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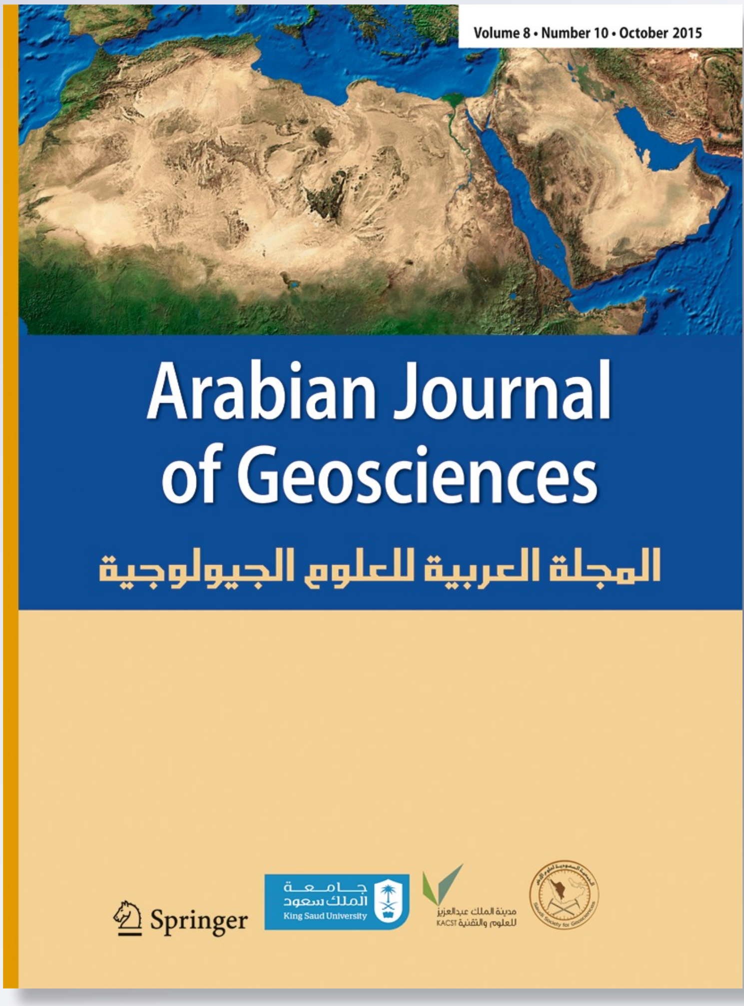

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ORIGINAL PAPER

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Abstract In Geodesy, the heights of points are normally orthometric heights measured above the geoid (an equipotential surface created by the earth masses and rotation which approximately coincides with the mean sea level) or the normal heights. It is necessary to transform the GNSS/GPS measured ellipsoidal heights (h) to classical physical heights (orthometric H/Normal H). The total gravity potential of the earth (W) is the summation of two components; gravitational potential (V) by earth masses and the centrifugal potential (Ω). The centrifugal potential is directly calculated, while the gravitational potential (V) needs to be modeled globally or locally using given measurements. The global models of the earth gravitational potential/gravity models (or so-called geoid models) are mostly given using spherical harmonics (SH). A modified approach of SH was defined to fit the use of SH for regional gravity/potential modeling called spherical cap harmonics (SCH). Due to the numerical difficulties of SCH, a simplified approach of SCH is selected to be used for a combined modeling of the earth potential using a variety of observations. This approach is called the Adjusted Spherical Cap harmonics.

Keywords Earth potential (W) · Gravitational potential (V) · Centrifugal potential (Ω) · Ellipsoidal heights (h) · Physical/orthometric height (H) · Normal height (H^N) · Geoid · Spherical harmonics · Spherical cap harmonics · Adjusted spherical cap harmonics

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Earth gravity potential

There are two types of forces (accelerations) affecting a point P on the Earth's surface, see Fig. 1. These types are the gravitational acceleration $\vec{g}_1$ due to the Earth's mass M and the centrifugal acceleration $\vec{z}$ due to the Earth's rotation. The total acceleration $\vec{g}$, representing the actual gravity vector, is the vector summation of both gravitational and centrifugal accelerations (Fan 2004):

$$\vec{g} = \vec{g}_1 + \vec{z} \tag{1}$$

The total earth potential (W) as result of the earth gravity $\vec{g}$ is the summation of potential related to the two acceleration components. These potential components are the gravitational potential V and the centrifugal potential Ω. This total gravity potential is given by:

$$W = V + \Omega \tag{2}$$

As the angular velocity ω of the Earth around its minor axis is $0.7292115 \times 10^{-4} s^{-1}$as defined by the GRS80 or WGS84 (Hofmann-Wellenhof and Moritz 2005), the centrifugal potential reads:

$$\Omega = 0.5\,\omega^2 r^2 \cos\overline{\phi} = \frac{1}{2}\omega^2\left(X^2 + Y^2\right) \tag{3}$$

In Eq. (3), $\overline{\phi}$ is spherical latitude of the point, r is the radial distance between the point P and the center of the earth.

The centrifugal potential at a point is directly calculated. In the other hand, the gravitational potential needs to be modeled using variety of observations. Most of global gravity models are using the spherical harmonics to model the gravitational potential (V). Examples of these gravity models are EGM2008, Eigen06c, and EGM96. The coefficients of these

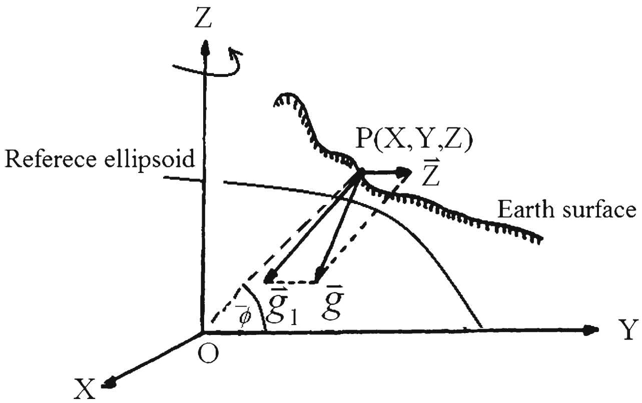

Fig. 1 The gravitational and centrifugal accelerations of the Earth (Younis 2013)

models are calculated using combinations of satellite gravity observations, satellite altimetry, terrestrial gravity, airborne gravity, marine gravity, and height fitting points. In terms of SH, the potential (V) reads (Fan 2004):

$$V(r,\bar{\phi},\lambda)=\frac{GM}{r}+\frac{GM}{r}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^{n}\sum_{m=0}^{n}\left(\bar{C}_{nm}\cos m\lambda+\bar{S}_{nm}\sin m\lambda\right)\bar{P}_{nm}\left(\sin\bar{\phi}\right) \quad (4)$$

In Eq. (4), GM is the gravitational constant of the earth. $\overline{C}_{nm}$ and $\overline{S}_{nm}$ are the fully normalized spherical harmonic coefficients. The values of m and n are the integer degree and order. $(r,\bar{\phi},\lambda)$are the spherical coordinates of the point. While $P^{-}_{nm}(\sin\overline{\phi})$ is the fully normalized Legendre function. $\overline{P}_{nm}(\sin\overline{\phi})$can be calculated by the recursive formulas (5), with the abbreviations $t=\sin\overline{\phi}$ and $u=\cos\overline{\phi}$ (Holmes and Featherstone 2002) as follows:

$$\overline{P}_{n,m}=a_{nm}t\overline{P}_{n-1,m}-b_{nm}\overline{P}_{n-2,m} \quad (5a)$$

$$a_{nm}=\sqrt{\frac{(2n-1)(2n+1)}{(n-m)(n+m)}} \quad (5b)$$

$$b_{nm}=\sqrt{\frac{(2n+1)(n+m-1)(n-m-1)}{(n-m)(n+m)(2n-3)}} \quad (5c)$$

$$\overline{P}_{0,0}=1\quad,\quad\overline{P}_{1,0}=\sqrt{3}t\quad,\overline{P}_{1,1}=\sqrt{3}u \quad (5d)$$

If n=m, then $\overline{P}_{n,m}$ reads:

$$\overline{P}_{m,m}=u\sqrt{\frac{2m+1}{2m}}\overline{P}_{m-1,\,m-1} \quad (5e)$$

Spherical Cap harmonics

The method of Spherical Cap Harmonics (SCH) (S'_{nm}, C'_{nm}) for modeling the gravity potential was introduced by Haines (1985a). This method is suitable to be used for modeling the gravitational potential V in a local cap area covering a region of interest on the sphere instead of the whole sphere, see Fig. 2. The position of points in the cap region is described by a spherical coordinates (α,θ,r) related to the cap pole. Here,α is the azimuth of the spherical line from the cap pole $\lambda_0,\overline{\phi}_0,R$ to the point.θ is the spherical distance from the cap pole $(\lambda_0,\overline{\phi}_0,R)$ to point P. The relationship between global coordinates and local coordinates reads (Younis et al 2011):

$$\tan\alpha=\frac{\cos\overline{\phi}\sin(\lambda-\lambda_0)}{\sin\overline{\phi}\cos\overline{\phi}_0-\cos\overline{\phi}\sin\overline{\phi}_0\cos(\lambda-\lambda_0)} \quad (6a)$$

$$\cos\theta=\sin\overline{\phi}\sin\overline{\phi}_0-\cos\overline{\varphi}\cos\overline{\varphi}_0\cos(\lambda-\lambda_0) \quad (6b)$$

The gravitational potential V in terms of SCH for a point P (r,α,θ)within the cap reads (Haines 1988):

$$V(r,\alpha,\theta)=\frac{GM}{r}\sum_{k=0}^{kmax}\left(\frac{R}{r}\right)^{n(k)}\sum_{m=0}^{k}\left(C'_{nm}\cos m\alpha+S'_{nm}\sin m\alpha\right)\overline{P}_{n(k),m}(\cos\theta) \quad (7)$$

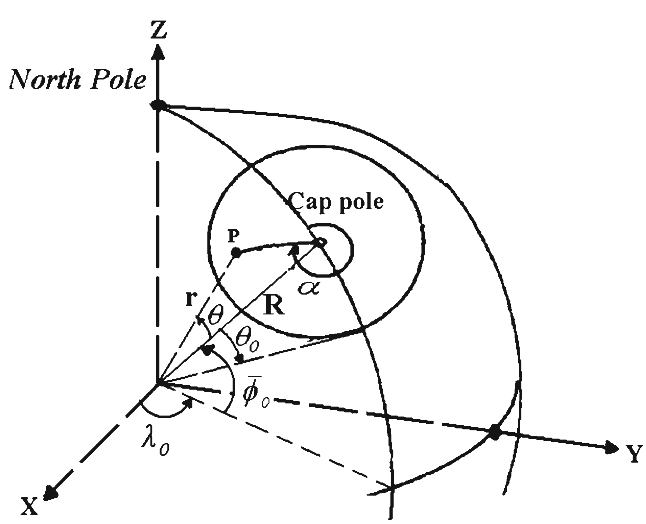

Fig. 2 Spherical cap area definition (Younis et al 2011)

In Eq. (7), the SCH have an integer order m and a real degree *n(k)*. The real degree *n(k)* is the root of the Legendre functions according to the orthogonality conditions (8a) and (8b) (Haines 1985b).

$$\left.\frac{dP_{n(k),m}(\cos\theta)}{d\theta}\right|_{\theta_0} = 0 \quad \text{for k−m} = \text{even} \tag{8a}$$

$$P_{n(k),m}(\cos\theta)\big|_{\theta_0} = 0 \quad \text{for k−m} = \text{odd} \tag{8b}$$

In Eq. (8a and 8b), k is the integer degree and m is the order. θ_0 is the angular spherical distance from the pole of the cap area to the boundary of the area of interest.

The Legendre function with the real degree *n(k)* and the integer m is not commonly available in the geodetic and the geophysical literature, as it is in the case of integer degree and order in Eq. (5a) to (5e). It is defined by an infinite power series, which must be elaborated iteratively introducing certain approximations (Haines 1988); these will introduce additional errors, otherwise, complex algorithms must be used (Oliver and Smith 1983). The search for the real degrees *n(k)* according to the conditions in Eq. (8a and 8b) must be performed. The algorithms to search for the roots of the Legendre functions are non-direct or iterative resulting on additional errors. These algorithms are also time consuming (De Santis et al. 1999). Furthermore, the calculations of Legendre functions and their derivatives with non-integer degrees are again a time consuming iterative process (Schneid 2006).

Adjusted spherical cap harmonics

The Adjusted Spherical Cap Harmonics is a modified approach of SCH that was introduced by De Santis (1992). Its aim is to avoid the iterative and approximate methods in SCH. This approach uses the well-known integer order and degree Legendre functions. The principle enlarges the cap area in Fig. 2 to a hemisphere using Eq. (9a) to (9d), where the pole of the hemisphere is also the pole of the cap itself (Franceschi and De Santis 1994). The scaling of the spherical cap coordinates (α,θ,r) to a hemisphere (adjusted cap) (α',θ',r') reads

$$s = \frac{0.5\pi}{\theta_0} \tag{9a}$$

$$\vartheta = s\ \theta \tag{9b}$$

$$\alpha' = \alpha \tag{9c}$$

$$r' = r \tag{9d}$$

According to the ASCH definition in Eq. (9), The new formula is similar to the conventional SH model in Eq. (4). Here, Legendre function of integer degree and order is used. Equation (7) is then modified to get the potential (V) using Eq. (10) (De Santis 1992).

$$V(r,\alpha,\vartheta) = \frac{GM}{r}\sum_{k=0}^{k\max}\left(\frac{R}{r}\right)^{n(k)}\sum_{m=0}^{k}\left(C'_{\mathrm{knm}}\cos m\alpha + S'_{\mathrm{km}}\sin m\alpha\right)\overline{P}_{\mathrm{km}}(\cos\vartheta) \tag{10}$$

The Legendre function $\overline{P}_{\mathrm{km}}(\cos\vartheta)$of integer degree k and order m in Eq. (5) is used. In Eq. (10), n is a real number. It can be calculated as function of k (*n(k)*). *n(k)* reads according to De Santis and Torta (1997):

$$n(k) = \sqrt{s^2k(k+1)+0.25} - 0.5 \tag{11}$$

In Eq. (11), s is the scale factor computed from Eq. (9a), k is the degree parameter in the ASCH model. For low degree and order ASCH models, there is an approximate formula of Eq. (11) that may be used, following De Santis et al. (1997) reading:

$$n(k) = s(k+0.5) \tag{12}$$

Compared to SCH in Eq. (7), the ASCH in Eq. (10) has the following advantages: First, the well-known Legendre function with its recursive formulas is used. Second, there is no need to search for the roots *n(k)* of Legendre functions and their derivatives according to the conditions in Eq. (8a) and (8a), which is time consuming (De Santis 1992).

Results of a combined solution

In the state of Baden–Württemberg in Germany, 15,000 terrestrial gravity points were available. These points are assumed to have measurement accuracy of 0.01 mGal (1 mGal=1×10^{-5}ms^{-2}) (Torge 2001). The radial component of gravity measurements is used. While the radial component of the gravity (g_r) is the first radial derivative of the gravitational potential (V) (Hofmann-Wellenhof and Moritz 2005):

$$g_r = \frac{\partial V}{\partial r} \tag{13}$$

In addition to the gravity data, 130 points were used as height fitting points. These have known normal/orthometric heights (H) (physical height). Their heights above the ellipsoid (h) were measured using GNSS. Their estimated measurement accuracy is 0.01–0.02 m. The height anomaly (ζ) is the difference between both heights ($\zeta=h_P-H^N$). The geoid height (N) is the difference between the ellipsoidal and the

Table 1 The residuals of the combined ASCH solution in Baden–Württemberg with maximum degree of 300

Parameter	Gravity points	Height fitting points
Number of points	13,671	129
RMSE	0.0032 mGal	0.8 cm
Maximum residual	0.032 mGal	3.3 cm
Minimum residual	0.041 mGal	-2.4 cm

orthometric heights ($N=h_P-H$). These anomalies can be transformed to potential values using the ellipsoidal normal potential (U) and the ellipsoidal normal gravity (γ). The ellipsoidal potential has two components, the gravitational potential (V') and the centrifugal potential (Ω). The relation between height anomaly (ζ)/geoid height (N) and potential read (Flury and Rummel 2009):

$$\zeta = \frac{T_{\mathrm{P}}}{\gamma_{\mathrm{q}}} \tag{14a}$$

$$N = \zeta + \frac{\overline{g}+\overline{\gamma}}{\overline{\gamma}}H \tag{14b}$$

In Eq. (14a and 14b), γ_{q} is the ellipsoidal normal gravity at the height of ($h=H^N$), $\overline{g}$ and $\overline{\gamma}$ are the mean gravity and normal gravity of the point, respectively. T_{P} is the disturbing potential at the point, which reads (Torge 2001):

$$T_{\mathrm{P}} = (V_{\mathrm{P}} + \Omega_{\mathrm{P}}) - \left(V'_{\mathrm{P}} + \Omega_{\mathrm{P}}\right) = W_{\mathrm{P}} - U_{\mathrm{P}} \tag{15}$$

Substituting Eq. (14) in Eq. (15), the potential of the height fitting point reads (Younis 2013):

$$V_{\mathrm{P}} = W_{\mathrm{P}} - \Omega = U_{\mathrm{P}} + \gamma_{\mathrm{q}}\zeta - \Omega \tag{16}$$

The calculation of the ASCH coefficients (S'_{nm}, C'_{nm}) in a least squares would need higher degree and order to achieve an ASCH model that represents the gravity measurement accuracy of 0.01 mGal or fitting points accuracy of 0.02 m. Here, we have to consider that having a maximum degree and order n of the model will have $(n+1)^2$ coefficients (S'_{nm}, C'_{nm}). To get more observations, global gravity models can be used as additional observations. The observations can be used directly as potential values (V) in 3D grid in the cap area or as a transformed model in the cap area. Here, the transformed coefficients with stochastic model are used as direct observations (Younis et al 2011).

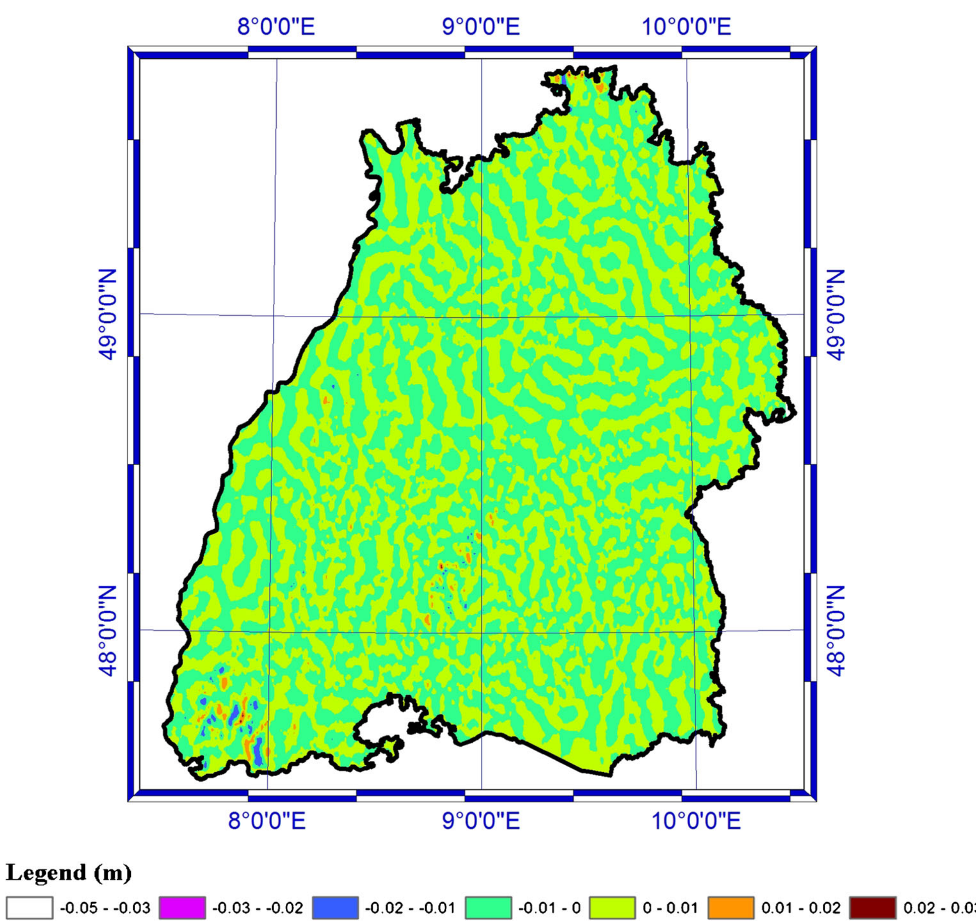

Fig. 3 The difference between height anomalies using ASCH combined model and the 1 cm-DFHRS-DB

The final solution was applied with a maximum degree and order of 300. The number of unknowns/coefficients (S'_{nm}, C'_{nm}) were 90,601. The maximum absolute residuals in gravity data was less than 0.05 mGal. The residuals of the height fitting points were less than 4 cm, see Table 1.

The height anomalies using Eq. (14) were calculated for a grid of points using the final model and the compared to the official anomalies from the height reference surface model of Baden–Württemberg (DFHRS-DB) as across validation of the model. The difference was all over the state in the range of (−2.5–3 cm). The differences are shown in Fig. 3.

Conclusions

The potential of the earth could be modeled in a local area using ASCH. The coefficients (S'_{nm}, C'_{nm}) of the Eq. (10) could be calculated using least squares solutions. This solution was applied using a combination of different data types as observations. These data are terrestrial gravity data, height fitting points, and global gravity models. Other types of observations can be used like deflections of vertical measured by astronomical methods and Zenith cameras as introduced and derived by Younis (2013). The use of ASCH enables to calculate the potential (V) at a given point in the cap area or any of its derived quantities; gravity, gravity anomalies, gravity disturbances, geoid heights, height anomalies, and the deflections of the vertical. These quantities can be estimated with a high accuracy as it was proofed in the example of state of Baden–Württemberg in Germany. These values are highly needed in geodetic, geophysical, and geological applications.

The use of ASCH modeling in Eq. (10) is applied similar to the conventional global SH in Eq. (4). Here, the Legendre function of integer degree and order is used. In this way, the problems and difficulties of the SCH in Eq. (7) and the boundary conditions in Eq. (8) to calculate the Legendre function with the real degree and integer order and their roots are avoided. In the other side, problems and instability of the calculations appear near the exterior border of the cap. That is because there are no available observations outside the cap, but all observations are completely inside the cap. A simple solution is applied by using a cap area larger the area of interest. For example, the state of Baden–Württemberg needs a cap opening angle of $\theta_0 = 1.425^{\circ}$. Instead, a cap opening angle of $\theta_0 = 1.7^{\circ}$ to avoid the problem near the boundary.

References

De Santis A (1992) Conventional spherical harmonic analysis for regional modeling of geomagnetic field. Geophys Res Lett 19(10):1065–1067

De Santis A, Torta J (1997) Spherical Cap harmonics analysis: a comment on its proper use for local gravity field representation. J Geod 71: 526–532

De Santis A, Chiappini M, Domonici G, Meloni A (1997) Regional geomagnetic field modeling: the contribution of the Instituto Nazionale de Geofisica, Annali Di Geophysica, Vol XL, No.5, 1161–1169

De Santis A, Torta J, Lowes F (1999) Spherical Cap harmonics revisited and their relationship to ordinary spherical harmonics. Phys Chem Earth A 24(11-12):935–941

Fan H (2004) Theoretical geodesy. Royal Institute of Technology, Stockholm

Flury J, Rummel R (2009) On the geoid-quasigeoid separation in mountain areas. J Geod 83(9):829–847

Franceschi G, De Santis A (1994) Ionospheric mapping by means of Spherical Harmonic Analysis: New Developments, Advances in Space Research, Issue 15

Haines G (1985a) Spherical Cap Harmonic Analysis, J Geophys Res, 90: B3, 2583-2591

Haines G (1985b) Spherical Cap Harmonic Analysis of Geomagnetic Secular Variation over Canada 1960-1983, J Geophys Res, 90 B14, 2593–2598

Haines G (1988) Computer programs for spherical cap harmonic analysis of potential and general fields. Comput & Geosci 14(4):413–447

Hofmann-Wellenhof B, Moritz H (2005) Physical geodesy. Springer-Verlag Wien, Austria

Holmes S, Featherstone W (2002) A unified approach to the clenshaw summation and recursive computation of very high degree and order normalized associated legendre functions. J Geod 76:279–299

Oliver F, Smith J (1983) Associated legendre functions on the cut. J Comput Phys 51:502–518

Schneid S (2006) Investigation of a Digital FEM Height Reference Surface as Vertical Reference Surface, PhD Thesis, University of Nottingham

Torge W (2001) Geodesy, 3rd edn. Walter de Gruyter, Berlin

Younis G (2013) Regional Gravity Field Modeling with Adjusted Spherical Cap Harmonics in an Integrated Approach, Schriftenreihe Fachrichtung Geodäsie der Technischen Universität Darmstadt (39). Darmstadt . ISBN 978-3-935631-28-0

Younis G, Jäger R, Becker M (2011) Transformation of global spherical harmonic models of the gravity field to a local adjusted spherical Cap harmonic model. AJGS. Springer, Berlin / Heidelberg. doi:10.1007/s12517-011-0352-1