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Dedications

I dedicate this work to

My beloved parents whose affection, love, encouragement, and prays of day and night make me able to get such success and honor.

My husband, a strong and gentle soul who supporting and encouraging me during all my master journal.

For my husband family, my sisters and brothers those who helped me to fly toward my dreams.

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Abstract

The variational iteration method (VIM) is a powerful method for solving a wide class of linear and nonlinear problems, first introduced by the Chinese mathematician He in 1999. This method is based on the use of Lagrange multiplier for evaluation of optimal value for parameters in a correction functional. The VIM has successfully been applied for a wide variety of scientific and engineering applications.

This thesis is concerned with the VIM for both ordinary and partial differential equations. Firstly, we present a brief introduction for the theory of calculus of variation, then the VIM is applied for ordinary differential equations. We consider both linear and nonlinear equations. In addition, a convergent analysis for a specific class of the differential equations is examined.

Furthermore, the VIM is applied to solve linear as well as nonlinear partial differential equations. In particular, the Laplace transform is used with the VIM to solve a class of partial differential equations.

Contents

Chapter 1

Introduction

The *Variation Iteration Method* (VIM) is one of the new powerful methods that capable of dealing with linear and nonlinear equations. This method is based on Lagrange multipliers and restricted variation, which make it simple and easier in calculations.

The VIM is distinguished from other efficient methods, such as A*domain Decomposition Method* (ADM) [31], *Runge-Kutta Method* [6], and also *Homotopy Analysis Method* (HAM) [5], that the VIM gives successive approximation that converge with high accuracy level to the exact solution if the solution exists.

The VIM does not require the presence of small parameters in the differential equations which would complicate the analytic calculations, and also it does not require that the nonlinearities be differentiable with respect to the dependent variable and its derivatives. In addition, the VIM provides efficient algorithm for analytic approximate solutions, see [19, 23].

Due to the importance of the VIM, considerable efforts have been devoted to the development and applications of this method. Firstly, The chinese mathematician

Ji-Huan He [16] used the VIM to solve several classical equations in view of Inokuti-Sekine-Mura considerations [16]. Then He used this method to approximate solutions for some linear problems [16, 18], nonlinear problems [5, 16, 18, 19, 34], and ordinary differential systems [5, 16, 18, 19, 34].

Nowadays, we find numerous applications for this method in solving various types of equations. In particular, in solving nonlinear ordinary differential equations, partial differential equations, integral equations and delay equations.

In [32] Wazwaz applied the VIM to solve linear and nonlinear ordinary differential equations. Altintan and Ugar used the variation iteration method for Sturm-Lioville differential equations in [3].

The VIM has been used to solve many physical models that are formulated by ordinary differential equations, from these model: the hybrid selection models [34], the Thomas-Fermi equation [34], unsteady flow of gas through a porous medium [34], the Riccati equation [34], mathematical pendulum [16], and a ball-bearing osscillator [16]. Moreover, Abbasbandy solved the quadratic Riccati differential equations by He's VIM with using Adomain's Polynomial [1].

As ordinary differential equations, the VIM is also applied for partial differential equations. Momani used the VIM to solve Helmholtz equation which is a second partial differential equation [21]. In addition, in [8], Bildik used the VIM, differential transform method and Adomain decomposition method for solving different types of nonlinear partial differential equations. In the study of Soliman and Abdou [2], they solved Burger's and coupled Burger's equations using the VIM.

In addition, the VIM has been used to solve linear and nonlinear differential equation with fractional orders by He's VIM, see the work of Odibat and Momani [24].

This thesis is mainly concerned with the VIM for ordinary and partial differential equations. We consider several forms of linear and nonlinear equations.

In Chapter 2, we present an introduction to the subject *calculus of variation*; since the technique of the VIM is based on the identification of Lagrange multiplier which is evaluated by using this theory. In Chapter 3, we present the principle of the VIM for solving ordinary differential equations (ODE's), in addition some scientific applications are considered. Chapter 4 is devoted to a specific form of ODE's. We find a correction functional for this form of ODE's. Moreover, the convergence of VIM is addressed for this form. In Chapter 5, we apply the VIM for partial differential equations (PDE's), then we use the Laplace transform with VIM to get an exact solution for some problems.

Chapter 2

Calculus of variation

The calculus of variation is concerned with maxima and minima theory, such that the main principles of calculus of variations is to find a function that makes certain integral smallest or largest possible, i.e. minimization or maximization.

It is expected that the first work on solving the problem of calculus of variation was due to the Queen Dido of Carthage, when she determined the largest land surrounded by bull's hide [22]. It seems that the first published article in this field was due to Newton. In his work in calculus of variation, he choosed which body shape that had the least resistance, for details see [22].

Later, calculus of variation has found numerous applications. In the seventeenth century, the brothers Johan and Jakob Bernoulli found the form of the curve that takes time as less as possible [22].

Nowadays, the theory of calculus of variation is considered as a comprehensive science. Many books and textbooks refer to this subject, see [7, 11, 20, 22].

This chapter is intended to be an introductory to the calculus of variation. At the begining, we present the main concepts of the subject, such as *functionals*, *stationary function*, *variation on functions*, and *the fundamental lemma in calculus of variation*. Then we present in details the theorems that depends on it. This theory is mainly used in the VIM as we will discuss in the rest of this thesis.

The material of this chapter is taken from [10, 14, 26].

To present the concept of the functionals, we consider the most famous example on this theory.

Example 2.1.
Consider the problem of finding the shortest curve C between two points in the plane. Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$ be any two point in a plane, and $A \neq B$, i.e. $x_1 \neq x_2$ and $y_1 \neq y_2$. Then the length of the curve between A and B is given by:

$$\begin{aligned} L &= \int_C ds \\ &= \int_{x_1}^{x_2} \sqrt{1 + y'^2}dx \end{aligned}$$

The solution of this problem is to find $y = f(x)$ that minimize the arc length functional among all admissible functions that satisfy the boundary conditions. It is clear that $L = L(f)$.

Figure 2-1: The shortest path from A to B is a straight line.

Definition 2.1. *A functional* is a correspondence between functions in some a well-defined class and the set of real numbers.

Now, let us consider the following examples.

Example 2.2. Let a and b be real numbers, and $y(x)$ is a bounded function on an interval $[a, b]$, and let

$$\Phi(y) = \min_{a \le x \le b} |y(x)|,$$

then $\Phi(y)$ is a functional since the result of this formula is a real number. i.e., the functional ϕ takes the function y as an input and produces a real number as an output.

Example 2.3. Suppose that

$$I(y) = \int_a^b F(x, y, y')dx, \tag{2.1}$$

Let the function F be a known function, and let y be a specified function. The domain of this functional is the space of a continuous functions with continuous first derivatives on an interval $[a, b]$, i.e., $y \in C^1[a, b]$). The functional such as (2.1) is mainly used in our work in this thesis.

Definition 2.2. *(Stationary function)*
A function g is stationary at the point x_0 when its first derivative at that point is 0, i.e.,

$$\left.\frac{dg(x)}{dx}\right|_{x=x_0} = 0.$$

Suppose that the class of functions S is a subset of C^1, and let x_1, x_2 be real numbers which satisfy

$$y(x_1) = y_1, \quad y(x_2) = y_2.$$

Now, consider the function $y + \epsilon\eta(x)$ such that y is in S and $\eta(x)$ is zero at x_1 and x_2 and is in S. i.e.

$$\eta(x_1) = \eta(x_2) = 0.$$

Hence, the sum $y + \epsilon\eta(x)$ is also in S for all $\epsilon > 0$.

Definition 2.3. *A First Variation* is the derivative of the functional, and defined as

$$\begin{aligned}\delta\phi(y,\eta) &= \lim_{\epsilon\to 0}\frac{\phi(y+\epsilon\eta)-\phi(y)}{\epsilon}\\ &= \frac{d}{d\epsilon}\phi(y+\epsilon\eta)\Big|_{\epsilon=0}.\end{aligned}$$

Let us consider the following example

Example 2.4. Suppose y be a real valued function, and let

$$\phi(y) = \int_a^b yy'dx$$

Then the first variation of $\phi(y)$ is

$$\begin{aligned}\delta\phi(y,\eta) &= \frac{d}{d\epsilon}\phi(y+\epsilon\eta)\Big|_{\epsilon=0}\\ &= \frac{d}{d\epsilon}\int_a^b (y+\epsilon\eta)(y'+\epsilon\eta')dx\Big|_{\epsilon=0}\\ &= \frac{d}{d\epsilon}\int_a^b (yy'+\epsilon y\eta'+\epsilon\eta y'+\epsilon^2\eta\eta')dx\Big|_{\epsilon=0}\\ &= \int_a^b \frac{d}{d\epsilon}(yy'+\epsilon y\eta'+\epsilon\eta y'+\epsilon^2\eta\eta')dx\Big|_{\epsilon=0}\\ &= \int_a^b [y\eta'+\eta y'+2\epsilon\eta\eta']dx\Big|_{\epsilon=0}\\ &= \int_a^b [y\eta'+\eta y']dx\\ &= \int_a^b (y\eta)'\,dx\end{aligned}$$

Definition 2.4. (*A neighbourhood*)
A function $y_0(x)$ is in the neighbourhood $N_\alpha(y)$ of the function $y(x)$, where $0 < \alpha \in \mathbb{R}$, if for all $x \in [a, b]$

$$\| y(x) - y_0(x) \| < \alpha$$

The extremal function is the function that makes variation zero. The fundamental problem of the calculus of variations is to find a function $y_0(x) \in S$ that optimize a given functional. The functional $I(y)$ takes an extremal value, maximum or minimum, with respect to all $y(x) \in S$ and belongs to the neighbourhood of $y_0(x)$, i.e. we aim to find $y_0(x)$ that minimizes or maximizes the integral $I(y)$ with respect to the neighbourhood $N_\alpha(y)$. For this purpose we present the next lemma.

Lemma 2.1. *(Fundamental lemma of the calculus of variation)*
Let $g(x)$ be continuous in $[a, b]$ and let $\eta(x)$ be an arbitrary function on $[a, b]$, where η, η', η'' are continuous and $\eta(a) = \eta(b) = 0$, If

$$\int_a^b g(x)\eta(x)dx = 0$$

for all such $\eta(x)$, then $g(x) = 0$ on $[a, b]$.

Proof.
Let c be an arbitrary real number. To the contrary, suppose that $g(x) > 0$ at $x = c$. Since $g(x)$ is continuous, there is a neighborhood $M = (\alpha_1, \alpha_2)$ of c, i.e. $\alpha_1 < c < \alpha_2$, in which $g(x) > 0$.
Consider

$$\eta(x) = \begin{cases} (x-\alpha_1)^3(\alpha_2 - x)^3, & \text{if } \alpha_1 < x < \alpha_2; \\ 0, & \text{elsewhere} \end{cases}$$

Then $g(x)\eta(x)$ has the same sign of $g(x)$, but

$$\int_a^b g(x)\eta(x)dx > 0$$

which is a contradiction. The proof is similar if we suppose $g(x) < 0$. □

For clarity, let us write the Euler-Lagrange Equation using the fundamental lemma in calculus of variation that we mentioned before.

Theorem 2.2. *(Euler-Lagrange Equations)*
Suppose that

$$I(y) = \int_a^b F(x, y, y')dx,$$

where $a, b, y(a), y(b)$ are given, $y''(x)$ is continuous, and F is a twice continuously differentiable function, then the extremal function $y = y_0(x)$ satisfies the equation

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

Proof.
Consider the functional

$$\Phi(\epsilon) = \int_a^b F(x, y, y')dx, \tag{2.2}$$

where $y = y_0 + \epsilon\eta(x)$, then $y' = y_0' + \epsilon\eta'$, and let y_0, and η be specified. Hence Φ is a function of ϵ with

$$\eta(a) = \eta(b) = 0.$$

Differentiating (2.2) with respect to ϵ, we get:

$$\begin{aligned}
\frac{d\Phi}{d\epsilon} &= \int_a^b \left[\frac{\partial F}{\partial y}\frac{\partial y}{\partial \epsilon} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \epsilon}\right] dx \\
&= \int_a^b [F_y\eta(x) + F_{y'}\eta'(x)]\, dx
\end{aligned}$$

Integrating the second term by parts, we have:

$$\begin{aligned}
\int_a^b F_{y'}\eta' dx &= \eta(x)F_{y'}\Big|_a^b - \int_a^b \eta(x)\frac{d}{dx}F_{y'}dx \\
&= 0 - \int_a^b \eta(x)\frac{d}{dx}F_{y'}dx \\
&= -\int_a^b \eta(x)\frac{d}{dx}F_{y'}dx.
\end{aligned}$$

And since at $y = y_0(x)$ we have an extremal value, thus

$$\left.\frac{d\phi}{d\epsilon}\right|_{\epsilon=0} = 0$$

Hence we get:

$$\begin{aligned}
\left.\frac{d\phi}{d\epsilon}\right|_{\epsilon=0} &= \int_a^b \left[F_y\eta(x) + F_{y'}\eta'(x)\right] dx \\
0 &= \int_a^b \left[F_y\eta(x) - \eta(x)\frac{d(F_{y'})}{dx}\right] dx \\
0 &= \int_a^b \eta(x)\left[F_y - \frac{d(F_{y'})}{dx}\right] dx
\end{aligned}$$

by the fundamental lemma of the calculus of variation 2.1, we have:

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

□

Example 2.5. Return to Example 2.1, in this example we interested in finding the shortest distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$, which is in the form:

$$L(y) = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx,$$

subject to the boundary conditions

$$y(x_1) = y_1,$$

$$y(x_2) = y_2.$$

Then

$$F(x, y, y') = \sqrt{1 + y'^2},$$

apply the Euler lagrange equation, we have:

$$\begin{aligned}
\frac{\partial F}{\partial y} &= 0, \\
\frac{\partial F}{\partial y'} &= \frac{y'}{\sqrt{1 + y'^2}}.
\end{aligned}$$

Hence,

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{y''}{(1 + y'^2)^{\frac{3}{2}}} = 0.$$

Thus, the shortest curve is described by the function that satisfies the differential equation

$$\frac{y''}{(1+y'^2)^{\frac{3}{2}}} = 0,$$

or

$$y'' = 0.$$

Integrating two times, we get:

$$\begin{aligned} y' &= c_1, \\ y &= c_1 x + c_2. \end{aligned}$$

Since y satisfies the boundary conditions, we get this solution:

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) + y_1$$

which is a straight line passes through the two points $A(x_1, y_1)$ and $B(x_2, y_2)$, as we expected and shown in figure2.1.

Chapter 3

Variational iteration method for ordinary differential equations

In this chapter we present the VIM for ordinary differential equations (ODE). At the begining we explain the basic methodology of the technique, then we apply the method for different forms of ordinary differential equations.

In Section 3.1, we present the methodology of the VIM for differential equations. Then, this method is used to solve linear ODE in Section 3.2 and nonlinear ODE in Section 3.3. The Section 3.4 is devoted to some famous physical models; these models involve the hybird selection model and the Riccati differential equation.

The VIM has been considered by many authers. For the material of this chapter, we refer to [3, 5, 10, 16, 18, 19, 27, 34]. In particular, we refer to the important work of He [16].

3.1 The VIM description

To illustrate the VIM technique, we consider the next general nonlinear system

$$L[u(t)] + N[u(t)] = g(t), \tag{3.1}$$

where L is a linear operator, N a nonlinear operator, and $g(t)$ is a given analytic function.

The VIM consider the correction functional for the system (3.1), as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,x)\,[L[u_n(x)] + N[u_n(x)] - g(x)]\,dx, \quad n \geq 0$$

where u_n is the *n-th* approximate solution, and we consider the nonlinear term $N[u_n(x)]$ as restricted variation, i.e., $\delta N[u_n(x)] = 0$, and λ is a general Lagrange multiplier which can be identified optimally by variational theory.

The main task of this method is to find the Lagrange multiplier $\lambda(t,x)$. Next we explain how to find $\lambda(t,x)$ for linear differential equations, then for nonlinear differential equations.

3.2 Linear differential equations

For linear problems the Lagrange multiplier can be exactly identified, hence the exact solution can be acquired by only one iteration.

To find $\lambda(t,x)$, we often use integration by parts. In particular, consider the following first order linear differential equation

$$u'(t) + a(t)u(t) = b(t),$$

with initial condition

$$u(0) = c.$$

Then the correction functional is identified as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,x)\left[\frac{du_n}{dx} + a(x)u_n(x) - b(x)\right] dx. \tag{3.2}$$

By taking the variation to the correction functional (3.2), and making it stationary, i.e. $\delta u_{n+1}(t) = 0$, we get:

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda(t,x)\left[\frac{du_n}{dx} + a(x)u_n(x) - b(x)\right] dx = 0,$$

notice that $\delta b(x) = 0$, then,

$$\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda(t,x)\frac{du_n}{dx}dx + \delta \int_0^t \lambda(t,x)a(x)u_n(x)dx - \delta \int_0^t \lambda(t,x)b(x)dx \\
0 &= \delta u_n(t) + \delta \int_0^t \lambda(t,x)\frac{du_n}{dx}dx + \delta \int_0^t \lambda(t,x)a(x)u_n(x)dx - \int_0^t \lambda(t,x)\delta b(x)dx \\
0 &= \delta u_n(t) + \delta \int_0^t \lambda(t,x)\frac{du_n}{dx}dx + \delta \int_0^t \lambda(t,x)a(x)u_n(x)dx
\end{aligned}$$

Integrating the first integral by parts, and using $\delta u_n(0) = 0$, we have

$$\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + [\lambda(t,x)\delta u_n(x)]_0^t - \int_0^t \frac{\partial \lambda}{\partial x}\delta u_n(x)dx + \int_0^t \lambda(t,x)a(x)\delta u_n(x)dx \\
0 &= \delta u_n(t) + \lambda(t,t)\delta u_n(t) - \lambda(t,0)\delta u_n(0) - \int_0^t \frac{\partial \lambda}{\partial x}\delta u_n(x)dx + \int_0^t \lambda(t,x)a(x)\delta u_n(x)dx \\
0 &= (1+\lambda(t,t))\delta u_n(t) + \int_0^t \left[-\frac{\partial \lambda}{\partial x} + a(x)\lambda(t,x)\right]\delta u_n(x).
\end{aligned}$$

Therefore, we have the following first order differential equation

$$-\frac{\partial \lambda}{\partial x} + a(x)\lambda(t,x) = 0 \tag{3.3}$$

with boundary condition

$$1 + \lambda(t,t) = 0$$

By separation of variables, we can identify the Lagrange multiplier as

$$\begin{aligned}\lambda(t,s) &= -\exp\left[\int_0^x a(s)ds - \int_0^t a(s)ds\right] \\ &= -\exp\left[\int_t^x a(s)ds\right]\end{aligned}$$

As a result, by substituting the value of $\lambda(t,x)$ into (3.2), we obtain:

$$u_{n+1}(t) = u_n(t) - \int_0^t e^{\int_t^x a(s)ds}\left[\frac{du_n}{dx} + a(x)u_n(x) - b(x)\right]dx \tag{3.4}$$

Let

$$u_0(t) = c\exp\left[-\int_0^t a(s)ds\right],$$

which is a solution of the homogeneous equation,

$$u' + a(t)u = 0,$$

with initial condition

$$u(0) = c.$$

Then by substituting into correction functional (3.4), we obtain

$$\begin{aligned}u_1(t) &= u_0(t) - \int_0^t e^{\int_t^x a(s)ds}\left[\frac{du_0}{dx} + a(x)u_0(x) - b(x)\right]dx \\ &= u_0(t) + \int_0^t b(x)e^{\int_t^0 a(s)ds}e^{\int_0^x a(s)ds}dx \\ &= c\ e^{-\int_0^t a(s)ds} + e^{-\int_0^t a(s)ds}\int_0^t b(x)e^{\int_0^x a(s)ds}dx \qquad (3.5)\end{aligned}$$

which is the exact solution. To illustrate this idea, let us consider the following example.

Example 3.1. Consider the differential equation

$$\frac{du}{dt} + \frac{u}{1+t^2} = \frac{\tan^{-1}(t)}{1+t^2}, \tag{3.6}$$

with initial condition

$$u(0) = 1. \tag{3.7}$$

Notice that

$$a(t) = \frac{1}{1+t^2}, \text{ and } b(t) = \frac{\tan^{-1}(t)}{1+t^2}.$$

To find the solution, we substitute into (3.5), so we get

$$\begin{aligned} u_1(t) &= e^{-\int_0^t \frac{1}{1+s^2}ds} + e^{-\int_0^t \frac{1}{1+s^2}ds} \int_0^t \frac{\tan^{-1}(x) e^{\int_0^x \frac{1}{1+s^2}ds}}{1+x^2} dx \\ &= e^{-\tan^{-1}(t)} + e^{-\tan^{-1}(t)} \int_0^t \frac{\tan^{-1}(x) e^{\tan^{-1}(x)}}{1+x^2} dx \\ &= e^{-\tan^{-1}(t)} + e^{-\tan^{-1}(t)} \int_0^{\tan^{-1}(t)} \xi e^{(\xi)} d\xi \end{aligned}$$

Integrating by parts, we obtain:

$$\begin{aligned} u_1(t) &= e^{-\tan^{-1}(t)} + e^{-\tan^{-1}(t)} \left[\xi e^{\xi} \Big|_0^{\tan^{-1}(t)} - \int_0^{\tan^{-1}(t)} e^{\xi} d\xi \right] \\ &= e^{-\tan^{-1}(t)} + e^{-\tan^{-1}(t)} \left[\tan^{-1}(t) e^{\tan^{-1}(t)} - e^{\tan^{-1}(t)} + 1 \right] \\ &= e^{-\tan^{-1}(t)} + e^{-\tan^{-1}(t)} + \tan^{-1}(t) - 1 \\ &= 2e^{-\tan^{-1}(t)} + \tan^{-1}(t) - 1 \end{aligned}$$

which is the exact solution for (3.6) satisfying the initial condition (3.7).

Next, let us consider the next second order linear ordinary differential equation,

$$u''(t) + \omega u'(t) + \omega^2 u(t) = b(t), \tag{3.8}$$

with initial condition

$$\begin{aligned} u(0) &= c, \\ u'(0) &= d. \end{aligned} \tag{3.9}$$

The correction functional is

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,x)\left[u_n''(x) + \omega u_n'(x) + \omega^2 u_n(x) - b(x)\right] dx, \quad n \geq 0.$$

Take the variation with respect to $u_n(x)$, this leads to

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda(t,x)\left[u_n''(x) + \omega u_n'(x) + \omega^2 u_n(x) - b(x)\right] dx \tag{3.10}$$

Applying the variation to (3.10), yields

$$\delta u_{n+1} = \delta u_n + \delta \int_0^t \lambda(t,x) u_n''(x) dx + \delta \int_0^t \lambda(t,x)\omega u_n'(x) dx + \delta \int_0^t \lambda(t,x)\omega^2 u_n(x) dx \tag{3.11}$$

Integrating the first two integral by parts, we obtain:

$$\begin{aligned} \int_0^t \lambda(t,x) u_n''(x) dx &= \lambda(t,t) u_n'(t) - \lambda(t,0) u_n'(0) - \left.\frac{\partial \lambda}{\partial x}\right|_{s=t} u_n(t) \\ &\quad + \left.\frac{\partial \lambda}{\partial x}\right|_{s=0} u_n(0) + \int_0^t \frac{\partial^2 \lambda}{\partial x^2} u_n(x) dx, \end{aligned}$$

and

$$\int_0^t \omega \lambda(t,x) u'(x) dx = \omega \lambda(t,t) u_n(t) - \omega \lambda(t,0) u_n(0) - \int_0^t \omega \frac{\partial \lambda}{\partial x} u(x) dx.$$

Or,

$$\begin{aligned}
\delta \int_0^t \lambda(t,x)u_n''(x)dx &= \lambda(t,t)\delta u_n'(t) - \lambda(t,0)\delta u_n'(0) - \left.\frac{\partial\lambda}{\partial x}\right|_{s=t} \delta u_n(t) \\
&\quad + \left.\frac{\partial\lambda}{\partial x}\right|_{s=0} \delta u_n(0) + \delta \int_0^t \frac{\partial^2\lambda}{\partial x^2} u_n(x)dx \\
&= \lambda(t,t)\delta u_n'(t) - \left.\frac{\partial\lambda}{\partial x}\right|_{s=t} \delta u_n(t) + \delta \int_0^t \frac{\partial^2\lambda}{\partial x^2} u_n(x)dx \qquad (3.12)
\end{aligned}$$

and

$$\begin{aligned}
\delta \int_0^t \omega\lambda(t,x)u'(x)dx &= \omega\lambda(t,t)\delta u_n(t) - \omega\lambda(t,0)\delta u_n(0) - \delta \int_0^t \frac{\partial\lambda}{\partial x}\omega u(x)dx \\
&= \omega\lambda(t,t)\delta u_n(t) - \delta \int_0^t \frac{\partial\lambda}{\partial x}\omega u(x)dx. \qquad (3.13)
\end{aligned}$$

Replacing the integrals in equation (3.11) by their values in equations (3.12) and (3.13), we have

$$\begin{aligned}
\delta u_{n+1} = \delta u_n + \lambda(t,t)\delta u_n'(t) &- \left.\frac{\partial\lambda}{\partial x}\right|_{x=t} \delta u_n(t) + \delta \int_0^t \frac{\partial^2\lambda}{\partial x^2} u_n(x)dx \\
&+ \omega\lambda(t,t)\delta u_n(t) - \delta \int_0^t \frac{\partial\lambda}{\partial x}\omega u(x)dx \\
&+ \int_0^t \lambda(t,x)\omega^2 u(x)dx,
\end{aligned}$$

i.e.

$$\begin{aligned}
\delta u_{n+1} &= \left[1 - \left.\frac{\partial\lambda}{\partial x}\right|_{s=t} + \omega\lambda(t,t)\right] \delta u_n(t) + \lambda(t,t)\delta u_n'(t) \\
&\quad + \delta \int_0^t \left[\frac{\partial^2\lambda}{\partial x^2} - \frac{\partial\lambda}{\partial x}\omega + \lambda\omega^2\right] u(x)dx.
\end{aligned}$$

Hence, we obtain the stationary conditions,

$$\frac{\partial^2\lambda}{\partial x^2} - \omega\frac{\partial\lambda}{\partial x} + \omega^2\lambda = 0, \tag{3.14}$$

$$1 - \left.\frac{\partial\lambda}{\partial x}\right|_{x=t} + \omega\lambda(t,t) = 0, \tag{3.15}$$

$$\lambda(t,t) = 0. \tag{3.16}$$

Where (3.14) is a second linear differential equation, and by solving it with respect to the two boundary condition (3.15) and (3.16), we get

$$\lambda(t,x) = -\frac{2}{\sqrt{3}\omega} e^{\frac{\omega}{2}(x-t)} \sin\left(\frac{\sqrt{3}}{2}\omega(t-x)\right). \tag{3.17}$$

Thus, for $n \geq 0$ the differential equation (3.8) has an iteration formula

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{-2}{\sqrt{3}\omega} e^{\frac{\omega}{2}(x-t)} \sin\left(\frac{\sqrt{3}}{2}\omega(t-x)\right) \left[u_n''(x) + \omega u_n'(x) + \omega^2 u_n(x) - b(x)\right] dx.$$

Let

$$u_0(t) = e^{-\frac{\omega}{2}t}\left[c \cos\left(\frac{\sqrt{3}}{2}\omega t\right) + \frac{2d + c\omega}{\sqrt{3}\omega} \sin\left(\frac{\sqrt{3}}{2}\omega t\right)\right],$$

which is a solution of the homogeneous equation

$$u''(t) + \omega u'(t) + \omega^2 u(t) = 0$$

with initial condition

$$\begin{aligned} u_0(0) &= c, \\ u_0'(0) &= d. \end{aligned}$$

Then,

$$\begin{aligned}
u_1(t) &= e^{-\frac{\omega}{2}t}\left(c\ \cos\left(\frac{\sqrt{3}}{2}\omega t\right) + \frac{2d+c\omega}{\sqrt{3}\omega}\ \sin\left(\frac{\sqrt{3}}{2}\omega t\right)\right) \\
&\quad + \int_0^t \frac{-2}{\sqrt{3}\omega} e^{\frac{\omega}{2}(x-t)} \sin\left(\frac{\sqrt{3}}{2}\omega(t-x)\right)(-b(x))\,dx \\
&= e^{-\frac{\omega}{2}t}\left(c\ \cos\left(\frac{\sqrt{3}}{2}\omega t\right) + \frac{2d+c\omega}{\sqrt{3}\omega}\ \sin\left(\frac{\sqrt{3}}{2}\omega t\right)\right) \\
&\quad + \frac{2}{\sqrt{3}\omega} e^{-\frac{\omega}{2}t} \int_0^t e^{\frac{\omega}{2}x} \sin\left(\frac{\sqrt{3}}{2}\omega(t-x)\right) b(x)dx \qquad (3.18)
\end{aligned}$$

which is the exact solution of (3.8) with initial condition (3.9).

In the next example we consider a specific second order linear differential equation.

Example 3.2. We aim to find the exact solution to the following second linear differential equation,

$$u''(t) + u'(t) + u(t) = \sin t,$$

with initial condition

$$\begin{aligned}
u_0(0) &= 0, \\
u_0'(0) &= 1.
\end{aligned}$$

notice that $c = 0$, $d = 1$, $\omega = 1$, and $b(t) = \sin t$, hence by substituting into (3.18), the exact solution is

$$\begin{aligned}
u_1(t) &= e^{\frac{-t}{2}}\left[0 + \frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right] + \frac{2}{\sqrt{3}}e^{\frac{-t}{2}}\int_0^t e^{\frac{x}{2}} sin\left[\frac{\sqrt{3}}{2}(t-x)\right]\sin(x)dx \\
&= \frac{2}{\sqrt{3}}e^{\frac{-t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right) + e^{\frac{-t}{2}}\cos(\frac{\sqrt{3}}{2}t) - \cos(t).
\end{aligned}$$

For more examples, see [3, 5, 16, 18, 19, 27]

3.3 Nonlinear differential equations

Unlike linear differential equations, where the Lagrange multiplier is exactly identified, the Lagrange multiplier in nonlinear differential equations are difficult to be identified. In order to overcome this difficulty, we apply restricted variation to nonlinear term.

Firstly, let us consider the first order linear differential equation of the form

$$u'(t) + \alpha u(t) = 0, \tag{3.19}$$

with initial condition

$$u(0) = c.$$

To best illustrate the restricted variation, we consider u_n as a restricted variation. Thus, the correction functional is

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,x)\left[u_n'(x) + \alpha u_n(x)\right] dx, \quad n \geq 0 \tag{3.20}$$

Take the variation with respect to $u_n(x)$, we get

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda(t,x)\left[u_n'(x) + \alpha u_n(x)\right] dx, \tag{3.21}$$

or equivalently,

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda(t,x) u_n'(x) dx + \int_0^t \lambda(t,x)\alpha\delta u_n(x) dx \tag{3.22}$$

since u_n is a restricted variation, $\delta u_n = 0$. By simplifying (3.22) , we get

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda(t,x) u_n'(x) dx \tag{3.23}$$

Integrating by parts, we obtain

$$\delta u_{n+1} = \delta u_n(t)\,[1+\lambda(t,t)] + \int_0^t \left[\frac{\partial \lambda}{\partial x}\right] \delta u(x)dx = 0. \tag{3.24}$$

We therefore obtain the next first order differential equation

$$\frac{\partial \lambda}{\partial x} = 0,$$

with boundary condition

$$1+\lambda(t,t) = 0.$$

We have

$$\lambda(t,x) = -1 \tag{3.25}$$

Thus, for (3.19) we have the following iteration formula

$$u_{n+1}(t) = u_n(t) - \int_0^t [u_n'(x) + \alpha u_n(x)]dx, \quad n \geq 0. \tag{3.26}$$

If the initial condition is

$$u(0) = 1$$

we begin with

$$u_0(t) = u(0) = 1.$$

Then by the above iteration formula (3.26) we have the next approximate solution

$$\begin{aligned}
u_1(t) &= 1-\alpha t \\
u_2(t) &= 1-\alpha t + \frac{\alpha^2}{2!}t^2 \\
u_3(t) &= 1-\alpha t + \frac{\alpha^2}{2!}t^2 - \frac{\alpha^3}{3!}t^3 \\
&\vdots \\
u_n(t) &= 1-\alpha t + \frac{\alpha^2}{2!}t^2 - \frac{\alpha^3}{3!}t^3 + \ldots + (-1)^n\frac{\alpha^n}{n!}t^n
\end{aligned}$$

Consequently, the solution can be obtained from

$$u = \lim_{n\to\infty} u_n$$

Observe that the approximate solutions converge to the exact solution $e^{-\alpha t}$, which can be obtained by the first iteration step when we don't applied the restricted variation to $u(t)$.

Consider the following second order linear differential equation, see [16].

$$u''(t) + \omega^2 u(t) = 0,$$

with initial conditions

$$\begin{aligned} u(0) &= a, \\ u'(0) &= b. \end{aligned}$$

The correction functional can be written as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,x)[u_n''(x) + \omega^2 u_n(x)]dx,$$

Taking the variation with respect to u_n, we obtain

$$\begin{aligned} \delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda(t,x)[u_n''(x) + \omega^2 u_n(x)]dx \\ &= \delta u_n(t) + \int_0^t \lambda(t,x)[\delta u_n''(x) + \omega^2 \delta u_n(x)]dx. \end{aligned}$$

Again to emphasize the idea of restricted variation, we deal with u_n as a restricted variation, i.e. $\delta u_n = 0$, hence,

$$\delta u_{n+1}(t) = \delta u_n(t) + \int_0^t \lambda(t,x)[\delta u_n''(x)]dx.$$

Integrating by parts, we have

$$\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \lambda(t,x)\delta u_n'(x)\Big|_0^t - \frac{\partial\lambda(t,x)}{\partial x}\delta u_n(x)\Big|_0^t + \int_0^t \frac{\partial^2\lambda(t,x)}{\partial x^2}\delta u_n(x)dx \\
&= \delta u_n(t) + \lambda(t,t)\delta u_n'(t) - \lambda(t,0)\delta u_n'(0) - \frac{\partial\lambda(t,x)}{\partial x}\Big|_{x=t}\delta u_n(t) \\
&\quad + \frac{\partial\lambda(t,x)}{\partial x}\Big|_{x=0}\delta u_n(0) + \int_0^t \frac{\partial^2\lambda(t,x)}{\partial x^2}\delta u_n(x)dx \\
&= \delta u_n(t) + \lambda(t,t)\delta u_n'(t) - \frac{\partial\lambda(t,x)}{\partial x}\Big|_{x=t}\delta u_n(t) + \int_0^t \frac{\partial^2\lambda(t,x)}{\partial x^2}\delta u_n(x)dx \\
&= \left(1 - \frac{\partial\lambda(t,x)}{\partial x}\Big|_{x=t}\right)\delta u_n(t) + \lambda(t,t)\delta u_n'(t) + \int_0^t \frac{\partial^2\lambda(t,x)}{\partial x^2}\delta u_n(x)dx.
\end{aligned}$$

Therefore, we have the next differential equation

$$\frac{\partial^2\lambda(t,x)}{\partial x^2} = 0,$$

with initial condition

$$\begin{aligned}
1 - \frac{\partial\lambda(t,x)}{\partial x}\Big|_{x=t} &= 0, \\
\lambda(t,t) &= 0.
\end{aligned}$$

Hence, the Lagrange multiplier

$$\lambda(t,x) = x - t.$$

So we get the following iteration formula

$$u_{n+1}(t) = u_n(t) + \int_0^t (x-t)[u_n''(x) + \omega^2 u_n(x)]dx. \tag{3.27}$$

Here, if we have boundary conditions

$$\begin{aligned} u(0) &= 1, \\ u(1) &= 0. \end{aligned}$$

And taking

$$u_0(t) = 1,$$

then by (3.27), we obtain the following approximate solutions

$$\begin{aligned} u_1(t) &= u_0(t) + \int_0^t (x-t)[u_0''(x) + \omega^2 u_0(x)]dx \\ &= 1 - \frac{1}{2!}\omega^2 t^2 \\ u_2(t) &= u_1(t) + \int_0^t (x-t)[u_1''(x) + \omega^2 u_1(x)]dx \\ &= 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4 \\ &\vdots \\ u_n(t) &= u_{n-1}(t) + \int_0^t (x-t)[u_{n-1}''(x) + \omega^2 u_{n-1}(x)]dx \\ &= 1 - \frac{1}{2!}\omega^2 t^2 + \frac{1}{4!}\omega^4 t^4 + \cdots + (-1)^n \frac{1}{(2n)!}\omega^{2n} t^{2n} \end{aligned}$$

Consequently, the approximate solutions can be obtained from

$$u = \lim_{n\to\infty} u_n$$

which converge to its exact solution $\cos(\omega t)$, although it can be found by only one iteration without considering $u(t)$ as a restricted variation.

Now, we consider the second nonlinear differential equation of the form

$$u''(t) - \mu^2 u(t) + N(u) = 0, \tag{3.28}$$

with initial condition

$$\begin{aligned} u(0) &= a, \\ u'(0) &= b. \end{aligned}$$

Where $N(u)$ is a nonlinear functional of u, thus the correction functional of (3.28) is

$$u_{n+1}(t) = u_n + \int_0^t \lambda(t,x)\left[u_n''(x) - \mu^2 u_n(x) + N(u)\right] dx, \quad n \geq 0. \tag{3.29}$$

In order to find the value of λ, we start by taking the variation with respect to u_n, we get

$$\delta u_{n+1}(t) = \delta u_n + \delta \int_0^t \lambda(t,x)\left[u_n''(x) - \mu^2 u_n(x) + N(u)\right] dx, \quad n \geq 0.$$

which is the same as

$$\delta u_{n+1}(t) = \delta u_n + \delta \int_0^t \lambda(t,x) u_n''(x) dx - \int_0^t \lambda(t,x)\mu^2 \delta u_n(x) dx + \int_0^t \lambda(t,x)\delta N(u) dx. \tag{3.30}$$

Applying the variation to Equation (3.30) and by using $\delta N(u) = 0$ due to being a restricted variation, we have

$$\delta u_{n+1}(t) = \delta u_n + \delta \int_0^t \lambda(t,x) u_n''(x) - \int_0^t \mu^2 \delta u_n(x) dx = 0. \tag{3.31}$$

Integrating the first integral by parts, we obtain:

$$\begin{aligned}\delta\int_0^t \lambda(t,x)u_n''(t) &= \lambda(t,x)\delta u_n'\Big|_0^t - \frac{\partial\lambda}{\partial x}\delta u_n\Big|_0^t + \int_0^t \frac{\partial^2\lambda}{\partial x^2}\delta u_n(x)dx \\ &= \lambda(t,t)\delta u_n'(t) - \frac{\partial\lambda}{\partial x}\Big|_{x=t}\delta u_n(t) + \int_0^t \frac{\partial^2\lambda}{\partial x^2}\delta u_n(x)dx. \quad (3.32)\end{aligned}$$

Substituting the integral in Equation (3.31) by the value of the integral (3.32) with simplifications, we get

$$\delta u_{n+1}(t) = \left(1 - \frac{\partial\lambda}{\partial x}\Big|_{x=t}\right)\delta u_n + \lambda(t,t)\delta u_n'(t) + \int_0^t \left[\frac{\partial^2\lambda}{\partial x^2} - \mu^2\right]\delta u_n(x)dx = 0. \quad (3.33)$$

thus, we have the following stationary conditions

$$\begin{aligned}\frac{\partial^2\lambda}{\partial x^2} - \mu^2 &= 0, \\ 1 - \frac{\partial\lambda}{\partial x}\Big|_{x=t} &= 0, \\ \lambda(t,t) &= 0.\end{aligned}$$

Which is a second differential equation, therefore

$$\lambda(t,x) = \frac{1}{2\mu}\left[e^{\mu(x-t)} - e^{\mu(t-x)}\right].$$

Hence, for this nonlinear differential equation (3.28) we have the next iteration formula for $n \geq 0$

$$\begin{aligned} u_{n+1}(t) &= u_n(t) + \int_0^t \frac{1}{2\mu}\left[e^{\mu(x-t)} - e^{\mu(t-x)}\right]\left[u_n''(x) - \mu^2 u_n(x) + N(u)\right] dx \\ &= u_n(t) + \frac{1}{2\mu}\int_0^t \left[e^{\mu(x-t)} - e^{\mu(t-x)}\right]\left[u_n''(x) - \mu^2 u_n(x) + N(u)\right] dx. \end{aligned} \tag{3.34}$$

Consequently, the solution is given by

$$u = \lim_{n\to\infty} u_n.$$

To understand this method more, let us consider the following example.

Example 3.3. Consider the second order nonlinear differential equation of the form

$$u''(t) - 4u(t) + 3e^{-t}u^2(t) = 0$$

with initial condition

$$\begin{aligned} u(0) &= 1, \\ u'(0) &= 1. \end{aligned}$$

where $\mu = 2$, and $N(u) = 3e^{-t}u^2$. By substitution into Equation (3.34), we have

$$u_{n+1}(t) = u_n + \frac{1}{4}\int_0^t \left[e^{2(x-t)} - e^{2(t-x)}\right]\left[u_n''(x) - 4u_n(x) + 3e^{-x}u_n^2(x)\right] dx, \quad n \geq 0.$$

Let

$$u_0 = \frac{3}{4}e^{2t} + \frac{1}{4}e^{-2t},$$

which is a solution of homogeneous problem

$$u''(t) - 4u(t) = 0$$

with initial condition

$$\begin{aligned} u(0) &= 1, \\ u'(0) &= 1. \end{aligned}$$

The calculations are made by Maple. Hence, some of these iterations are

$$\begin{aligned}
u_1(t) &= \frac{3}{4}e^{2t} + \frac{1}{4}e^{-2t} + \frac{1}{4}\int_0^t \left[e^{2(x-t)} - e^{2(t-x)}\right]\left[u_0''(x) - 4u_0(x) + 3e^{-x}u_0^2(x)\right]dx \\
&= \frac{-27}{80}e^{3t} - \frac{1}{112}e^{-5t} + \frac{3}{8}e^{-t} - \frac{1}{10}e^{-2t} + \frac{15}{14}e^{2t} \\
u_2(t) &= u_1(t) + \frac{1}{4}\int_0^t \left[e^{2(x-t)} - e^{2(t-x)}\right]\left[u_1''(x) - 4u_1(x) + 3e^{-x}u_1^2(x)\right]dx \\
&= \frac{23189}{18200}e^{2t} + \frac{109}{840}e^{-2t} - \frac{135}{196}e^{3t} - \frac{3}{14}e^{-t} - \frac{1}{700}e^{-5t} - \frac{729}{44800}e^{5t} + \frac{81}{448}e^{4t} \\
&\quad - \frac{81}{320}e^{t} - \frac{1971}{22400}e^{-3t} + \frac{369}{15680}e^{-4t} + \frac{1}{2240}e^{-7t} \\
&\quad - \frac{1}{11200}e^{-8t} - \frac{1}{489216}e^{-11t} + \frac{7317}{11200} \\
u_3(t) &= u_2(t) + \frac{1}{4}\int_0^t \left[e^{2(x-t)} - e^{2(t-x)}\right]\left[u_2''(x) - 4u_2(x) + 3e^{-x}u_2^2(x)\right]dx \\
&= 2.073823681e^{t} + 0.2797284903e^{2t} - 1.329173824e^{3t} + 0.4669899641e^{4t} \\
&\quad -0.1347686824e^{5t} + 0.02723737062e^{6t} - 0.003673725629e^{7t} \\
&\quad +0.0002942093830e^{8t} - 0.00001031643291e^{9t} + 1.220019747te^{2t} \\
&\quad -0.4521273348te^{-2t} + 0.9957208461e^{-t} - 0.4035948124e^{-2t} \\
&\quad -0.1931599616e^{-3t} + 0.04649291977e^{-4t} - 0.01219616573e^{-5t} \\
&\quad +0.003143193519e^{-6t} - 0.0009339086803e^{-7t} + 0.0001942198151e^{-8t} \\
&\quad -0.00001948210854e^{-9t} - 0.000002552428526e^{-10t} + 0.000002529733078e^{-11t} \\
&\quad -7.297205516\ 10^{-7}e^{-12t} + 8.36695192610^{-8}e^{-13t} + 4.302924524\ 10^{-9}e^{-14t} \\
&\quad -7.588523839\ 10^{-9}e^{-15t} + 2.094369615\ 10^{-9}e^{-16t} - 1.453914417\ 10^{-10}e^{-17t} \\
&\quad +1.533678621\ 10^{-11}e^{-19t} - 2.765269028\ 10^{-12}e^{-20t} - 2.387594922\ 10^{-14}e^{-23t} \\
&\quad -0.8160933459.
\end{aligned}$$

$$\vdots$$

We obviously see that we have a large amount of calculations, so we don't write all iterations of this method, and consequently, the solution can be obtained by

$$u = \lim_{n\to\infty} u_n.$$

In Figure (3-1) we compare the fourth and fifth iteration with the exact solution $u(t) = e^t$.

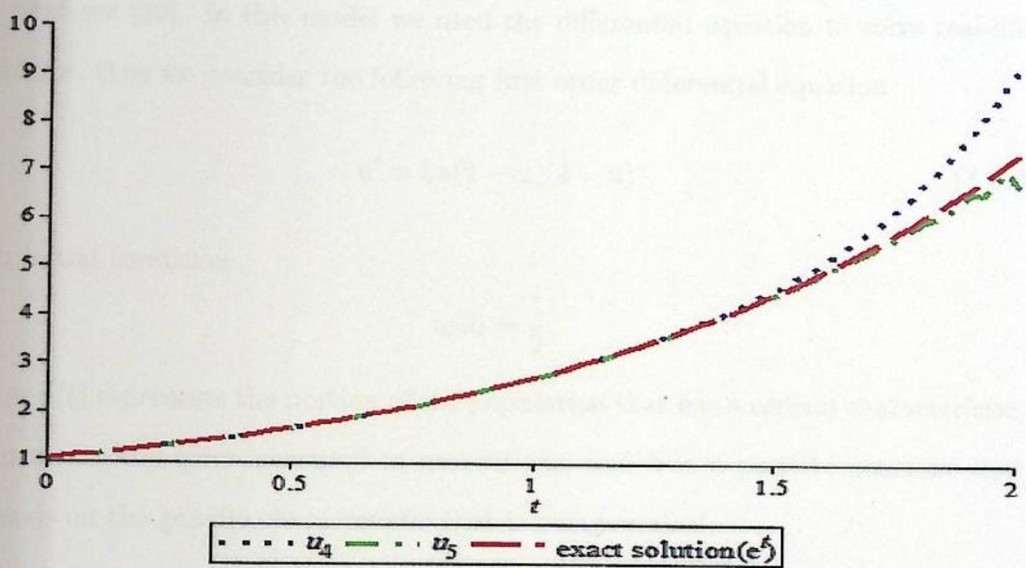

Figure 3-1: Comparing the fourth and fifth iteration obtained by the VIM by the exact solution e^t.

For more examples, see [16, 18, 19, 34].

3.4 Scientific applications

In this section, we discuss some applications of the VIM on nonlinear differential equations and two models of interest which are considered and solved by the VIM. Namely, the hybrid selection model, and the Riccati equation, see Wazwaz [34].

The hybrid selection model

In boilogy, the hybrid is a result of an interbreeding between two animal species or plant species. Hybrid selection is a very important factors to understand and consider each growing season. We are studying a population of animals or plants to determine how quickly of a specific characteristic will pass from one generation to the next see [30]. In this model we used the differential equation to solve real-life problems, thus we consider the following first order differential equation

$$u' = ku(1-u)(2-u) \tag{3.35}$$

with initial condition

$$u(0) = \frac{1}{2},$$

where $u(t)$ represents the portion of the population that has a certain characteristic, t represents the time measured in generations, and k is a positive constant that depends on the genetic characteristic that is being studied.

According to the VIM, the correction functional is

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,x)\left[u_n'(x) - ku_n(x)(1-u_n(x))(2-u_n(x))\right]dx \tag{3.36}$$

Taking the variation with respect to u_n, we get

$$\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \delta\int_0^t \lambda(t,x)\left[u_n'(x) - ku_n(x)(1-u_n(x))(2-u_n(x))\right]dx \\
&= \delta u_n(t) + \delta\int_0^t \lambda(t,x)u_n'(x)dx - \delta\int_0^t ku_n(x)(1-u_n(x))(2-u_n(x))dx \\
&= \delta u_n(t) + \delta\int_0^t \lambda(t,x)u_n'(x)dx - \int_0^t \delta ku_n(x)(1-u_n(x))(2-u_n(x))dx
\end{aligned}$$

Since

$$ku_n(x)(1-u_n(x))(2-u_n(x))$$

is a restrict variable, then

$$\delta ku_n(x)(1-u_n(x))(2-u_n(x))=0.$$

Hence,

$$\delta u_{n+1}(t)=\delta u_n(t)+\delta\int_0^t \lambda(t,x)u_n'(x)dx.$$

Integrate the integral by parts and with $\delta u_n(0)=0$, we obtain

$$\begin{aligned}\delta u_{n+1}(t) &= \delta u_n(t)+\lambda(t,t)\delta u_n(t)-\lambda(t,0)\delta u_n(0)-\int_0^t \frac{\partial\lambda}{\partial x}\delta u_n(x)dx\\ &= \delta u_n(t)+\lambda(t,t)\delta u_n(t)-\int_0^t \frac{\partial\lambda}{\partial x}\delta u_n(x)dx.\end{aligned}$$

By simplifying, we have

$$u_{n+1}(t)=(1+\lambda(t,t))\,\delta u_n(t)-\int_0^t \frac{\partial\lambda}{\partial x}\delta u_n(x)dx=0.$$

Therefore, we have the following first order differential equation

$$\frac{\partial\lambda}{\partial x}=0, \tag{3.37}$$

subject to boundary condition

$$1+\lambda(t,t)=0.$$

Solving (3.37), we obtain

$$\lambda(t,x)=-1. \tag{3.38}$$

By substituting (3.38) into (3.36)

$$u_{n+1}(t) = u_n(t) - \int_0^t [u_n'(x) - ku_n(x)(1 - u_n(x))(2 - u_n(x))]\, dx. \tag{3.39}$$

Let

$$u_0(t) = \frac{1}{2},$$

which is a solution of the homogeneous differential equation

$$u' = 0,$$

subject to the initial condition

$$u(0) = \frac{1}{2}.$$

Hence, by Maple we have

$$\begin{aligned}
u_1(t) &= u_0(t) - \int_0^t [u_0'(x) - ku_0(x)(1 - u_0(x))(2 - u_0(x))]\, dx. \\
&= \frac{1}{2} + \frac{3}{8}kt \\
u_2(t) &= u_1(t) - \int_0^t [u_1'(x) - ku_1(x)(1 - u_1(x))(2 - u_1(x))]\, dx. \\
&= \frac{1}{2} + \frac{3}{8}kt - \frac{3}{64}(kt)^2 - \frac{9}{128}(kt)^3 + \frac{27}{2048}(kt)^4 \\
u_3(t) &= u_2(t) - \int_0^t [u_2'(x) - ku_2(x)(1 - u_2(x))(2 - u_2(x))]\, dx. \\
&= \frac{1}{2} + \frac{3}{8}kt - \frac{3}{64}(kt)^2 - \frac{17}{256}(kt)^3 - \frac{63}{2048}(kt)^4 + \frac{27}{25600}(kt)^5 - \frac{567}{65536}(kt)^6 \\
&\quad + \frac{1917}{1835008}(kt)^7 + \frac{1701}{2097152}(kt)^8 - \frac{1377}{4194304}(kt)^9 + \frac{729}{67108864}(kt)^{10} \\
&\quad + \frac{45927}{2952790016}(kt)^{11} - \frac{6561}{2147483648}(kt)^{12} + \frac{19683}{111669149696}(kt)^{13} \\
&\vdots
\end{aligned}$$

The iterations u_1, u_2, $\cdots$, are approximations for the solution. In fact, the exact solution is

$$u(t) = \frac{\sqrt{1+3e^{2kt}}-1}{\sqrt{1+3e^{2kt}}}.$$

We plot the fourth and fifth iterations with comparing them by the exact solution $u(t)$, see Figure 3.2.

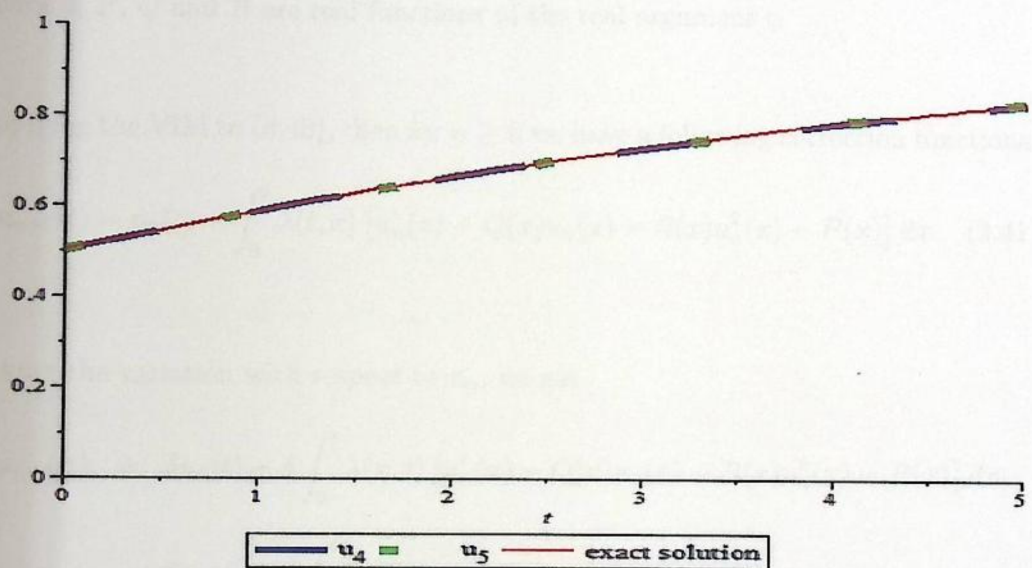

Figure 3-2: Comparing the fourth and fifth iteration obtained by the VIM with the exact solution, with $k = 0.25$.

The Riccati differential equation

The Riccati differential equation is a first order nonlinear differential equation that used in different problem of mathematics and physics, and named after the Italian mathematician Jacopo Francesco Riccati.

The Riccati equation can be solved by using classical numeric methods, for example, the forward Euler method, and Runge-Kutta method. In this section, we aim to find the solution of the Riccati differential equation by using the VIM [5, 34].

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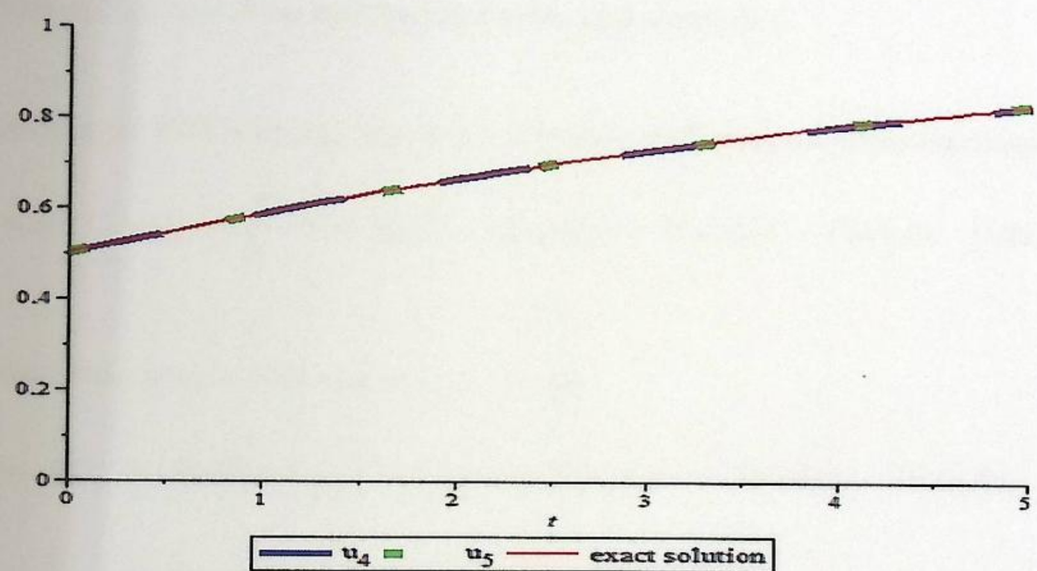

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Consider the following Riccati equation

$$\frac{du}{dt} = Q(t)u + R(t)u^2 + P(t), \tag{3.40}$$

subject to the initial condition

$$y(0) = a.$$

where u, P, Q and R are real functions of the real argument t.

Applying the VIM to (3.40), then for $n \geq 0$ we have a following correction functional

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,x)\left[u_n'(x) - Q(x)u_n(x) - R(x)u_n^2(x) - P(x)\right] dx. \tag{3.41}$$

Taking the variation with respect to u_n, we get

$$\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda(x,t)\left[u_n'(x) - Q(x)u_n(x) - R(x)u_n^2(x) - P(x)\right] dx \\
&= \delta u_n(t) + \delta \int_0^t \lambda u_n'(x)dx - \delta \int_0^t \lambda Q(x)u_n(x)dx \\
&\qquad - \delta \int_0^t \lambda R(x)u_n^2(x)dx - \delta \int_0^t \lambda P(x)dx \\
&= \delta u_n(t) + \delta \int_0^t \lambda u_n'(x)dx - \int_0^t \lambda Q(x)\delta u_n(x)dx \\
&\qquad - \int_0^t \lambda R(x)\delta u_n^2(x)dx - \int_0^t \lambda \delta P(x)dx
\end{aligned}$$

We will consider $Q(x)u_n(x)$ to be a restricted variation for being easier in the calculations and we also know that u_n^2 is a restricted variation. And by using the fact

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$$\begin{aligned}
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&= \delta u_n(t) + \delta \int_0^t \lambda u_n'(x)dx - \delta \int_0^t \lambda Q(x)u_n(x)dx \\
&\qquad - \delta \int_0^t \lambda R(x)u_n^2(x)dx - \delta \int_0^t \lambda P(x)dx \\
&= \delta u_n(t) + \delta \int_0^t \lambda u_n'(x)dx - \int_0^t \lambda Q(x)\delta u_n(x)dx \\
&\qquad - \int_0^t \lambda R(x)\delta u_n^2(x)dx - \int_0^t \lambda \delta P(x)dx
\end{aligned}$$

We will consider $Q(x)u_n(x)$ to be a restricted variation for being easier in the calculations and we also know that u_n^2 is a restricted variation. And by using the fact that $\delta P(x) = 0$, then

$$\begin{aligned} Q(x)\delta u_n(x) &= 0, \\ \delta u_n^2 &= 0, \\ \delta u_n(0) &= 0. \end{aligned}$$

Hence,

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda u_n'(x)dx$$

Integrating the integral by parts, we get

$$\delta u_{n+1}(t) = \delta u_n(t) + \lambda(t,t)\delta u_n(t) - \lambda(t,0)\delta u_n(0) - \int_0^t \frac{\partial \lambda}{\partial x}\delta u_n(x)dx$$

After simplifying, the last equation can be written as

$$\delta u_{n+1}(t) = (1 + \lambda(t,t))\delta u_n(t) - \int_0^t \frac{\partial \lambda}{\partial x}\delta u_n(x)dx = 0.$$

Hence, we have the following first order linear differential equation

$$\frac{\partial \lambda}{\partial x} = 0, \tag{3.42}$$

with condition

$$1 + \lambda(t,t) = 0 \tag{3.43}$$

By Solving (3.42) with respect to the initial condition (3.43), we get

$$\lambda(x,t) = -1. \tag{3.44}$$

By substituting (3.44) into (3.41), we obtain

$$u_{n+1}(t) = u_n(t) - \int_0^t \left[u_n'(x) - Q(x)u_n(x) - R(x)u_n^2(x) - P(x)\right] dx, \quad n \geq 0 \tag{3.45}$$

And consequently, we can be obtain the solution from

$$u(t) = \lim_{n\to\infty} u_n(t).$$

Example 3.4. We close this section by applying the VIM to the next Riccti equation

$$u'(t) = u^2(t) - 2tu(t) + t^2 + 1, \quad u(0) = \frac{1}{2}$$

Note that $Q(t) = -2t$, $R(t) = 1$, and $p(t) = t^2 + 1$

According to (3.44), we have

$$\lambda(x,t) \;=\; -1.$$

And (3.45) becomes

$$u_{n+1}(t) = u_n(t) - \int_0^t \left[u_n'(x) + 2xu_n(x) - u_n^2(x) - x^2 - 1\right] dx, \tag{3.46}$$

consider

$$u_0(t) = \frac{1}{2},$$

which is a solution of the homogenous differential equation

$$u'(t) = 0$$

subject to the initial condition

$$u_0(0) = \frac{1}{2}.$$

Then the iteration formula (3.46) gives a following successive approximation

$$
\begin{aligned}
u_1(t) &= u_0(t) - \int_0^t \left[u_0'(x) + 2xu_0(x) - u_0^2(x) - x^2 - 1\right] dx \\
&= \frac{1}{2} + \frac{5}{4}t - \frac{1}{2}t^2 + \frac{1}{3}t^3 \\
u_2(t) &= u_1(t) - \int_0^t \left[u_1'(x) + 2xu_1(x) - u_1^2(x) - x^2 - 1\right] dx \\
&= \frac{1}{2} + \frac{5}{4}t + \frac{1}{8}t^2 - \frac{7}{48}t^3 + \frac{1}{48}t^4 + \frac{1}{12}t^5 - \frac{1}{18}t^6 + \frac{1}{63}t^7 \\
u_3(t) &= u_2(t) - \int_0^t \left[u_2'(x) + 2xu_2(x) - u_2^2(x) - x^2 - 1\right] dx \\
&= \frac{1}{2} + \frac{5}{4}t + \frac{1}{8}t^2 + \frac{1}{16}t^3 - \frac{1}{48}t^4 - \frac{7}{960}t^5 + \frac{11}{1152}t^6 + \frac{29}{16128}t^7 \\
&\quad + \frac{23}{64512}t^8 - \frac{481}{145152}t^9 + \frac{143}{60480}t^{10} - \frac{13}{18144}t^{12} \\
&\quad + \frac{1}{2268}t^{13} - \frac{1}{7938}t^{14} + \frac{1}{59535}t^{15} \\
&\vdots
\end{aligned}
$$

In the same mannar, we can get all iterations, especially the fourth and fifth iteration that we plot in Figure3.3, in this figure we compare these iterations with the exact solution.

And consequently, we have the solution from

$$
\begin{aligned}
u(t) &= \lim_{n\to\infty} u_n(t) \\
&= t + \frac{1}{2-t}, |t| < 2.
\end{aligned}
$$

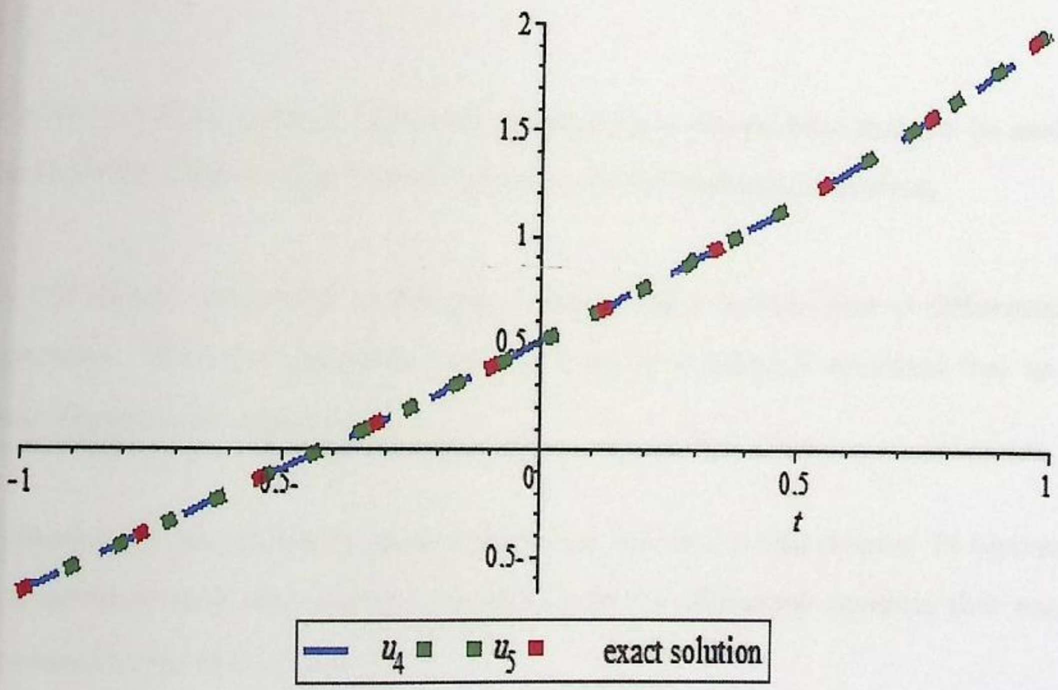

Figure 3-3: Comparing the fourth and fifth iterations obtained by the VIM with the exact solution.

Chapter 4

On the convergence analysis

It is not possible to put all differential equations in a general form that can be used for the VIM. Thus we don't have a global study for convergence analysis.

In this chapter we present convergence analysis for a certain class of differential equations. This class represents a general form for differential equations that appear frequently in applications.

In Section 4.1, we present the general form that was used in this chapter. In Section 4.2, we state some theorems for convergence of the differential equation that was discussed in the first section.

4.1 Iteration formula

In this section, we apply the VIM to the differential equations of the form

$$u^{(m)} = f(u, u', u'', \cdots, u^{(m)}) \tag{4.1}$$

Differential equations of the form (4.1) have been the focus of many studies due to their frequent appearance in various applications see [5, 34].

As mentioned in the last chapter, if we employ the VIM to this system, we get the following correction functional for $n \geq 0$

$$u_{n+1}(t) = u_n(t) + (-1)^m \int_0^t \frac{(x-t)^{m-1}}{(m-1)!} \left[u_n^{(m)} - f(u_n, u_n', u_n'', \cdots, u_n^{(m)})\right] dx. \quad (4.2)$$

Proof.
This functional is obtained by mathematical induction:

1. For $m = 1$: if we have an equation of the form

$$u'(t) = f(u, u'), \quad (4.3)$$

we aim to prove that Equation (4.3) has the following correction functional

$$u_{n+1}(t) = u_n(t) - \int_0^t [u_n'(x) - f(u_n, u_n')] \, dx, \quad n \geq 0. \quad (4.4)$$

Firstly, the equation (4.3) has the following correction functional

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(x,t) \, [u_n'(x) - f(u_n, u_n')] \, dx.$$

Take the variation with respect to u_n, we have

$$\begin{aligned} \delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda(t,x) \, [u_n'(x) - f(u_n, u_n')] \, dx \\ &= \delta u_n(t) + \delta \int_0^t \lambda(t,x) u_n'(x) dx - \delta \int_0^t \lambda(t,x) f(u_n, u_n') dx \\ &= \delta u_n(t) + \delta \int_0^t \lambda(t,x) u_n'(x) dx - \int_0^t \lambda(t,x) \delta f(u_n, u_n') dx \end{aligned}$$

since $f(u_n, u_n')$ are a restricted variable, then $\delta f(u_n, u_n') = 0$, we get

$$\begin{aligned} u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda(t,x) u_n'(x) dx \\ &= \delta u_n(t) + \lambda(t,t)\delta u_n(t) - \lambda(t,0)\delta u_n(0) - \int_0^t \frac{\partial \lambda(t,x)}{\partial x} \delta u_n dx \\ &= (1 + \lambda(t,t))\, \delta u_n(t) - \int_0^t \frac{\partial \lambda(t,x)}{\partial x} \delta u_n dx. \end{aligned}$$

So, we have the following stationary conditions

$$\begin{aligned} \frac{\partial \lambda}{\partial x} &= 0, \\ 1 + \lambda(t,t) &= 0. \end{aligned}$$

According to these conditions, we have that

$$\lambda(t,x) = -1.$$

Hence, for (4.3) we have the following iteration formula

$$u_{n+1}(t) = u_n(t) - \int_0^t \left[u_n'(x) - f(u_n, u_n')\right] dx, \quad n \geq 0. \tag{4.5}$$

Thus Equation (4.2) is true for $m = 1$.

2. In this step, we assume that (4.2) true for $m = k$, i.e. if we consider the k^{th} order differential equation

$$u^{(k)} = f(u, u', u'', \cdots, u^{(k)}) \tag{4.6}$$

we get the correction functional

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,x) \left[u_n^{(k)}(x) - f(u, u', u'', \cdots, u^{(k)})\right] dx, \quad n \geq 0.$$

Now, by taking the variation with respect to $u_n(t)$, we get

$$\begin{aligned}\delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda(t,x)\left[u_n^{(k)}(x) - f(u,u',u'',\cdots,u^{(k)})\right]dx \\ &= \delta u_n(t) + \int_0^t \lambda(t,x)\delta\left[u_n^{(k)}(x) - f(u,u',u'',\cdots,u^{(k)})\right]dx \\ &= \delta u_n(t) + \int_0^t \lambda(t,x)\delta u_n^{(k)}(x)dx - \int_0^t \lambda(t,x)\delta f(u,u',u'',\cdots,u^{(k)})dx.\end{aligned}$$

since $\delta f(u,u',u'',\cdots,u^{(k)}) = 0$, then

$$\delta u_{n+1}(t) = \delta u_n(t) + \int_0^t \lambda(t,x)\delta u_n^{(k)}(x)dx.$$

By our assumption, the value of the Lagrange multiplier that makes $u_{n+1}(t)$ stationary, i.e. $\delta u_{n+1}(t) = 0$, is in the form

$$\lambda(t,x) = \frac{(-1)^k(x-t)^{k-1}}{(k-1)!}.$$

Thus, we get the following iteration formula

$$u_{n+1}(t) = u_n(t) + (-1)^k \int_0^t \frac{(x-t)^{k-1}}{(k-1)!}\left[u_n^{(k)} - f(u_n,u_n',u_n'',\cdots,u_n^{(k)})\right]dx, \;\; n \geq 0. \tag{4.7}$$

3. We aim now to prove Equation (4.2) for $m = k+1$, i.e. for the $(k+1)^{th}$ differential equation of the form

$$u^{(k+1)} = f(u,u',u'',\cdots,u^{(k+1)}) \tag{4.8}$$

The correction functional reeds

$$u_{n+1}(t) = u_n(t) + (-1)^{k+1} \int_0^t \frac{(x-t)^k}{k!}\left[u_n^{(k)} - f(u_n,u_n',u_n'',\cdots,u_n^{(k)})\right]dx, \;\; n \geq 0.$$

Now Equation (4.8) has the correction functional

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t,x)\left[u_n^{(k+1)} - f(u_n, u_n', u_n'', \cdots, u_n^{(k+1)})\right] dx, \quad n \geq 0. \tag{4.9}$$

Take the variation with respect to u_n, we obtain

$$\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda(t,x)\left[u_n^{(k+1)} - f(u_n, u_n', u_n'', \cdots, u_n^{(k+1)})\right] dx \\
&= \delta u_n(t) + \delta \int_0^t \lambda(t,x)\left[u_n^{(k+1)} - f(u_n, u_n', u_n'', \cdots, u_n^{(k+1)})\right] dx \\
&= \delta u_n(t) + \delta \int_0^t \lambda(t,x) u_n^{(k+1)} dx - \delta \int_0^t \lambda(t,x) f(u_n, u_n', u_n'', \cdots, u_n^{(k+1)}) dx \\
&= \delta u_n(t) + \delta \int_0^t \lambda(t,x) u_n^{(k+1)} dx - \int_0^t \lambda(t,x) \delta f(u_n, u_n', u_n'', \cdots, u_n^{(k+1)}) dx
\end{aligned}$$

By the fact of restricted variation, $\delta f(u_n, u_n', u_n'', \cdots, u_n^{(k+1)}) = 0$, we have

$$\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \delta \int_0^t \lambda(t,x) u_n^{(k+1)} dx \\
&= \delta u_n(t) + \lambda(t,t)\delta u^{(k)}(t) - \lambda(t,0)\delta u^{(k)}(0) - \delta \int_0^t \frac{\partial \lambda}{\partial x} u_n^{(k)} dx \\
&= \delta u_n(t) + \lambda(t,t)\delta u^{(k)}(t) - \delta \int_0^t \frac{\partial \lambda}{\partial x} u_n^{(k)} dx
\end{aligned}$$

In order to make u_{n+1} stationary, i.e. $\delta u_{n+1} = 0$, we aim to find $\lambda(t,x)$ that satisfies the following conditions

$$\begin{aligned}
\lambda(t,t) &= 0, \\
\delta u_n(t) - \delta \int_0^t \frac{\partial \lambda}{\partial x} u_n^{(k)} dx &= 0.
\end{aligned} \tag{4.10}$$

According to step 2, The solution of (4.10) is given by

$$-\frac{\partial \lambda}{\partial x} = \frac{(-1)^k (x-t)^{k-1}}{(k-1)!} \tag{4.11}$$

with initial condition

$$\lambda(t,t) = 0$$

Hence,

$$\lambda(t,x) = \frac{(-1)^{k+1}(x-t)^k}{(k)!}.$$

Thus, if we have a differential equation of the form (4.8), we have the next correction functional

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{(-1)^{k+1}(x-t)^k}{(k)!} \left[u_n^{(k+1)} - f(u_n, u_n', u_n'', \cdots, u_n^{(k+1)})\right] dx.$$

Therefore, Equation (4.2) is true for all $m \geq 1$. □

For convergence, we can select the zeroth approximation by using the initial conditions, i.e. In Equation (4.1), we should use $u_0(t)$ as follows:

$$u_0(t) = u(0) + tu'(0) + \frac{t^2}{2!}u''(0) + \cdots + \frac{t^{k-1}}{(k-1)!}u^{(k-1)}(0),$$

see Odibat [23].

Example 4.1. Consider the following third order nonlinear differential equation

$$u'''(t) + e^t u^2(t) = 0$$

subject to initial condition

$$\begin{aligned} u(0) &= 1, \\ u'(0) &= -1, \\ u''(0) &= 1. \end{aligned}$$

For this problem, we have the next iteration formula

$$u_{n+1}(t) = u_n(t) - \int_0^t \frac{(x-t)^2}{2!}\left[u_n''' + e^x u_n^2(x)\right] dx.$$

Choose

$$u_0(t) = 1 - t + \frac{t^2}{2!}.$$

Hence,

$$\begin{aligned}
u_1(t) &= u_0(t) - \int_0^t \frac{(x-t)^2}{2!}\left[u_0''' + e^x u_0^2(x)\right] dx \\
&= 182 + 70t + 10t^2 - (1/4)e^t t^4 + 4e^t t^3 - 29e^t t^2 + 110e^t t - 181e^t \\
u_2(t) &= u_1(t) - \int_0^t \frac{(x-t)^2}{2!}\left[u_1''' + e^x u_1^2(x)\right] dx \\
&= 4953.831576 + 11106.03125e^{2t} - 4894.862826e^t + 5227.200732e^{3t}t \\
&\quad - 149.3148148e^{3t}t^4 + 763.0973937e^{3t}t^3 - 1.734567901t^6e^{3t} \\
&\quad + 19.76954733t^5e^{3t} + 0.9259259259e - 1t^7e^{3t} - 0.2314814815e \\
&\quad - 2e^t t^8 - 2268.375e^{2t}t + .625e^{2t}t^6 - 11.25e^{2t}t^5 + 84.1875e^{2t}t^4 \\
&\quad - 295.25e^{2t}t^3 - 2572.928212e^{3t}t^2 + 502.3125e^{2t}t^2 - 640e^t t - 100e^t t^4 \\
&\quad - 200e^t t^3 - 3140e^t t^2 - 11164.e^t + 103.0837334t^2 + 1316.700246t \\
&\;\;\vdots
\end{aligned}$$

4.2 Convergence of the method

In this section, we study the convergence of the iteration method that was presented in the previous section. The results in this section are due to Odibat [23].

Consider the general nonlinear problem

$$u^{(m)} = f(u, u', u'', \cdots, u^{(m)}) \tag{4.12}$$

with initial condition

$$\begin{aligned} u(0) &= c_0, \\ u'(0) &= c_1, \\ u''(0) &= c_2, \\ &\vdots \\ u^{m-1}(0) &= c_{m-1}. \end{aligned}$$

Then the correction functional for Equation (4.12) is

$$u_{n+1}(t) = u_n(t) + (-1)^m \int_0^t \frac{(x-t)^{m-1}}{(m-1)!} \left[u_n^{(m)} - f(u_n, u_n', u_n'', \cdots, u_n^{(m)})\right] dx, \tag{4.13}$$

and the exact solution can be obtained as

$$u(t) = \lim_{n\to\infty} u_n(t).$$

To study the convergence of this method, we define a new operator

$$B[w] = \int_0^t \frac{(-1)^m (x-t)^{m-1}}{(m-1)!} \left[w^{(m)} - f(w, w', w'', \cdots, w^{(m)}\right] dx, \tag{4.14}$$

hence, Equation (4.13) becomes

$$u_{n+1}(t) = u_n(t) + B[u_n], \tag{4.15}$$

or,

$$B[u_n] = u_{n+1}(t) - u_n(t).$$

Define the following components $v_k, k = 0, 1, 2, \ldots$.

$$\begin{aligned}
v_0 &= u_0 \\
v_1 &= B[v_0] = B[u_0] = u_1 - u_0 \\
v_2 &= B[v_0 + v_1] = B[u_1] = u_2 - u_1 \\
&\vdots \\
v_{k+1} &= B[v_0 + v_1 + v_2 + \ldots + v_k] = B[u_k] = u_{k+1} - u_k
\end{aligned} \tag{4.16}$$

The limit of the sequence will be the solution $u(t)$ if the series of v_k is convergent, since its a telescoping series.

$$\begin{aligned}
u(t) &= \lim_{k\to\infty} u_k(t) \\
&= \lim_{k\to\infty} \left(\sum_{j=0}^{k} v_j \right) \\
&= \sum_{j=0}^{\infty} v_j.
\end{aligned}$$

The initial approximation $v_0 = u_0$ can be selected if it satisfies the initial conditions of the problem. In this thesis, we use the initial values

$$u^{(k)}(0) = c_k, k = 0, 1, 2, \ldots, m-1$$

in our selection of v_0, so that can be as follow

$$v_0 = \sum_{k=0}^{m-1} \frac{c_k}{k!} t^k.$$

The results of this section are given in next theorems.

Theorem 4.1. *Let B, defined in Equation (4.14), be an operator from a Hilbert space Y to Y, then the series solution*

$$u(t) = \sum_{k=0}^{\infty} v_k(t)$$

converges if there exist $0 < \gamma < 1$ such that

$$\| v_{k+1} \| \leq \gamma \| v_k \|,$$

i.e.

$$\| B[v_0 + v_1 + \ldots + v_{k+1}] \| \leq \gamma \| B[v_0 + v_1 + \ldots + v_k] \|, \quad k = 0, 1, 2 \ldots.$$

Proof. Firstly, define the sequence of partial sums $\{S_n\}_{n=0}^{\infty}$ as,

$$\begin{aligned} S_0 &= v_0 \\ S_1 &= v_0 + v_1 \\ S_2 &= v_0 + v_1 + v_2 \\ &\vdots \\ S_n &= v_0 + v_1 + v_2 + \ldots + v_n. \end{aligned}$$

In order to prove that $\{S_n\}_{n=0}^{\infty}$ is convergent, we want to prove that it is a Cauchy sequence in the Hilbert space Y, so we consider that

$$\| S_{n+1} - S_n \| = \| v_{n+1} \| \leq \gamma \| v_n \| \leq \gamma^2 \| v_{n-1} \| \leq \ldots \leq \gamma^{n+1} \| v_0 \|$$

Hence, for $n, k \in \mathbb{N}, n \geq k$, and by using triangle inequality, we obtain

$$\begin{aligned} \| S_n - S_k \| &= \| (S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \ldots + (S_{k+1} - S_k) \| \\ &\leq \| S_n - S_{n-1} \| + \| S_{n-1} - S_{n-2} \| + \ldots + \| S_{k+1} - S_k \| \\ &\leq \gamma^n \| v_0 \| + \gamma^{n-1} \| v_0 \| + \gamma^{n-2} \| v_0 \| + \ldots + \gamma^{k+1} \| v_0 \| \\ &= \left(\gamma^{n-k-1} + \gamma^{n-k-2} + \ldots + \gamma + 1 \right) \gamma^{k+1} \| v_0 \| . \end{aligned}$$

But

$$\gamma^{n-k-1} + \gamma^{n-k-2} + \ldots + \gamma + 1 = \frac{1 - \gamma^{n-k}}{1 - \gamma} \gamma^{k+1},$$

thus

$$\| S_n - S_k \| \leq \frac{1 - \gamma^{n-k}}{1 - \gamma} \gamma^{k+1} \| v_0 \| .$$

Since $0 < \gamma < 1$, we have

$$\lim_{n,k \to \infty} \| S_n - S_k \| = 0$$

which implies that the sequence $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space Y, and therefore implies that $u(t) = \sum_{k=0}^{\infty} v_k(t)$ converges. □

Theorem 4.2. *If the series solution $u(t) = \sum_{k=0}^{\infty} v_k(t)$ converges, then it is an exact solution for* (4.12).

Proof. Assume that the series solution converges, say

$$\psi(t) = \sum_{k=0}^{\infty} v_k(t),$$

then

$$\lim_{k \to \infty} v_k = 0.$$

By telescoping we have

$$\sum_{k=0}^{n} (v_{k+1} - v_k) = v_{n+1} - v_0.$$

Let n goes to ∞, we get

$$\sum_{k=0}^{\infty} (v_{k+1} - v_k) = \lim_{n \to \infty} v_{n+1} - v_0 = -v_0.$$

Differentiating both sides m times, we obtain

$$\frac{d^m}{dt^m}\left[\sum_{k=0}^{\infty}(v_{k+1}-v_k)\right] = -\frac{d^m}{dt^m}v_0,$$

$$\sum_{k=0}^{\infty}\frac{d^m}{dt^m}\left[v_{k+1}-v_k\right] = -\frac{d^m}{dt^m}v_0.$$

Since

$$v_0=\sum_{k=0}^{m-1}\frac{c_k}{k!}t^k,$$

hence, the largest power is $m-1$, i.e.

$$\frac{d^m}{dt^m}v_0=0. \tag{4.17}$$

We therefore obtain

$$\sum_{k=0}^{\infty}\frac{d^m}{dt^m}\left[v_{k+1}-v_k\right]=0. \tag{4.18}$$

Indeed, let

$$F(v_k)=f\left([v_0+v_1+\ldots+v_k],[v_0+v_1+\ldots+v_k]',\ldots,\frac{d^m}{dt^m}[v_0+v_1+\ldots+v_k]\right),$$

where $k\geq 0$ from (4.14), we have for $n\geq 1$

$$\begin{aligned}\frac{d^m}{dt^m}\left[v_{k+1}-v_k\right] &= \frac{d^m}{dt^m}\left[B\left[\sum_{j=0}^{k}v_j(t)\right]-B\left[\sum_{j=0}^{k-1}v_j(t)\right]\right]\\ &= \frac{d^m}{dt^m}\int_0^t\frac{(-1)^m(x-t)^{m-1}}{(m-1)!}\left[\frac{d^m}{dx^m}\left(\sum_{j=0}^{k}v_j(x)\right)-F(v_k)\right]dx\\ &\quad -\frac{d^m}{dt^m}\int_0^t\frac{(-1)^m(x-t)^{m-1}}{(m-1)!}\left[\frac{d^m}{dx^m}\left(\sum_{j=0}^{k-1}v_j(x)\right)-F(v_{k-1})\right]dx.\end{aligned}$$

Thus,

$$\frac{d^m}{dt^m}\left[v_{k+1}-v_k\right]$$

is equal to

$$\frac{d^m}{dt^m}\int_0^t \frac{(-1)^m(x-t)^{m-1}}{(m-1)!}\left[\frac{d^m}{dx^m}\sum_{j=0}^{k} v_j(x) - \frac{d^m}{dx^m}\sum_{j=0}^{k-1} v_j(x) - F(v_k) + F(v_{k-1})\right] dx.$$

Since the $m^{th}-$derivative is a left inverse to the $m^{th}-$fold integral,we get

$$\begin{aligned}
\frac{d^m}{dt^m}\left[v_{k+1}-v_k\right] &= \left[\frac{d^m}{dt^m}\sum_{j=0}^{k} v_j(t) - \frac{d^m}{dt^m}\sum_{j=0}^{k-1} v_j(t) - F(v_k) + F(v_{k-1})\right] \\
&= \left[\sum_{k=0}^{k}\frac{d^m}{dt^m}v_k(t) - \sum_{k=0}^{k-1}\frac{d^m}{dt^m}v_k(t) - F(v_k) + F(v_{k-1})\right] \\
&= \frac{d^m v_k}{dt^m} - F(v_k) + F(v_{k-1})
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\sum_{k=0}^{n}\frac{d^m}{dt^m}\left[v_{k+1}-v_k\right] &= \frac{d^m}{dt^m}\left[v_1-v_0\right] + \sum_{k=1}^{n}\left(\frac{d^m v_k}{dt^m} - F(v_k) + F(v_{k-1})\right) \\
&= \frac{d^m}{dt^m}v_1(t) - \frac{d^m}{dt^m}v_0(t) + \sum_{k=1}^{n}\left(\frac{d^m v_k}{dt^m} - F(v_k) + F(v_{k-1})\right)
\end{aligned}$$

According to (4.14) and (4.17), we have

$$\begin{aligned}
\frac{d^m}{dt^m}v_0 &= 0, \\
\frac{d^m}{dt^m}v_1(x) &= \frac{d^m}{dt^m}B[v_0] \\
&= \frac{d^m}{dt^m}\left[\int_0^t \frac{(-1)^m(x-t)^{m-1}}{(m-1)!}\left[\frac{d^m}{dx^m}v_0 - F(v_0)\right] dx\right] \\
&= \frac{d^m}{dt^m}v_0 - F(v_0).
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{k=0}^{n} \frac{d^m}{dt^m}\left[v_{k+1}-v_k\right] &= \frac{d^m v_0}{dt^m} - F(v_0) \\
&\quad + \frac{d^m v_1}{dt^m} - F(v_1) + F(v_0) \\
&\quad + \frac{d^m v_2}{dt^m} - F(v_2) + F(v_1) \\
&\quad \vdots \\
&\quad + \frac{d^m v_n}{dt^m} - F(v_n) + F(v_{n-1}) \\
&= \frac{d^m}{dt^m}\sum_{k=0}^{n} v_k - F(v_n) \\
&= \frac{d^m}{dt^m}\sum_{k=0}^{n} v_k - f\left(\sum_{k=0}^{n} v_k, \sum_{k=0}^{n} v_k', \ldots, \sum_{k=0}^{n} \frac{d^m}{dt^m} v_k\right)
\end{aligned}$$

If n goes to ∞ and by using (4.18), we obtain

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{d^m}{dt^m}\left[v_{k+1}-v_k\right] &= \frac{d^m}{dt^m}\sum_{k=0}^{\infty} v_k - f\left(\sum_{k=0}^{\infty} v_k, \sum_{k=0}^{\infty} v_k', \ldots, \sum_{k=0}^{\infty} \frac{d^m}{dt^m} v_k\right) \\
0 &= \frac{d^m}{dt^m}\sum_{k=0}^{\infty} v_k - f\left(\sum_{k=0}^{\infty} v_k, \sum_{k=0}^{\infty} v_k', \ldots, \sum_{k=0}^{\infty} \frac{d^m}{dt^m} v_k\right)
\end{aligned}$$

Therefore,

$$\frac{d^m}{dt^m}\sum_{k=0}^{\infty} v_k - f\left(\sum_{k=0}^{\infty} v_k, \frac{d}{dt}\sum_{k=0}^{\infty} v_k, \ldots, \frac{d^m}{dt^m}\sum_{k=0}^{\infty} v_k\right) = 0. \tag{4.19}$$

Hence, from (4.19) we can observe that

$$\sum_{k=0}^{\infty} v_k$$

is the exact solution to the nonlinear problem

$$u^{(m)} = f(u, u', u'', \cdots, u^{(m)}).$$

$\square$

Theorem 4.3. *Suppose that the series $\sum_{k=0}^{\infty} v_k$ is convergent to the solution $u(t)$. If we use $\sum_{k=0}^{i} v_k$ to be an approximation to the solution $u(t)$, then we can estimate the maximum error $E_i(t)$ as*

$$E_i(t) \leq \frac{\gamma^{i+1}}{1-\gamma} \parallel v_0 \parallel$$

Proof. From Theorem (4.1), we have

$$\parallel S_n - S_i \parallel \leq \frac{1-\gamma^{n-i}}{1-\gamma} \gamma^{i+1} \parallel v_0 \parallel, \quad n \geq i.$$

If $n \to \infty$, then

$$\lim_{n\to\infty} S_n = u(t),$$

hence

$$\parallel u(t) - S_i \parallel \leq \lim_{n\to\infty} \frac{1-\gamma^{n-i}}{1-\gamma} \gamma^{i+1} \parallel v_0 \parallel .$$

Since $0 < \gamma < 1$, then

$$\lim_{n\to\infty} (1-\gamma^{n-i}) = 1.$$

Therefore,

$$\parallel u(t) - S_i \parallel \leq \frac{\gamma^{i+1}}{1-\gamma} \parallel v_0 \parallel .$$

$\square$

Remark 4.1. *The series solution $\sum_{k=0}^{i} v_k$ converges to the exact solution $u(t)$ if there exist $0 < \gamma < 1$, such that*

$$\parallel B[v_0 + v_1 + \ldots + v_{k+1}] \parallel \leq \gamma \parallel B[v_0 + v_1 + \ldots + v_k] \parallel,$$

equivalently

$$\begin{aligned} \| v_{k+1} \| &\leq \gamma \| v_k \| \\ \frac{\| v_{k+1} \|}{\| v_k \|} &\leq \gamma \end{aligned}$$

If we define

$$\begin{cases} \beta_k = \frac{\|v_{k+1}\|}{\|v_k\|}, & \text{if } \| v_k \| \neq 0; \\ 0, & \| v_k \| = 0 \end{cases}$$

Then the series solution $\sum_{k=0}^{i} v_k$ *converge to the exact solution* $u(t)$ *if* $\beta_k < 1$ *for all* $k \geq 0$.

Remark 4.2. *If* $\beta_k > 1$, *which is defined in the last remark, for* $0 \leq k \leq m$ *then the series solution* $\sum_{k=0}^{i} v_k$ *converges to the exact solution* $u(t)$, *i.e.*

$$\begin{aligned} \beta_k &\geq 1, \text{if } 0 \leq k \leq m \\ \beta_k &< 1, \text{if } k \geq m \end{aligned}$$

the first finite terms don't affect to the convergence of the series solution.

Example 4.2. To explain the convergence of this method, let us consider the following example of the second order linear differential equation.

$$u''(t) + u(t) = 0, \quad 0 \leq t \leq 1,$$

subject to the initial condition

$$\begin{aligned} u(0) &= 0, \\ u'(0) &= 1. \end{aligned}$$

Then we have the next iteration formula

$$
\begin{aligned}
v_0 &= t, \\
v_1 &= \int_0^t (x-t)\,[v_0''(x) + v_0(x)]\,dx \\
&= \frac{1}{3!}t^3 \\
v_2(t) &= \int_0^t (x-t)\,[(v_0+v_1)'' + v_0(x) + v_1(x)]\,dx \\
&= \frac{1}{5!}t^5 \\
v_3(t) &= \int_0^t (x-t)\,[(v_0+v_1+v_2)'' + v_0(x) + v_1(x) + v_2(x)]\,dx \\
&= \frac{1}{7!}t^7 \\
&\vdots \\
v_k(t) &= \int_0^t (x-t)\,[(v_0+v_1+v_2+v_3+\ldots+v_{k-1})'' + v_0+v_1+v_2+v_3+\ldots+v_{k-1}]\,dx \\
&= \frac{(-1)^k}{(2k+1)!}t^{2k+1}.
\end{aligned}
$$

Observe that the obtained solution $\sum_{k=0}^{i} v_k$ converges to the exact solution

$$u(t) = \sin(t).$$

Moreover, by computing β_k, we get

$$
\begin{aligned}
\beta_0 &= \frac{\| v_1 \|}{\| v_0 \|} \\
&= \frac{\| t^3/3! \|}{\| t \|} = \frac{1}{3!}
\end{aligned}
$$

$$\begin{aligned}\beta_1 &= \frac{\| v_2 \|}{\| v_1 \|} \\ &= \frac{\| t^5/5! \|}{\| t^3/3! \|} = \frac{3!}{5!} \\ &\vdots \\ \beta_k &= \frac{\| v_{k+1} \|}{\| v_k \|} \\ &= \frac{\| t^{2k+3}/(2k+3)! \|}{\| t^{2k+1}/(2k+1)! \|} = \frac{(2k+3)!}{(2k+1)!}.\end{aligned}$$

Where

$$\| v_k \| = \sup_{t \in (0,1)} |v_k(t)|,$$

since $\beta_k < 1$ for all $k \geq 0$, then the VIM is convergent to the exact solution $\sin(t)$

Remark 4.3. *If β_k are not less than 1 for all $k \geq 0$, then we can't say that the VIM is divergent. So the VIM may be convergent or divergent.*

Example 4.3. Now, we consider the next nonlinear differential equation,

$$u'(t) + u^3(t) = t^3 + 3t^2 + 3t + 2 \;\;,\;\; 0 < t \leq 1. \tag{4.20}$$

subject to the initial condition,

$$u(0) = 1.$$

Then, the iteration formula can be constructed by Maple as

$$
\begin{aligned}
v_0 &= 1,\\
v_1 &= -\int_0^t [v_0'(x) + v_0^3(x) - x^3 - 3x^2 - 3x - 2]dx\\
&= \frac{1}{4}t^4 + t^3 + \frac{3}{2}t^2 + t\\
v_2 &= -\int_0^t [v_0'(x) + v_1'(x) + (v_0(x)+v_1(x))^3 - x^3 - 3x^2 - 3x - 2]dx\\
&= \frac{-3}{2}t^2 - \frac{5}{2}t^3 - \frac{13}{4}t^4 - \frac{18}{5}t^5 - \frac{27}{8}t^6 - \frac{21}{8}t^7 - \frac{27}{16}t^8 - \frac{7}{8}t^9 - \frac{11}{32}t^{10} - \frac{3}{32}t^{11}\\
&\qquad - \frac{1}{64}t^{12} - \frac{1}{832}t^{13}\\
v_3 &= -\int_0^t [v_0'(x) + v_1'(x) + v_2'(x) + (v_0(x)+v_1(x)+v_2(x))^3 - x^3 - 3x^2 - 3x - 2]dx\\
&= 1.5t^3 + 4.125t^4 + 7.2t^5 + 8.925t^6 + 7.478571429t^7 + 2.334375t^8 - 5.2875t^9 - 12.71375t^{10}\\
&\quad - 16.89136364t^{11} - 15.7196875t^{12} - 8.909375t^{13} + 1.940748626t^{14} + 14.01398077t^{15}\\
&\quad + 24.35538552t^{16} + 30.85885817t^{17} + 32.77327524t^{18} + 30.63266352t^{19}\\
&\quad + 25.78150781t^{20} + 19.77938444t^{21} + 13.92748320t^{22} + 9.033640259t^{23}\\
&\quad + 5.405086200t^{24} + 2.982433143t^{25} + 1.514958742t^{26} + .7061743020t^{27}\\
&\quad + 0.300677872t^{28} + 0.1162297175t^{29} + 0.04047732381t^{30} + 0.01257978025t^{31}\\
&\quad + 0.003449120606t^{32} + 0.0008227105677t^{33} + 0.0001678015353t^{34}\\
&\quad + 0.00002863054445t^{35} + 0.000003968940449t^{36} + 4.28871290210^{-7}t^{37}\\
&\quad + 3.38582597510^{-8}t^{38} + 1.73632101310^{-9}t^{39} + 4.34080253310^{-11}t^{40}\\
&\ \vdots
\end{aligned}
$$

Clearly, we observe that the obtained solution $\sum_{k=0}^{i} v_k$ does not converge to the exact solution $u(t) = 1 + t$. Moreover, by computing β_k, Where

$$\| v_k \| = \sup_{t\in(0,1)} |v_k(t)|,$$

we get

$$\begin{aligned}
\beta_0 &= \frac{\| v_1 \|}{\| v_0 \|} \\
&= 3.75 \\
\beta_1 &= \frac{\| v_2 \|}{\| v_1 \|} \\
&= 5.297821 \\
\beta_2 &= \frac{\| v_3 \|}{\| v_2 \|} \\
&= 9.373479 \\
\beta_3 &= \frac{\| v_4 \|}{\| v_3 \|} \\
&= 439.133224 \\
&\vdots
\end{aligned}$$

In this example, β_k are not less than 1 for all $k \geq 0$. Hence, we prove the convergence by the following way. We have the following iteration formula for (4.20)

$$u_{n+1}(t) = u_n(t) - \int_0^t \left[u_n'(x) + u_n^3(x) - x^3 - 3x^2 - 3x - 2\right] dx, \quad n \geq 1 \qquad (4.21)$$

with $u_0(t) = 1$. Subtract $u(t)$ from both sides of (4.21), we obtain

$$u_{n+1}(t) - u(t) = u_n(t) - u(t) - \int_0^t \left[u_n'(x) + u_n^3(x) - x^3 - 3x^2 - 3x - 2\right] dx$$

Add and subtract $u'(x)$ in the integral, we get

$$\begin{aligned}
u_{n+1}(t) - u(t) &= u_n(t) - u(t) - \int_0^t \left[u_n'(x) - u'(x) + u'(x) + u_n^3(x) - x^3 - 3x^2 - 3x - 2\right] dx \\
&= u_n(t) - u(t) - \int_0^t \left[(u_n(x) - u(x))' + u'(x) + u_n^3(x) - x^3 - 3x^2 - 3x - 2\right] dx
\end{aligned}$$

Since $u(t)$ is the exact solution, then

$$u'(t) = t^3 + 3t^2 + 3t + 2 - u^3(t).$$

Let $E_n(t) = u_n(t) - u(t)$, so we have

$$\begin{aligned}
E_{n+1}(t) &= E_n(t) - \int_0^t \left[E_n'(x) + x^3 + 3x^2 + 3x + 2 - u^3(x) + u_n^3(x) - x^3 - 3x^2 - 3x - 2\right] dx \\
&= E_n(t) - \int_0^t \left[E_n'(x) - u^3(x) + u_n^3(x)\right] dx \\
&= E_n(t) - \int_0^t E_n'(x)dx - \int_0^t \left(u_n^3(x) - u^3(x)\right) dx \\
&= E_n(t) - E_n(t) + E_n(0) - \int_0^t \left(u_n^3(x) - u^3(x)\right) dx \\
&= E_n(0) - \int_0^t \left(u_n^3(x) - u^3(x)\right) dx
\end{aligned}$$

We know that

$$\begin{aligned}
E_n(0) &= u_n(0) - u(0) \\
&= 0
\end{aligned}$$

Hence, we have

$$E_{n+1}(t) = -\int_0^t \left(u_n^3(x) - u^3(x)\right) dx$$

Operating with the L^2-norm on both sides of the last equation, we obtain

$$\begin{aligned}
\| E_{n+1}(t) \|_{L^2} &= \| -\int_0^t \left(u_n^3(x) - u^3(x)\right) dx \|_{L^2} \\
&\leq \int_0^t \left(\| u_n^3(x) - u^3(x) \|_{L^2}\right) dx \qquad (4.22)
\end{aligned}$$

Applying the mean value theorem to the integral (4.22), then we have

$$\begin{aligned}
\| E_{n+1}(t) \|_{L^2} &\leq \int_0^t 3L \| u_n(x) - u(x) \|_{L^2} dx \\
&\leq 3L \int_0^t \| E_n(x) \|_{L^2} dx \qquad (4.23)
\end{aligned}$$

Where

$$L = max_{x\in[0,1]}(\mid \overline{u}(x) \mid)^2$$

Then, from inequality (4.23), and by letting $M = 3L$, we get

$$\| E_{n+1}(t) \|_{L^2} \leq M \int_0^t \| E_n(x) \|_{L^2} \, dx$$

By induction and using

$$\| E_0(x) \|_{L^2} \leq \| E_0(t) \|_\infty \quad ,$$

we obtain

$$\begin{aligned}
\| E_1(t) \|_{L^2} &\leq M \int_0^t \| E_0(x) \|_{L^2} \, dx \\
&\leq M \| E_0(t) \|_\infty \int_0^t dx = M \| E_0(t) \|_\infty \, t, \\
\| E_2(t) \|_{L^2} &\leq M \int_0^t \| E_1(x) \|_{L^2} \, dx \\
&\leq M^2 \| E_0(t) \|_\infty \int_0^t x dx = M^2 \| E_0(t) \|_\infty \frac{t^2}{2}, \\
&\vdots \\
\| E_{n+1}(t) \|_{L^2} &\leq M \int_0^t \| E_n(x) \|_{L^2} \, dx \\
&\leq M^{n+1} \| E_0(t) \|_\infty \int_0^t \frac{x^n}{n!} dx = M^{n+1} \| E_0(t) \|_\infty \frac{t^{n+1}}{(n+1)!},
\end{aligned}$$

where $\| E_0(t) \|_\infty = max_{t\in[0,1]} \mid E_0(t) \mid$. And,

$$\begin{aligned}
\| E_0(t) \|_\infty &= \| u_0(t) - u(t) \|_\infty \\
&= \| 1 - u(t) \|_\infty \\
&\leq \| 1 \|_\infty + \| u(t) \|_\infty \\
&= 1 + max_{t\in[0,1]} |u(t)|.
\end{aligned}$$

According to be $u(x)$ the exact solution of (4.20), then $u(x) \in C^2[0,1]$, hence it is

Where

$$L = max_{x\in[0,1]}(|\, \overline{u}(x)\,|)^2$$

Then, from inequality (4.23), and by letting $M = 3L$, we get

$$\| E_{n+1}(t) \|_{L^2} \leq M \int_0^t \| E_n(x) \|_{L^2}\, dx$$

By induction and using

$$\| E_0(x) \|_{L^2} \leq \| E_0(t) \|_{\infty} \;,$$

we obtain

$$\begin{aligned}
\| E_1(t) \|_{L^2} &\leq M \int_0^t \| E_0(x) \|_{L^2}\, dx \\
&\leq M \| E_0(t) \|_{\infty} \int_0^t dx = M \| E_0(t) \|_{\infty}\, t, \\
\| E_2(t) \|_{L^2} &\leq M \int_0^t \| E_1(x) \|_{L^2}\, dx \\
&\leq M^2 \| E_0(t) \|_{\infty} \int_0^t x dx = M^2 \| E_0(t) \|_{\infty} \frac{t^2}{2}, \\
&\vdots \\
\| E_{n+1}(t) \|_{L^2} &\leq M \int_0^t \| E_n(x) \|_{L^2}\, dx \\
&\leq M^{n+1} \| E_0(t) \|_{\infty} \int_0^t \frac{x^n}{n!} dx = M^{n+1} \| E_0(t) \|_{\infty} \frac{t^{n+1}}{(n+1)!},
\end{aligned}$$

where $\| E_0(t) \|_{\infty} = max_{t\in[0,1]} \mid E_0(t) \mid$. And,

$$\begin{aligned}
\| E_0(t) \|_{\infty} &= \| u_0(t) - u(t) \|_{\infty} \\
&= \| 1 - u(t) \|_{\infty} \\
&\leq \| 1 \|_{\infty} + \| u(t) \|_{\infty} \\
&= 1 + max_{t\in[0,1]}|u(t)|.
\end{aligned}$$

According to be $u(x)$ the exact solution of (4.20), then $u(x) \in C^2[0,1]$, hence it is bounded and therefore $E_0(t)$ is also bounded. Let $P = max_{t\in[0,1]}|u(t)|$, So we obtain

$$\| E_{n+1}(t) \|_{L^2} \leq M^{n+1}(1+P)\frac{t^{n+1}}{(n+1)!}, \tag{4.24}$$

as $n \to \infty$, the sequence in the right hand side of (4.24) converge uniformly to 0, it follows that

$$\| E_{n+1}(t) \|_{L^2} \to 0$$

which means that $u_n(t)$ converge uniformly to $u(t) = 1 + t$.

Chapter 5

Variational iteration method for partial differential equations

In this chapter, we present a good strategy which is the VIM for solving linear and nonlinear partial differential equations (PDE). As ODE's, we use the Lagrange multiplier and the concept of restricted variables for solving a PDE's problems.

In Section 5.1, we present the methodology of the VIM for linear partial differential equations by giving some illustrative examples. In section 5.2, we applied the VIM for a certain equation, which is the Heat and Wave equations. In Section 5.3, we illustrate the VIM for nonlinear partial differential equation. Moreover, in Section 5.4, we find the approximate solution by using laplace transform together with VIM.

5.1 Linear partial differential equations

The material of this section is mainly taken from [4, 13, 15]. We explain the method of the VIM in PDE's by considering the next linear third order PDE's in one dimension

$$u_t + a\ u_x + b\ u_{xxx} = g(x,t), \quad k_0 < x < k_1, \quad t > 0, \quad a, b > 0, \tag{5.1}$$

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5.1 Linear partial differential equations

The material of this section is mainly taken from [4, 13, 15]. We explain the method of the VIM in PDE's by considering the next linear third order PDE's in one dimension

$$u_t + a\ u_x + b\ u_{xxx} = g(x,t), \quad k_0 < x < k_1, \quad t \geq 0, \quad a, b \geq 0, \tag{5.1}$$

where a, b, k_0, and k_1 are real numbers, and g is a source term. Assume the following conditions

$$\begin{aligned} u(x,0) &= f(x), \\ u(0,t) &= h_0(t), \\ u_x(0,t) &= h_1(t), \\ u_{xx}(0,t) &= h_2(t). \end{aligned}$$

By constructing a correction functional, we obtain

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,s)\left[(u_n)_s + a(u_n)_x + b(u_n)_{xxx} - g(x,s)\right] ds,$$

Taking the variation with respect to $u_n(x,t)$, we get

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(t,s)\left[(u_n)_s + a(u_n)_x + b(u_n)_{xxx} - g(x,s)\right] ds,$$

or equivalently,

$$\begin{aligned} \delta u_{n+1}(x,t) &= \delta u_n(x,t) + \int_0^t \lambda(t,s)\delta\left[(u_n)_s + a(u_n)_x + b(u_n)_{xxx} - g(x,s)\right] ds \\ &= \delta u_n(x,t) + \int_0^t \lambda(t,s)\left[\delta(u_n)_s + a\delta(u_n)_x + b\delta(u_n)_{xxx} - \delta g(x,s)\right] ds. \end{aligned} \tag{5.2}$$

We deal with $(u_n)_x$, $(u_n)_{xxx}$, and $g(x,s)$ as a restrict variables, which means

$$\begin{aligned} \delta(u_n)_x &= 0, \\ \delta(u_n)_{xxx} &= 0, \\ \delta g(x,s) &= 0. \end{aligned}$$

Equation (5.2) becomes as

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \lambda(t,s)\delta(u_n)_s ds.$$

Using integration by parts, we obtain

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \lambda(t,t)\delta u_n(x,t) - \int_0^t \frac{\partial}{\partial s}\lambda(t,s)\ \delta u_n ds.$$

Hence, we obtain the following stationary conditions

$$\begin{aligned} 1+\lambda(t,t) &= 0, \\ \frac{\partial\lambda(t,s)}{\partial s} &= 0. \end{aligned}$$

So we get

$$\lambda(t,s) = -1.$$

Therefore, we have the following iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[(u_n)_s + a(u_n)_x + b(u_n)_{xxx} - g(x,s)\right] ds. \tag{5.3}$$

Taking

$$u_0(x,t) = u(x,0) = f(x)$$

as an initial approximation, then we have the next iteration formula

$$\begin{aligned} u_1(x,t) &= u_0(x,t) - \int_0^t \left[(u_0)_s + a(u_0)_x + b(u_0)_{xxx} - g(x,s)\right] ds \\ u_2(x,t) &= u_1(x,t) - \int_0^t \left[(u_1)_s + a(u_1)_x + b(u_1)_{xxx} - g(x,s)\right] ds \\ &\vdots \end{aligned}$$

Consequently, the exact solution can be obtained by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

Let us consider the following example to illustrate the accuracy of this method comparing with the exact solution.

Example 5.1. In this example, we consider a nonhomogeneous third order PDE of one dimension.

$$u_t + u_{xxx} = -\sin(x)\sin(t) - \cos(x)\cos(t), \quad 0 < x < 1, \quad t > 0,$$

with initial condition

$$u(x,0) = \sin(x).$$

By (5.3), and using the following initial approximation

$$u_0(x,t) = \sin(x).$$

By Maple, we have the next iterations

$$\begin{aligned}
u_1(x,t) &= u_0(x,t) - \int_0^t [(u_0)_s + (u_0)_{xxx} + \sin(x)\sin(s) + \cos(x)\cos(s))]\,ds \\
&= \sin(x)\cos(t) + \cos(x)\,(t - \sin(t)) \\
u_2(x,t) &= u_1(x,t) - \int_0^t [(u_1)_s + (u_1)_{xxx} + \sin(x)\sin(s) + \cos(x)\cos(s))]\,ds \\
&= \sin(x)\cos(t) + \sin(x)\left(1 - (1/2)t^2 - \cos(t)\right) \\
u_3(x,t) &= u_2(x,t) - \int_0^t [(u_2)_s + (u_2)_{xxx} + \sin(x)\sin(s) + \cos(x)\cos(s))]\,ds \\
&= \sin(x)\cos(t) + \cos(x)\left(t - (1/6)t^3 - \sin(t)\right) \\
u_4(x,t) &= u_3(x,t) - \int_0^t [(u_3)_s + (u_3)_{xxx} + \sin(x)\sin(s) + \cos(x)\cos(s))]\,ds \\
&= \sin(x)\cos(t) + \sin(x)\left(1 - (1/2)t^2 + (1/24)t^4 - \cos(t)\right) \\
u_5(x,t) &= u_4(x,t) - \int_0^t [(u_4)_s + (u_4)_{xxx} + \sin(x)\sin(s) + \cos(x)\cos(s))]\,ds \\
&= \sin(x)\cos(t) + \cos(x)\left(t - (1/6)t^3 + (1/120)t^5 - \sin(t)\right) \\
&\vdots
\end{aligned}$$

x/t	0.2	0.3	0.4	0.6
0.1	$2.49 * 10^{-9}$	$4.314 * 10^{-8}$	$3.2273 * 10^{-7}$	0.00000549898
0.3	$2.4 * 10^{-9}$	$4.14 * 10^{-8}$	$3.098 * 10^{-7}$	0.0000052798
0.6	$2.1 * 10^{-9}$	$3.57 * 10^{-8}$	$2.677 * 10^{-7}$	0.0000045613
0.9	$1.6 * 10^{-9}$	$2.69 * 10^{-8}$	$2.017 * 10^{-7}$	0.0000034354

Table 5.1: The absolute error between u_5 and the exact solution.

Note that

$$1 - (1/2)t^2 + (1/24)t^4$$

is a fourth order approximation of $\cos(t)$ in the fourth iteration,

and

$$t - (1/6)t^3 + (1/120)t^5$$

is a fifth order approximation of $\sin(t)$ in the fifth iteration, and so on.

According to that, this iterations is an approximation of the exact solution

$$u(x,t) = \sin(x)\cos(t),$$

where the second term in the approximations tends to zero. Table (5.1) shows the absolute error between u_5 and the exact solution.

As we apply the VIM to one dimension, we can apply it for two, three, and more. For more details, consider the following example to determine the approximate solution.

Example 5.2. In this example, we consider a second order PDE of two dimensions, see [15].

$$u_t = u_{xx} + u_{yy}, \quad 0 < t < 1,$$

over the region $\Omega = [0,1] \times [0,1]$ with initial and boundary conditions

$$\begin{aligned} u &= 0 \text{ on } \partial\Omega, \\ u(x,y,0) &= \sin \pi x \sin \pi y, \quad 0 < x, y < 1. \end{aligned}$$

Then, the correction functional is

$$u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^t \lambda(t,s)\left[(u_n)_s - (u_n)_{xx} - (u_n)_{yy}\right] ds.$$

Taking the variation with respect to $u_n(x,y,t)$

$$\begin{aligned}
\delta u_{n+1}(x,y,t) &= \delta u_n(x,y,t) + \delta \int_0^t \lambda(t,s)\left[(u_n)_s - (u_n)_{xx} - (u_n)_{yy}\right] ds \\
&= \delta u_n(x,y,t) + \int_0^t \lambda(t,s)\delta(u_n)_s ds - \int_0^t \lambda(t,s)\delta(u_n)_{xx} ds \\
&\qquad - \int_0^t \lambda(t,s)\delta(u_n)_{yy} ds.
\end{aligned}$$

We consider $(u_n)_{xx}$ and $(u_n)_{yy}$ as restrict variables, i.e.

$$\delta(u_n)_{xx} = (u_n)_{yy} = 0.$$

Thus

$$\delta u_{n+1}(x,y,t) = \delta u_n(x,y,t) + \delta \int_0^t \lambda(t,s)(u_n)_s ds$$

Integrating by parts, we obtain

$$\begin{aligned}
\delta u_{n+1}(x,y,t) &= \delta u_n(x,y,t) + \lambda(t,s)\delta u_n(x,y,s)\Big|_0^t + \int_0^t \frac{\partial\lambda(t,s)}{\partial s}\delta u_n ds \\
&= \delta u_n(x,y,t) + \lambda(t,t)\delta u_n(x,y,t) \\
&\qquad - \lambda(t,0)\delta u_n(x,y,0) + \int_0^t \frac{\partial\lambda(t,s)}{\partial s}\delta u_n ds \\
&= \delta u_n(x,y,t) + \lambda(t,t)\delta u_n(x,y,t) + \int_0^t \frac{\partial\lambda(t,s)}{\partial s}\delta u_n ds \\
&= (1+\lambda(t,t))\delta u_n(x,y,t) + \int_0^t \frac{\partial\lambda(t,s)}{\partial s}\delta u_n ds
\end{aligned}$$

Thus, we have the next first order differential equation

$$\frac{\partial\lambda(t,s)}{\partial s} = 0$$

with boundary condition

$$1 + \lambda(t, t) = 0$$

Then

$$\lambda(t, s) = -1.$$

The correction functional can be

$$u_{n+1}(x, y, t) = u_n(x, y, t) - \int_0^t [(u_n)_s - (u_n)_{xx} - (u_n)_{yy}]\, ds$$

Taking

$$u_0(x, y, t) = \sin \pi x \cos \pi y$$

So,

$$\begin{aligned}
u_1(x, y, t) &= u_0(x, y, t) - \int_0^t [(u_0)_s - (u_0)_{xx} - (u_0)_{yy}]\, ds \\
&= (1 - 2\pi^2 t) \sin \pi x \cos \pi y \\
u_2(x, y, t) &= u_1(x, y, t) - \int_0^t [(u_1)_s - (u_1)_{xx} - (u_1)_{yy}]\, ds \\
&= \left(1 - 2\pi^2 t + \frac{2^2}{2!}\pi^4 t^2\right) \sin(\pi x) \sin(\pi y) \\
u_3(x, y, t) &= u_2(x, y, t) - \int_0^t [(u_2)_s - (u_2)_{xx} - (u_2)_{yy}]\, ds \\
&= \left(1 - 2\pi^2 t + \frac{2^2}{2!}\pi^4 t^2 - \frac{2^3}{3!}\pi^6 t^3\right) \sin(\pi x) \sin(\pi y) \\
u_4(x, y, t) &= u_3(x, y, t) - \int_0^t [(u_3)_s - (u_3)_{xx} - (u_3)_{yy}]\, ds \\
&= \left(1 - 2\pi^2 t + \frac{2^2}{2!}\pi^4 t^2 - \frac{2^3}{3!}\pi^6 t^3 + \frac{2^4}{4!}\pi^8 t^4\right) \sin(\pi x) \sin(\pi y) \\
u_5(x, y, t) &= u_4(x, y, t) - \int_0^t [(u_4)_s - (u_4)_{xx} - (u_4)_{yy}]\, ds \\
&= \left(1 - 2\pi^2 t + \frac{2^2}{2!}\pi^4 t^2 - \frac{2^3}{3!}\pi^6 t^3 + \frac{2^4}{4!}\pi^8 t^4 - \frac{2^5}{5!}\pi^{10} t^5\right) \sin(\pi x) \sin(\pi y)
\end{aligned}$$

$$\vdots$$

$$\begin{aligned} u_n(x,y,t) &= u_{n-1}(x,y,t) - \int_0^t [(u_{n-1})_s - (u_{n-1})_{xx} - (u_{n-1})_{yy}]\, ds \\ &= \Big(1 - 2\pi^2 t + 2\pi^4 t^2 - \frac{4}{3}\pi^6 t^3 + \frac{2}{3}\pi^8 t^4 - \frac{4}{15}\pi^{10} t^5 + \cdots \\ &\qquad + (-1)^n \frac{2^n}{n!}\pi^{2n} t^n\Big) \sin(\pi x)\sin(\pi y) \end{aligned}$$

which clearly converges to the exact solution

$$u(x,y,t) = e^{-2\pi^2 t}\sin(\pi x)\sin(\pi y)$$

5.2 Scientific applications (heat and wave equations)

In this section, we apply the VIM for the heat and wave equations.

Heat equation

Consider the following basic equation of one dimensional heat flow

$$u_t = \alpha^2 u_{xx}, \quad 0 \le x \le L, \quad 0 \le t \le \infty. \tag{5.4}$$

Where $u(x,t)$ represents the temperature as a function of space and time, u_t is the rate of change in temperature with respect to the time, and u_{xx} is the concavity of the temperature profile $u(x,t)$.

The boundary conditions describing the physical nature of our problem on the boundaries, and can be as follow

$$
\begin{aligned}
u(0,t) &= a, \\
u(L,t) &= b, \quad 0 \leq t.
\end{aligned}
$$

The initial conditions describing the physical phenomena at the start of the experiment, and is given as

$$u(x,0) = f(x), \quad 0 \leq x \leq L$$

Applying the VIM to Equation (5.4), we get the next correction functional

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,s)\left((u_n)_s(x,s) - \alpha^2 (u_n)_{xx}(x,s)\right) ds, \quad n \geq 0.$$

Taking the variation with respect to u_n, we obtain

$$
\begin{aligned}
\delta u_{n+1}(x,t) &= \delta u_n(x,t) + \delta \int_0^t \lambda(t,s)\left((u_n)_s(x,s) - \alpha^2 (u_n)_{xx}(x,s)\right) ds \\
&= \delta u_n(x,t) + \int_0^t \lambda(t,s)\delta\left((u_n)_s(x,s) - \alpha^2 (u_n)_{xx}(x,s)\right) ds \\
&= \delta u_n(x,t) + \int_0^t \lambda(t,s)\left(\delta(u_n)_s(x,s) - \alpha^2 \delta(u_n)_{xx}(x,s)\right) ds
\end{aligned}
$$

Here, $(u_n)_{xx}$ is restricted variable, i.e., $\delta(u_n)_{xx} = 0$. Thus,

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \lambda(t,s)\delta(u_n)_s(x,s) ds$$

Integrating by parts, we obtain

$$\begin{aligned}
\delta u_{n+1}(x,t) &= \delta u_n(x,t) + \lambda(t,s)\delta u_n(x,s)\Big|_0^t - \int_0^t \frac{\partial \lambda}{\partial s}\delta u_n(x,s)ds \\
&= \delta u_n(t) + \lambda(t,t)\delta u_n(x,t) - \lambda(t,0)\delta u_n(x,0) - \int_0^t \frac{\partial \lambda}{\partial s}\delta u_n(s)ds \\
&= (1+\lambda(t,t))\,\delta u_n(x,t) - \int_0^t \frac{\partial \lambda}{\partial s}\delta u_n(x,s)ds
\end{aligned}$$

Then, we have this differential equation of the first order

$$\frac{\partial \lambda}{\partial s} = 0$$

with respect to the next condition

$$1 + \lambda(t,t) = 0.$$

Thus

$$\lambda(t,s) = -1.$$

The correction functional will be

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left((u_n)_s(x,s) - \alpha^2 (u_n)_{xx}(x,s)\right) ds, \quad n \geq 0.$$

And the zeroth approximation is

$$u_0(x,t) = f(x).$$

Consequently, the exact solution can be obtained by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

Let us consider the following example

Example 5.3. Consider the following heat equation

$$u_t = u_{xx}, \quad 0 \le x \le 2, \quad t \ge 0.$$

with respect to the conditions

$$\begin{aligned} u(0,t) = u(2,t) &= 0, \\ u(x,0) &= 3\sin(2\pi x). \end{aligned}$$

According to the VIM, we obtain the following iteration formula

$$\begin{aligned}
u_0(x,t) &= 3\sin(2\pi x) \\
u_1(x,t) &= u_0(x,t) - \int_0^t ((u_0)_s(x,s) - (u_0)_{xx}(x,s))\, ds \\
&= 3\sin(2\pi x)\left(1 - 4\pi^2 t\right) \\
u_2(x,t) &= u_1(x,t) - \int_0^t ((u_1)_s(x,s) - (u_1)_{xx}(x,s))\, ds \\
&= 3\sin(2\pi x)\left(1 - 4\pi^2 t + \frac{16}{2!}\pi^4 t^2\right) \\
u_3(x,t) &= u_2(x,t) - \int_0^t ((u_2)_s(x,s) - (u_2)_{xx}(x,s))\, ds \\
&= 3\sin(2\pi x)\left(1 - 4\pi^2 t + \frac{16}{2!}\pi^4 t^2 - \frac{64}{3!}\pi^6 t^3\right) \\
&\vdots \\
u_n(x,t) &= u_{n-1}(x,t) - \int_0^t \left((u_{n-1})_s(x,s) - \alpha^2 (u_{n-1})_{xx}(x,s)\right) ds \\
&= 3\sin(2\pi x)\left(1 - 4\pi^2 t + \frac{16}{2!}\pi^4 t^2 - \frac{64}{3!}\pi^6 t^3 + \ldots + (-1)^n \frac{(4)^n}{n!}\pi^n t^n\right),
\end{aligned}$$

and the exact solution can be

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

According to these iteration, which is an approximation of the exact solution

$$u(x,t) = 3\sin(2\pi x)e^{-4\pi^2 t}.$$

Wave equation

The wave equation is a partial differential equation of second order that describes a wave such as sound waves, water waves and light waves. In this section we apply the VIM to wave equation that study a small vibration on the string, see [33].

Consider the following wave equation of second order

$$u_{tt} = c^2 u_{xx}, \quad 0 \le x \le L, \quad 0 \le t \le \infty. \tag{5.5}$$

Where $u(x,t)$ is the displacement of the string from equilbrium, c is the velocity of propagation, and $u_{tt}(x,t)$ represents the vertical acceleration of the string at a point.

The boundary conditions represent that the end points of the vibration string are fixed, and can be identified as follow

$$u(0,t) = u(L,t) = 0, \qquad t \ge 0.$$

The initial condition indicates the initial displacement and the initial velocity of any point at the time $t = 0$. In this wave equation we have the next initial condition

$$\begin{aligned} u(x,0) &= f(x), \\ u_t(x,0) &= g(x), \quad 0 \leq x \leq L. \end{aligned}$$

The correction functional of the wave equation (5.5) is identified as follow

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,s)\left((u_n)_{ss} - c^2(u_n)_{xx}\right) ds \quad n \geq 0.$$

Taking the variation with respect to $u_n(t)$, yields

$$\begin{aligned} \delta u_{n+1}(x,t) &= \delta u_n(x,t) + \delta \int_0^t \lambda(t,s)\left((u_n)_{ss} - c^2(u_n)_{xx}\right) ds \\ &= \delta u_n(x,t) + \int_0^t \lambda(t,s)\delta\left((u_n)_{ss} - c^2(u_n)_{xx}\right) ds \\ &= \delta u_n(x,t) + \int_0^t \lambda(t,s)\left(\delta(u_n)_{ss} - c^2\delta(u_n)_{xx}\right) ds \end{aligned}$$

We deal with u_{xx} as a restrict variable, i.e.

$$\delta(u_n)_{xx} = 0.$$

thus, we have

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \int_0^t \lambda(t,s)\delta(u_n)_{ss} ds.$$

Integrating by parts two times, we obtain

$$\begin{aligned}
\delta u_{n+1}(x,t) &= \delta u_n(x,t) + \lambda(t,s)\delta(u_n)_s(x,s)\Big|_0^t - \frac{\partial \lambda}{\partial s}\delta(u_n)(x,s)\Big|_0^t + \int_0^t \frac{\partial^2 \lambda}{\partial s^2}\delta(u_n)(x,s)ds \\
&= \delta u_n(x,t) + \lambda(t,t)\delta(u_n)_s(x,t) - \lambda(t,0)\delta(u_n)_s(x,0) - \frac{\partial \lambda}{\partial s}\Big|_{s=t}\delta(u_n)(x,t) \\
&\qquad + \frac{\partial \lambda}{\partial s}\Big|_{s=0}\delta(u_n)(x,0) + \int_0^t \frac{\partial^2 \lambda}{\partial s^2}\delta(u_n)(x,s)ds \\
&= \delta u_n(x,t) + \lambda(t,t)\delta(u_n)_s(x,t) - \frac{\partial \lambda}{\partial s}\Big|_{s=t}\delta(u_n)(x,t) + \int_0^t \frac{\partial^2 \lambda}{\partial s^2}\delta(u_n)(x,s)ds \\
&= \left(1 - \frac{\partial \lambda}{\partial s}\Big|_{s=t}\right)\delta u_n(x,t) + \lambda(t,t)\delta(u_n)_s(x,t) + \int_0^t \frac{\partial^2 \lambda}{\partial s^2}\delta(u_n)(x,s)ds.
\end{aligned}$$

Thus, we have the next second order partial differential equation

$$\frac{\partial^2 \lambda}{\partial s^2} = 0,$$

according to conditions

$$\begin{aligned}
\lambda(t,t) &= 0, \\
1 - \frac{\partial \lambda}{\partial s}\Big|_{s=t} &= 0.
\end{aligned}$$

So,

$$\lambda(t,s) = s - t.$$

And the correction functional will be

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (s-t)\left((u_n)_{ss} - c^2 (u_n)_{xx}\right) ds \quad n \geq 0.$$

The zeroth approximation is

$$u_0(x,t) = f(x) + tg(x).$$

Consequently, the exact solution can be obtained by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

Let us consider the following example

Example 5.4. Consider the following wave equation, see [33]

$$u_{tt} = u_{xx}, \quad 0 \leq x \leq \pi, t \geq 0$$

with respect to the conditions

$$\begin{aligned} u(0,t) = u(\pi,t) &= 0, \\ u(x,0) &= 0, \\ u_t(x,0) &= \sin x. \end{aligned}$$

Taking

$$u_0(x,t) = t \sin x.$$

The successive iteration will be

$$
\begin{aligned}
u_1(x,t) &= u_0(x,t) + \int_0^t (s-t)\left((u_0)_{ss} - (u_0)_{xx}\right) ds \\
&= \sin x \left(t - \frac{1}{6}t^3\right) \\
u_2(x,t) &= u_1(x,t) + \int_0^t (s-t)\left((u_1)_{ss} - (u_1)_{xx}\right) ds \\
&= \sin x \left(t - \frac{1}{6}t^3 + \frac{1}{120}t^5\right) \\
u_3(x,t) &= u_2(x,t) + \int_0^t (s-t)\left((u_2)_{ss} - (u_2)_{xx}\right) ds \\
&= \sin x \left(t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7\right) \\
&\;\;\vdots \\
u_n(x,t) &= u_{n-1}(x,t) + \int_0^t (s-t)\left((u_{n-1})_{ss} - (u_{n-1})_{xx}\right) ds \\
&= \sin x \left(t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + \ldots + (-1)^n \frac{1}{(2n+1)!}t^{(2n+1)}\right).
\end{aligned}
$$

Consequently, the exact solution can be obtained by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

Note that this iteration is an approximation of

$$u(x,t) = \sin x \sin t,$$

which is an exact solution.

5.3 Nonlinear partial differential equations

In this section, we employ the VIM for solving nonlinear partial differential equations, see [25, 28, 29]

Consider the following nonlinear gas dynamic equations

$$u_t + \frac{1}{2}(u^2)_x = u(1-u), \quad 0 \le x \le 1, \;\; t \ge 0$$

with condition:

$$u(x,0) = g(x),$$

where $u(x,t)$ represent the velocity, see [29].

By constructing correction functional with using Lagrange multiplier, we get

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,s)\left[(u_n)_s + \frac{1}{2}(u_n^2)_x - u_n + u_n^2)\right] ds,$$

Taking the variation with respect to $u_n(t)$, we obtain

$$\begin{aligned} \delta u_{n+1}(x,t) &= \delta u_n(x,t) + \delta\int_0^t \lambda(t,s)\left[(u_n)_s + \frac{1}{2}(u_n^2)_x - u_n + u_n^2)\right] ds \\ &= \delta u_n(x,t) + \int_0^t \lambda(t,s)\delta\left[(u_n)_s + \frac{1}{2}(u_n^2)_x - u_n + u_n^2)\right] ds \\ &= \delta u_n(x,t) + \int_0^t \lambda(t,s)\left[\delta(u_n)_s + \frac{1}{2}\delta(u_n^2)_x - \delta u_n + \delta u_n^2)\right] ds. \end{aligned}$$

Note that $(u_n^2)_x$ and u_n^2 are restricted variables, i.e.

$$\begin{aligned} \delta(u_n^2)_x &= 0, \\ \delta u_n^2 &= 0, \end{aligned}$$

Now, we have

$$\begin{aligned}\delta u_{n+1}(x,t) &= \delta u_n(x,t) + \int_0^t \lambda(t,s)\left[\delta(u_n)_s - \delta u_n\right] ds \\ &= \delta u_n(x,t) + \int_0^t \lambda(t,s)\delta(u_n)_s ds - \int_0^t \lambda(t,s)\delta u_n ds\end{aligned}$$

Integrating the first integral by parts, we get

$$\begin{aligned}\delta u_{n+1}(x,t) &= \delta u_n(x,t) + \lambda(t,s)\delta u_n(x,s)\Big|_0^t - \int_0^t \frac{\partial \lambda}{\partial s}(s,t)\delta u_n ds - \int_0^t \lambda \delta u_n ds \\ &= \delta u_n(x,t) + \lambda(t,t)\delta u_n(x,t) - \lambda(t,0)\delta u_n(x,0) \\ &\qquad - \int_0^t \frac{\partial \lambda}{\partial s}(t,s)\delta u_n ds - \int_0^t \lambda \delta u_n ds \\ &= \Big(1+\lambda(t,t)\Big)\delta u_n(x,t) + \int_0^t \left[-\frac{\partial \lambda(s,t)}{\partial s} - \lambda(s,t)\right]\delta u_n ds.\end{aligned}$$

According to the last equations, we have the following first order differential equation

$$-\frac{\partial \lambda(s,t)}{\partial s} - \lambda(s,t) = 0.$$

with condition

$$1 + \lambda(t,t) = 0.$$

Thus, we have

$$\lambda(t,s) = -e^{t-s}.$$

Now, we have the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t e^{t-s}\left[(u_n)_s + \frac{1}{2}(u_n^2)_x - u_n + u_n^2)\right] ds, \tag{5.6}$$

and the initial approximation can be chosen such that it satisfies the initial condition.

Then the exact solution can be obtained by

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

To demonstrate the effectiveness of the method, we consider the next example.

Example 5.5. Consider the nonlinear partial differential equation

$$u_t + \frac{1}{2}(u^2)_x = u(1-u), \qquad 0 \le x \le 1, \quad t \ge 0.$$

with the next initial condition

$$u(x,0) = e^{-x}.$$

We can use it as initial approximation as

$$u_0(x,t) = u(x,0) = e^{-x}.$$

According to (5.6), we obtain the following successive approximation

$$\begin{aligned} u_1(x,t) &= u_0(x,t) - \int_0^t e^{t-s}\left[(u_0)_s + \frac{1}{2}(u_0^2)_x - u_0 + u_0^2\right] ds \\ &= e^{t-x} \\ u_2(x,t) &= u_1(x,t) - \int_0^t e^{t-s}\left[(u_1)_s + \frac{1}{2}(u_1^2)_x - u_1 + u_1^2\right] ds \\ &= e^{t-x} \\ &\vdots \end{aligned}$$

It is clear that these iterations converge to the exact solution

$$u(x,t) = e^{t-x}.$$

We can use the VIM to solve special equations such as Kawahara equation as we can see in the next example.

Example 5.6. Consider the following meromorphic travelling wave equation which called the Kawahara equation, see Saadatmandi and Dehghan [28]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}-\frac{\partial^5 u}{\partial x^5}=0. \tag{5.7}$$

subject to the initial condition

$$u(x,0)=f(x), \quad x\in\mathbb{R}$$

where u is a scalar real valued function.

The Kawahara equation models the plasma waves and the capillary gravity water waves, we aim to find an approximate solution for this equation using the VIM. For (5.7), we have the next correction functional

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda(t,s)\left[\frac{\partial u_n}{\partial s}+u_n\frac{\partial u_n}{\partial x}+\frac{\partial^3 u_n}{\partial x^3}-\frac{\partial^5 u_n}{\partial x^5}\right]ds$$

By taking the variation with respect to u_n, we obtain

$$\begin{aligned}\delta u_{n+1}(x,t) &= \delta u_n(x,t)+\delta\int_0^t \lambda(t,s)\left[\frac{\partial u_n}{\partial s}+u_n\frac{\partial u_n}{\partial x}+\frac{\partial^3 u_n}{\partial x^3}-\frac{\partial^5 u_n}{\partial x^5}\right]ds\\ &= \delta u_n(x,t)+\int_0^t \lambda(t,s)\left[\delta\frac{\partial u_n}{\partial s}+\delta\left(u_n\frac{\partial u_n}{\partial x}\right)+\delta\frac{\partial^3 u_n}{\partial x^3}-\delta\frac{\partial^5 u_n}{\partial x^5}\right]ds\end{aligned}$$

we deal with

$$u_n\frac{\partial u_n}{\partial x},\ \frac{\partial^3 u_n}{\partial x^3},\ \text{and}\ \frac{\partial^5 u_n}{\partial x^5}$$

as restricted variables, i.e.

$$\begin{aligned}\delta(u_n\frac{\partial u_n}{\partial x}) &= 0,\\ \delta\frac{\partial^3 u_n}{\partial x^3} &= 0,\\ \delta\frac{\partial^5 u_n}{\partial x^5} &= 0.\end{aligned}$$

Then, we have

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\int_0^t \lambda(t,s)\left[\delta\frac{\partial u_n}{\partial s}\right]ds.$$

Integrating by parts, we get

$$\begin{aligned}
\delta u_{n+1}(x,t) &= \delta u_n(x,t) + \lambda(t,s)\delta u_n(s)\Big|_0^t - \int_0^t \frac{\partial}{\partial s}\lambda(s,t)\delta u_n ds \\
&= \delta u_n(x,t) + \lambda(t,t)\delta u_n(t) - \lambda(t,0)\delta u_n(0) - \int_0^t \frac{\partial}{\partial s}\lambda(t,s)\delta u_n ds \\
&= (1+\lambda(t,t))\delta u_n(t) - \int_0^t \frac{\partial}{\partial s}\lambda(t,s)\delta u_n ds
\end{aligned}$$

Thus we have this first order PDE

$$\frac{\partial}{\partial s}\lambda(t,s) = 0,$$

with respect to

$$1 + \lambda(t,t) = 0.$$

Hence,

$$\lambda(t,s) = -1,$$

and the iteration formula is

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[\frac{\partial u_n}{\partial s} + u_n\frac{\partial u_n}{\partial x} + \frac{\partial^3 u_n}{\partial x^3} - \frac{\partial^5 u_n}{\partial x^5}\right] ds.$$

We can use initial condition as an initial approximation, i.e.

$$u_0(x,t) = u(x,0) = f(x)$$

and clearly the exact solution can be

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

If for example, we take

$$u(x,0) = -\frac{72}{169} + \frac{105}{169}\mathrm{sech}^4\left(\frac{1}{2\sqrt{13}}x\right)$$

x/t	0.1	0.2	0.3	0.4	0.5
0.1	$2.2 * 10^{-9}$	$3.89 * 10^{-8}$	$1.957 * 10^{-7}$	$6.076 * 10^{-7}$	0.0000014478
0.2	$2.1 * 10^{-9}$	$3.82 * 10^{-8}$	$1.911 * 10^{-7}$	$5.924 * 10^{-7}$	0.0000014116
0.3	$2.3 * 10^{-9}$	$3.70 * 10^{-8}$	$1.828 * 10^{-7}$	$5.681 * 10^{-7}$	0.0000013519
0.4	$2.1 * 10^{-9}$	$3.44 * 10^{-8}$	$3.44 * 10^{-8}$	$5.340 * 10^{-7}$	0.0000012696
0.5	$1.8 * 10^{-9}$	$3.18 * 10^{-8}$	$1.595 * 10^{-7}$	$4.920 * 10^{-7}$	0.0000011674

Table 5.2: The absolute error between u_3 and the exact solution.

The successive approximation will be made by Maple as

$$\begin{aligned}
u_0(t) &= -\frac{72}{169} + \frac{105}{169}\mathrm{sech}^4(\frac{1}{2\sqrt{13}}x) \\
u_1(x,t) &= u_0(x,t) - \int_0^t \left[\frac{\partial u_0}{\partial s} + u_0\frac{\partial u_0}{\partial x} + \frac{\partial^3 u_0}{\partial x^3} - \frac{\partial^5 u_0}{\partial x^5}\right] ds \\
&= -0.4260355030 + 0.6213017751\mathrm{sech}^4(\frac{1}{2\sqrt{13}}x) \\
&\quad - 0.2875367250\mathrm{sech}^4(\frac{1}{2\sqrt{13}}x)\tanh(\frac{1}{2\sqrt{13}}x)t \\
&\quad + 0.2141230931\mathrm{sech}^8(\frac{1}{2\sqrt{13}}x)\tanh(\frac{1}{2\sqrt{13}}x)t \\
&\quad + 0.4282461860\mathrm{sech}^4(\frac{1}{2\sqrt{13}}x)\tanh^3(\frac{1}{2\sqrt{13}}x)t \\
&\quad - 0.2141230930\mathrm{sech}^4(\frac{1}{2\sqrt{13}}x)\tanh^5(\frac{1}{2\sqrt{13}}x)t \\
&\vdots
\end{aligned}$$

It is clear that we have a huge number of computations, we write only one iteration. Consequently, the approximate solution will converge to the exact solution, i.e.

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

Note that the exact solution is

$$u(x,t) = \frac{-72}{169} + \frac{105}{169}sech^4\left(\frac{1}{2\sqrt{13}}(x + \frac{36}{169}t)\right).$$

Table (5.6) represent the absolute error between u_3 and the exact solution.

For more examples, see [9, 12, 25, 28, 29].

x/t	0.1	0.2	0.3	0.4	0.5
0.1	$2.2 * 10^{-9}$	$3.89 * 10^{-8}$	$1.957 * 10^{-7}$	$6.076 * 10^{-7}$	0.0000014478
0.2	$2.1 * 10^{-9}$	$3.82 * 10^{-8}$	$1.911 * 10^{-7}$	$5.924 * 10^{-7}$	0.0000014116
0.3	$2.3 * 10^{-9}$	$3.70 * 10^{-8}$	$1.828 * 10^{-7}$	$5.681 * 10^{-7}$	0.0000013519
0.4	$2.1 * 10^{-9}$	$3.44 * 10^{-8}$	$3.44 * 10^{-8}$	$5.340 * 10^{-7}$	0.0000012696
0.5	$1.8 * 10^{-9}$	$3.18 * 10^{-8}$	$1.595 * 10^{-7}$	$4.920 * 10^{-7}$	0.0000011674

Table 5.2: The absolute error between u_3 and the exact solution.

The successive approximation will be made by Maple as

$$\begin{aligned}
u_0(t) &= -\frac{72}{169} + \frac{105}{169}\text{sech}^4(\frac{1}{2\sqrt{13}}x) \\
u_1(x,t) &= u_0(x,t) - \int_0^t \left[\frac{\partial u_0}{\partial s} + u_0 \frac{\partial u_0}{\partial x} + \frac{\partial^3 u_0}{\partial x^3} - \frac{\partial^5 u_0}{\partial x^5} \right] ds \\
&= -0.4260355030 + 0.6213017751\text{sech}^4(\frac{1}{2\sqrt{13}}x) \\
&\quad - 0.2875367250\text{sech}^4(\frac{1}{2\sqrt{13}}x)\tanh(\frac{1}{2\sqrt{13}}x)t \\
&\quad + 0.2141230931\text{sech}^8(\frac{1}{2\sqrt{13}}x)\tanh(\frac{1}{2\sqrt{13}}x)t \\
&\quad + 0.4282461860\text{sech}^4(\frac{1}{2\sqrt{13}}x)\tanh^3(\frac{1}{2\sqrt{13}}x)t \\
&\quad - 0.2141230930\text{sech}^4(\frac{1}{2\sqrt{13}}x)\tanh^5(\frac{1}{2\sqrt{13}}x)t \\
&\quad \vdots
\end{aligned}$$

It is clear that we have a huge number of computations, we write only one iteration. Consequently, the approximate solution will converge to the exact solution, i.e.

$$u(x,t) = \lim_{n\to\infty} u_n(x,t).$$

Note that the exact solution is

$$u(x,t) = \frac{-72}{169} + \frac{105}{169}\text{sech}^4\left(\frac{1}{2\sqrt{13}}(x + \frac{36}{169}t)\right)$$

Table (5.3) represent the absolute error between u_3 and the exact solution.

For more examples, see [9, 17, 25, 28, 29].

5.4 Laplace variational iteration method

In this section we use a new method for solving PDE's which is the Laplace variational iteration method. The main concept of this method is to find the value of Lagrange multiplier using an alternative Laplace correction functional and express the integral as a convolution, see Hilal and Elzaki [17]. Let present the following definition for the Laplace transform.

Definition 5.1. let f be a function defined for $t \geqslant 0$. Then the integral

$$\ell[f(t)] = \int_0^\infty e^{-\xi t} f(t)dt = F(\xi)$$

is said to be the Laplace transform of f provided the integral converges.

Consider the following second order differential equation

$$\frac{\partial u(x,t)}{\partial t} + N[u(x,t)] = g(x,t),$$

where N is a nonlinear operater, and $g(x,t)$ is a known function. Hence, by constructing a correction functional, we get

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,s)\left[\frac{\partial u_n}{\partial s} + N[u(x,s)] - g(x,s)\right] ds.$$

Taking the variation with respect to $u_n(x,t)$, we obtain

$$\begin{aligned}
\delta u_{n+1}(x,t) &= \delta u_n(x,t) + \delta \int_0^t \lambda(t,s)\left[\frac{\partial u_n}{\partial s} + N[u(x,s)] - g(x,s)\right] ds \\
&= \delta u_n(x,t) + \int_0^t \lambda(t,s)\delta\left[\frac{\partial u_n}{\partial s} + N[u(x,s)] - g(x,s)\right] ds \\
&= \delta u_n(x,t) + \int_0^t \lambda(t,s)\left[\delta\frac{\partial u_n}{\partial s} + \delta N[u(x,s)] - \delta g(x,s)\right] ds
\end{aligned}$$

As we study before, we have $N[u(x,s)]$ and $g(x,s)$ as a restrict variables, i.e.

$$\begin{aligned}\delta N[u(x,s)] &= 0,\\ \delta g(x,s) &= 0.\end{aligned}$$

Let

$$\overline{\lambda}(t-s,s)=\lambda(t,s).$$

So,

$$\delta u_{n+1}(x,t)=\delta u_n(x,t)+\delta\int_0^t \overline{\lambda}(t-s,s)\frac{\partial u_n}{\partial s}ds$$

by taking the Laplace transform with respect to t, the correction functional will be

$$\begin{aligned}\ell[\delta u_{n+1}(x,t)] &= \ell[\delta u_n(x,t)]+\ell\left[\delta\int_0^t \overline{\lambda}(t-s,s)\frac{\partial u_n}{\partial s}ds\right]\\ &= \ell[\delta u_n(x,t)]+\delta\ell\left[\int_0^t \overline{\lambda}(t-s,s)\frac{\partial u_n}{\partial s}ds\right]\\ &= \ell[\delta u_n(x,t)]+\delta\ell\left[\overline{\lambda}(t,s)*\frac{\partial u_n}{\partial s}\right]\end{aligned}$$

which $*$ is a single convolution with respect to t. Then,

$$\begin{aligned}\ell[\delta u_{n+1}(x,t)] &= \ell[\delta u_n(x,t)]+\delta\ell\left[\overline{\lambda}(t,s)\right]\ell\left[\frac{\partial u_n}{\partial s}\right]\\ &= \ell[\delta u_n(x,t)]+\delta\ell\left[\overline{\lambda}(t,s)\right][\xi\ell[u_n]]\\ &= \ell[\delta u_n(x,t)]+\ell\left[\overline{\lambda}(t,s)\right][\xi\ell[\delta u_n]] \qquad (5.8)\end{aligned}$$

To find the value of $\overline{\lambda}(t,s)$, $\ell[\delta u_{n+1}(x,t)]$ should be 0. According to this condition, (5.8) will be

$$0=\left[1+\ell[\overline{\lambda}(t,s)]\xi\right],$$

hence,

$$\ell[\overline{\lambda}(t,s)] = \frac{-1}{\xi},$$

and by Laplace inverse, we get

$$\overline{\lambda}(t,s) = -1$$

also

$$\lambda(t,s) = \overline{\lambda}(t-s,s) = -1$$

So, the successive iteration will be

$$\ell[u_{n+1}(x,t)] = \ell[u_n(x,t)] - \ell\left[\int_0^t \left[\frac{\partial u_n}{\partial s} + N[u(x,s)] - g(x,s)\right] ds\right].$$

Consider the following example

Example 5.7. Let us consider the following first order homogeneous ordinary differential equation

$$\frac{\partial u}{\partial t} - 2tu = 0,$$

with initial condition

$$u(x,0) = 1.$$

Then according to the Laplace transform, we have the next Laplace iteration correction functional

$$\begin{aligned} \ell[u_{n+1}(x,t)] &= \ell[u_n(x,t)] + \ell\left[\int_0^t - \left[\frac{\partial u_n}{\partial s} - 2su_n(x,s)\right] ds\right]. \\ &= \ell[u_n(x,t)] + \ell[-1]\ell\left[\frac{\partial u_n}{\partial t} - 2tu_n(t)\right] \end{aligned}$$

Taking

$$u_0(t) = u(x,0) = 1,$$

we obtain

$$\begin{aligned}
\ell[u_1(x,t)] &= \ell[u_0(x,t)] + \ell[-1]\ell\left[\frac{\partial u_0}{\partial t} - 2tu_0(t)\right] \\
&= \ell[1] + \ell[-1]\ell[-2t] \\
&= \frac{1}{\xi} + \frac{-1}{\xi}\frac{-2}{\xi^2} \\
&= \frac{1}{\xi} + \frac{2}{\xi^3}.
\end{aligned}$$

Then, the inverse transform yields

$$u_1(t) = 1 + t^2.$$

by the same way, we obtain

$$\begin{aligned}
\ell[u_2(x,t)] &= \ell[u_1(x,t)] + \ell[-1]\ell\left[\frac{\partial u_1}{\partial t} - 2tu_1(t)\right] \\
&= \frac{1}{\xi} + \frac{2}{\xi^3} + \frac{-1}{\xi}\ell[-2t^3] \\
&= \frac{1}{\xi} + \frac{2}{\xi^3} + \frac{-1}{\xi}\left(\frac{-12}{\xi^4}\right) \\
&= \frac{1}{\xi} + \frac{2}{\xi^3} + \frac{12}{\xi^5}
\end{aligned}$$

Also, by inverse transform, we get

$$u_2(t) = 1 + t^2 + \frac{1}{2!}t^4.$$

If we continue at the same manner, we have

$$u_n(t) = 1 + t^2 + \frac{t^4}{2!} + \ldots + \frac{t^{2n}}{n!},$$

which is an approximation to the exact solution

$$u(t) = e^{t^2}.$$

For more examples, see Hilal and Elzaki [17].

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