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GENERATING STATISTICAL DISTRIBUTIONS USING FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In a recent paper of Dixit and Ujlayan (UD), a new fractional derivative is introduced as a convex combination of the function and its first derivative; that is

$$D^\alpha f(x) = (1-\alpha)f(x) + \alpha f'(x).$$

In this article, a new technique of generating fractional continuous probability distributions by solving UD fractional differential equations that associated to well-known continous probability distributions is presented. In particular, the UD fractional probability distributions for the Exponential, Pareto, Lomax, and Levy distributions are generated. Finally, a real data application is considered for investigating the usefulness of the new fractional distributions. The results reveal that the proposed new fractional distribution performs better than the baseline distribution.

1. Introduction

In the science of modelling data, finding the best-suited distribution to fit the data being studied is of major concern. For example, studies showed that Rayleigh distribution can be used to model the individual ocean wave heights [1, 2]; Pareto distribution can be used to model extreme data such as the size of freak waves, the highest one-day rainfall in one year, etc. [3].

Several studies in the literature investigated new techniques that aim to produce new distributions to get more flexible data fitting. One of these techniques is to add a new parameter(s) to an existing distribution. The author in [4] developed skew

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normal distribution by adding the skewness parameter to the normal distribution. The authors in [5] introduced exponentiated Weibull distribution by raising the cumulative distribution function of the Weibull random variable to a new parameter. For more studies see [6], [7], [8], and the references therein.

Another technique to produce a new probability distribution is to consider the solution of a specific differential equation. The author in [9] was first to introduce a system of probability distributions by solving of differential equation. The Pearson probability distribution $f(x)$ is defined to be any solution to the following differential equation

$$\frac{f'(x)}{f(x)} + \frac{x-d}{ax^2+bx+c} = 0, \tag{1.1}$$

in which a, b, c, and d are distributional parameters. The solution of this differential equation is given by

$$f(x) = A \exp\left\{ \int \frac{d-x}{ax^2+bx+c} dx \right\}, \tag{1.2}$$

where A is the constant of integration. Letting $a = b = 0$, the differential equation produces the Normal probability distribution with mean equals $-d$ and variance equals c. For more details and some other distributions that can be produced by this differential equation see [9, 10, 11], and [12].

Another system of probability distributions was developed by [13]. The Burr probability distribution $f(x) = F'(x)$ is defined to be any solution to the following differential equation

$$\frac{F'(x)}{F(x)\,(1-F(x))} - g(x) = 0, \tag{1.3}$$

where $g(x)$ is any chosen non-negative function that makes $F(x)$ increases over its support and $0 \leq F(x) \leq 1$.

The solution of this differential equation is given by

$$F(x) = \frac{1}{1+e^{-\int g(x)dx}}. \tag{1.4}$$

The solution depends on the choice $g(x)$. For a list of different Burr types distributions and some more details see [12], [13], and [14].

As a generalization of ordinary differential equation, fractional differential equation is discussed in literature under different types of differential operator. Indeed, many researchers have been defined many types of fractional derivatives. The most well-known ones are Riemann-Liouville and the Caputo definitions.

(1) Riemann-Liouville definition. For $\alpha \in [n-1, n)$, the α-derivative of f is

$$D_{t_0}^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} \frac{f(x)}{(t-x)^{\alpha-n+1}} dx. \tag{1.5}$$

(2) Caputo definition. For $\alpha \in [n-1, n)$, the α-derivative of f is

$$D_{t_0}^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx. \tag{1.6}$$

However, the following are some of the setbacks of these definitions:

(1) The Riemann-Liouville derivative does not satisfy $D_a^{\alpha}(1) = 0$, while the Caputo definition does.

(2) The two definitions do not satisfy the product rule of differentiation: $D_a^{\alpha}(fg) = f\left(D_a^{\alpha} g\right) + g\left(D_a^{\alpha} f\right)$.

(3) The two definitions do not satisfy the quotient rule of differentiation: $D_a^{\alpha}\left(\frac{f}{g}\right) = \frac{g(D_a^{\alpha} f) - f(D_a^{\alpha} g)}{g^2}$.

(4) The two definitions do not satisfy the chain rule:
$D_a^{\alpha}(f \circ g) = f^{(\alpha)}(g(t)) g^{(\alpha)}(t)$.

(5) The two definitions do not satisfy the index rule: $D^{\alpha} D^{\beta}(f) = D^{\alpha+\beta}(f)$, in general.

The authors in [15] have introduced a new fractional derivative called the conformable fractional derivative. For $0 < \alpha < 1$, the conformable fractional derivative of f, denoted by T_{α}, is defined by

$$T_{\alpha} f(t) = \lim_{h \to 0} \frac{f\left(t + h t^{1-\alpha}\right) - f(t)}{h}, \tag{1.7}$$

which is a natural extension of the derivative of order 1. It can be represented as

$$T_{\alpha} f(t) = t^{1-\alpha} f'(t). \tag{1.8}$$

It worth to note that the conformable fractional derivative satisfies almost all the classical properties of the usual first derivative.

Recently, conformable fractional differential equations are used to generate new probability distributions. For instance, [16] have generated some conformable fractional probability distributions, including fractional exponential distribution, fractional Lomax distribution, fractional Levy distribution, fractional Rayleigh distribution, and fractional gamma distribution. The properties of the conformable fractional Rayleigh probability distribution and the conformable gamma distribution have been investigated by [17] and [18], respectively. More recently, [19] investigated the conformable Lomax probability distribution and its properties.

Dixit and Ujlayan in [20] and [21] have defined a UD fractional derivative as a convex combination of the function and its first derivative, where $(D^{\alpha} f)(t) = (1-\alpha)f(t) + \alpha f'(t)$ for $\alpha \in (0,1]$. In addition, they have studied the main properties and results of this fractional differential operator.

Actually, the problem is, in spite of having a number of definitions to deal with a derivative of arbitrary order, to compute such derivative is very tough. One has to get a numerical solution while working with mathematical modeling involving fractional derivative. This point motivated us to use the UD fractional derivative which is easy at a glance of computation, analytic, and possess almost all the properties of the classical derivative. Therefore, the UD fractional differential operator reduces the complexity in the calculation to solve the fractional ordered differential equation and provides an analytic result. The advantage of the proposed definition is that it simply converts a fractional derivative into a convex combination of the function itself and its ordinary derivative without involving any variable, and finally the equation reduces to the classical differential equation which can be solved with existing known methods.

In this paper, UD fractional differential equation will be used to generate new fractional distributions based on some existing probability distributions.

The rest of the paper is organized as follows. In Section 2, the UD fractional derivative is described and some of its properties are presented. The UD fractional probability distribution functions for the exponential, Pareto, Levy, and Lomax distributions are developed in Sections 3, 4, 5, 6, respectively. Real data application is considered in Section 7. Section 8 concludes the paper.

2. The UD Derivatie

In this section, the definition of UD fractional derivative and some of its properties are presented. This section is mainly based on [21].

Definition 2.1. For a given function $f:[0,\infty)\to\mathbb{R}$ and $\alpha\in[0,1]$, the UD derivative of order α is defined as

$$D^{\alpha}f(x)=\lim_{\epsilon\to 0}\frac{e^{\epsilon(1-\alpha)}f\left(xe^{\frac{\epsilon\alpha}{x}}\right)-f(x)}{\epsilon} \tag{2.1}$$

If this limit exists, then $D^{\alpha}f(x)$ is called the UD derivative of f for $\alpha\in[0,1]$, with the understanding that $D^{\alpha}f(x)=\frac{d^{\alpha}f(x)}{dx^{\alpha}}$. Also, if f is UD differentiable in the interval $(0,x)$ for $x>0$ and $\alpha\in[0,1]$ such that $\lim_{x\to 0^+}f^{\alpha}(x)$ exist then,

$$f^{\alpha}(0)=\lim_{x\to 0^+}f^{\alpha}(x) \tag{2.2}$$

Theorem 2.1. *Let $f:[0,\infty)\to\mathbb{R}$ be a differentiable function and $\alpha\in[0,1]$. Then, f is UD differentiable, and*

$$D^{\alpha}f(x)=(1-\alpha)f(x)+\alpha f'(x). \tag{2.3}$$

Proof. By Definition 2.1, we have

$$\begin{aligned}
D^{\alpha}f(x)&=\lim_{\varepsilon\to 0}\frac{e^{\varepsilon(1-\alpha)}f\left(xe^{\frac{\varepsilon\alpha}{x}}\right)-f(x)}{\varepsilon}\\
&=\lim_{\varepsilon\to 0}\frac{\{1+\varepsilon(1-\alpha)+o\left(\varepsilon^2\right)\}\left[f\left\{x+\varepsilon\alpha+o\left(\varepsilon^2\right)\right\}\right]-f(x)}{\varepsilon}\\
&=\lim_{\varepsilon\to 0}\frac{\{1+\varepsilon(1-\alpha)\}\left[f(x)+f'(x)\{\varepsilon\alpha\}\right]-f(x)}{\varepsilon}\\
&=\lim_{\varepsilon\to 0}\frac{f(x)+\varepsilon(1-\alpha)f(x)+\varepsilon\alpha f'(x)-f(x)}{\varepsilon}\\
&=(1-\alpha)f(x)+\alpha f'(x),
\end{aligned}$$

where $\alpha\in[0,1]$. □

Remark 1. *The UD derivatives of order $\alpha,\alpha\in[0,1]$, of some elementary real-valued differentiable functions in $[0,\infty)$, can be given as follow:*

1. $D^{\alpha}(\lambda)=(1-\alpha)\lambda$ *for all constants* $\lambda\in\mathbb{R}$.
2. $D^{\alpha}\left((ax+b)^n\right)=(1-\alpha)(ax+b)^n+an\alpha(ax+b)^{n-1}$ *for all* $a,b\in\mathbb{R}$.
3. $D^{\alpha}\left(e^{ax+b}\right)=((1-\alpha)+a\alpha)e^{ax+b}$ *for all* $a,b\in\mathbb{R}$.

(4) $D^\alpha(\sin(ax+b)) = (1-\alpha)\sin(ax+b) + a\alpha\cos(ax+b)$ *for all* $a, b \in \mathbb{R}$.

(5) $D^\alpha(\cos(ax+b)) = (1-\alpha)\cos(ax+b) - a\alpha\sin(ax+b)$ *for all* $a, b \in \mathbb{R}$.

(6) $D^\alpha(\log(ax+b)) = (1-\alpha)\log(ax+b) + a\alpha(ax+b)^{-1}$ *for all* $a, b \in \mathbb{R}$.

Theorem 2.2. *Let f and g be two differentiable functions in $[0,\infty)$ and $0 \le \alpha, \gamma \le 1$, then the following properties hold:*

(1) *Linearity:* $D^\alpha(af+bg) = aD^\alpha f + bD^\alpha g$ *for all* $a, b \in \mathbb{R}$.
(2) *Product rule:* $D^\alpha(fg) = (D^\alpha f)\, g + \alpha(Dg)f$.
(3) *Quotient rule:* $D^\alpha\left(\frac{f}{g}\right) = \frac{(D^\alpha f)g - \alpha(Dg)f}{g^2}$, *provided* $g(x) \neq 0$ *for all* $x \in [0,\infty)$.
(4) *Chain rule:* $D^\alpha(f\circ g)(t) = (1-\alpha)(f\circ g)(t) + \alpha f'(g(t))g'(t)$. *So, the classical chain rule* ($D^\alpha_a(f\circ g) = f^{(\alpha)}(g(t))g^{(\alpha)}(t)$) *does not satisfied here.*
(5) *Commutativity:* $D^\alpha\left(D^\gamma\right) f = D^\gamma\left(D^\alpha\right) f$.

Proof. The proof of the first 4 parts is straightforward. Here is the proof of Part 5. Using Equation 2.3, we get

$$\begin{aligned} D^\alpha\left(D^\gamma\right) f &= (1-\alpha)(1-\gamma)f + \alpha(1-\gamma)f' + \gamma(1-\alpha)f' + \alpha\gamma f'' \\ &= D^\gamma\left(D^\alpha\right) f. \end{aligned}$$

This completes the proof. □

Remark 2. *The UD derivative of order $\alpha, \alpha \in [0,1]$, as given in Definition 2.1. violets the Leibnitz's rule for fractional derivatives, $D^\alpha(fg) \neq gD^\alpha f + fD^\alpha g$. It also violets the law of indices, $D^\alpha\left(D^\gamma\right) f \neq D^{\alpha+\gamma} f$.*

Definition 2.2. A fractional derivative D^α has a conformable property if $D^\alpha(t) \to f'(t)$ when $\alpha \to 1$.

Remark 3. *Equation 2.3 asserts that the UD derivative of order $\alpha, \alpha \in [0,1]$, of a differentiable function $f : [0,\infty) \to \mathbb{R}$, is a convex combination of the function and the first derivative itself. Also, $D^\alpha f(x) = f(x)$, for $\alpha = 0$ and $D^\alpha f(x) = f'(x)$, for $\alpha = 1$, i.e., the UD derivative posses conformable property.*

Theorem 2.3. *Let the function f be bounded in $[0,\infty)$. If f is UD differentiable for some $\alpha \in [0,1]$ at $x = a$, then f is continuous at $x = a$.*

Proof. We need to show that $\lim_{\epsilon\mapsto 0} f(x+\epsilon\alpha) = f(x)$.

$$\begin{aligned}
\lim_{\epsilon\mapsto 0} f(x+\epsilon\alpha) - f(x) &= \lim_{\epsilon\mapsto 0}\left(\frac{(1+\epsilon(1-\alpha))f(x+\epsilon\alpha) - \epsilon(1-\alpha)f(x+\epsilon\alpha) - f(x)}{\epsilon}\right)\epsilon \\
&= \lim_{\epsilon\mapsto 0}\left(\frac{(1+\epsilon(1-\alpha))f(x+\epsilon\alpha) - f(x)}{\epsilon}\right)\epsilon - \lim_{\epsilon\mapsto 0}\epsilon(1-\alpha)f(x+\epsilon\alpha) \\
&= \lim_{\epsilon\mapsto 0}\left(D^{\alpha}f\right)\epsilon - \lim_{\epsilon\mapsto 0}\epsilon(1-\alpha)f(x+\epsilon\alpha) \\
&= 0\,(\text{as } f \text{ is not unbounded for all } 0 \le x < \infty)
\end{aligned}$$

□

As a consequence, one can use fractional UD Calculus in probability theory. Indeed, it can be used as a tool in the determination of the structural form of probability distributions and in parameter estimation of the probability distributions, amongst other uses. Continuous probability density function can be expressed as ordinary differential equation whose solution is the probability density function, and conversely a UD fractional version of this ordinary differential equation can be solved to obtain a (new) fractional continuous probability distribution that is identical with the original one when $\alpha = 1$.

3. UD Fractional Exponential Distribution

A random variable X is said to have an exponential distribution if its probability density function is given by

$$f(x;\lambda) = \lambda e^{-\lambda x}, \quad x > 0,\ \lambda > 0. \tag{3.1}$$

Let $y = \lambda e^{-\lambda x}$, then the first derivative of y is given by

$$y' = -\lambda(\lambda e^{-\lambda x}) = -\lambda y. \tag{3.2}$$

This gives the following first order ordinary differential equation (DE)

$$y' + \lambda y = 0. \tag{3.3}$$

Now, consider the α-order DE with respect to the UD derivative as follows.

$$y^{(\alpha)} + \lambda y = 0$$

$$(1-\alpha)y + \alpha y' + \lambda y = 0$$

$$\alpha y' + (1-\alpha+\lambda)y = 0$$

$$y' + \left(\frac{1-\alpha+\lambda}{\alpha}\right) y = 0. \tag{3.4}$$

Equation 3.4 is linear first order DE with integrating factor

$$\nu(x) = e^{\int\left(\frac{1-\alpha+\lambda}{\alpha}\right)dx} = e^{\left(\frac{1-\alpha+\lambda}{\alpha}\right)x}. \tag{3.5}$$

The general solution is

$$y = \frac{A}{\nu(x)} = Ae^{\frac{\alpha-(\lambda+1)}{\alpha}x}. \tag{3.6}$$

Thus, the new probability distribution will be

$$f_\alpha(x) = Ae^{\frac{\alpha-(\lambda+1)}{\alpha}x}, \tag{3.7}$$

where the normalizing constant A can be found by solving the following equation

$$\int_0^\infty f_\alpha(x)dx = 1. \tag{3.8}$$

Thus,

$$A = \frac{(\lambda+1)-\alpha}{\alpha}, \quad \alpha \le \lambda + 1. \tag{3.9}$$

It is worth to observe that the restriction $(\alpha \le \lambda + 1)$ is needed for the integration to be convergent, but for the fractional derivative α must be between 0 and 1, and this is a subset of $(\alpha \le \lambda + 1)$. Thus, the integration is also convergent over $(0 < \alpha < 1)$.

Finally, the new probability distribution can be written as

$$f_\alpha(x) = \frac{(\lambda+1)-\alpha}{\alpha} e^{\frac{\alpha-(\lambda+1)}{\alpha}x}, \quad x > 0,\ \lambda > 0,\ 0 < \alpha < 1. \tag{3.10}$$

It is clear that, $f_\alpha(x)$ is again an exponential distribution with $\lambda^* = \frac{(\lambda+1)-\alpha}{\alpha}$. Note that

$$\lim_{\alpha \to 1^-} f_\alpha(x) = \lambda e^{-\lambda x} = f(x). \tag{3.11}$$

4. UD Fractional Pareto Distribution

The Pareto distribution with shape parameter k and scale parameter λ is given by

$$f(x;k,\lambda)=\frac{k\lambda^k}{x^{k+1}},\quad x>\lambda,\ \lambda>0,\ k>0. \tag{4.1}$$

Let $y=k\lambda^k x^{-(k+1)}$, then the first derivative of y is given by

$$y'=-(k+1)k\lambda^k x^{-(k+1)-1}=-\frac{k+1}{x}y. \tag{4.2}$$

This gives the following first order ordinary differential equation

$$y'+\left(\frac{k+1}{x}\right)y=0. \tag{4.3}$$

Now, consider the α-order DE with respect to the UD derivative as follows.

$$\begin{aligned} y^{(\alpha)}+\left(\frac{k+1}{x}\right)y&=0\\ (1-\alpha)y+\alpha y'+\left(\frac{k+1}{x}\right)y&=0\\ \alpha y'+\left((1-\alpha)+\left(\frac{k+1}{x}\right)\right)y&=0\\ y'+\left(\frac{(1-\alpha)}{\alpha}+\left(\frac{k+1}{\alpha x}\right)\right)y&=0. \end{aligned} \tag{4.4}$$

Equation 4.4 is linear first order DE with integrating factor

$$\nu(x)=e^{\int\left(\frac{(1-\alpha)}{\alpha}+\left(\frac{k+1}{\alpha x}\right)\right)dx}=e^{\left(\frac{1-\alpha}{\alpha}\right)x+\left(\frac{k+1}{\alpha}\right)\ln x}=x^{\frac{k+1}{\alpha}}e^{\left(\frac{1-\alpha}{\alpha}\right)x}. \tag{4.5}$$

The general solution is

$$y=\frac{A}{\nu(x)}=Ax^{-\frac{k+1}{\alpha}}e^{\left(\frac{\alpha-1}{\alpha}\right)x}. \tag{4.6}$$

Thus, the new probability distribution will be

$$f_\alpha(x)=Ax^{-\frac{k+1}{\alpha}}e^{\left(\frac{\alpha-1}{\alpha}\right)x}, \tag{4.7}$$

where the normalizing constant A can be found by solving the following equation

$$\int_\lambda^\infty f_\alpha(x)dx=1. \tag{4.8}$$

Thus,

$$\int_\lambda^\infty Ax^{-\frac{k+1}{\alpha}}e^{\left(\frac{\alpha-1}{\alpha}\right)x}dx=1. \tag{4.9}$$

By substitution, $u = \frac{x}{\lambda}$, the integral becomes

$$\int_1^{\infty} \frac{Ae^{\left(\frac{\alpha-1}{\alpha}\lambda\right)u}}{\lambda^{\frac{k+1}{\alpha}-1}u^{\frac{k+1}{\alpha}}}du = 1, \tag{4.10}$$

which can be rewritten as

$$A\lambda^{1-\frac{k+1}{\alpha}} \int_1^{\infty} \frac{e^{-\left(\frac{1-\alpha}{\alpha}\lambda\right)u}}{u^{\frac{k+1}{\alpha}}}du = 1. \tag{4.11}$$

Employing Eq. 06.34.02.0001.01 of [22], the following result is obtained.

$$A\lambda^{1-\frac{k+1}{\alpha}} E_{\frac{k+1}{\alpha}}\left(\left(\frac{1}{\alpha}-1\right)\lambda\right) = 1, \tag{4.12}$$

where E is the generalized exponential integral function and is given by

$$E_{\nu}(z) = \int_1^{\infty} \frac{e^{-zt}}{t^{\nu}}dt. \tag{4.13}$$

Thus,

$$A = \frac{1}{\lambda^{1-\frac{k+1}{\alpha}} E_{\frac{k+1}{\alpha}}\left(\left(\frac{1}{\alpha}-1\right)\lambda\right)}. \tag{4.14}$$

Therefore, the new probability distribution can be written as

$$f_{\alpha}(x) = \frac{x^{-\frac{k+1}{\alpha}} e^{\left(\frac{\alpha-1}{\alpha}\right)x}}{\lambda^{1-\frac{k+1}{\alpha}} E_{\frac{k+1}{\alpha}}\left(\left(\frac{1}{\alpha}-1\right)\lambda\right)}, \quad x > \lambda > 0,\ k > 0,\ 0 < \alpha < 1. \tag{4.15}$$

5. UD Fractional Levy Distribution

The Levy distribution with location parameter μ and scale parameter c is given by

$$f(x;\mu,c) = \sqrt{\frac{c}{2\pi}}\frac{e^{-\frac{c}{2(x-\mu)}}}{(x-\mu)^{3/2}}, \quad x > \mu,\ \mu > 0,\ c > 0. \tag{5.1}$$

Let $y = \sqrt{\frac{c}{2\pi}}\frac{e^{-\frac{c}{2(x-\mu)}}}{(x-\mu)^{3/2}}$, then the first derivative of y is given by

$$\begin{aligned} y' &= \sqrt{\frac{c}{2\pi}}\left(\frac{ce^{-\frac{c}{2(x-\mu)}}}{2(x-\mu)^{7/2}} - \frac{3e^{-\frac{c}{2(x-\mu)}}}{2(x-\mu)^{5/2}}\right) \\ &= \sqrt{\frac{c}{2\pi}}\frac{e^{-\frac{c}{2(x-\mu)}}}{(x-\mu)^{3/2}}\left(\frac{c}{2(x-\mu)^2} - \frac{3}{2(x-\mu)}\right) \\ &= y\left(\frac{c-3(x-\mu)}{2(x-\mu)^2}\right). \end{aligned} \tag{5.2}$$

This gives the following first order ordinary differential equation

$$2(x-\mu)^2 y' - (c - 3(x-\mu))\, y = 0. \tag{5.3}$$

Now, consider the α-order DE with respect to the UD derivative as follows.

$$2(x-\mu)^2 y^{(\alpha)} - (c - 3(x-\mu))\, y = 0$$

$$2(x-\mu)^2 \left((1-\alpha)y + \alpha y'\right) - (c - 3(x-\mu))\, y = 0$$

$$2(1-\alpha)(x-\mu)^2 y + 2\alpha(x-\mu)^2 y' - (c - 3(x-\mu))\, y = 0$$

$$y' + \left(\frac{1-\alpha}{\alpha} - \frac{c-3(x-\mu)}{2\alpha(x-\mu)^2}\right) y = 0. \tag{5.4}$$

Equation 5.4 is linear first order DE with integrating factor

$$\begin{aligned}\nu(x) &= e^{\int\left(\frac{1-\alpha}{\alpha} - \frac{c-3(x-\mu)}{2\alpha(x-\mu)^2}\right)dx}\\ &= e^{\frac{(1-\alpha)}{\alpha}x + \frac{c}{2\alpha(x-\mu)} + \frac{3}{2\alpha}\ln(x-\mu)}\\ &= (x-\mu)^{\frac{3}{2\alpha}} e^{\frac{(1-\alpha)}{\alpha}x + \frac{c}{2\alpha(x-\mu)}}.\end{aligned} \tag{5.5}$$

The general solution is

$$y = A(x-\mu)^{-\frac{3}{2\alpha}} e^{\frac{(\alpha-1)}{\alpha}x - \frac{c}{2\alpha(x-\mu)}}. \tag{5.6}$$

Thus, the new probability distribution will be

$$f_\alpha(x) = A(x-\mu)^{-\frac{3}{2\alpha}} e^{\frac{(\alpha-1)}{\alpha}x - \frac{c}{2\alpha(x-\mu)}}, \tag{5.7}$$

where the normalizing constant A can be found by solving the following equation

$$\int_\mu^\infty f_\alpha(x)dx = 1. \tag{5.8}$$

Thus,

$$\int_\mu^\infty A(x-\mu)^{-\frac{3}{2\alpha}} e^{\frac{(\alpha-1)}{\alpha}x - \frac{c}{2\alpha(x-\mu)}}\, dx = 1. \tag{5.9}$$

By substitution, $u = x - \mu$, the integral becomes

$$\int_0^\infty A u^{-\frac{3}{2\alpha}} e^{\frac{(\alpha-1)}{\alpha}(u+\mu) - \frac{c}{2\alpha u}}\, du = 1. \tag{5.10}$$

Which can be rewitten as

$$A e^{\frac{(\alpha-1)}{\alpha}\mu} \int_0^\infty u^{-\left(\frac{3}{2\alpha}-1\right)-1} e^{\frac{(\alpha-1)}{\alpha}\left(u + \frac{c/2(1-\alpha)}{u}\right)} du = 1. \tag{5.11}$$

Employing [23, Eq. 8.432.7], the following result is obtained.

$$2Ae^{\frac{(\alpha-1)}{\alpha}\mu}\left(\frac{c}{2(1-\alpha)}\right)^{\frac{1}{2}-\frac{3}{4\alpha}} K_{\frac{3}{2\alpha}-1}\left(\frac{\sqrt{2c(1-\alpha)}}{\alpha}\right) = 1, \tag{5.12}$$

where $K_\nu(\cdot)$ denotes the modified Bessel function of the second kind of order ν and is given by

$$K_\nu(z) = \frac{z^\nu}{2}\int_0^\infty t^{-\nu-1} e^{-\frac{1}{2}\left(t+\frac{z^2}{t}\right)} dt. \tag{5.13}$$

Thus,

$$A = \frac{e^{\frac{(1-\alpha)}{\alpha}\mu}\left(\frac{c}{2(1-\alpha)}\right)^{\frac{3}{4\alpha}-\frac{1}{2}}}{2K_{\frac{3}{2\alpha}-1}\left(\frac{\sqrt{2c(1-\alpha)}}{\alpha}\right)}. \tag{5.14}$$

Finally, the new probability distribution is written as

$$f_\alpha(x) = \frac{\left(\frac{c}{2(1-\alpha)}\right)^{\frac{3}{4\alpha}-\frac{1}{2}}}{2K_{\frac{3}{2\alpha}-1}\left(\frac{\sqrt{2c(1-\alpha)}}{\alpha}\right)}(x-\mu)^{-\frac{3}{2\alpha}} e^{\frac{(\alpha-1)(x-\mu)}{\alpha}-\frac{c}{2\alpha(x-\mu)}}, \quad x>\mu>0,\, c>0, 0<\alpha<1. \tag{5.15}$$

6. UD Fractional Lomax Distribution

The probability density function for the Lomax distribution with shape parameter β and scale parameter λ is given by

$$f(x;\beta,\lambda) = \frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}, \quad x>0,\ \beta>0,\ \lambda>0. \tag{6.1}$$

Let $y = \frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}$, then the first derivative of y is given by

$$\begin{aligned} y' &= -\frac{\beta(\beta+1)}{\lambda^2}\left(1+\frac{x}{\lambda}\right)^{-(\beta+2)} \\ &= -\frac{\beta(\beta+1)}{\lambda^2}\left(1+\frac{x}{\lambda}\right)^{-(\beta+1)}\left(1+\frac{x}{\lambda}\right)^{-1} \\ &= -\frac{(\beta+1)}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-1} y. \end{aligned} \tag{6.2}$$

This gives the following first order ordinary differential equation

$$y' + \frac{(\beta+1)}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-1} y = 0. \tag{6.3}$$

Now, consider the α-order DE with respect to the UD derivative as follows.

$$\begin{aligned} y^{(\alpha)} + \frac{(\beta+1)}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-1} y &= 0 \\ \alpha y' + \left((1-\alpha) + \frac{(\beta+1)}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-1}\right) y &= 0 \\ y' + \left(\frac{(1-\alpha)}{\alpha} + \frac{(\beta+1)}{\alpha\lambda}\left(1+\frac{x}{\lambda}\right)^{-1}\right) y &= 0. \end{aligned} \tag{6.4}$$

Equation 6.4 is linear first order DE with integrating factor

$$
\begin{aligned}
\nu(x) &= e^{\int\left(\frac{(1-\alpha)}{\alpha}+\frac{(\beta+1)}{\alpha\lambda}\left(1+\frac{x}{\lambda}\right)^{-1}\right)dx} \\
&= e^{\frac{(1-\alpha)}{\alpha}x+\frac{(\beta+1)}{\alpha}\ln\left(1+\frac{x}{\lambda}\right)} \\
&= e^{\frac{(1-\alpha)}{\alpha}x}\left(1+\frac{x}{\lambda}\right)^{\frac{(\beta+1)}{\alpha}}. \qquad (6.5)
\end{aligned}
$$

The general solution of Equation 6.4 is

$$y = \frac{A}{\nu(x)} = Ae^{\frac{(\alpha-1)}{\alpha}x}\left(1+\frac{x}{\lambda}\right)^{-\frac{(\beta+1)}{\alpha}}. \qquad (6.6)$$

Thus, the new probability distribution will be

$$f_\alpha(x) = Ae^{\frac{(\alpha-1)}{\alpha}x}\left(1+\frac{x}{\lambda}\right)^{-\frac{(\beta+1)}{\alpha}}, \qquad (6.7)$$

where the normalizing constant A can be found by solving the following equation

$$\int_0^\infty f_\alpha(x)dx = 1. \qquad (6.8)$$

Thus,

$$\int_0^\infty Ae^{\frac{(\alpha-1)}{\alpha}x}\left(1+\frac{x}{\lambda}\right)^{-\frac{(\beta+1)}{\alpha}}dx = 1. \qquad (6.9)$$

By substitution, $u = 1+\frac{x}{\lambda}$, the integral becomes

$$\int_1^\infty Ae^{\frac{(\alpha-1)}{\alpha}\lambda(u-1)}u^{-\frac{(\beta+1)}{\alpha}}\lambda du = 1. \qquad (6.10)$$

Equivalently,

$$A\lambda e^{\frac{(1-\alpha)}{\alpha}\lambda}\int_1^\infty e^{-\frac{(1-\alpha)}{\alpha}\lambda u}u^{-\frac{(\beta+1)}{\alpha}}du = 1. \qquad (6.11)$$

Again, by substitution, $t = \frac{(1-\alpha)}{\alpha}\lambda u$, the integral becomes

$$A\lambda e^{\frac{(1-\alpha)}{\alpha}\lambda}\left(\frac{\alpha}{(1-\alpha)\lambda}\right)^{1-\left(\frac{\beta+1}{\alpha}\right)}\int_{\left(\frac{1-\alpha}{\alpha}\right)\lambda}^\infty e^{-t}t^{-\left(\frac{\beta+1}{\alpha}\right)}dt = 1.$$

Employing [23, Eq. 8.350.2], the integral becomes

$$A\lambda e^{\frac{(1-\alpha)}{\alpha}\lambda}\left(\frac{\alpha}{(1-\alpha)\lambda}\right)^{1-\left(\frac{\beta+1}{\alpha}\right)}\Gamma\left(1-\frac{\beta+1}{\alpha}, \frac{1-\alpha}{\alpha}\lambda\right) = 1, \qquad (6.12)$$

where $\Gamma(\cdot,\cdot)$ is the upper incomplete gamma function and is given by

$$\Gamma(\zeta, x) = \int_x^\infty e^{-t}t^{\zeta-1}dt. \qquad (6.13)$$

Thus,

$$A = \frac{1}{\lambda e^{\frac{(1-\alpha)}{\alpha}\lambda}\left(\frac{\alpha}{(1-\alpha)\lambda}\right)^{1-\left(\frac{\beta+1}{\alpha}\right)}\Gamma\left(1-\frac{\beta+1}{\alpha},\frac{1-\alpha}{\alpha}\lambda\right)}. \tag{6.14}$$

Finally, the new probability distribution can be written as

$$f_\alpha(x) = \frac{e^{\frac{\alpha-1}{\alpha}(x+\lambda)}\left(1+\frac{x}{\lambda}\right)^{-\frac{\beta+1}{\alpha}}}{\lambda\left(\frac{\alpha}{(1-\alpha)\lambda}\right)^{1-\left(\frac{\beta+1}{\alpha}\right)}\Gamma\left(1-\frac{\beta+1}{\alpha},\frac{1-\alpha}{\alpha}\lambda\right)}, \quad x>0,\ \beta>0,\ \lambda>0,\ 0<\alpha<1. \tag{6.15}$$

7. Data Application

In this section, a real-life data set is analyzed for the purpose of illustration to show the usefulness and flexibility of the new UD fractional Levy distribution. The data set represents the fatigue life of 6061-T6 aluminum coupons cut parallel with the direction of rolling and oscillated at 18 cycles per second (See Table 1). This data set is fitted to the Levy and UD fractional Levy distributions. The MLEs, the Akaike information criterion (AIC), and Bayesian information criterion (BIC) are used as an assessment metrics for the two distributions. Moreover, Kolmogorov-Smirnov (K-S) test is used to check the goodness of fit. The values of the assessment metrics and the p-value of the K-S test for Fatigue data are evaluated and the results are reported in Table 2. It can be concluded that the UD fractional Levy distribution provides a superior fit for the Fatigue data to the Levy distribution. Moreover, Figure 2 depicts the histogram of Fatigue data plotted against the probability density function of a fitted Levy distribution (continuous red line), and a UD fractional Levy distribution (dashed blue line). Visually it is easy to see that the modified Levy distribution is the best fit for this data set.

Table 1. Fatigue data [24]

70	107	114	124	130	133	138	142	151	162	212	90	108	114	124
130	134	138	142	152	163	96	108	116	124	131	134	139	144	155
163	97	108	119	124	131	134	139	144	156	164	99	109	120	124
131	134	141	145	157	166	100	109	120	128	131	134	141	146	157
166	103	112	120	128	131	136	142	148	157	168	104	112	121	129
132	136	142	148	157	170	104	113	121	129	132	137	142	149	158
174	105	114	123	130	132	138	142	151	159	196				

TABLE 2. The MLEs, -2LL, AIC, BIC, and the p-values of K-S statistic for Fatigue data set under Levy and UD fractional Levy distributions

Distribution	Parameter estimates			-2LL	AIC	BIC	K-S
Levy (μ, c)	66.45	52.42	–	1140.32	1144.32	1149.55	$< 2.2e-16$
UD Fractional Levy (μ, c, α)	0.94	43.56	999.98	**924.59**	**930.59**	**938.43**	**0.1344**

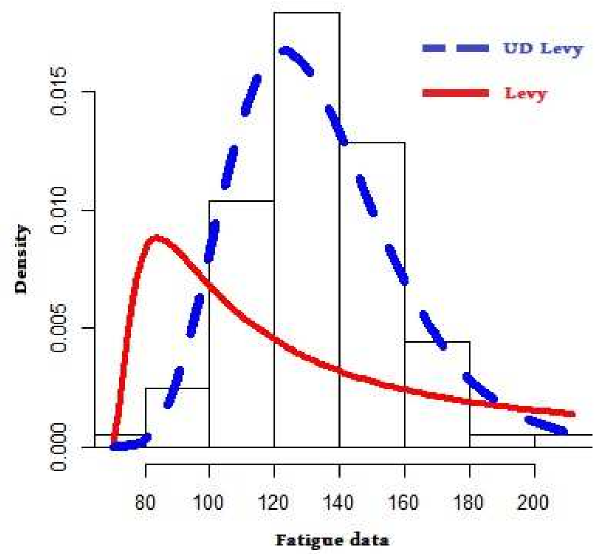

FIGURE 1. Histogram of the Fatigue data plotted against the density function of Levy (continuous red line) and UD fractional Levy (dashed blue line)

8. CONCLUSION

It is shown in this paper that UD fractional differential equations can be considered as a new technique to generate continuous fractional probability distributions. The fractional differentail equation is obtained from an existing probability distribution with number of parameters k, and the solution of this differential equations yeilds a (new) probability distribution with $k+1$ parameters. The resulting distribution could belongs to the same family of the baseline distribution as in the exponential distribution, or could belongs to a different family. Moreover, it is investigated that the new proposed fractional Levy distribution equips better fits than Levy itself by using a real data application.

As a future work, one may consider the new obtained fractional distributions and investigate their properties, including the effect of the new additional parameter (α) on the shape of the new obtained probability density and cumulative distribution functions, and other measures such as moments, skewness, kurtosis, entropy, etc. In addition, parameter estimation can be investigated.

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