last but not the least, I would like to thank my family especially my parents for their continuous and endless support throughout all my life.

Abstract

The fractional calculus is a theory of integrals and derivatives of arbitrary (i.e., non-integer) order. And it is considered as a natural extension of classical calculus. Thus there are many preserved basic properties between them. This thesis, consisting of four chapters, explores the concept and definition of fractional calculus.

In this thesis, a brief history and definition of fractional calculus are given. Two definitions of fractional derivative are considered, namely the Riemann-Liouville and the Caputo definitions of the fractional derivative. Some illustrative examples are included. Further we present some basic properties with proofs. Finally, present some fractional differential equations with an emphasis on the Laplace transform of the fractional derivative.

Contents

Chapter 1

Introduction

Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals of non-integer orders. The subject of fractional calculus has gained considerable popularity and importance during the past three decades.

The history of fractional calculus started almost at the same time when classical calculus was established. The theory of derivative of non-integer order goes back to Leibniz in September 1695 [16], where the idea of semiderivative was suggested. Leibniz introduce the symbol

$$\frac{d^n}{dx^n} f(x) \quad n \in \mathbb{N}$$

to denote the n-th derivative of a function f. He reported this in a letter to L'Hopital. L'Hopital replied "what does $\frac{d^n}{dx^n} f(x)$ mean if $n = \frac{1}{2}$".

Following L'Hopital's and Liebniz's first inquisition, fractional calculus was primarily a study reserved for the best minds in mathematics. Fourier, Euler, Liouville, Riemann, Laplace are among the many who have provided important contributions with fractional calculus and the mathematical consequences up to the middle of our century .[2, 3, 12, 14, 15, 16, **?**]

The easiest access to the idea of the non-integer differential and integral operators studied in the field of fractional calculus is given by Cauchy's well known representation of an n-fold integral [19]

$$D^{-n}y(x) = \int_0^x \int_0^{x_{n-1}} \dots \int_0^{x_1} y(x_0)\ dx_0 \dots dx_{n-2}\ dx_{n-1} \tag{1.1}$$

Which is given by single integration

$$D^{-n}y(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} y(t)\ dt, \quad n \in \mathbb{N}, x \in \mathbb{R}_+ \tag{1.2}$$

Where D^{-n} is the n-fold integral operator, to obtain a definition of a non-integer order integral we replace $(n-1)!$ with Euler's gamma function $\Gamma(n)$

$$D^{-\alpha}y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t)\ dt, \quad \alpha, x \in \mathbb{R}_+ \tag{1.3}$$

The fractional derivative of order α can be defined by combining the standard differential operator with a fractional integral. In this thesis two different definitions of the fractional derivative are presented. The first definition, in which the fractional integral is applied before differentiating, is called the Riemann-Liouville fractional derivative. The second, in which the fractional integral is applied afterwards, is called the Caputo derivative. These two forms of the fractional derivative each behave a bit differently, as we will see. More details are in [1, 6, 13, 15].

The Laplace transform method has been applied for solving the fractional ordinary differential equations with constant coefficients. The solutions are expressed in terms of Mittage-Leffller functions [10, 20].

This thesis contains four chapters in which we summaries the fractional differentiation.

Chapter 1 was the introduction chapter which contained the history and the main motivation for the idea of the fractional calculus.

In Chapter 2 review on fractional calculus are considered. Also, we talk about the relation between fractional calculus and classical calculus. In addition we present the fractional integral; definition, examples, properties and some rules with its proofs.

In Chapter 3 two popular definitions of fractional derivatives are presented. The main properties and rules of compositions are given.

Chapter 4 presents solving of some fractional differential equations with Riemann-Liouville and Caputo sense by using the method of Laplace transform.

Chapter 2

Fractional Calculus

In this chapter, we interested in some important functions, which are inherently tied to fractional calculus. The Gamma function plays the role of the generalized factorial, the Beta function is necessary to compute fractional derivatives of power functions; the Mittag-Leffler functions appear in the solution of linear fractional differential equations. More information about these functions can be found in [4, 11]. Also we present some basic facts about Laplace transform and its properties. The basic definition and the concepts of fractional calculus are introduced in [10, 20]. In addition we present the relation between the classical calculus and fractional calculus. We discuss Riemann-Liouville fractional integral, which is the most popularized in the world of fractional calculus, introduce its definition, examples and properties. More details can be found in [14, 15].

2.1 Special Functions

Before looking at the definitions of fractional calculus we will first introduce and discuss some special functions, including Gamma and Beta functions and its properties. We also talk about incomplete gamma function, Digamma function, Mittag-leffler function and Mellin-Ross function. This set of functions plays an important role in the theory of fractional calculus as we will see later.

2.1.1 Gamma Function

The simple interpretation of the *Gamma function* is simply the generalization of the factorial for all real numbers.

Definition 2.1. [4] The function $z \to \Gamma(z)$, $Re\ z > 0$, defined by

$$\Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t}\, dt\ , Re\ z > 0. \tag{2.1}$$

is called the *Gamma function*.

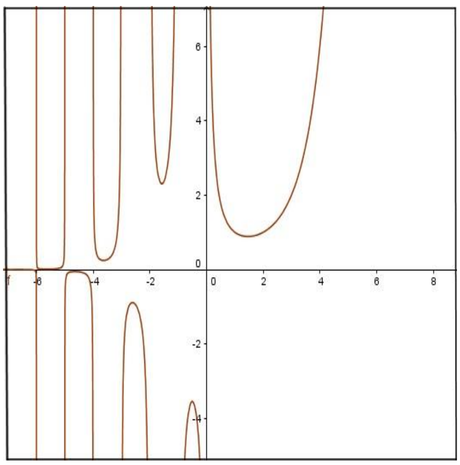

Figure 2.1: Gamma Function .

Properties of Gamma function

Here we will give the most important properties of the Gamma function:

- $\Gamma(1) = \Gamma(2) = 1.$
- $\Gamma(n) = (n-1)!,\ n \in \mathbb{N}.$
- $\Gamma(z+1) = z\Gamma(z),\ z \in \mathbb{R}^+.$

  Proof: To show part 3,

  Using integration by parts, we obtain:

$$\begin{aligned}
\Gamma(z+1) &= \int_0^\infty t^z e^{-t}\, dt \\
&= [-t^z e^{-t}]_0^\infty + \int_0^\infty z t^{z-1} e^{-t}\, dt \\
&= \lim_{t\to\infty} (-t^z e^{-t}) - (0e^0) + z \int_0^\infty t^{z-1} e^{-t}\, dt
\end{aligned}$$

  But when $t \to \infty$, $(-t^z e^{-t}) \to 0$.

$$\begin{aligned}
&= z \int_0^\infty t^{z-1} e^{-t}\, dt \\
&= z\Gamma(z)
\end{aligned}$$

$\square$

Example 2.1. Let us take an example to solve for $\Gamma\left(\frac{1}{2}\right)$

By definition of the Gamma function (2.1) we have

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2}\ e^{-t}\, dt$$

Let $t = y^2$, then $dt = 2ydy$, we have

$$\Gamma\left(\frac{1}{2}\right) = 2\int\limits_{0}^{\infty} e^{-y^2}\,dy \tag{2.2}$$

we can write (2.2) as

$$\Gamma\left(\frac{1}{2}\right) = 2\int\limits_{0}^{\infty} e^{-x^2}\,dx \tag{2.3}$$

Thus, if we multiplying (2.2) by (2.3) we get

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int\limits_{0}^{\infty}\int\limits_{0}^{\infty} e^{-(x^2+y^2)}\,dxdy \tag{2.4}$$

equation (2.4) is a multiple integral. We can solve it by using polar coordinates as follows

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int\limits_{0}^{\pi/2}\int\limits_{0}^{\infty} e^{-r^2}\,rdrd\theta = \pi$$

thus,

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

2.1.2 Beta Function

Definition 2.2. [4] The function $(x, y) \to \beta(x, y)$, $Re\ x > 0, Re\ y > 0$, defined by

$$\beta(x, y) = \int\limits_{0}^{1} t^{x-1}(1-t)^{y-1}\,dt \tag{2.5}$$

is called the *Beta function*.

There is a relation between Gamma and Beta functions given in the relation:

$$\beta(x, y) = \frac{\Gamma(x)\ \Gamma(y)}{\Gamma(x+y)}. \tag{2.6}$$

It should also be mentioned that the Beta function is symmetric, i.e.,

$$\beta(x, y) = \beta(y, x).$$

2.1.3 Incomplete Gamma Function

In mathematics, the Upper Incomplete Gamma function and Lower Incomplete Gamma function are types of special functions, which defined respectively as follows

Definition 2.3. [4] for $x, Re\ z > 0$,

$$\Gamma(z,x) = \int_x^\infty e^{-t}\ t^{z-1}\,dt. \tag{2.7}$$

and

$$\gamma(z,x) = \int_0^x e^{-t}\ t^{z-1}\,dt. \tag{2.8}$$

$\Gamma(z,x)$, $\gamma(z,x)$ are called the incomplete Gamma functions related to x.

Note that

$$\Gamma(z,x) + \gamma(z,x) = \Gamma(z). \tag{2.9}$$

Definition 2.4. [12] for $Re(z) > 0$, the *Incomplete Gamma Function* denoted by $\gamma^*(z,x)$ defined as

$$\gamma^*(z,x) = \frac{1}{\Gamma(z)\ x^z}\gamma(z,x) = \frac{1}{\Gamma(z)\ x^z}\int_0^x e^{-t}\ t^{z-1}\,dt. \tag{2.10}$$

2.1.4 Digamma Function

The *Digamma function* is a special function, which is given by the logarithmic derivative of the gamma function as follows

$$\Psi(x) = \frac{d}{dx}\ln(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{2.11}$$

we can find that in [5] the following relation

$$\int_0^1 x^{\mu-1}(1-x)^{v-1} \ln x \; dx = \beta(\mu, v)[\Psi(\mu) - \Psi(\mu+v)]. \tag{2.12}$$

which we will use later in solving some examples.

2.1.5 The Mittag-Leffler Function

The *Mittag-Leffler Function* is a special function, which is named after a Swedish mathematician who defined and studied it in 1903. It is a direct generalization of the exponential function e^x [11]. This function is important in the theory of the fractional calculus. The *Mittag-Leffler* function is defined in terms of a power series as

$$E_\alpha(x) = \textstyle\sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}, \;\; \alpha > 0 \tag{2.13}$$

$$E_{\alpha,\beta}(x) = \textstyle\sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)}, \;\; \alpha > 0, \;\; \beta > 0 \tag{2.14}$$

Where (2.13) is called a one parameter function of Mittag-Leffler type; whereas (2.14) is called a two parameter function of Mittag-Leffler type. Observe that (2.14) is a generalization of (2.13). i.e., when $\beta = 1$,

$$E_\alpha(x) = E_{\alpha,1}(x)$$

Note that $E_{\alpha,\beta}(0) = 1$. Also, for some specific values of α and β, the Mittag-Leffler function reduces to some familiar functions. For example,

$$E_{1,1}(x) = \textstyle\sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x \tag{2.15}$$

$$E_{1,2}(x) = \textstyle\sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+2)} = \frac{1}{x} \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} = \frac{e^x - 1}{x} \tag{2.16}$$

2.1.6 The Mellin-Ross Function

The *Mellin-Ross function*, arises when finding the fractional derivative of an exponential. The function is closely related to both the incomplete Gamma and Mittag-Leffler functions [11]. Its definition is given by

$$E_t(z, a) = t^z \; e^{at} \; \gamma^*(z, t) \tag{2.17}$$

which is can also written as

$$E_t(z, a) = t^z \; \textstyle\sum_{k=0}^{\infty} \dfrac{(at)^k}{\Gamma(k + z + 1)} = t^z \; E_{1,z+1}(at) \tag{2.18}$$

2.1.7 Euler's Constant

Euler's constant is a mathematical constant recurring in analysis and number theory, usually denoted by the γ. It is defined as the limiting difference between the harmonic series and the natural logarithm:

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) \tag{2.19}$$

and has the numerical value $\gamma = 0.577...$

2.2 Laplace Transform

In this section we interested in Laplace transform definition, properties, some theorems and examples.

Definition 2.5. [21] Let f be a function defined for $t \geq 0$. Then the integral

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) \; dt = \lim_{A\to\infty} \int_0^{A} e^{-st} f(t) \; dt$$

is said to be the *Laplace transform* of f, provided that the integral converges.

Definition 2.6. [21] we say a function $f(t)$ is exponentially bounded or of exponential order if there exists non negative numbers a, k and M such that

$$|f(t)| \leq ke^{at},\ t \geq M. \tag{2.20}$$

Definition 2.7. A function f is said to be piecewise continuous on a bounded interval if it has a finite number of discontinuities and the left and right limits exist at each discontinuity. It is said to be piecewise continuous on $[0,\infty]$ if it is piecewise continuous on every bounded subinterval $I \subset [0,\infty]$.

Theorem 2.1. *[21] Suppose f is piecewise continuous on $[0,\infty]$ and exponentially bounded. Then $\mathscr{L}\{f\} = F(s)$ exists for all $s > a$.*

Next, we take some examples of finding Laplace transform for particular cases.

Example 2.2. Find Laplace transform of $f(t) = t^n, n \in \mathbb{N}$

Solution:

By definition, the Laplace transform of $f(t)$ is

$$F(s) = \mathscr{L}\{f(t)\} \quad = \quad \lim_{A\to\infty} \int_0^A e^{-st} t^n \, dt$$

Integration by parts

$$\begin{aligned} &= \lim_{A\to\infty}\left[t^n \frac{e^{-st}}{-s}\Big|_0^A + \frac{n}{s}\int\limits_0^A e^{-st}t^{n-1}\ dt \right] \\ &= \frac{n}{s}\mathscr{L}\{t^{n-1}\} \end{aligned}$$

By induction method we obtain

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0.$$

Example 2.3. Find Laplace transform of the function $f(t) = e^{at}$.

Solution:

The Laplace transform of the function $f(t)$ is

$$\begin{aligned} F(s) = \mathscr{L}\{f(t)\} &= \lim_{A\to\infty}\int\limits_0^A e^{-st}e^{a}t\ dt \\ &= \lim_{A\to\infty}\int\limits_0^A e^{-(s-a)t}\ dt \\ &= \lim_{A\to\infty}\frac{-1}{s-a}e^{-(s-a)t}\Big|_0^A \\ &= \frac{1}{s-a} \end{aligned}$$

• **Properties of Laplace transform**

1. **Linearity**

$$\mathscr{L}\{c_1 f(t) + c_2 g(t)\} = c_1\mathscr{L}\{f(t)\} + c_2\mathscr{L}\{g(t)\} \tag{2.21}$$

The proof follows by definition.

2. **First Derivative**

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0) \tag{2.22}$$

Proof:

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\ dt$$

Integrating by parts

$$\begin{aligned} &= e^{-st} f(t)\Big|_0^\infty + \int_0^\infty s e^{-st} f(t)\ dt \\ &= s\mathscr{L}\{f(t)\} - f(0) \end{aligned}$$

3. **Higher order Derivative**

$$\mathscr{L}\{f^{(n)}(t)\} = s^n \mathscr{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - ... - s f^{(n-2)}(0) - f^{(n-1)}(0) \tag{2.23}$$

Table 2.2 gives a summary of some useful Laplace transform pairs. We will Notice that the Mittag-Leffler function is very important. Next we will give a theorem and proof it for one relation.

Theorem 2.2. *[20]*

$$\mathscr{L}^{-1}\left[\frac{s^{\alpha-\beta}}{s^\alpha - a}\right] = t^{\beta-1} E_{\alpha,\beta}(at^\alpha), \qquad |a| < |s^\alpha|, \quad \alpha, \beta \in \mathbb{R}_+ \tag{2.24}$$

Proof:

Take the Laplace transform for the right side

$$\mathscr{L}\{t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})\} = \int_0^{\infty} e^{-st}t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})\ dt$$

using (2.14) we have

$$\begin{aligned}\mathscr{L}\{t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})\} &= \int_0^{\infty} e^{-st}t^{\beta-1}\sum_{k=0}^{\infty}\frac{(at^{\alpha})^k}{\Gamma(\alpha k+\beta)}\ dt \\ &= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\alpha k+\beta)}\int_0^{\infty} e^{-st}t^{\alpha k+\beta-1}\ dt \\ &= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\alpha k+\beta)}\mathscr{L}\{t^{\alpha k+\beta-1}\}\end{aligned}$$

We know that $\mathscr{L}\{t^{\alpha}\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, using it we have

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\alpha k+\beta)}\mathscr{L}\{t^{\alpha k+\beta-1}\} &= \sum_{k=0}^{\infty}\frac{a^k}{\Gamma(\alpha k+\beta)}\frac{\Gamma(\alpha k+\beta)}{s^{\alpha k+\beta}} \\ &= \frac{1}{s^{\beta}}\sum_{k=0}^{\infty}\left(\frac{a}{s^{\alpha}}\right)^k\end{aligned}$$

Here we obtain a geometric series, which is converges as $n \to \infty$ for $\left|\frac{a}{s^{\alpha}}\right| < 1$

$$\frac{1}{s^{\beta}}\sum_{k=0}^{\infty}\left(\frac{a}{s^{\alpha}}\right)^k = \frac{s^{\alpha-\beta}}{(s^{\alpha}-a)}$$

Thus, we get

$$\mathscr{L}^{-1}\left[\frac{s^{\alpha-\beta}}{s^{\alpha}-a}\right] = t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})$$

$\square$

Figure 2.2: Table 2.2: Laplace transform pairs [8].

$Y(s)$	$y(t) = \mathcal{L}^{-1}\{Y(s)\}$
$\frac{1}{s^{\alpha}}$	$\frac{t^{\alpha-1}}{\Gamma(\alpha)}$
$\frac{1}{(s+a)^{\alpha}}$	$\frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-at}$
$\frac{1}{s^{\alpha}-a}$	$t^{\alpha-1} E_{\alpha,\alpha}(at^{\alpha})$
$\frac{s^{\alpha}}{s(s^{\alpha}+a)}$	$E_{\alpha}(-at^{\alpha})$
$\frac{a}{s(s^{\alpha}+a)}$	$1 - E_{\alpha}(-at^{\alpha})$
$\frac{1}{s^{\alpha}(s-a)}$	$t^{\alpha} E_{1,\alpha+1}(at)$
$\frac{s^{\alpha-\beta}}{s^{\alpha}-a}$	$t^{\beta-1} E_{\alpha,\beta}(at^{\alpha})$
$\frac{s^{\alpha-\beta}}{(s-a)^{\alpha}}$	$\frac{t^{\beta-1}}{\Gamma(\beta)} F_1(\alpha;\beta;at)$
$\frac{1}{(s-a)(s-b)}$	$\frac{1}{a-b}(e^{at} - e^{bt})$

2.3 Review on Fractional Calculus

Fractional Calculus is a generalization of the classical calculus, It is a field of mathematics study, that grows out of the traditional definitions of the calculus integral and derivative operators. It has been used successfully in various fields of science and engineering.[2, **?**, 15]

Fractional Calculus has its origin in the question of the extension of meaning. In generalized integration and differentiation the question of the extension of meaning is: Can the meaning of derivatives of integer order $\frac{d^n y}{dx^n}$ be extended to have meaning where n is any number; real, irrational, fractional or complex? This question was first advanced by L'Hopital in a letter dated September 30th, 1695. When he wrote to Leibniz asked him "what the result would be if $n = \frac{1}{2}$". Leibniz replied "it will lead to a paradox". But he added "From this apparent paradox, someday it would lead to useful consequences"[15].

Thereafter, many known mathematicians contributed to the theory of *Fractional Calculus* over the years. Among them Euler, Laplace, Fourier, Lacroix, Abel, Riemann and Liouville, Weyl, Leibniz, Grunwald and Letnikov [2, 3, 12, 14, 15, 16, **?**]. Each used their own notation and they found many concepts of a non-integer order integral or derivative. The field of *Fractional Calculus* had been studied extensively by many researcher for name few [12, 13, 14, 15, 16].

In 1730 Euler mentioned interpolating between integral orders of a derivative. In 1812 Laplace defined a fractional derivative by means of an integral, and the first discussion of a derivative of fractional order appeared in a calculus written by Lacroix in 1819 [3].

• **Lacroix Definition** [17]

Lacroix develops a formula for *fractional* differentiation for the n-th derivative of x^m where m is a positive integer, by induction.
starting with

$$y = x^m,$$

the first derivative is as usual

$$\frac{dy}{dx} = mx^{m-1}.$$

Repeating this n-times we get

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!}x^{m-n}, m \geq n.$$

Which, after replacing the factorials with the gamma function, leads to

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}.$$

Thus, let $m = 1$, and replace n by $\frac{1}{2}$, one obtains the order $\frac{1}{2}$ of the function x

$$\frac{d^{1/2} y}{dx^{1/2}} = \frac{\Gamma(2)}{\Gamma(3/2)}x^{1/2} = \frac{2}{\sqrt{\pi}}\sqrt{x}.$$

• **Fourier's Definition** [12]

Fourier's talked about *fractional derivative* of arbitrary order. His integral representation of $f(x)$ is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha)\ d\alpha \int_{-\infty}^{\infty} cos[p(x-\alpha) + \frac{n\pi}{2}]\ dp$$

He obtained the definition of *fractional* operation from this formula as follow

$$\frac{d^n}{dx^n}cos[p(x-\alpha)] = p^n cos[p(x-\alpha) + \frac{n\pi}{2}]$$

Replacing n by u where u is arbitrary (positive or negative), getting

$$\frac{d^u}{dx^u}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha)\ d\alpha \int_{-\infty}^{\infty} p^u cos[p(x-\alpha)+\frac{u\pi}{2}]\ dp.$$

• **Abel's Definition** [12]

Abel in 1823 was the first who use the *fractional operations*, he applied the *fractional* calculus in the solution of an integral equation which arises in the formulation of tautochrone problem [13]. Explains the tautochrone problem as "The tautochrone problem is the problem of determining the shape of the curve such that the time of descent of a frictionless point mass sliding down the curve under the action of gravity is independent of the starting point"[13]. In this problem the time of slide is a known constant K such that

$$\begin{aligned} K &= \int_0^x (x-t)^{-1/2} f(t)\,dt \\ &= \sqrt{\pi}\int_0^x \frac{1}{\Gamma\left(\frac{1}{2}\right)}(x-t)^{-1/2} f(t)\,dt \\ &= \sqrt{\pi}\frac{d^{-1/2}}{dx^{-1/2}}f(x) \end{aligned}$$

then he differentiated both sides of the equation to order $\frac{1}{2}$, obtained

$$\frac{d^{1/2}}{dx^{1/2}}K = \sqrt{\pi}f(x)$$

• **Liouville's First Definition** [12, 17]

Liouville who made the first major study of *fractional calculus*, give two definitions in this area. He was successful in applying his definitions to problems in theory. His first definition comes from

$$D^n(e^{ax}) = a^n e^{ax} \quad , n \in \mathbb{Z}^+$$

which he generalized it for derivatives of any arbitrary order v

$$D^v(e^{ax}) = a^v e^{ax}$$

Thus, for any function $f(x)$ can be expressed into a series $\sum_{n=0}^{\infty} c_n e^{a_n x}$ where $Re\ a_n > 0, \forall n$. The *fractional derivative* of $f(x)$ is

$$D^v f(x) = \sum_{n=0}^{\infty} c_n (a_n)^v e^{a_n x}.$$

• **Liouville's Second Definition** [12, 17]

Liouville second definition based on the definite integral related to Gamma integral

$$I = \int_0^{\infty} u^{a-1} e^{-xu}\, du \ , x > 0 \ , a > 0$$

If we make a substitution $t = xu$, then $dt = xdu$ and we get

$$\begin{aligned} I &= \int_0^{\infty} \left(\frac{t}{x}\right)^{a-1} e^{-t} \frac{1}{x}\, dt \\ &= x^{-a} \int_0^{\infty} t^{a-1} e^{-t}\, dt \\ &= x^{-a}\Gamma(a). \end{aligned}$$

Thus,

$$x^{-a} = \frac{1}{\Gamma(a)} \int_0^{\infty} u^{a-1} e^{-xu}\, du$$

Hence we can find the *fractional derivative* of x^{-a} for $a > 0$. By taking D^v for both sides above, we have

$$\begin{aligned} D^v x^{-a} &= \frac{1}{\Gamma(a)} D^v \left(\int_0^{\infty} u^{a-1} e^{-xu}\, du \right) \\ &= \frac{1}{\Gamma(a)} \int_0^{\infty} u^{a-1} (-1)^v u^v e^{-xu}\, du \end{aligned}$$

$$= \frac{(-1)^v}{\Gamma(a)} \int_0^\infty u^{a+v-1} e^{-xu}\, du$$

Substitute $t = xu$ again, we get $dt = xdu$, then

$$= \frac{(-1)^v}{\Gamma(a)} \int_0^\infty \left(\frac{t}{x}\right)^{a+v-1} e^{-t}\, \frac{dt}{x}$$

Hence,

$$D^v x^{-a} = \frac{(-1)^v \Gamma(v+a)}{\Gamma(a)} x^{-a-v} \quad ,a > 0\ ,x > 0.$$

We note that Liouville first definition is restricted just for the functions which can be expressed as a trigonometric series, whereas the second definition is useful for functions of the type x^{-a} for $a > 0$. Observe that both are definitions of *fractional derivative*.

• **Riemann's Definition** [12, 17]

Riemann; when he was a student he developed his theory of *fractional integration*. His definition is given as

$$D^{-v} f(x) = \frac{1}{\Gamma(v)} \int_c^x (x-t)^{v-1} f(t)\ dt + \Psi(x)$$

Where $\Psi(x)$ is a complementary function. For the confusion in the lower limit of integration c, he added the function $\Psi(x)$. "The complementary function is essentially an attempt to provide a measure of the deviation from the law of exponents. For example, this law, as mentioned later, is ${}_cD_x^{-\mu}{}_cD_x^{-v} f(x) ={}_c\ D_x^{-\mu-v} f(x)$and is valid when the lower terminals c are equal. Riemann was concerned with a measure of deviation for the case ${}_cD_x^{-\mu}{}_{c'}D_x^{-v} f(x)$ when $c \neq c'$" [13].

• **Riemann-Liouville's Definition** [12]

The first calculation of the definition of the Riemann-Liouville of the *fractional integral* was in 1869, N. Ya. Sonin published a paper, "On differentiation with arbitrary index"[18]. The definition is based on the *Cauchy formula* for the n^{th} integral, is given by

$$ {}_cD_x^{-v} f(x) = \frac{1}{\Gamma(v)} \int\limits_c^x (x-t)^{v-1} f(t)\ dt \quad , Re\ x > 0 $$

If we compare it with Riemann definition. We see that they are the same without the complementary function $\Psi(x)$. Later we will talk in details about Riemann-Liouville's Definition, because it is the most popular definition in *fractional calculus.*

2.4 Relation Between Fractional Calculus and Classical Calculus

This section shows the relation between the fractional calculus and the classical calculus. In addition we present some challenges to fractional calculus.

Since the fractional calculus is a generalization of the classical calculus, so it is -classical calculus- considered as a part of the fractional one, because when we are dealing with the real-order operation (which are differentiation or integration), this already include the integer-order operation.

As we know about the classical calculus, derivative and integral are uniquely determined, which also applies for the fractional integral, whereas there are a several definitions for a fractional derivative, which are inconsistent. Oftentimes, some definitions are not equivalent to each other, which makes the situation more complicated.

2.5 Fractional Integration

In this section we present the fractional integration topic. Specifically Riemann-Liouville fractional integral, definition, and properties. This subject had been studied extensively by [8, 12, 19].

2.5.1 Definition of Riemann-Liouville Fractional Integral

We begin this section by stating Cauchy formula for repeated integration [19].

Theorem 2.3. *(Cauchy formula for repeated integration). Let f be some continuous function on the interval [a,b]. The n-th repeated integral of f based at a,*

$$f^{(-n)}(x) = \int_a^x \int_a^{\tau_1} \dots \int_a^{\tau_{n-1}} f(\tau_n)\ d\tau_n\ d\tau_{n-1}\ \dots d\tau_2\ d\tau_1,$$

is given by single integration:

$$f^{(-n)}(x) = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\ dt$$

Proof:

We use mathematical induction to prove this theorem, the case $n = 1$. We have

$$\int_a^x f(\tau_1)\ d\tau_1 = \frac{1}{(0)!} \int_a^x (x-t)^0 f(t)\ dt$$

Thus, the statement holds for $n = 1$. Let the statement holds for some arbitrary n. We will prove it for $n + 1$

$$\begin{aligned} f^{(-(n+1))}(x) &= \int_a^x \int_a^{\tau_1} \dots \int_a^{\tau_n} f(\tau_{n+1})\ d\tau_{n+1}\ d\tau_n\ \dots d\tau_2\ d\tau_1 \\ &= \frac{1}{(n-1)!} \int_a^x \int_a^{\tau_1} (\tau_1 - t)^{n-1} f(t)\ dt d\tau_1 \end{aligned}$$

$$= \frac{1}{(n-1)!} \int_a^x \int_t^x (\tau_1 - t)^{n-1} f(t) \, d\tau_1 dt$$

$$= \frac{1}{(n)!} \int_a^x ((x-t)^n - (t-t)^n) f(t) \, dt$$

$$= \frac{1}{(n)!} \int_a^x (x-t)^n f(t) \, dt$$

□

From this formula the definition of *fractional integral* is constructed, so we can take an integral of any real degree. Replacing $(n-1)!$ by $\Gamma(n)$ and the power n in the integrand with some $\alpha \in \mathbb{R}_+$, we have Riemann-Liouville fractional integral.

Definition 2.8. [19] (Riemann-Liouville Operator). Let f be a continuous function with $\alpha \in \mathbb{R}_+$ and $x \in \mathbb{R}$. The fractional integral of order α is defined as:

$$D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt \tag{2.25}$$

2.5.2 Examples of Riemann-Liouville Fractional Integration

Example 2.4. Power Function

Suppose we want to find the fractional integral for the power function $f(x) = x^m$, $m \in \mathbb{N}$.

By definition of Riemann-Liouville fractional integral we have

$$D^{-\alpha} x^m = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^m \, dt$$

Letting $t = ux$, $0 \leq u \leq 1$, then $dt = xdu$, we get

$$D^{-\alpha} x^m = \frac{1}{\Gamma(\alpha)} \int_0^1 (x-ux)^{\alpha-1} (ux)^m \, xdu$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \int_0^1 x^{\alpha+m}(1-u)^{\alpha-1}u^m \; du \\
&= \frac{x^{\alpha+m}}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1}u^m \; du \\
&= \frac{x^{\alpha+m}}{\Gamma(\alpha)} \beta(\alpha, m+1) \\
&= \frac{x^{\alpha+m}}{\Gamma(\alpha)} \frac{\Gamma(\alpha)\Gamma(m+1)}{\Gamma(\alpha+m+1)} \\
&= \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} x^{\alpha+m}
\end{aligned}
$$

Thus, the fractional integral of a power function is

$$
D^{-\alpha}x^m = \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} x^{\alpha+m}. \tag{2.26}
$$

Example 2.5. Find the fractional integral with $\alpha = 1$ ”classical integral” for the function $f(x) = x^2$

Solution:

Using equation (2.26) we obtain

$$
\begin{aligned}
D^{-1}x^2 &= \frac{\Gamma(2+1)}{\Gamma(1+2+1)} x^{1+2} \\
&= \frac{1}{3} x^3
\end{aligned}
$$

Example 2.6. Constant Function

To find the fractional integral of any constant function K we obtain the following

$$
\begin{aligned}
D^{-\alpha}K &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} K \; dt \\
&= \frac{K}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \; dt
\end{aligned}
$$

Let $t = xu$, then $dt = xdu$, for $0 \leq u \leq$, we obtain

$$\begin{aligned} D^{-\alpha}K &= \frac{K}{\Gamma(\alpha)} \int\limits_0^1 (x - xu)^{\alpha-1}\, xdu \\ &= \frac{Kx^{\alpha}}{\Gamma(\alpha)} \int\limits_0^1 (1-u)^{\alpha-1}\, du \\ &= \frac{Kx^{\alpha}}{\Gamma(\alpha)} \beta(1,\alpha) \\ &= \frac{Kx^{\alpha}}{\Gamma(\alpha)} \frac{\Gamma(1)\Gamma(\alpha)}{\Gamma(\alpha+1)} \\ &= \frac{K}{\Gamma(\alpha+1)} x^{\alpha} \end{aligned}$$

Hence, we can say that for any constant function K the Riemann-Liouville fractional integral is given by

$$D^{-\alpha}K = \frac{K}{\Gamma(\alpha+1)} x^{\alpha}. \tag{2.27}$$

Example 2.7. If we want to find the traditional integral for any constant function K we have

$$\int\limits_0^x K dt = Kx$$

Which we can get by substitute $\alpha = 1$ in (2.27)

Example 2.8. Exponential Function

Suppose we wish to find the fractional integral of the exponential function e^{ax}.

$$D^{-\alpha}e^{ax} = \frac{1}{\Gamma(\alpha)} \int\limits_0^x (x-t)^{\alpha-1} e^{at}\, dt$$

Using the change of the variable by letting $t = x(1-u)$, for $0 \leq u \leq 1$ and $dt = -xdu$. We obtain

$$\begin{aligned} D^{-\alpha}e^{ax} &= \frac{1}{\Gamma(\alpha)}\int\limits_{1}^{0}(x-x(1-u))^{\alpha-1}e^{ax(1-u)}\,(-x)du \\ &= \frac{1}{\Gamma(\alpha)}\int\limits_{0}^{1}x^{\alpha}u^{\alpha-1}e^{ax}e^{-axu}\,du \\ &= \frac{x^{\alpha}e^{ax}}{\Gamma(\alpha)}\int\limits_{0}^{1}u^{\alpha-1}e^{-axu}\,du \end{aligned}$$

Substituting $r = axu$, $dr = axdu$, we have

$$\begin{aligned} D^{-\alpha}e^{ax} &= \frac{x^{\alpha}e^{ax}}{\Gamma(\alpha)}\int\limits_{0}^{ax}\left(\frac{r}{ax}\right)^{\alpha-1}e^{-r}\frac{1}{ax}\,dr \\ &= \frac{x^{\alpha}e^{ax}}{\Gamma(\alpha)}\frac{1}{(ax)^{\alpha}}\int\limits_{0}^{ax}r^{\alpha-1}e^{-r}\,dr \\ &= x^{\alpha}e^{ax}\gamma^{*}(\alpha, ax) \;\; \textit{using equation } (2.10) \\ &= E_{x}(\alpha, a) \;\; \textit{using equation } (2.17) \end{aligned}$$

Thus, we have that

$$D^{-\alpha}e^{ax} = E_{x}(\alpha, a). \tag{2.28}$$

Example 2.9. If we take $\alpha = 2$ in equation (2.28) we obtain

$$\begin{aligned} D^{-2}e^{ax} &= \frac{1}{\Gamma(2)}\int\limits_{0}^{x}(x-t)^{2-1}e^{at}\,dt \\ &= \frac{e^{ax}}{a^{2}} + \frac{x}{a} - \frac{1}{a^{2}} \end{aligned}$$

Which is the same when we integrate e^{ax} double integral.

Example 2.10. Find the fractional integral of the function $\ln x$.

Solution:

By using definition of fractional integral we obtain

$$\begin{aligned} D^{-\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\; dt \\ D^{-\alpha} \ln x &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \ln t\; dt \end{aligned}$$

Let $t = xu$, we have $dt = xdu$, for $0 \le u \le 1$, we then obtain

$$\begin{aligned} D^{-\alpha} \ln x &= \frac{1}{\Gamma(\alpha)} \int_0^1 (x-xu)^{\alpha-1} \ln(xu)\; xdu \\ &= \frac{x^\alpha}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} \ln(xu)\; du \\ &= \frac{x^\alpha}{\Gamma(\alpha)} \left[\int_0^1 (1-u)^{\alpha-1} \ln x\; du + \int_0^1 (1-u)^{\alpha-1} \ln u\; du \right] \\ &= \frac{x^\alpha}{\Gamma(\alpha)} \ln x \beta(1,\alpha) + \frac{x^\alpha}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} \ln u\; du \end{aligned}$$

By using the property of the logarithms and using equation (2.19). By equation (2.12) we get

$$\begin{aligned} D^{-\alpha} \ln x &= \frac{x^\alpha}{\Gamma(\alpha)} \frac{\Gamma(1)\Gamma(\alpha)}{\Gamma(\alpha+1)} \ln x + \frac{x^\alpha}{\Gamma(\alpha)} \beta(1,\alpha)[\Psi(1) - \Psi(\alpha+1)] \\ &= \frac{x^\alpha}{\Gamma(\alpha+1)} \ln x + \frac{x^\alpha}{\Gamma(\alpha)} \frac{\Gamma(1)\Gamma(\alpha)}{\Gamma(\alpha+1)} [-\gamma - \Psi(\alpha+1)] \\ &= \frac{x^\alpha}{\Gamma(\alpha+1)} \ln x + \frac{x^\alpha}{\Gamma(\alpha+1)} [-\gamma - \Psi(\alpha+1)] \\ &= \frac{x^\alpha}{\Gamma(\alpha+1)} [\ln x - \gamma - \Psi(\alpha+1)] \end{aligned}$$

Example 2.11. In this example we aim to have an equation to find the fractional integral of the functions of the form $g(x) = xf(x)$. By definition we have

$$\begin{aligned}
D^{-\alpha}g(x) &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}g(t)\ dt \\
&= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}tf(t)\ dt \\
&= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}[x-(x-t)]f(t)\ dt \\
&= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}xf(t)\ dt \ - \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha}f(t)\ dt \\
&= \frac{x}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\ dt \ - \frac{\alpha}{\alpha\Gamma(\alpha)}\int_0^x (x-t)^{\alpha}f(t)\ dt \\
&= xD^{-\alpha}f(x) - \frac{\alpha}{\Gamma(\alpha+1)}\int_0^x (x-t)^{\alpha}f(t)\ dt \\
&= xD^{-\alpha}f(x) - \alpha D^{-(\alpha+1)}f(x)
\end{aligned}$$

From this example we can find the fractional integral of any function in the form $xf(x)$. So we can say that this is a general example.

2.5.3 Properties of Riemann-Liouville Fractional Integral

Now, let us talk about the properties of the Riemann-Liouville fractional integral.

- $D^0 f(x) = f(x)$, i.e., $D^0 = I$ where I is the identity operator.

- **Linearity**

  Lemma 2.4. *[19] Let $\alpha > 0, C, K \in \mathbb{R}$, and let f and g be functions such that their fractional integrals exist. Then*

$$D^{-\alpha}[Cf(x) + Kg(x)] = CD^{-\alpha}f(x) + KD^{-\alpha}g(x) \tag{2.29}$$

Proof:

$$\begin{aligned}
D^{-\alpha}[Cf(x)+Kg(x)] &= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}[Cf(t)+Kg(t)]\ dt \\
&= \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}[Cf(t)]\ dt + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}[Kg(t)]\ dt \\
&= \frac{C}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\ dt + \frac{K}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}g(t)\ dt \\
&= CD^{-\alpha}f(x) + KD^{-\alpha}g(x).
\end{aligned}$$

□

- **The Law of Exponents for Fractional Integrals**

Lemma 2.5. *[8] Let $\alpha, \mu > 0$, and let f be continuous function such that their fractional integral exist. Then*

$$D^{-\mu}[D^{-\alpha}f(x)] = D^{-(\alpha+\mu)}f(x) = D^{-\alpha}[D^{-\mu}f(x)] \tag{2.30}$$

Before we give a proof, we will present Dirichlet formula, because we have to use it in the proof.

Dirichlet's formula [8] Let h be jointly continuous on the Euclidean plane, and let μ, α be positive numbers. Then

$$\int_0^t \int_0^x (t-x)^{\mu-1}(x-y)^{\alpha-1}h(x,y)\ dxdy = \int_0^t \int_y^t (t-x)^{\mu-1}(x-y)^{\alpha-1}h(x,y)\ dydx \tag{2.31}$$

We will use a special case of a Dirichlet's formula when $h(x,y) = f(y)$, then (2.31) takes the form

$$\int_0^t \int_0^x (t-x)^{\mu-1}(x-y)^{\alpha-1}f(y)\ dxdy = \beta(\mu,\alpha)\int_0^t (t-y)^{\mu+\alpha-1}f(y)\ dy \tag{2.32}$$

Now, we prove the lemma.

Proof:

By definition of Riemann-Liouville fractional integral we have

$$\begin{aligned}
D^{-\mu}[D^{-\alpha}f(x)] &= \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}[D^{-\alpha}f(t)]\ dt \\
&= \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}f(y)\ dy\right] dt \\
&= \frac{1}{\Gamma(\mu)\Gamma(\alpha)}\int_0^x\int_0^t (x-t)^{\mu-1}(t-y)^{\alpha-1}f(y)\ dydt \\
&= \frac{1}{\Gamma(\mu)\Gamma(\alpha)}\beta(\mu,\alpha)\int_0^x (x-y)^{\mu+\alpha-1}f(y)\ dy \\
&= \frac{1}{\Gamma(\mu+\alpha)}\int_0^x (x-y)^{\mu+\alpha-1}f(y)\ dy \\
&= D^{-(\alpha+\mu)}f(x).
\end{aligned}$$

□

- **The Derivatives of the Fractional Integral and the Fractional Integral of the Derivatives**

  We will now consider the cases when we differentiate the fractional integral of order α, or when we take the $\alpha+1$ fractional integral of the first derivative. Next we have a theorem about this cases.

Theorem 2.6. *[12] Let f be continuous on* $[0,\infty)$*, and let* $\alpha > 0$*.*

1. *If Df is integrable over any subinterval in* $[0,\infty)$*. Then*

$$D^{-\alpha-1}[Df(x)] = D^{-\alpha}f(x) - \frac{f(0)}{\Gamma(\alpha+1)}x^{\alpha} \tag{2.33}$$

2. *If Df is continuous on* $[0,\infty)$*, then for* $x > 0$

$$D[D^{-\alpha}f(x)] = D^{-\alpha}[Df(x)] + \frac{f(0)}{\Gamma(\alpha)}x^{\alpha-1} \tag{2.34}$$

Proof:

1. By definition we have

$$D^{-\alpha-1}[Df(x)] = \frac{1}{\Gamma(\alpha+1)} \int_0^x (x-t)^{\alpha} Df(t)\ dt$$

integrating by parts with

$$\begin{aligned} u &= (x-t)^{\alpha} & dv &= Df(t)dt \\ du &= -\alpha(x-t)^{\alpha-1}dt & v &= f(t) \end{aligned}$$

$$\begin{aligned} D^{-\alpha-1}[Df(x)] &= \frac{1}{\Gamma(\alpha+1)} \left[(x-t)^{\alpha} f(t) \Big|_0^x + \int_0^x \alpha(x-t)^{\alpha-1} f(t)\ dt \right] \\ &= \frac{1}{\Gamma(\alpha+1)} \left[-x^{\alpha} f(0) + \alpha \int_0^x (x-t)^{\alpha-1} f(t)\ dt \right] \\ &= D^{-\alpha} f(x) - \frac{f(0)}{\Gamma(\alpha+1)} x^{\alpha} \end{aligned}$$

□

2. By definition,

$$D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\ dt$$

Let $t = x - u^{\lambda}$ where $\lambda = \dfrac{1}{\alpha}$, then $dt = (-\lambda u^{\lambda-1})du$ for $0 \le u \le x^{\alpha}$, we have then

$$D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x^{\alpha}}^{0} (u^{\lambda})^{\alpha-1} f(x-u^{\lambda})(-\lambda u^{\lambda-1})\ du$$

$$= \frac{\lambda}{\Gamma(\alpha)} \int_0^{x^\alpha} u^{\lambda(\alpha-1)} u^{\lambda-1} f(x-u^\lambda)\ du$$

$$= \frac{1}{\alpha\Gamma(\alpha)} \int_0^{x^\alpha} f(x-u^\lambda)\ du$$

$$= \frac{1}{\Gamma(\alpha+1)} \int_0^{x^\alpha} f(x-u^\lambda)\ du$$

Now, taking the operator D for both sides, getting

$$D[D^{-\alpha} f(x)] = \frac{d}{dx}\left[\frac{1}{\Gamma(\alpha+1)} \int_0^{x^\alpha} f(x-u^\lambda)\ du\right]$$

Here, we need to use the Leibniz's Integral Rule which states that

$$\frac{d}{dx}\left[\int_0^{b(x)} f(x,t)\ dt\right] = f(x,b(x))b'(x) + \int_0^{b(x)} \frac{\partial}{\partial x} f(x,t)\ dt \tag{2.35}$$

Therefore,

$$\begin{aligned} D[D^{-\alpha} f(x)] &= \frac{1}{\Gamma(\alpha+1)}\left[f(x-(x^\alpha)^\lambda)\alpha x^{\alpha-1} + \int_0^{x^\alpha} \frac{\partial}{\partial x} f(x-u^\lambda)\ du\right] \\ &= \frac{1}{\Gamma(\alpha+1)}\left[f(0)\alpha x^{\alpha-1} + \int_0^{x^\alpha} \frac{\partial}{\partial x} f(x-u^\lambda)\ du\right] \end{aligned}$$

refer to the assumption $(x-u^\lambda) = t$, henceforth,

$$D[D^{-\alpha} f(x)] = \frac{\alpha f(0)}{\Gamma(\alpha+1)} x^\alpha + \frac{1}{\Gamma(\alpha+1)} \int_x^0 \frac{\partial}{\partial x} f(t) \left(\frac{1}{-\lambda} u^{1-\lambda}\right)\ dt$$

Simplify it we have

$$\begin{aligned} D[D^{-\alpha} f(x)] &= \frac{f(0)}{\Gamma(\alpha)} x^{\alpha} + \frac{\alpha}{\Gamma(\alpha+1)} \int\limits_0^x (x-t)^{\alpha-1} \frac{\partial}{\partial x} f(t)\, dt \\ &= \frac{f(0)}{\Gamma(\alpha)} x^{\alpha} + \frac{1}{\Gamma(\alpha)} \int\limits_0^x (x-t)^{\alpha-1} \frac{\partial}{\partial x} f(t)\, dt \\ &= \frac{f(0)}{\Gamma(\alpha)} x^{\alpha} + D^{-\alpha}[Df(x)] \end{aligned}$$

□

Hence, we conclude that

$$D[D^{-\alpha} f(x)] \neq D^{-\alpha}[Df(x)]. \tag{2.36}$$

Chapter 3

Fractional Differentiation

In chapter 2, we introduced the fractional integral of a function $f(x)$, we presented it before fractional differentiation because the definition of the fractional derivatives depends on the definition of the fractional integration.

In this chapter, we introduce the definition of the fractional derivative, specifically *Riemann Liouville* definition and *Caputo* definition since they are the most common and used in fractional calculus world. We will give some examples of fractional derivatives for elementary functions. In addition we present the most important properties and its proofs. Finally we will derive some relations for composition of the fractional derivative. More details about this subject can be found in [7, 9, 15].

3.1 Riemann Liouville Fractional Derivative

In this section, we are interested in *Riemann Liouville* fractional derivative, firstly; its definition, then present some examples, after this we will introduce its properties. Then we will derive some rules for the composition of fractional derivative.

3.1.1 Definition of Riemann-Liouville Fractional Derivative

The Riemann-Liouville fractional derivative can be defined using the definition of the Riemann-Liouville fractional integral, it is an ordinary derivative of the fractional integral i.e.,

$$\begin{aligned} D[D^{-(1-\alpha)}f(x)] &= D[D^{-1}D^{-(1-\alpha-1)}f(x)] \\ &= D[D^{-1}D^{\alpha}f(x)] \\ &= D^{\alpha}f(x) \end{aligned}$$

Hence,

$$D[D^{-(1-\alpha)}f(x)] = D^{\alpha}f(x)$$

Now, using the definition of the fractional integral we get

$$\begin{aligned} D^{\alpha}f(x) &= \frac{d}{dx}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^x (x-t)^{-\alpha}f(t)\ dt\right] \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha}f(t)\ dt \end{aligned}$$

If we differentiate the fractional integral n-times so we have

$$D^{\alpha}f(x) = \underbrace{\frac{d}{dx}\frac{d}{dx}\frac{d}{dx}...\frac{d}{dx}}_{n} D^{-(n-\alpha)}f(x)$$

Hence,

$$D^{\alpha}f(x) = D^{n}[D^{-(n-\alpha)}f(x)] \quad , n-1 \leq \alpha < n \tag{3.1}$$

Now, we can give the definition of the Riemann-Liouville fractional derivative

Definition 3.1. [15] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. The *Riemann-Liouville* fractional derivative of order α of a function $f(x)$ is given by:

$$D^{\alpha} f(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\, dt, & n-1 \le \alpha < n \\ \frac{d^n}{dx^n} f(x), & \alpha = n \in \mathbb{N} \end{cases} \tag{3.2}$$

Where $\Gamma(\alpha)$ denotes Eulers Gamma function.

3.1.2 Examples of Riemann-Liouville Fractional Derivative

Example 3.1. Constant function

If $f(x) = K$, where K is a constant, then $D^{\alpha} f(x) = \frac{K}{\Gamma(1-\alpha)} x^{-\alpha}$

Solution:

By definition of Riemann-Liouville fractional derivative we have

$$\begin{aligned} D^{\alpha} f(x) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\, dt \\ D^{\alpha} K &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_0^x \frac{K}{(x-t)^{\alpha-n+1}}\, dt \end{aligned}$$

Let $t = xu$, $0 \le u \le 1$ and $dt = xdu$, then

$$\begin{aligned} D^{\alpha} K &= \frac{K}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_0^1 (x-xu)^{n-\alpha-1}\, xdu \\ &= \frac{K}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} x^{n-\alpha} \int\limits_0^1 (1-u)^{n-\alpha-1}\, du \\ &= \frac{K}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} x^{n-\alpha} \beta(1, n-\alpha) \\ &= \frac{K}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} x^{n-\alpha} \frac{\Gamma(n-\alpha)\Gamma(1)}{\Gamma(n-\alpha+1)} \\ &= \frac{K}{\Gamma(n-\alpha+1)} \frac{d^n}{dx^n} x^{n-\alpha} \end{aligned}$$

$$= \frac{K}{\Gamma(n-\alpha+1)} \frac{\Gamma(n-\alpha+1)}{\Gamma(1-\alpha)} x^{-\alpha}$$
$$= \frac{K}{\Gamma(1-\alpha)} x^{-\alpha}$$

Thus, we have establish that

$$D^{\alpha} K = \frac{K}{\Gamma(1-\alpha)} x^{-\alpha} \tag{3.3}$$

From this example we can say that, the fractional derivative of a constant is not zero by Riemann-Liouville definition. Note that it is inconsistent result, since the result is a function of x.

Example 3.2. Power Function

Suppose we wish to find the fractional derivative of the power function $f(x) = x^m$, $m \geq 0$.

Solution:

By the definition of Riemann-Liouville fractional derivative we obtain

$$D^{\alpha}(x)^m = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{(t)^m}{(x-t)^{\alpha-n+1}}\, dt$$

set $t = ux$ for $0 \leq u \leq 1$, $dt = xdu$, we get

$$\begin{aligned} D^{\alpha}(x)^m &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^1 (xu)^m (x(1-u))^{n-\alpha-1}\, xdu \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} x^{m+n-\alpha} \int_0^1 u^m (1-u)^{n-\alpha-1}\, du \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} x^{m+n-\alpha} \beta(m+1, n-\alpha) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} x^{m+n-\alpha} \frac{\Gamma(m+1)\Gamma(n-\alpha)}{\Gamma(m+n-\alpha+1)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(m+1)}{\Gamma(m+n-\alpha+1)} \frac{d^n}{dx^n} x^{m+n-\alpha} \\
&= \frac{\Gamma(m+1)}{\Gamma(m+n-\alpha+1)} \frac{\Gamma(m+n-\alpha+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha} \\
&= \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}
\end{aligned}
$$

In the above example which is known as the power rule we obtain

$$D^\alpha x^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha} \quad , m \geq 0 \tag{3.4}$$

From this formula we can find the fractional derivative of any polynomial, by taking fractional derivatives of each term separately.

Example 3.3. Find the second derivative of the function $f(x) = x^2$.

Solution:

Using equation (3.4) we obtain

$$
\begin{aligned}
D^2 x^2 &= \frac{\Gamma(2+1)}{\Gamma(2-2+1)} x^{2-2} \\
&= 2
\end{aligned}
$$

Which is the same result when we differentiate x^2 twice in classical derivative.

Example 3.4. Let us take $f(x) = x^2$ as an example to find some of its fractional derivatives

1. if $\alpha = \frac{1}{3}$ then $D^{1/3}x^2 = \frac{\Gamma(3)}{\Gamma(8/3)} x^{5/3}$
2. if $\alpha = \frac{1}{2}$ then $D^{1/2}x^2 = \frac{\Gamma(3)}{\Gamma(5/2)} x^{3/2}$
3. if $\alpha = \frac{3}{4}$ then $D^{3/4}x^2 = \frac{\Gamma(3)}{\Gamma(9/4)} x^{5/4}$

Figure 3.1 illustrated these results.

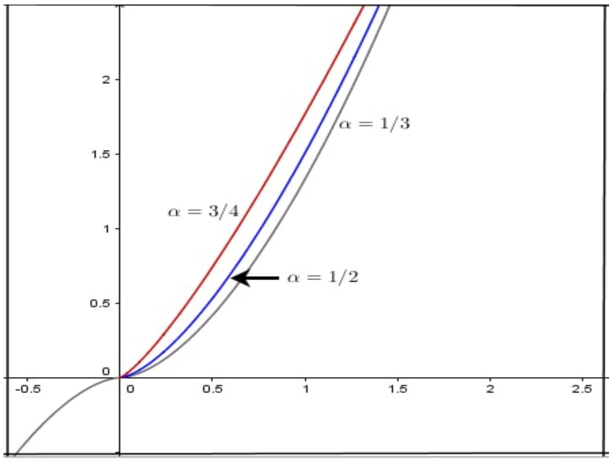

Figure 3.1: Fractional Derivatives of $f(x) = x^2$.

Example 3.5. Exponential Function

Suppose we want to find the fractional derivative of the exponential function e^{ax}.

Solution:

We want to solve this example by using equation (3.1), because it is more fitting. So we have

$$\begin{aligned} D^\alpha e^{ax} &= D^n[D^{-(n-\alpha)}e^{ax}] \\ &= D^n\left[\frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{e^{(at)}}{(x-t)^{\alpha-n+1}}\,dt\right] \\ &= D^n\left[\frac{1}{\Gamma(n-\alpha)}\int_0^x e^{(at)}(x-t)^{n-\alpha-1}\,dt\right] \end{aligned}$$

using the change of variable by letting $t = x(1-u)$, for $0 \le u \le 1$ and $dt = -xdu$.

$$\begin{aligned} D^\alpha e^{ax} &= D^n\left[\frac{1}{\Gamma(n-\alpha)}\int_0^1 (x-x(1-u))^{n-\alpha-1}e^{ax(1-u)}\,xdu\right] \\ &= D^n\left[\frac{1}{\Gamma(n-\alpha)}\int_0^1 x^{n-\alpha}u^{n-\alpha-1}e^{ax}e^{-axu}\,du\right] \\ &= D^n\left[\frac{e^{ax}x^{n-\alpha}}{\Gamma(n-\alpha)}\int_0^1 u^{n-\alpha-1}e^{-axu}\,du\right] \end{aligned}$$

Using the substitution $r = axu$, $dr = axdu$. then

$$\begin{aligned}
D^{\alpha}e^{ax} &= D^{n}\left[\frac{x^{n-\alpha}e^{ax}}{\Gamma(n-\alpha)}\int\limits_{0}^{ax}\left(\frac{r}{ax}\right)^{n-\alpha-1}e^{-r}\,\frac{dr}{ax}\right] \\
&= D^{n}\left[\frac{x^{n-\alpha}e^{ax}}{\Gamma(n-\alpha)}\frac{1}{(ax)^{n-\alpha}}\int\limits_{0}^{ax}r^{n-\alpha-1}e^{-r}\,dr\right] \\
&= D^{n}\left[e^{ax}x^{n-\alpha}\gamma^{*}(n-\alpha,ax)\right] \quad \textit{using equation } (2.10) \\
&= D^{n}\left[E_{x}(n-\alpha,a)\right] \quad \textit{using equation } (2.17)
\end{aligned}$$

Recall theorem (2.6) in chapter 2 we have

$$D^{-\alpha-1}[Df(x)] = D^{-\alpha}f(x) - \frac{f(0)}{\Gamma(\alpha+1)}x^{\alpha}$$

and

$$D[D^{-\alpha}f(x)] = D^{-\alpha}[Df(x)] + \frac{f(0)}{\Gamma(\alpha)}x^{\alpha-1}$$

Thus,

$$D^{-\alpha-1}[De^{ax}] = D^{-\alpha-1}[ae^{ax}] = D^{-\alpha}e^{ax} - \frac{f(0)}{\Gamma(\alpha+1)}x^{\alpha}$$

and

$$D[D^{-\alpha}e^{ax}] = D^{-\alpha}[ae^{ax}] + \frac{f(0)}{\Gamma(\alpha)}x^{\alpha-1}$$

Using the fractional integral of exponential

$$aE_{x}(\alpha+1,a) = E_{x}(\alpha,a) - \frac{f(0)}{\Gamma(\alpha+1)}x^{\alpha}$$

$$D[E_{x}(\alpha,a)] = aE_{x}(\alpha,a) + \frac{f(0)}{\Gamma(\alpha)}x^{\alpha-1}$$

If we replace α by $\alpha - 1$ in the first equation we have

$$aE_{x}(\alpha,a) = E_{x}(\alpha-1,a) - \frac{f(0)}{\Gamma(\alpha)}x^{\alpha-1}$$

Substitute $aE_{x}(\alpha,a)$ in the second equation, we get

$$D[E_{x}(\alpha,a)] = E_{x}(\alpha-1,a) - \frac{f(0)}{\Gamma(\alpha)}x^{\alpha-1} + \frac{f(0)}{\Gamma(\alpha)}x^{\alpha-1}$$

Thus,

$$D[E_x(\alpha, a)] = E_x(\alpha - 1, a)$$

similar, we can obtain

$$D^n[E_x(n-\alpha, a)] = E_x(n-\alpha-n, a) = E_x(-\alpha, a)$$

3.1.3 Properties of Riemann Liouville Fractional Derivative

Now after talking about the definition and examples, we are interested in study some basic properties of Riemann-Liouville fractional derivative and their proofs.

- **Representation**

  For $n-1 \leq \alpha < n$, $n \in \mathbb{N}$, and $f(x)$ be such that $D^\alpha f(x)$ exists, then

  $$D^\alpha f(x) = D^n[D^{-(n-\alpha)} f(x)] \tag{3.5}$$

  This shows that the Riemann-Liouville fractional derivative is equivalent to the composition of nth differentiation after $(n-\alpha)$ integration.

- **Linearity**

  Lemma 3.1. *[15] Let $n-1 \leq \alpha < n$, $n \in \mathbb{Z}^+$, $\alpha, \lambda \in \mathbb{C}$ and the functions f(x) and g(x) be such that both D^αf(x) and D^αg(x) exist. The Riemann Liouville fractional derivative is a linear operator, i.e.,*

  $$D^\alpha(\lambda f(x) + g(x)) = \lambda D^\alpha f(x) + D^\alpha g(x) \tag{3.6}$$

Proof:

By the definition of Riemann Liouville fractional derivative we have

$$
\begin{aligned}
D^{(\alpha)} f(x) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\, dt \\
D^{\alpha}(\lambda f(x) + g(x)) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_0^x \frac{(\lambda f(x) + g(x))}{(x-t)^{\alpha-n+1}}\, dt \\
&= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_0^x \frac{(\lambda f(x))}{(x-t)^{\alpha-n+1}}\, dt \\
&+ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_0^x \frac{g(x)}{(x-t)^{\alpha-n+1}}\, dt \\
&= \frac{\lambda}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_0^x \frac{f(x)}{(x-t)^{\alpha-n+1}}\, dt \\
&+ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_0^x \frac{g(x)}{(x-t)^{\alpha-n+1}}\, dt \\
&= \lambda D^{\alpha} f(x) + D^{\alpha} g(x)
\end{aligned}
$$

□

- **Interpolation**

Lemma 3.2. *[7] Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, and $f(x)$ be such that $D^{\alpha} f(x)$ exists, then the following is hold for the Riemann-Liouville fractional derivative*

$$\lim_{\alpha \to n} D^{\alpha} f(x) = f^{(n)}(x) \tag{3.7}$$

$$\lim_{\alpha \to n-1} D^{\alpha} f(x) = f^{(n-1)}(x) \tag{3.8}$$

Proof:

$$D^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\, dt$$

using integration by parts, then

$$\begin{array}{cc} u = f(t) & dv = (x-t)^{n-\alpha-1}dt \\ du = f'(t)dt & v = \frac{-(x-t)^{n-\alpha}}{(n-\alpha)} \end{array}$$

Thus,

$$\begin{aligned} D^\alpha f(x) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left[\frac{-f(t)}{(n-\alpha)}(x-t)^{n-\alpha} \Big|_0^x + \int_0^x \frac{f'(t)}{(n-\alpha)(x-t)^{n-\alpha}} \, dt \right] \\ &= \frac{1}{\Gamma(n-\alpha+1)} \frac{d^n}{dx^n} \left[f(0)x^{n-\alpha} + \int_0^x \frac{f'(t)}{(x-t)^{n-\alpha}} \, dt \right] \end{aligned}$$

Now, for $\alpha \to n$ and $\alpha \to n-1$ respectively we have

$$\lim_{\alpha \to n} D^\alpha f(x) = \frac{d^n}{dx^n} \left(f(0) + f(t) \Big|_0^x \right) = f^{(n)}(x)$$

and

$$\begin{aligned} \lim_{\alpha \to n-1} D^\alpha f(x) &= \frac{d^n}{dx^n} \left[\left(f(0)x + f(t)(x-t) \Big|_0^x \right) + \int_0^x f(t) \, dt \right] \\ &= \frac{d^n}{dx^n} \int_0^x f(t) \, dt \\ &= f^{(n-1)}(x). \end{aligned}$$

□

- **Non-commutation**

Lemma 3.3. *[7] Suppose that $n-1 < \alpha < n$, $m, n \in \mathbb{N}$, and the function $f(x)$ is such that $D^\alpha f(x)$ exists. Then in general*

$$D^m D^\alpha f(x) = D^{\alpha+m} f(x) \neq D^\alpha D^m f(x) \tag{3.9}$$

Proof:

We want to proof it using equation (3.1)

$$\begin{aligned}
D^{\alpha} f(x) &= D^{n}[D^{-(n-\alpha)} f(x)] \\
D^{m} D^{\alpha} f(x) &= D^{m} D^{n}[D^{-(n-\alpha)} f(x)] \\
&= D^{n} D^{m}[D^{-(n-\alpha)} f(x)] \\
&= D^{n} D^{m} D^{-m}[D^{-(n-\alpha-m)} f(x)] \\
&= D^{n}[D^{-(n-\alpha-m)} f(x)] \\
&= D^{\alpha+m} f(x)
\end{aligned}$$

□

To show that Riemann-Liouville derivative is not commutative we will give a counter example

Example 3.6. Take $f(x) = x$, $m = 2$, $\alpha = 1/3$. Then

$$D^{1/3} D^{2} x = 0$$

Where

$$\begin{aligned}
D^{2} D^{1/3} x &= D^{2}\left[\frac{\Gamma(1+1)}{\Gamma(1-1/3+1)} x^{1-1/3}\right] \\
&= \frac{-1}{3\Gamma(2/3)} x^{-4/3}
\end{aligned}$$

Thus, Rimann-Liouville derivative is not commute.

3.1.4 Composition with Riemann-Liouville Fractional Derivative

- **Fractional derivative of fractional integral**

Lemma 3.4. *[9] Let $n-1 \leq \alpha < n$, $m-1 \leq \beta < m$. $n, m \in \mathbb{N}$, and $f(x)$ be such that $D^{\alpha} f(x)$ exists, then we have two cases:*

1. $\beta \geq \alpha \geq 0$

$$\begin{aligned} D^{\alpha}[D^{-\beta} f(x)] &= D^{\alpha}[D^{-\alpha} D^{-(\beta-\alpha)} f(x)] \\ &= D^{\alpha-\beta} f(x) \end{aligned}$$

2. $\alpha > \beta \geq 0$

$$\begin{aligned} D^{\alpha}[D^{-\beta} f(x)] &= \frac{d^n}{dx^n}[D^{-(n-\alpha)}(D^{-\beta} f(x)] \\ &= \frac{d^n}{dx^n}[D^{\alpha-\beta-n}] \\ &= D^{\alpha-\beta} f(x) \end{aligned}$$

Both cases give the same result, so we have

$$D^{\alpha}[D^{-\beta} f(x)] = D^{\alpha-\beta} f(x), \quad \alpha, \beta > 0 \tag{3.10}$$

- **Fractional derivative of the fractional integral of the same order**

Corollary 3.5. *Let $n-1 \leq \alpha < n$, $n \in \mathbb{N}$, and $f(x)$ be such that $D^{\alpha} f(x)$ exists, then*

$$D^{\alpha}[D^{-\alpha} f(x)] = f(x) \tag{3.11}$$

Proof:

$$D^{\alpha}[D^{-\alpha} f(x)] = D^n[D^{-(n-\alpha)}(D^{-\alpha} f(x))]$$

$$
\begin{aligned}
&= D^n[D^{-n}f(x)] \\
&= f(x)
\end{aligned}
$$

□

- **Fractional integral of the fractional derivative**

Lemma 3.6. *[9] Let $n-1 \leq \alpha < n$, $m-1 \leq \beta < m$. $m, n \in \mathbb{N}$, and $f(x)$ be such that $D^{\alpha}f(x)$ exists, we have the same for $\beta \geq \alpha \geq 0$ and $\alpha > \beta \geq 0$, then*

$$
D^{-\alpha}[D^{\beta}f(x)] = D^{-\alpha+\beta}f(x) - \sum_{k=1}^{m} D^{\beta-k}f(0)\frac{x^{\alpha-k}}{\Gamma(\alpha-k+1)} \tag{3.12}
$$

Proof:

$$
\begin{aligned}
D^{-\alpha}[D^{\beta}f(x)] &= D^{-(\alpha-\beta)}D^{-\beta}[D^{\beta}f(x)] \\
&= D^{\beta-\alpha}[D^{-\beta}D^{\beta}f(x)] \\
&= D^{\beta-\alpha}\left[\frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}D^{\beta}f(t)\ dt\right] \\
&= D^{\beta-\alpha}\left[\frac{1}{\Gamma(\beta+1)}\frac{d}{dx}\int_0^x (x-t)^{\beta}\frac{d^n}{dx^n}D^{-(n-\beta)}f(t)\ dt\right]
\end{aligned}
$$

Now integrating by parts

$$
\begin{aligned}
u &= (x-t)^{\beta} & dv &= \tfrac{d^n}{dx^n}D^{-(n-\beta)}f(t)\ dt \\
du &= -\beta(x-t)^{\beta-1}dt & v &= \tfrac{d^{n-1}}{dx^{n-1}}D^{-(n-\beta)}f(t)
\end{aligned}
$$

Repeating this process n times we get

$$
\begin{aligned}
D^{-\beta}D^{\beta}f(x) &= \frac{d}{dx}D^{-(\beta+1-n)}[D^{-(n-\beta)}f(x)] - \sum_{k=1}^{n} D^{\beta-k}f(0)\frac{x^{\beta-k}}{\Gamma(\beta-k+1)} \\
&= f(x) - \sum_{k=1}^{n} D^{\beta-k}f(0)\frac{x^{\beta-k}}{\Gamma(\beta-k+1)}
\end{aligned}
$$

Thus,

$$\begin{aligned} D^{-\alpha}[D^{\beta}f(x)] &= D^{\beta-\alpha}\left[f(x)-\sum_{k=1}^{m}D^{\beta-k}f(0)\frac{x^{\alpha-k}}{\Gamma(\alpha-k+1)}\right] \\ &= D^{-\alpha+\beta}f(x)-\sum_{k=1}^{m}D^{\beta-k}f(0)D^{\beta-\alpha}\frac{x^{\alpha-k}}{\Gamma(\alpha-k+1)} \end{aligned}$$

By using the rule of the power function we obtain

$$\begin{aligned} &= D^{-\alpha+\beta}f(x)-\sum_{k=1}^{m}D^{\beta-k}f(0)\frac{\Gamma(\alpha-k+1)}{\Gamma(\alpha-k+1)}\frac{x^{\alpha-k}}{\Gamma(\alpha-k+1)} \\ &= D^{-\alpha+\beta}f(x)-\sum_{k=1}^{m}D^{\beta-k}f(0)\frac{x^{\alpha-k}}{\Gamma(\alpha-k+1)} \end{aligned}$$

□

- **Fractional integral of the fractional derivative of the same order**

Corollary 3.7. *[9] Let $n-1 \leq \alpha < n$, $n \in \mathbb{N}$, and $f(x)$ be such that $D^{\alpha}f(x)$ exists, then*

$$D^{-\alpha}D^{\alpha}f(x) = f(x)-\sum_{k=1}^{n}D^{\alpha-k}f(0)\frac{x^{\alpha-k}}{\Gamma(\alpha-k+1)} \tag{3.13}$$

- **Fractional derivative of fractional derivative**

Lemma 3.8. *[9] Let $n-1 \leq \alpha < n$, $n-1 \leq \beta < n$ $n, m \in \mathbb{N}$, and $f(x)$ be such that $D^{\alpha}f(x)$ exists, then*

$$D^{\alpha}[D^{\beta}f(x)] = D^{\alpha+\beta}f(x)-\sum_{k=1}^{m}D^{\beta-k}f(0)\frac{x^{-\alpha-k}}{\Gamma(1-\alpha-k)} \tag{3.14}$$

Proof:

$$D^{\alpha}[D^{\beta}f(x)] \quad = \quad D^{n}[D^{-(n-\alpha)}(D^{\beta}f(x))]$$

Using (3.12)

$$\begin{aligned}
&= D^n[D^{-(n-\alpha)+\beta} f(x) - \sum_{k=1}^{m} D^{\beta-k} f(0) \frac{x^{n-\alpha-k}}{\Gamma(n-\alpha-k+1)} \\
&= D^{\alpha+\beta} f(x) - \sum_{k=1}^{m} D^{\beta-k} f(0) \frac{\Gamma(n-\alpha-k+1)}{\Gamma(n-\alpha-k+1)} \frac{x^{-\alpha-k}}{\Gamma(-\alpha-k+1)} \\
&= D^{\alpha+\beta} f(x) - \sum_{k=1}^{m} D^{\beta-k} f(0) \frac{x^{-\alpha-k}}{\Gamma(-\alpha-k+1)}
\end{aligned}$$

□

- **Laplace transform of Riemann-Liouville fractional derivative**

Lemma 3.9. *[15] Suppose that $F(s)$ is the Laplace transform of $f(x)$. Then the Laplace transform of the Riemann-Liouville fractional differential operator of order α is given by*

$$\mathscr{L}\{D^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k [D^{\alpha-k-1} f(t)]_{t=0} \tag{3.15}$$

Proof:

Using equation (3.1) which is

$$D^\alpha f(x) = D^n[D^{-(n-\alpha)} f(x)]$$

Take Laplace transform, obtain

$$\begin{aligned}
\mathscr{L}\{D^\alpha f(t)\} &= \mathscr{L}\{D^n[D^{-(n-\alpha)} f(t)]\} \\
&= s^n \mathscr{L}\{D^{-(n-\alpha)} f(t)\} - \sum_{k=0}^{n-1} s^{n-k-1} D^k [D^{-(n-\alpha)} f(t)]_{t=0} \\
&= s^n [s^{-(n-\alpha)} F(s)] - \sum_{k=0}^{n-1} s^{n-k-1} D^{k-(n-\alpha)} f(0) \\
&= s^\alpha F(s) - \sum_{k=0}^{n-1} s^{n-k-1} D^{k-(n-\alpha)} f(0)
\end{aligned}$$

□

3.2 Caputo Fractional Derivative

In this section we will define Caputo fractional derivative and give some examples. then we study its properties. More details are in [7, 15].

3.2.1 Definition of Caputo Fractional Derivative

Caputo fractional derivative is defined using the definition of Riemann-Liouville fractional integral, the idea is fractional integrating the derivative of a function not a function itself, i.e.,

$$D^{-(1-\alpha)}[Df(x)] = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{\alpha} \frac{d}{dt} f(t)\ dt$$

Also,

$$D^{-(2-\alpha)}[D^2 f(x)] = \frac{1}{\Gamma(2-\alpha)} \int_0^x (x-t)^{\alpha-1} \frac{d^2}{dt^2} f(t)\ dt$$

Repeating this n times we obtain

$$D_*^{\alpha} f(x) = D^{-(n-\alpha)} \underbrace{\frac{d}{dx}\frac{d}{dx}\frac{d}{dx}\cdots\frac{d}{dx}}_{n} f(x)$$

Hence,

$$D_*^{\alpha} f(x) = D^{-(n-\alpha)}[D^n f(x)] \tag{3.16}$$

Thus, we can now introduce the definition of the Caputo fractional derivative.

Definition 3.2. [15] Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. The *Caputo* fractional derivative of order α of a function $f(x)$ is given by:

$$D_*^{\alpha} f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int\limits_0^x \dfrac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}\, dt, & n-1 \leq \alpha < n \\ \dfrac{d^n}{dx^n} f(x), & \alpha = n \in \mathbb{N} \end{cases} \tag{3.17}$$

3.2.2 Examples of Caputo Fractional Derivative

Example 3.7. Constant Function

Suppose we want to find Caputo derivative for any constant function K. By definition we have

$$\begin{aligned} D_*^\alpha f(x) &= \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}\,dt \\ D_*^\alpha K &= \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{K^{(n)}}{(x-t)^{\alpha-n+1}}\,dt \\ &= 0 \end{aligned}$$

Thus, we have

$$D_*^\alpha K = 0 \tag{3.18}$$

we can say that the fractional derivative of any constant function using Caputo definition is consistent. Since it is equal to zero.

Example 3.8. Power Function

Assume we aim to find the Caputo derivative of a power function $f(x) = x^m$, $m \geq 0$.

Solution:

By Caputo definition we have

$$\begin{aligned} D_*^\alpha f(x) &= \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}\,dt \\ D_*^\alpha x^m &= \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{(t^m)^{(n)}}{(x-t)^{\alpha-n+1}}\,dt \end{aligned}$$

But we know that

$$\frac{d^n}{dx^n} x^m = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}$$

So our Caputo derivative is given by:

$$\begin{aligned}
D_*^\alpha x^m &= \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \frac{\Gamma(m+1)}{\Gamma(m-n+1)} t^{m-n}\,dt \\
&= \frac{\Gamma(m+1)}{\Gamma(n-\alpha)\Gamma(m-n+1)} \int_0^x (x-t)^{n-\alpha-1} t^{m-n}\,dt
\end{aligned}$$

put $t = xu$, $0 \leq u \leq 1$, and $dt = xdu$

$$\begin{aligned}
D_*^\alpha x^m &= \frac{\Gamma(m+1)}{\Gamma(n-\alpha)\Gamma(m-n+1)} \int_0^1 (x-xu)^{n-\alpha-1} (xu)^{m-n}\,xdu \\
&= \frac{\Gamma(m+1)}{\Gamma(n-\alpha)\Gamma(m-n+1)} \int_0^1 x^{n-\alpha-1}(1-u)^{n-\alpha-1} u^{m-n} x^{m-n+1}\,du \\
&= \frac{\Gamma(m+1)}{\Gamma(n-\alpha)\Gamma(m-n+1)} \int_0^1 x^{m-\alpha}(1-u)^{n-\alpha-1} u^{m-n}\,du \\
&= \frac{\Gamma(m+1)}{\Gamma(n-\alpha)\Gamma(m-n+1)} x^{m-\alpha} \int_0^1 (1-u)^{n-\alpha-1} u^{m-n}\,du \\
&= \frac{\Gamma(m+1)}{\Gamma(n-\alpha)\Gamma(m-n+1)} x^{m-\alpha} \beta(m-n+1, n-\alpha) \\
&= \frac{\Gamma(m+1)}{\Gamma(n-\alpha)\Gamma(m-n+1)} x^{m-\alpha} \frac{\Gamma(m-n+1)\Gamma(n-\alpha)}{\Gamma(m-\alpha+1)} \\
&= \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}
\end{aligned}$$

Hence,

$$D_*^\alpha x^m = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha} \tag{3.19}$$

We can find the Caputo derivative of any polynomial by this equation taking each term separately.

Example 3.9. Find the Caputo Derivative of the function $f(x) = x^3$ with $\alpha = 2$.

Solution:

$$
\begin{aligned}
D_*^{\alpha} x^m &= \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha} \\
D_*^{2} x^3 &= \frac{\Gamma(3+1)}{\Gamma(3-2+1)} x^{3-2} \\
&= 6x.
\end{aligned}
$$

Example 3.10. If we want to find the Caputo derivative of the exponential function e^{ax} we have by definition

$$
\begin{aligned}
D_*^{\alpha} f(x) &= \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}\, dt \\
D_*^{\alpha} e^{ax} &= \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{\frac{d^n}{dt^n} e^{at}}{(x-t)^{\alpha-n+1}}\, dt \\
&= \frac{a^n}{\Gamma(n-\alpha)} \int_0^x \frac{e^{at}}{(x-t)^{\alpha-n+1}}\, dt \\
&= a^n E_x(n-\alpha, a)
\end{aligned}
$$

3.2.3 Properties of Caputo Fractional Derivative

In this section we will study the properties of Caputo fractional derivative and its proofs.

- **Representation**

  For $n-1 \le \alpha < n$, $n \in \mathbb{N}$, and $f(x)$ be such that $D_*^{\alpha} f(x)$ exists, then

  $$D_*^{\alpha} f(x) = D^{-(n-\alpha)}[D^n f(x)] \tag{3.20}$$

  This show that the Caputo fractional derivative is equivalent to the composition of $(n-\alpha)$ integration after nth differentiation.

- **Linearity**

Lemma 3.10. *[7] Let $n-1 \leq \alpha < n$, $n \in \mathbb{Z}^+$, $\alpha, \lambda \in \mathbb{C}$ and the functions f(x) and g(x) be such that both D_*^αf(x) and D_*^αg(x) exist. The Caputo fractional derivative is a linear operator,i.e.,*

$$D_*^\alpha(\lambda f(x) + g(x)) = \lambda D_*^\alpha f(x) + D_*^\alpha g(x) \tag{3.21}$$

Proof:

By the definition of Caputo fractional derivative we have

$$\begin{aligned}
D_*^\alpha f(x) &= \frac{1}{\Gamma(n-\alpha)} \int\limits_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}\, dt \\
D_*^\alpha(\lambda f(x) + g(x)) &= \frac{1}{\Gamma(n-\alpha)} \int\limits_0^x \frac{(\lambda f(x) + g(x))^{(n)}}{(x-t)^{\alpha-n+1}}\, dt \\
&= \frac{1}{\Gamma(n-\alpha)} \int\limits_0^x \frac{(\lambda f^{(n)}(x))}{(x-t)^{\alpha-n+1}}\, dt + \frac{1}{\Gamma(n-\alpha)} \int\limits_0^x \frac{g^{(n)}(x)}{(x-t)^{\alpha-n+1}}\, dt \\
&= \frac{\lambda}{\Gamma(n-\alpha)} \int\limits_0^x \frac{f^{(n)}(x)}{(x-t)^{\alpha-n+1}}\, dt + \frac{1}{\Gamma(n-\alpha)} \int\limits_0^x \frac{g^{(n)}(x)}{(x-t)^{\alpha-n+1}}\, dt \\
&= \lambda D_*^\alpha f(x) + D_*^\alpha g(x)
\end{aligned}$$

□

- **Interpolation**

Lemma 3.11. *[15] Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, and $f(x)$ be such that $D_*^\alpha f(x)$ exist, then the following is hold for the Caputo fractional derivative*

$$\lim_{\alpha \to n} D_*^\alpha f(x) = f^{(n)}(x), \tag{3.22}$$

$$\lim_{\alpha \to n-1} D_*^\alpha f(x) = f^{(n-1)}(x) - f^{(n-1)}(0). \tag{3.23}$$

Proof:

$$D_*^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}\, dt$$

By using integration by parts then we have

$$\begin{aligned} u &= f^{(n)}(t) & dv &= (x-t)^{n-\alpha-1}dt \\ du &= f^{(n+1)}(t)dt & v &= \frac{-(x-t)^{n-\alpha}}{(n-\alpha)} \end{aligned}$$

Thus,

$$\begin{aligned} D_*^\alpha f(x) &= \frac{1}{\Gamma(n-\alpha)} \left[\frac{-f^{(n)}(t)}{(n-\alpha)}(x-t)^{n-\alpha}\Bigg|_0^x + \int_0^x \frac{f^{(n+1)}(t)}{(n-\alpha)(x-t)^{\alpha-n}}\, dt \right] \\ &= \frac{1}{\Gamma(n-\alpha+1)} \left[f^{(n)}(0)x^{n-\alpha} + \int_0^x \frac{f^{(n+1)}(t)}{(x-t)^{\alpha-n}}\, dt \right] \end{aligned}$$

Now, for $\alpha \to n$ and $\alpha \to n-1$ respectively we have

$$\lim_{\alpha \to n} D_*^\alpha f(x) = (f^{(n)}(0) + f^{(n)}(t)\Big|_0^x) = f^{(n)}(x)$$

and

$$\begin{aligned} \lim_{\alpha \to n-1} D_*^\alpha f(x) &= \left(f^{(n)}(0)x + f^{(n)}(t)(x-t)\Big|_0^x \right) + \int_0^x f^{(n)}(t)\, dt \\ &= f^{(n-1)}(t)\Big|_0^x \\ &= f^{(n-1)}(x) - f^{(n-1)}(0). \end{aligned}$$

□

- **Non-commutation**

Lemma 3.12. *[7] Suppose that $n-1 < \alpha < n$, $m, n \in \mathbb{N}$, and the function $f(x)$ is such that $D_*^\alpha f(x)$ exists. Then in general*

$$D_*^\alpha D^m f(x) = D_*^{\alpha+m} f(x) \neq D^m D_*^\alpha f(x) \tag{3.24}$$

Proof:

$$\begin{aligned} D_*^\alpha D^m f(x) &= D^{-(n-\alpha)}[D^n D^m f(x)] \\ &= D^{-(n-\alpha)}[D^m D^n f(x)] \\ &= D^{-(n-\alpha)} D^m D^{-m}[D^m D^n f(x)] \\ &= D^{-(n-\alpha-m)}[D^n f(x)] \\ &= D_*^{m+\alpha} \end{aligned}$$

□

Now we give counter example to show that the Caputo derivative is not commute

Example 3.11. Take $f(x) = x^2$, $\alpha = \frac{1}{2}$, $m = 3$. Then

$$D_*^{1/2} D_*^3 x^2 = 0$$

But

$$\begin{aligned} D_*^3 D_*^{1/2} x^2 &= D_*^3 \left[\frac{\Gamma(2+1)}{\Gamma(2-1/2+1)} x^{2-1/2} \right] \\ &= \frac{-2}{5\sqrt{\pi}} x^{-3/2} \end{aligned}$$

Thus, Caputo derivative is not commute.

- **Fractional integral of fractional derivative of the same order**

Lemma 3.13. *Let $n-1 \leq \alpha < n$, $n \in \mathbb{N}$, and $f(x)$ be such that $D_*^\alpha f(x)$ exists, then we have*

$$D^{-\alpha}[D_*^\alpha f(x)] = f(x) - \sum_{k=1}^{n} D^{\alpha-k} f(0) \frac{x^{\alpha-k}}{\Gamma(\alpha-k+1)} \tag{3.25}$$

Proof:

$$
\begin{aligned}
D^{-\alpha}[D_*^{\alpha} f(x)] &= D^{-\alpha}[D^{-(n-\alpha)}(D^n f(x))] \\
&= D^{-\alpha}[D^{\alpha} f(x)] \\
&= f(x) - \sum_{k=1}^{n} D^{\alpha-k} f(0) \frac{x^{\alpha-k}}{\Gamma(\alpha-k+1)} \quad \textit{using equation } (3.13).
\end{aligned}
$$

□

- **Fractional derivative of fractional integral of the same order**

Lemma 3.14. *Let $n-1 \leq \alpha < n$, $n \in \mathbb{N}$, and $f(x)$ be such that $D_*^{\alpha} f(x)$ exists, then we have*

$$D_*^{\alpha}[D^{-\alpha} f(x)] = f(x) \tag{3.26}$$

Proof:

$$
\begin{aligned}
D_*^{\alpha}[D^{-\alpha} f(x)] &= D^{-(n-\alpha)}[D^n (D^{-\alpha}) f(x)] \\
&= D^{\alpha}[D^{-\alpha} f(x)] \\
&= f(x) \quad \textit{using equation } (3.11).
\end{aligned}
$$

□

- **Laplace transform of Caputo fractional derivative**

Lemma 3.15. *[15] Suppose that $F(s)$ is the Laplace transform of $f(x)$. Then the Laplace transform of the Caputo fractional differential operator of order α is given by*

$$\mathscr{L}\{D_*^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0) \tag{3.27}$$

Proof:

We will use the equation

$$D_*^\alpha f(x) = D^{-(n-\alpha)}[D^n f(x)]$$

Take Laplace transform

$$\begin{aligned}
\mathscr{L}\{D_*^\alpha f(t)\} &= \mathscr{L}\{D^{-(n-\alpha)}[D^n f(x)]\} \\
&= s^{-(n-\alpha)}\mathscr{L}\{D^n f(t)\} \\
&= s^{-(n-\alpha)}\left(s^n F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)\right) \\
&= s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)
\end{aligned}$$

□

- **Relation between Riemann-Liouville and Caputo fractional derivative**

Theorem 3.16. *[7] Let $x > 0$, $\alpha \in \mathbb{R}$, $n-1 \leq \alpha < n \in \mathbb{N}$. Then the following relation between the Riemann-Liouville and Caputo derivative hold*

$$D_*^\alpha f(x) = D^\alpha f(x) - \sum_{k=0}^{n-1} \frac{x^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(0) \tag{3.28}$$

Proof:

We will begin our proof with the definition of Riemann-Liouville derivative, then we integrating by parts as follows

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\, dt$$

Now, let

$$\begin{aligned} u &= f(t) & dv &= (x-t)^{n-\alpha-1}dt \\ du &= f'(t)dt & v &= \frac{-(x-t)^{n-\alpha}}{(n-\alpha)} \end{aligned}$$

Thus, we have

$$\begin{aligned} &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\left[\frac{-(x-t)^{n-\alpha}f(t)}{n-\alpha}\bigg|_0^x + \int_0^x \frac{f'(t)(x-t)^{n-\alpha}}{n-\alpha}\,dt\right] \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\left[\frac{x^{n-\alpha}f(0)}{n-\alpha} + \int_0^x \frac{f'(t)(x-t)^{n-\alpha}}{n-\alpha}\,dt\right] \end{aligned}$$

Integrating by parts $n-1$ times we get

$$\begin{aligned} &= \frac{d^n}{dx^n}\left[\sum_{k=0}^{n-1}\frac{x^{n+k-\alpha}f^{(k)}(0)}{\Gamma(n+k-\alpha+1)} + \frac{1}{\Gamma(2n-\alpha)}\int_0^x (x-t)^{2n-\alpha-1}f^{(n)}(t)\,dt\right. \\ &= \sum_{k=0}^{n-1}\frac{x^{k-\alpha}f^{(k)}(0)}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}f^{(n)}(t)\,dt \\ &= \sum_{k=0}^{n-1}\frac{x^{k-\alpha}f^{(k)}(0)}{\Gamma(k-\alpha+1)} + D_*^{\alpha}f(x) \end{aligned}$$

$\square$

Chapter 4

Fractional Differential Equations

After introducing the definition of the Riemann-Liouville and Caputo fractional derivative, some examples, properties and relations. Also finding the Laplace transform of these definitions, we will now study fractional differential equations (FDEs), that is the study of equations involving fractional derivatives, and solving it by applying the Laplace transform.

Before solving some fractional differential equations, we will give sufficient conditions for existence and uniqueness of solutions. We will introduce only the existence and uniqueness theorem for a continuous case of general linear fractional differential equations (LFDEs) with a Riemann-Liouville and Caputo fractional derivatives.

For the initial value problems for fractional differential equations with fractional derivatives in the Riemann-Liouville sense, it should be given as (bounded) initial values of the fractional integral $D^{-(n-\alpha)}$ and of its integer derivatives of order $k = 1, 2, ..., n$. For fractional differential equations with fractional derivatives in the Caputo sense, only the initial values of the function and its integer derivatives of order $k = 0, 1, ..., n-1$ are required.

The Laplace transform of the Caputo fractional derivative is a generalization of the Laplace transform of the integer-order derivative, where n is replaced by α. The same does not hold for the Riemann-Liouville case. This property is an important advantage of the Caputo operator over the Riemann-Liouville operator. For more details see [10, 20, 15].

4.1 The Existence and Uniqueness Theorem

In this section, we will introduce only the existence and uniqueness theorem for a continuous case of general linear fractional differential equations

Linear Fractional Differential Equations

Consider the following initial-value problem

$$D^{\alpha_m} y(t) + \sum_{k=1}^{m-1} p_k(t) D^{\alpha_{m-k}} y(t) + p_m(t) y(t) = f(t), \tag{4.1}$$

$$D^{\alpha_k - r} y(t)\Big|_{t=0} = b_k, \tag{4.2}$$

Where $n-1 \leq \alpha_m < n$, $\alpha_m > \alpha_{m-1} > \alpha_{m-2} > \ldots > \alpha_2 > \alpha_1 > 0$, $k = 1, \ldots, m$, $0 < t < T < \infty$ and $r = -[-\alpha_k]$. And $f(t) \in L_1(0, T)$, i.e.

$$\int_0^T |f(t)|\, dt < \infty.$$

Theorem 4.1. *[15]* ***(Existence and Uniqueness for LFDEs)***
If $f(t) \in L_1(0, T)$, and $p_j(t) (j = 1, \ldots, n)$ are continuous functions in the closed interval $[0, T]$, the the initial value problem (4.1)-(4.2) *has a unique solution $y(t) \in L_1(0, T)$.*

The proof of this theorem can be found in details in [15] page 124.

4.2 Linear Fractional Differential Equations (LFDE)

In this section, we will apply the Laplace transform to solve some fractional order differential equations, which is one of the most powerful methods of solving LFDEs with constant coefficients.

Definition 4.1. [12] A linear homogeneous fractional differential equation with constant coefficients is an equation of the form

$$D^{\alpha_m}y(t) + b_1 D^{\alpha_{m-1}}y(t) + b_2 D^{\alpha_{m-2}}y(t) + ... + b_m D^{\alpha_0}y(t) = 0 \tag{4.3}$$

Where α_i's are real numbers with $\alpha_m > \alpha_{m-1} > \alpha_{m-2} > ... > \alpha_0$, and the b_i's are constants.

Definition 4.2. A linear non-homogeneous fractional differential equation with constant coefficients is an equation of the form

$$D^{\alpha_m}y(t) + b_1 D^{\alpha_{m-1}}y(t) + b_2 D^{\alpha_{m-2}}y(t) + ... + b_m D^{\alpha_0}y(t) = h(t) \tag{4.4}$$

Where α_i's are real numbers with $\alpha_m > \alpha_{m-1} > \alpha_{m-2} > ... > \alpha_0$, and the b_i's are constants.

4.2.1 Fractional Differential Equations with Riemann-Liouville Derivative

Example 4.1. Suppose we want to find the solution of the initial value problem in the following form

$$D^{\alpha}y(t) = 0, \quad (t > 0); \tag{4.5}$$

$$D^{\alpha-k}y(t)\big|_{t=0} = b_k, \quad (k = 1, 2, ..., n), \tag{4.6}$$

where $n - 1 \leq \alpha < n$.

Solution:

To solve this problem, taking Laplace transform for equation (4.5) getting

$$\mathscr{L}\{D^{\alpha}y(t)\} = 0$$

Which implies

$$s^{\alpha}Y(s) - \sum_{k=1}^{n} s^{k-1}[D^{\alpha-k}y(t)]_{t=0} = 0$$

Solving it with respect to $Y(s)$, we have

$$Y(s) = \sum_{k=1}^{n} \frac{s^{k-1}}{s^{\alpha}}[D^{\alpha-k}y(t)]_{t=0}$$

Substitute the initial conditions in (4.6)

$$Y(s) = \sum_{k=1}^{n} \frac{1}{s^{\alpha-k+1}} b_k$$

Take Laplace inverse, and using table 2.2 we have

$$y(t) = \sum_{k=1}^{n} \frac{t^{\alpha-k}}{\Gamma(\alpha - k + 1)} b_k$$

So we can say that the solution of the equation (4.5) with initial conditions (4.6) is of the form

$$y(t) = \sum_{k=1}^{n} \frac{t^{\alpha-k}}{\Gamma(\alpha - k + 1)} b_k \tag{4.7}$$

Example 4.2. Solve the differential equation

$$D^{4/3}y(t) = 0,$$

with initial conditions $D^{1/3}y(0) = b_1$ and $D^{-2/3}y(0) = b_2$

Solution:

Taking the Laplace transform of both sides of the equation we have

$$\mathscr{L}\{D^{4/3}y(t)\} = 0$$

which implies that

$$s^{4/3}Y(s) - \sum_{k=1}^{2} s^{k-1}[D^{4/3-k}y(t)]_{t=0} = 0$$

which is equal

$$s^{4/3}Y(s) - D^{1/3}y(0) - sD^{-2/3}y(0) = 0$$

Solving for $Y(s)$. Then it becomes

$$Y(s) = \frac{D^{1/3}y(0) + sD^{-2/3}y(0)}{s^{4/3}}$$

substituting the initial conditions we have

$$Y(s) = \frac{b_1}{s^{4/3}} + \frac{b_2 s}{s^{4/3}}$$

Finally, we find the inverse Laplace of $Y(s)$ and conclude that the solution is given as follows

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left[\frac{b_1}{s^{4/3}}\right] + \mathscr{L}^{-1}\left[\frac{b_2 s}{s^{4/3}}\right] \\ &= b_1\frac{t^{1/3}}{\Gamma(4/3)} + b_2\frac{t^{-2/3}}{\Gamma(1/3)} \end{aligned}$$

Example 4.3. Let us consider the initial value problem for a homogeneous fractional differential equation under non-zero initial conditions

$$D^{\alpha}y(t) - \lambda y(t) = 0, \quad (t > 0); \tag{4.8}$$

$$D^{\alpha-k}y(t)\Big|_{t=0} = b_k, \quad (k = 1, 2, ..., n), \tag{4.9}$$

where $n - 1 \leq \alpha < n$.

Solution:

To solve this initial value problem we will use the Laplace transform method as we said before, so applying the Laplace transform to the fractional differential equation (4.8) we have by

$$\mathscr{L}\{D^{\alpha}y(t)\} - \mathscr{L}\{\lambda y(t)\} = 0$$

which becomes

$$s^{\alpha}Y(s) - \sum_{k=0}^{n-1} s^{k}[D^{\alpha-k-1}y(t)]_{t=0} - \lambda Y(s) = 0$$

but this is same as

$$s^{\alpha}Y(s) - \sum_{k=1}^{n} s^{k-1}[D^{\alpha-k}y(t)]_{t=0} - \lambda Y(s) = 0$$

this equation can be solved with respect to $Y(s)$ as follows

$$Y(s) = \sum_{k=1}^{n} \frac{s^{k-1}}{s^{\alpha} - \lambda}[D^{\alpha-k}y(t)]_{t=0}$$

Now substituting the initial conditions in (4.9) we have

$$Y(s) = \sum_{k=1}^{n} b_k \frac{s^{k-1}}{s^{\alpha} - \lambda}$$

Taking the Laplace inverse and using the table of Laplace transform pairs 2.2 we obtain

$$y(t) = \sum_{k=1}^{n} b_k t^{\alpha-k} E_{\alpha,\alpha-k+1}(\lambda t^{\alpha})$$

Hence, the fractional differential equation (4.8) with initial conditions (4.9) has its solution of the form

$$y(t) = \sum_{k=1}^{n} b_k t^{\alpha-k} E_{\alpha,\alpha-k+1}(\lambda t^{\alpha}) \tag{4.10}$$

Example 4.4. Solve the following fractional differential equation

$$D^{2/5}y(t) - 5y(t) = 0,$$

$$D^{2/5-1}y(0) = b_1$$

Solution:

Taking Laplace transform for each term

$$\mathscr{L}\{D^{2/5}y(t)\} - \mathscr{L}\{5y(t)\} = 0$$

Then we obtain

$$s^{2/5}Y(s) - D^{2/5-1}y(0) - 5Y(s) = 0$$

Solving with respect to $Y(s)$ and substituting the initial condition

$$Y(s) = \frac{b_1}{s^{2/5} - 5}$$

After taking inverse Laplace transform and using table 2.2 we have

$$y(t) = b_1 t^{-3/5} E_{2/5,2/5}(5t^{2/5})$$

Example 4.5. Let us consider the following equation

$$D^{\alpha_2}y(t)+\lambda_1 D^{\alpha_1}y(t)+\lambda_2 y(t)=0, \quad (t>0); \tag{4.11}$$

$$D^{\alpha_2-k}y(t)\Big|_{t=0} = b_k, \quad (k=1,2,...,n), \tag{4.12}$$

$$D^{\alpha_1-j}y(t)\Big|_{t=0} = c_j, \quad (j=1,2,...,m), \tag{4.13}$$

where $n-1\le\alpha_2<n$, and $m-1\le\alpha_1<m$.

Solution:

Take Laplace transform for each term in equation (4.11), we have

$$\mathscr{L}\{D^{\alpha_2}y(t)\}+\mathscr{L}\{\lambda_1 D^{\alpha_1}y(t)\}+\mathscr{L}\{\lambda_2 y(t)\}=0$$

Then it becomes

$$s^{\alpha_2}Y(s)-\sum_{k=0}^{n-1}s^k[D^{\alpha_2-k-1}y(t)]_{t=0}+\lambda_1 s^{\alpha_1}Y(s)-\lambda_1\sum_{j=0}^{m-1}s^j[D^{\alpha_1-j-1}y(t)]_{t=0}+\lambda_2 Y(s)=0$$

Which is the same of

$$s^{\alpha_2}Y(s)-\sum_{k=1}^{n}s^{k-1}[D^{\alpha_2-k}y(t)]_{t=0}+\lambda_1 s^{\alpha_1}Y(s)-\lambda_1\sum_{j=1}^{m}s^{j-1}[D^{\alpha_1-j}y(t)]_{t=0}+\lambda_2 Y(s)=0$$

Solve it with respect to $Y(s)$ we obtain

$$Y(s)=\frac{\sum_{k=1}^{n}s^{k-1}[D^{\alpha_2-k}y(t)]_{t=0}+\lambda_1\sum_{j=1}^{m}s^{j-1}[D^{\alpha_1-j}y(t)]_{t=0}}{s^{\alpha_2}+\lambda_1 s^{\alpha_1}+\lambda_2}$$

Substitute the initial conditions in (4.12)

$$Y(s)=\frac{\sum_{k=1}^{n}s^{k-1}b_k+\lambda_1\sum_{j=1}^{m}s^{j-1}c_j}{s^{\alpha_2}+\lambda_1 s^{\alpha_1}+\lambda_2}$$

Put $D_2=\sum_{k=1}^{n}s^{k-1}b_k$ and $D_1=\sum_{j=1}^{m}s^{j-1}c_j$

Thus we have

$$
\begin{aligned}
Y(s) &= \frac{D_2+\lambda_1 D_1}{s^{\alpha_2}+\lambda_1 s^{\alpha_1}+\lambda_2} \\
&= \frac{D_2+\lambda_1 D_1}{s^{\alpha_2}+\lambda_1 s^{\alpha_1}}.\frac{1}{1+\frac{\lambda_2}{s^{\alpha_2}+\lambda_1 s^{\alpha_1}}} \\
&= \frac{D_2 s^{-\alpha_1}+\lambda_1 D_1 s^{-\alpha_1}}{s^{\alpha_2-\alpha_1}+\lambda_1}.\sum_{i=0}^{\infty}\left(\frac{-\lambda_2}{s^{\alpha_2}+\lambda_1 s^{\alpha_1}}\right)^i \\
&= \frac{D_2 s^{-\alpha_1}+\lambda_1 D_1 s^{-\alpha_1}}{s^{\alpha_2-\alpha_1}+\lambda_1}.\sum_{i=0}^{\infty}\left(\frac{-\lambda_2}{s^{\alpha_2}+\lambda_1 s^{\alpha_1}}\right)^i\left(\frac{s^{-\alpha_1}}{s^{-\alpha_1}}\right)^i \\
&= \frac{D_2 s^{-\alpha_1}+\lambda_1 D_1 s^{-\alpha_1}}{s^{\alpha_2-\alpha_1}+\lambda_1}.\sum_{i=0}^{\infty}\frac{(-\lambda_2 s^{-\alpha_1})^i}{(s^{\alpha_2-\alpha_1}+\lambda_1)^i} \\
&= \sum_{i=0}^{\infty}(-\lambda_2)^i\frac{D_2 s^{-\alpha_1-\alpha_1 i}+\lambda_1 D_1 s^{-\alpha_1-\alpha_1 i}}{(s^{\alpha_2-\alpha_1}+\lambda_1)^{i+1}} \\
&= \sum_{i=0}^{\infty}(-\lambda_2)^i\frac{\sum_{k=1}^{n} s^{k-1} b_k s^{-\alpha_1-\alpha_1 i}+\lambda_1\sum_{j=1}^{m} s^{j-1} c_j s^{-\alpha_1-\alpha_1 i}}{(s^{\alpha_2-\alpha_1}+\lambda_1)^{i+1}} \\
&= \sum_{i=0}^{\infty}(-\lambda_2)^i\frac{\sum_{k=1}^{n} s^{k-1-\alpha_1-\alpha_1 i} b_k+\lambda_1\sum_{j=1}^{m} s^{j-1-\alpha_1-\alpha_1 i} c_j}{(s^{\alpha_2-\alpha_1}+\lambda_1)^{i+1}} \\
&= \sum_{i=0}^{\infty}(-\lambda_2)^i\left[\sum_{k=1}^{n}\frac{s^{k-1-\alpha_1-\alpha_1 i}}{(s^{\alpha_2-\alpha_1}+\lambda_1)^{i+1}} b_k+\lambda_1\sum_{j=1}^{m}\frac{s^{j-1-\alpha_1-\alpha_1 i}}{(s^{\alpha_2-\alpha_1}+\lambda_1)^{i+1}} c_j\right] \\
&= \sum_{i=0}^{\infty}(-\lambda_2)^i\left[\sum_{k=1}^{n}\frac{s^{k-1-\alpha_2 i-\alpha_2}}{(1+\lambda_1 s^{\alpha_1-\alpha_2})^{i+1}} b_k+\lambda_1\sum_{j=1}^{m}\frac{s^{j-1-\alpha_2 i-\alpha_2}}{(1+\lambda_1 s^{\alpha_1-\alpha_2})^{i+1}} c_j\right] \\
&= \sum_{i=0}^{\infty}(-\lambda_2)^i\left[\sum_{k=1}^{n} b_k s^{k-1-\alpha_2 i-\alpha_2}\sum_{r=0}^{\infty}\binom{i+r}{r}\left(-\lambda_1 s^{\alpha_1-\alpha_2}\right)^r\right. \\
&\qquad\left.+\lambda_1\sum_{j=1}^{m} c_j s^{j-1-\alpha_2 i-\alpha_2}\sum_{z=0}^{\infty}\binom{i+z}{z}\left(-\lambda_1 s^{\alpha_1-\alpha_2}\right)^z\right] \\
&= \sum_{i=0}^{\infty}(-\lambda_2)^i\left[\sum_{k=1}^{n} b_k\sum_{r=0}^{\infty}\binom{i+r}{r}(-\lambda_1)^r s^{\alpha_1 r-\alpha_2 r-\alpha_2 i-\alpha_2-1+k}\right. \\
&\qquad\left.+\lambda_1\sum_{j=1}^{m} c_j\sum_{z=0}^{\infty}\binom{i+z}{z}(-\lambda_1)^z s^{\alpha_1 z-\alpha_2 z-\alpha_2 i-\alpha_2-1+j}\right]
\end{aligned}
$$

Then taking Laplace inverse transform, we have a solution $y(t)$ given in the form

$$\begin{aligned} y(t) &= \sum_{i=0}^{\infty} \frac{(-\lambda_2)^i}{i!} \left[\sum_{k=1}^{n} b_k \sum_{r=0}^{\infty} \frac{\Gamma(i+r+1)(-\lambda_1)^r}{\Gamma(\alpha_2 r + \alpha_2 i + \alpha_2 - \alpha_1 r - k + 1)} . \frac{t^{\alpha_2 r + \alpha_2 i + \alpha_2 - \alpha_1 r - k}}{r!} \right. \\ & \left. + \lambda_1 \sum_{j=1}^{m} c_j \sum_{z=0}^{\infty} \frac{\Gamma(i+z+1)(-\lambda_1)^z}{\Gamma(\alpha_2 z + \alpha_2 i + \alpha_2 - \alpha_1 z - j + 1)} . \frac{t^{\alpha_2 z + \alpha_2 i + \alpha_2 - \alpha_1 z - j}}{z!} \right] \end{aligned}$$

Example 4.6. Consider the initial value problem for a non-homogeneous fractional differential equation under non-zero initial conditions

$$D^{\alpha} y(t) - \lambda y(t) = h(t), \quad (t > 0); \tag{4.14}$$

$$D^{\alpha-k} y(t)\big|_{t=0} = b_k, \quad (k = 1, 2, ..., n), \tag{4.15}$$

where $n - 1 \leq \alpha < n$.

Solution:

In order to solve this fractional equation, we take Laplace transform to each term as follows

$$\mathscr{L}\{D^{\alpha} y(t)\} - \mathscr{L}\{\lambda y(t)\} = \mathscr{L}\{\ h(t)\}$$

Then we have

$$s^{\alpha} Y(s) - \sum_{k=0}^{n-1} s^k [D^{\alpha-k-1} y(t)]_{t=0} - \lambda Y(s) = H(s)$$

Solve it with respect to $Y(s)$ and substitute the initial conditions we obtain

$$Y(s) = \frac{H(s)}{s^{\alpha} - \lambda} + \sum_{k=1}^{n} \frac{s^{k-1}}{s^{\alpha} - \lambda} b_k$$

Take the inverse Laplace transform and using table 2.2 we get a solution $y(t)$ as follows

$$y(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^{\alpha}) h(\tau)\ d\tau + \sum_{k=1}^{n} b_k t^{\alpha-k} E_{\alpha,\alpha-k+1}(\lambda t^{\alpha}) \tag{4.16}$$

4.2.2 Fractional Differential Equations with Caputo Derivative

Example 4.7. Let us consider the initial value problem as follows

$$D_*^\alpha y(t) = 0, \quad (t > 0); \tag{4.17}$$

$$y^{(k)}(0) = b_k, \quad (k = 0, 1, 2, ..., n-1), \tag{4.18}$$

where $n - 1 \leq \alpha < n$.

Solution:

Take Laplace transform for equation (4.17)

$$\mathscr{L}\{D_*^\alpha y(t)\} = 0$$

which implies

$$s^\alpha Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} y^{(k)}(0) = 0$$

Solve it wiht respect to $Y(s)$ we have

$$Y(s) = \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^\alpha} y^{(k)}(0)$$

Substituting the initial conditions (4.18) we get

$$Y(s) = \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^\alpha} b_k$$

Now take Laplace inverse we obtain a solution $y(t)$ given in the form

$$y(t) = \sum_{k=0}^{n-1} b_k \frac{t^k}{\Gamma(k+1)} \tag{4.19}$$

Example 4.8. Consider the initial value problem for a homogeneous fractional differential equation under non-zero initial conditions

$$D_*^\alpha y(t) - \lambda y(t) = 0, \quad (t > 0); \tag{4.20}$$

$$y^{(k)}(0) = b_k, \quad (k = 0, 1, ..., n - 1), \tag{4.21}$$

where $n - 1 \leq \alpha < n$.

Solution:

To solve this fractional equation, take Laplace transform as follows

$$\mathscr{L}\{D_*^\alpha y(t)\} - \mathscr{L}\{\lambda y(t)\} = 0$$

Then it becomes

$$s^\alpha Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} y^{(k)}(0) - \lambda Y(s) = 0$$

Solve it with respect to $Y(s)$ we get

$$Y(s) = \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^\alpha - \lambda} y^{(k)}(0)$$

Substitute the initial conditions in (4.21) we obtain

$$Y(s) = \sum_{k=0}^{n-1} \frac{s^{\alpha-k-1}}{s^\alpha - \lambda} b_k$$

Now taking Laplace inverse and using table 2.2, we have a solution as follow

$$y(t) = \sum_{k=0}^{n-1} b_k t^k E_{\alpha,k+1}(\lambda t^\alpha) \tag{4.22}$$

Example 4.9. A nearly simple harmonic vibration equation

$$D_*^\alpha y(t) + \omega^2 y(t) = 0, \quad (t > 0); \tag{4.23}$$

$$y^{(k)}(0) = b_k, \quad (k = 0, 1), \tag{4.24}$$

where $1 \leq \alpha < 2$. ω is the angular frequency.

Solution:

Using the result in the previous example, has its solution given by

$$y(t) = b_0 E_{\alpha,1}(-\omega^2 t^\alpha) + b_1 t E_{\alpha,2}(-\omega^2 t^\alpha) \tag{4.25}$$

Example 4.10. Let us consider the following equation

$$D_*^{\alpha_2} y(t) + \lambda_1 D_*^{\alpha_1} y(t) + \lambda_2 y(t) = 0, \quad (t > 0); \tag{4.26}$$

$$y^{(k)}(0) \;\; = \;\; b_k, \quad (k = 0, 1, ..., m-1, ..., n-1), \tag{4.27}$$

where $n - 1 \leq \alpha_2 < n$, and $m - 1 \leq \alpha_1 < m$.

Solution:

Take Laplace transform for each term in equation (4.26)

$$\mathscr{L}\{D_*^{\alpha_2} y(t)\} + \mathscr{L}\{\lambda_1 D_*^{\alpha_1} y(t)\} + \mathscr{L}\{\lambda_2 y(t)\} = 0$$

which implies

$$s^{\alpha_2} Y(s) - \sum_{k=0}^{n-1} s^{\alpha_2 - k - 1} y^{(k)}(0) + \lambda_1 s^{\alpha_1} Y(s) - \lambda_1 \sum_{k=0}^{m-1} s^{\alpha_1 - k - 1} y^{(k)}(0) + \lambda_2 Y(s) = 0$$

Solving with respect to $Y(s)$

$$Y(s) = \frac{\sum_{k=0}^{n-1} s^{\alpha_2 - k - 1} y^{(k)}(0) + \sum_{k=0}^{m-1} \lambda_1 s^{\alpha_1 - k - 1} y^{(k)}(0)}{s^{\alpha_2} + \lambda_1 s^{\alpha_1} + \lambda_2}$$

Substituting the initial conditions we have

$$Y(s) = \frac{\sum_{k=0}^{n-1} s^{\alpha_2 - k - 1} b_k + \lambda_1 \sum_{k=0}^{m-1} s^{\alpha_1 - k - 1} b_k}{s^{\alpha_2} + \lambda_1 s^{\alpha_1} + \lambda_2}$$

Solving it we have

$$\begin{aligned}
Y(s) &= \frac{\sum_{k=0}^{n-1} s^{\alpha_2-k-1}b_k + \lambda_1\sum_{k=0}^{m-1} s^{\alpha_1-k-1}b_k}{s^{\alpha_2}+\lambda_1 s^{\alpha_1}+\lambda_2} \\
&= \frac{\sum_{k=0}^{n-1} s^{\alpha_2-k-1}b_k + \lambda_1\sum_{k=0}^{m-1} s^{\alpha_1-k-1}b_k}{s^{\alpha_2}+\lambda_1 s^{\alpha_1}} . \frac{1}{1+\frac{\lambda_2}{s^{\alpha_2}+\lambda_1 s^{\alpha_1}}} \\
&= \frac{\left[\sum_{k=0}^{n-1} s^{\alpha_2-k-1}b_k + \lambda_1\sum_{k=0}^{m-1} s^{\alpha_1-k-1}b_k\right]s^{-\alpha_1}}{s^{\alpha_2-\alpha_1}+\lambda_1} . \sum_{i=0}^{\infty}\left(\frac{-\lambda_2}{s^{\alpha_2}+\lambda_1 s^{\alpha_1}}\right)^i \\
&= \frac{\left[\sum_{k=0}^{n-1} s^{\alpha_2-k-1}b_k + \lambda_1\sum_{k=0}^{m-1} s^{\alpha_1-k-1}b_k\right]s^{-\alpha_1}}{s^{\alpha_2-\alpha_1}+\lambda_1} . \sum_{i=0}^{\infty}\left(\frac{-\lambda_2}{s^{\alpha_2}+\lambda_1 s^{\alpha_1}}\right)^i\left(\frac{s^{-\alpha_1}}{s^{-\alpha_1}}\right)^i \\
&= \frac{\left[\sum_{k=0}^{n-1} s^{\alpha_2-k-1}b_k + \lambda_1\sum_{k=0}^{m-1} s^{\alpha_1-k-1}b_k\right]s^{-\alpha_1}}{s^{\alpha_2-\alpha_1}+\lambda_1} . \sum_{i=0}^{\infty}\frac{(-\lambda_2 s^{-\alpha_1})^i}{(s^{\alpha_2-\alpha_1}+\lambda_1)^i} \\
&= \sum_{i=0}^{\infty}(-\lambda_2)^i\frac{\left[\sum_{k=0}^{n-1} s^{\alpha_2-k-1}b_k + \lambda_1\sum_{k=0}^{m-1} s^{\alpha_1-k-1}b_k\right]s^{-\alpha_1-\alpha_1 i}}{(s^{\alpha_2-\alpha_1}+\lambda_1)^{i+1}} \\
&= \sum_{i=0}^{\infty}(-\lambda_2)^i\left[\sum_{k=0}^{n-1}\frac{(s^{\alpha_2-k-1-\alpha_1-\alpha_1 i})b_k}{(s^{\alpha_2-\alpha_1}+\lambda_1)^{i+1}} + \lambda_1\sum_{k=0}^{m-1}\frac{s^{\alpha_1-k-1-\alpha_1-\alpha_1 i}b_k}{(s^{\alpha_2-\alpha_1}+\lambda_1)^{i+1}}\right] \\
&= \sum_{i=0}^{\infty}(-\lambda_2)^i\left[\sum_{k=0}^{n-1}\frac{s^{-k-1-\alpha_2 i}}{(1+\lambda_1 s^{\alpha_1-\alpha_2})^{i+1}}b_k + \lambda_1\sum_{k=0}^{m-1}\frac{s^{-k-1-\alpha_2 i-\alpha_2+\alpha_1}}{(1+\lambda_1 s^{\alpha_1-\alpha_2})^{i+1}}b_k\right] \\
&= \sum_{i=0}^{\infty}(-\lambda_2)^i\left[\sum_{k=0}^{n-1} b_k s^{-k-1-\alpha_2 i}\sum_{r=0}^{\infty}\binom{i+r}{r}\left(-\lambda_1 s^{\alpha_1-\alpha_2}\right)^r\right. \\
&\qquad \left. + \lambda_1\sum_{k=0}^{m-1} b_k s^{-k-1-\alpha_2 i-\alpha_2+\alpha_1}\sum_{z=0}^{\infty}\binom{i+z}{z}\left(-\lambda_1 s^{\alpha_1-\alpha_2}\right)^z\right] \\
&= \sum_{i=0}^{\infty}(-\lambda_2)^i\left[\sum_{k=0}^{n-1} b_k\sum_{r=0}^{\infty}\binom{i+r}{r}(-\lambda_1)^r s^{\alpha_1 r-\alpha_2 r-k-1-\alpha_2 i}\right. \\
&\qquad \left. + \lambda_1\sum_{k=0}^{m-1} b_k\sum_{z=0}^{\infty}\binom{i+z}{z}(-\lambda_1)^z s^{\alpha_1 z-\alpha_2 z-k-1-\alpha_2 i-\alpha_2+\alpha_1}\right]
\end{aligned}$$

After taking Laplace inverse transform

$$\begin{aligned}
y(t) &= \sum_{i=0}^{\infty}\frac{(-\lambda_2)^i}{i!}\left[\sum_{k=0}^{n-1} b_k\sum_{r=0}^{\infty}\frac{\Gamma(i+r+1)(-\lambda_1)^r}{\Gamma(\alpha_2 r+\alpha_2 i+k-1-\alpha_1 r)} . \frac{t^{\alpha_2 r+\alpha_2 i+k-\alpha_1 r-2}}{r!}\right. \\
&\quad \left. + \lambda_1\sum_{k=0}^{m-1} b_k\sum_{z=0}^{\infty}\frac{\Gamma(i+z+1)(-\lambda_1)^z}{\Gamma(\alpha_2 z+\alpha_2 i+\alpha_2+k-1-\alpha_1 z-\alpha_1)} . \frac{t^{\alpha_2 z+\alpha_2 i+\alpha_2+k-\alpha_1 z-\alpha_1-2}}{z!}\right]
\end{aligned}$$

Bibliography

[1] Bologna, M. (2005). *Short Introduction to fractional calculus.* Lecture Notes, 41-54.

[2] Dalir, M., Bashour, M. (2010). *Applications of fractional calculus.* Applied Mathematical Sciences, 4(21), 1021-1032.

[3] David, S. A., Linares, J. L., Pallone, E. M. J. A. (2011). *Fractional order calculus: historical apologia, basic concepts and some applications.* Revista Brasileira de Ensino de Fsica, 33(4), 4302-4302.

[4] Freeden, W., Gutting, M. (2013). *Special functions of mathematical (geo-) physics.* Springer Science Business Media.

[5] Gradshteyn, I. S., Ryzhik, I. M. (2014). *Table of integrals, series, and products.* Academic press.

[6] Guce, I. K. (2013). *On fractional derivatives: the non-integer order of the derivative.* International Journal of Scientific Engineering, Research, 4(3), 1.

[7] Ishteva, M. (2005). *Properties and applications of the Caputo fractional operator.* MSc. Thesis.

[8] Kimeu, J. M. (2009). *Fractional calculus: Definitions and applications.* Masters Theses, 115.

[9] Kisela, T. (2008). *Fractional differential equations and their applications.* (Doctoral dissertation, Diploma thesis, 2008.

[10] Lin, S. D., Lu, C. H. (2013). *Laplace transform for solving some families of fractional differential equations and its applications.* Advances in Difference Equations, 2013(1), 137.

[11] Mathai, A. M., Haubold, H. J. (2008). *Special functions for applied scientists.* Springer Science+ Business Media.

[12] Miller, D. A. (2004). *Fractional Calculus.* Minor Thesis part of PHD.

[13] Miller, Kenneth S., and Bertram Ross (1993). *"An introduction to the fractional calculus and fractional differential equations."*

[14] Oldham, K., Spanier, J. (1974). *The fractional calculus theory and applications of differentiation and integration to arbitrary order.* (Vol. 111). Elsevier.

[15] Pandey, V. (2016). *Physical and Geometrical Interpretation of Fractional Derivatives in Viscoelasticity and Transport Phenomena.* (Doctoral dissertation, University of Oslo).

[16] Podlubny, I. (1998). *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.* (Vol. 198). Academic press.

[17] Ross, B. (1977). *The development of fractional calculus 16951900.* Historia Mathematica, 4(1), 75-89.

[18] Ross, B. (1975). *A brief history and exposition of the fundamental theory of fractional calculus.* In Fractional calculus and its applications (pp. 1-36). Springer Berlin Heidelberg.

[19] Sonin, N. Ya. *On differentiation with arbitrary index.* Moscow Matem. Sbornik 6.1 (1869): 1-38.

[20] Vance, D. (2014). *Fractional Derivatives and Fractional Mechanics.*

[21] Yousif, E. A., Alawad, F. A. (2012). *Laplace Transform Method Solution of Fractional Ordinary Differential Equations.*

[22] Zill, D. G. *Differential equations with boundary-value problems.* Nelson Education.